

# NOTES ON COMPLEX NUMBERS

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ABSTRACT. Some notes on complex numbers for Math 53, a course in dynamical systems given at Swarthmore College during Spring 2011.

## 1. HISTORY AND MOTIVATION

1.1. **Natural numbers.** Numbers are invented to quantify things. The most primitive set of numbers is the set of *natural numbers*, denoted  $\mathbb{N} = \{0, 1, 2, \dots\}$ . (Some people start the natural numbers with 1, so for them  $\mathbb{N}$  is  $\{1, 2, 3, \dots\}$ .) What works well about the natural numbers is that they count whole numbers of objects, and that we can talk about adding and multiplying two natural numbers. What doesn't work well is that we can't always talk about subtracting two natural numbers, e.g. there is no natural number which is equal to  $3 - 7$ . Restated, you cannot solve the equation  $x + 7 = 3$  if your universe of discourse is the natural numbers.

1.2. **Integers.** To fix the subtraction problem, the *integers* are invented (or discovered, depending on your philosophical perspective). This set is denoted  $\mathbb{Z}$  and is the set  $\{\dots, -2, -1, 0, 1, 2, \dots\}$ . The integers do all the things the natural numbers do, and you can also subtract any two integers and get an integer as your answer. Unfortunately division doesn't work well with integers, e.g. there is no integer  $x$  which solves the equation  $2x = 3$ .

1.3. **Rational numbers.** To fix this, we consider the *rational numbers*, denoted  $\mathbb{Q}$  and described as the set of all fractions  $\frac{p}{q}$  where  $p$  and  $q$  are integers with no common divisors, and  $q \neq 0$ . What is great about the rational numbers is that you get all the pros of the integers and you can also divide one rational number by another (as long as the divisor is not zero) and get a rational number as your answer. What is not great about the rational numbers is that they are *incomplete*. Essentially this means that sequences of rational numbers that converge don't necessarily converge to rational numbers. For example, the sequence  $3, 3.1, 3.14, 3.1415, \dots$  is a sequence of rational numbers that converges to  $\pi$ , which is irrational. More formally, the Monotone Convergence Theorem for sequences (which says that a bounded, monotone sequence of numbers must converge) is invalid if your universe of discourse is the rational numbers.

1.4. **Real numbers.** To fix this, we take the *real numbers*, denoted  $\mathbb{R}$  and described as the set of all limits of bounded, convergent sequences of rational numbers (this sentence is actually a bit of a lie, but it's okay if you believe it). This set  $\mathbb{R}$  is complete (in that the Monotone Convergence Theorem holds) and the arithmetic operations of addition, subtraction, multiplication and division work well. However, the real numbers are unsatisfactory because polynomial equations with real

coefficients can't always be solved; e.g. the equation  $x^2 + 1 = 0$  has no solution if your universe of discourse is  $\mathbb{R}$ .

**1.5. Complex numbers.** To solve the polynomial equation above, we invent a solution to  $x^2 + 1 = 0$  called  $i$  and create the *complex numbers*, denoted  $\mathbb{C}$ . As a set, we have

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$$

and so formally, the complex numbers are a vector space of dimension 2 over the reals with basis  $\{1, i\}$ . However, the complex numbers are more than just vectors, because we define multiplication of complex numbers by setting  $i^2 = -1$  (see the next section). The complex numbers are great: they have addition, subtraction, multiplication, division (even by zero, as we'll see later), they are complete, and all polynomials with coefficients that are complex numbers have solutions which are complex numbers (this fact is called the Fundamental Theorem of Algebra). The only drawback of complex numbers is that they are not ordered (i.e. you can't say one complex number is less than or greater than another one). But this turns out not to be too much of a problem.

Historically speaking, complex numbers were first invented by Cardano in the 1500s as a "trick" to find real solutions to cubic equations with real coefficients. Basically, square roots of negative numbers pop up in Cardano's method but eventually drop out to give real answers. Cardano called these square roots of negative numbers "imaginary". The name stuck, but in fact complex numbers are no more imaginary than the numbers 1 or 2. We now know that complex numbers are essential in electrical engineering and for the theoretical formulation of quantum mechanics and special relativity.

Notice that  $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ . It is natural to wonder if there is an even bigger set than  $\mathbb{C}$  that is even better than  $\mathbb{C}$ . The answer is no; problems arise if you try this (in particular multiplication stops being commutative, so division is a big problem).

## 2. ARITHMETIC OF COMPLEX NUMBERS

**Definition 2.1.** *The set of complex numbers  $\mathbb{C}$  is defined by*

$$\mathbb{C} = \{z = x + iy : x, y \in \mathbb{R}\}.$$

*Given a complex number  $z = x + iy$ , the real part of  $z$ , denoted  $\Re(z)$ , is  $x$ , and the imaginary part of  $z$ , denoted  $\Im(z)$ , is  $y$ . Note that for  $z \in \mathbb{C}$ ,  $\Re(z)$  and  $\Im(z)$  **are real numbers**.*

For example,  $\Im(1 - 2i)$  is  $-2$ , not  $-2i$ . Usually a complex number is denoted by  $z, w$ , or a Greek letter like  $\zeta$  (zeta) or  $\xi$  (xi) or  $\omega$  (omega). The letters  $s, t, u, v, x$  and  $y$  should not be used to denote complex numbers; they connote real numbers. In particular it is always understood with complex numbers that "z" means the complex number  $z = x + iy$ . Although many proofs and definitions in this handout involve writing complex numbers like  $z$  as  $x + iy$ , in general you want to avoid immediately thinking of a complex number as  $x + iy$ . Just think of it as  $z$ .

**Definition 2.2.** *The complex conjugate of  $z = x + iy \in \mathbb{C}$  is  $\bar{z} = x - iy$ .*

Addition and subtraction in  $\mathbb{C}$  are defined in the “obvious way” by combining like terms. For example,

$$(2 - 3i) + (1 + i) = 3 - 2i \quad \text{and} \quad (-3 + i) - (2 + 7i) = -5 - 6i.$$

Multiplication is defined by distributing terms, together with the law that  $i^2 = -1$ . For example:

$$(2 + 5i)(-1 - 2i) = -2 - 5i - 4i - 10i^2 = -2 - 9i + 10 = 8 - 9i.$$

Division is trickier; to divide one complex number by a nonzero complex number, what you do is multiply through the numerator and denominator of the fraction by the conjugate of the denominator. An example:

$$(1 + i) \div (3 - 4i) = \frac{1 + i}{3 - 4i} = \frac{(1 + i)(3 + 4i)}{(3 - 4i)(3 + 4i)} = \frac{-1 + 7i}{25} = \frac{-1}{25} + \frac{7}{25}i.$$

The operations thus defined satisfy all the elementary arithmetic properties: addition and multiplication are commutative and associative, addition and multiplication have identity elements ( $0 = 0 + 0i$  and  $1 = 1 + 0i$  respectively); every element has an additive inverse; every nonzero element has a reciprocal; the distributive property holds. Moreover, we see the following:

**Proposition 2.3.** *Let  $z \in \mathbb{C}$ . Then  $\Re(z) = \frac{z + \bar{z}}{2}$  and  $\Im(z) = \frac{z - \bar{z}}{2i}$ .*

*Proof.* Every complex number  $z$  can be written as  $z = x + iy$ . For the first statement,

$$\frac{z + \bar{z}}{2} = \frac{(x + iy) + (x - iy)}{2} = \frac{2x}{2} = x = \Re(z).$$

The other statement is similar. □

**Proposition 2.4.** *Let  $z_1, z_2 \in \mathbb{C}$ . Then  $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$  and  $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ .*

*Proof.* Write  $z = x + iy$ . If you worked out both sides of the equations in terms of  $x$  and  $y$ , you would see the left- and right- hand sides of the equations are equal. □

### 3. GEOMETRIC INTERPRETATION OF $\mathbb{C}$

We think of the complex number  $z = x + iy$  as the vector  $(x, y)$  in a plane. Thus the complex numbers form a plane (in the same way that the real numbers form a line). The “ $x$ -axis” of this plane is called the *real axis* and the “ $y$ -axis” of this plane is called the *imaginary axis*. Thinking of complex numbers as vectors, addition of complex numbers corresponds to “head-to-tail” or “parallelogram” addition of vectors, i.e.

$$(2 - 3i) + (1 + i) = 3 - 2i$$

is essentially the same as

$$(2, -3) + (1, 1) = (3, -2).$$

The main goal of this section is to explain the geometric interpretation of multiplication. We consider two examples. For the first example, suppose  $z = x + iy \in \mathbb{C}$ . Then  $2z = 2(x + iy) = 2x + i(2y)$  and it should be clear that  $2z$ , thought of as a vector, points in the same direction as  $z$  but is twice as long. Similarly,  $5z$  points in the same direction as  $z$  but is 5 times as long as  $z$  and  $-\frac{2}{3}z$  points in the opposite direction as  $z$  and is  $2/3$  as long as  $z$ .

For the second example, suppose again that  $z = x + iy \in \mathbb{C}$ . Multiply this  $z$  by  $i$  to get  $iz = i(x + iy) = ix - y = -y + ix$ . In vector language, multiplication by  $i$  takes the vector  $(x, y)$  and sends it to  $(-y, x)$ . In linear algebra terms, this corresponds to taking the vector  $(x, y)$  and multiplying it (on the left) by the matrix  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Either by recognizing this matrix as the matrix of a linear transformation which rotates the plane, or by looking at the picture below, we see that multiplication by  $i$  rotates points in  $\mathbb{C}$  by  $\pi/2$  counterclockwise.

We see from these two examples that multiplication in  $\mathbb{C}$  is related to stretching and/or rotation of vectors. These ideas are best expressed using polar coordinates, so we now discuss how to represent complex numbers with polar coordinates.

### 3.1. Polar coordinates.

**Definition 3.1.** *The absolute value a.k.a. norm a.k.a. modulus of a complex number  $z = x + iy$  is  $|z| = \sqrt{x^2 + y^2}$ . (Observe the norm of any complex number is a nonnegative real number, and that if  $|z| = 0$ , then  $z = 0$ ).*

**Definition 3.2.** *The (Euclidean) distance between two complex numbers  $z_1$  and  $z_2$  is  $|z_1 - z_2|$ .*

This notion of distance coincides with the usual notion of distance on the plane. The norm of a complex number is its distance from zero, so this generalizes the notion of absolute value of a real number.

Observe that if we think of  $z$  as a vector, then  $\bar{z}$  is the vector obtained by reflecting  $z$  through the real axis. In particular, this means  $\overline{\bar{z}} = z$  and  $|\bar{z}| = |z|$ .

This notion of distance can be used to describe circles in the complex plane. The set  $\{z : |z - z_0| = r\}$  is a circle centered at the complex number  $z_0$  with radius  $r$ ; the set  $\{z : |z - z_0| < r\}$  is the set of points inside (but not on) this circle; the set  $\{z : |z - z_0| > r\}$  is the set of points outside (but not on) this circle. In particular the set  $\{|z| = r\}$  is a circle of radius  $r$  centered at 0.

**Proposition 3.3.** *Let  $z_1, z_2 \in \mathbb{C}$ . Then  $|z_1 z_2| = |z_1| |z_2|$ .*

*Proof.* Write  $z_1 = x_1 + iy_1$  (same for  $z_2$ ), if you work out both sides in terms of the  $x$ s and  $y$ s you would see that they are equal.  $\square$

As a special case of this, let  $z_2 = -1$  to obtain  $|-z| = |-1||z| = 1|z| = |z|$ . So  $|z_1 - z_2| = |z_2 - z_1|$ , i.e. the distance between two complex numbers doesn't depend on the order in which you subtract them.

**Proposition 3.4** (Triangle Inequality). *Let  $z_1, z_2 \in \mathbb{C}$ . Then  $|z_1 + z_2| \leq |z_1| + |z_2|$ .*

*Proof.* Write  $z_1 = x_1 + iy_1$  (same for  $z_2$ ), work out both sides, do a lot of complicated algebra, you end up with an inequality that is always true (it is something like  $a^2 \geq 0$  where  $a$  is some quantity that must be real).  $\square$

**Theorem 3.5.** *Let  $z \in \mathbb{C}$ . Then  $z\bar{z} = |z|^2$  (and therefore  $|z| = \sqrt{z\bar{z}}$ ). (In particular this means  $z\bar{z}$  is real and nonnegative for any  $z \in \mathbb{C}$ .)*

*Proof.* Write  $z = x + iy$ , work out both sides and show they are equal.  $\square$

This theorem allows us to restate division in  $\mathbb{C}$  as follows: given  $z_1, z_2 \in \mathbb{C}$ , we have

$$z_1 \div z_2 = \frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}.$$

As a special case of this, we have a formula for reciprocals. If  $z \neq 0$ , then

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}.$$

In particular, if  $|z| = 1$ , then  $z^{-1} = \bar{z}$ .

**Definition 3.6.** *Let  $z = x + iy \in \mathbb{C}$ . The argument of  $z$ , denoted  $\arg(z)$ , is any angle  $\theta$  (in radians) such that  $x = |z| \cos \theta$  and  $y = |z| \sin \theta$ .*

Given  $z$ , you can solve for  $\theta = \arg z$  by setting  $\theta = \arctan(y/x)$  if  $x \neq 0$ ; if  $x = 0$  then  $\theta = \pi/2$  if  $y > 0$  and  $\theta = -\pi/2$  if  $y < 0$ . Notice that arguments are only defined up to multiples of  $2\pi$ .

**Definition 3.7.** *The polar coordinates of a complex number  $z$  are  $(r, \theta)$  where  $r = |z|$  and  $\theta = \arg z$ .*

If  $z$  has polar coordinates  $(r, \theta)$ , then  $z = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta)$ ; we shorthand this expression and write  $z = r \operatorname{cis} \theta$  (although there is a better way to write this coming later).

**Theorem 3.8.** *Suppose  $z_1 = r_1 \operatorname{cis} \theta_1$  and  $z_2 = r_2 \operatorname{cis} \theta_2$  (this means  $r_1 = |z_1|, r_2 = |z_2|$ ). Then  $z_1 z_2 = r_1 r_2 \operatorname{cis} (\theta_1 + \theta_2)$ .*

*Proof.* Coming in Section 5. □

This theorem tells us how to interpret multiplication geometrically in  $\mathbb{C}$ . Given two complex numbers, if those numbers are multiplied, then the “moduli multiply” (since  $|z_1 z_2| = |z_1| |z_2|$ ) and the “arguments add” (since this theorem implies  $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ ).

4. TOPOLOGY OF  $\mathbb{C}$ 

Recall that the distance between two complex numbers  $z_1$  and  $z_2$  is  $d(z_1, z_2) = |z_1 - z_2|$ . From this notion of distance we can describe open and closed sets as we have done for the real numbers and sequence spaces.

**Definition 4.1.** Given  $z \in \mathbb{C}$  and  $\epsilon > 0$ , the neighborhood of  $z$  with radius  $\epsilon$ , denoted  $N_\epsilon(z)$ , is the set  $N_\epsilon(z) = \{w \in \mathbb{C} : d(w, z) < \epsilon\} = \{w \in \mathbb{C} : |w - z| < \epsilon\}$ .

A subset  $U \subseteq \mathbb{C}$  is called *open* if every point in  $U$  has a neighborhood of some positive radius contained entirely in  $U$ . (Essentially this means  $U$  does not contain any of its boundary.) A subset  $V \subseteq \mathbb{C}$  is called *closed* if the complement of  $V$  is open. (Essentially this means  $V$  contains its boundary.) The *closure* of a set  $W \subset \mathbb{C}$ , denoted  $\overline{W}$ , is the smallest closed subset of  $\mathbb{C}$  containing  $W$ . A subset of  $\mathbb{C}$  is called *bounded* if it is contained in some neighborhood (perhaps of very large, but finite radius) of 0.

**4.1. The Riemann sphere.** An amazing thing about  $\mathbb{C}$  is that it can be thought of as not only a plane but a sphere. To do this, take the complex plane and imagine that it is lying horizontally (like on a table or desk). Mark the origin and the real and imaginary axes on this plane; then imagine a sphere of radius  $1/2$  tangent to this plane at the origin, sitting on top of the plane. Imagine that this sphere is translucent. Define a map from the complex numbers to the sphere by the following procedure: given  $z \in \mathbb{C}$ , go to the point  $z$  on the plane and shoot a laser beam from  $z$  to the point at the top of the sphere (call the top of the sphere  $(0, 0, 1)$  for now). It will necessarily be the case that the laser beam strikes the sphere somewhere between  $z$  and  $(0, 0, 1)$ ; call this point  $z^*$ . The map from  $\mathbb{C}$  to the sphere takes each  $z$  to the corresponding  $z^*$  and is called *stereographic projection*. (See the picture on the next page.)

Here is a picture describing the stereographic projection. In fact this map is a homeomorphism between  $\mathbb{C}$  and the set of points on the sphere other than the north pole  $(0, 0, 1)$  (that is, it is 1-1, onto, continuous, and has continuous inverse):

We think of the complex number  $z$  in the plane as being no different than the complex number  $z^*$  on the sphere (in fact, we drop the  $*$  notation and think of the point on the sphere as just  $z$ ). Furthermore, suppose you start at 0 and draw any ray emanating from 0 in your plane. This corresponds on the sphere to a semicircle starting at the bottom of the sphere, wrapping around to the top. Also, the stereographic projection preserves lines and circles in the following sense: if you draw any line on your plane, that line corresponds to a circle on the sphere passing through  $(0, 0, 1)$ . If you draw any circle on your plane, that circle corresponds to a circle on the sphere. In particular this makes the following identifications:

PLANE		SPHERE
0	$\leftrightarrow$	bottom of sphere
real axis	$\leftrightarrow$	“left and right side” great circle of sphere
imaginary axis	$\leftrightarrow$	“front and back” great circle of sphere
complex numbers of modulus 1	$\leftrightarrow$	“equator” of sphere
$\infty$	$\leftrightarrow$	top of sphere

Amazingly, what this means is that if we take the spherical model for the complex numbers, we can treat  $\infty$  as a number! (We cannot do this for real numbers, ever.)

**Definition 4.2.** *The Riemann sphere, denoted  $\widehat{\mathbb{C}}$ , is the set  $\mathbb{C} \cup \{\infty\}$ , viewed as a sphere where the north pole of the sphere is  $\infty$  and the rest of the sphere corresponds to the set of complex numbers as described above.*

On the Riemann sphere, there is a natural notion of distance:

**Definition 4.3.** Given  $z, w \in \widehat{\mathbb{C}}$  (including  $z$  or  $w = \infty$ ), define the (chordal) distance between  $z$  and  $w$ , denoted  $d(z, w)$  to be the Euclidean distance (in 3-D space) of the chord connecting the points  $z$  and  $w$ .

Note that the maximum chordal distance between any two points is 1 (the diameter of the sphere). From this notion of distance we can define neighborhoods and open and closed subsets of the sphere, in the same way we described these objects in  $\mathbb{R}$ ,  $\{0, 1\}^{\mathbb{N}}$  and the planar model of  $\mathbb{C}$ :

**Definition 4.4.** Given  $z \in \widehat{\mathbb{C}}$  and  $\epsilon > 0$ , the neighborhood of  $z$  with radius  $\epsilon$ , denoted  $N_\epsilon(z)$ , is the set

$$N_\epsilon(z) = \{w \in \widehat{\mathbb{C}} : d(w, z) < \epsilon\}.$$

A neighborhood of  $z$  on the sphere looks like a contact lens-shaped region centered at  $z$ :

A subset  $U \subseteq \widehat{\mathbb{C}}$  is called *open* if every point in  $U$  has a neighborhood of some positive radius contained entirely in  $U$ . (Essentially this means  $U$  does not contain any of its boundary.) A subset  $V \subseteq \widehat{\mathbb{C}}$  is called *closed* if the complement of  $V$  is open. (Essentially this means  $V$  contains its boundary.) The *closure* of a set  $W \subset \widehat{\mathbb{C}}$ , denoted  $\bar{W}$ , is the smallest closed subset of  $\widehat{\mathbb{C}}$  containing  $W$ .

Open (resp. closed), bounded subsets of  $\mathbb{C}$  correspond to open (resp. closed) subsets of  $\widehat{\mathbb{C}}$  not containing the north pole  $\infty$ . Open (resp. closed), unbounded subsets of  $\mathbb{C}$  correspond to open (resp. closed) subsets of  $\widehat{\mathbb{C}}$  containing the north pole  $\infty$ . Lines in  $\mathbb{C}$  correspond to circles in  $\widehat{\mathbb{C}}$  passing through the north pole; circles in  $\mathbb{C}$  correspond to circles in  $\widehat{\mathbb{C}}$  which do not pass through the north pole.

**4.2. Arithmetic with  $\infty$ .** As mentioned earlier,  $\infty$  can be treated as a number if we work with the Riemann sphere rather than the complex numbers. Arithmetic operations that are consistently defined with  $\infty$  are:

- $\infty \pm$  (any finite complex number)  $= \infty$
- $\infty \cdot$  (any nonzero finite complex number)  $= \infty$
- $\infty \cdot \infty = \infty$
- (any nonzero finite complex number)  $\div 0 = \infty$
- (any nonzero finite complex number)  $\div \infty = 0$

The following expressions are indeterminate; if you encounter them in a limit you would need to use L'Hopital's rule or the equivalent to evaluate them:

$$\frac{\infty}{\infty} \quad \infty \cdot 0 \quad \infty + \infty \quad \infty - \infty \quad \frac{0}{0}$$

An important thing to keep in mind about complex infinity is that there is no distinction between  $+\infty$  or  $-\infty$  or  $i\infty$ , etc. When we say  $z \rightarrow \infty$ , we mean  $|z| \rightarrow \infty$ , that is, that on the sphere, we are letting points get closer and closer to the north pole from an arbitrary direction along the sphere.

## 5. FUNCTIONS ON $\mathbb{C}$ AND $\widehat{\mathbb{C}}$

We can define basic functions on  $\mathbb{C}$  using arithmetic operations, e.g.  $f(z) = z^2$  and  $f(z) = \frac{4z^3+1}{-z^4-2}$  make sense. To define non-arithmetic functions like exponentials and trig functions, we use power series (because power series are made up only of addition, subtraction, multiplication and division, and all these operations are already defined for complex numbers). In particular, it is a fact that any power series that converges for all real numbers also converges for all complex numbers. So we define, for all complex numbers  $z$ :

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

From this, you can show that all the usual trigonometric and exponential identities that hold for real numbers also hold for complex numbers. More importantly, we have:

**Theorem 5.1** (Euler's Identity). *For any  $z \in \mathbb{C}$ ,  $e^{iz} = \cos z + i \sin z$ .*

*Proof.* Write the power series for both sides of this equation; they are identical.  $\square$

As an important consequence, we see that if  $z$  has polar coordinates  $(r, \theta)$ , then

$$z = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta) = re^{i\theta}.$$

Now Theorem 3.8 follows immediately because if  $z_1 = r_1 e^{i\theta_1}$  and  $z_2 = r_2 e^{i\theta_2}$ , then by elementary properties of exponentials,  $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$ . In particular, if we write  $z = re^{i\theta}$  where  $r \geq 0$  and  $\theta \in \mathbb{R}$ , this means  $(r, \theta)$  are the polar coordinates of  $z$ , so  $r = |z|$  and  $\theta = \arg z$ .

**Definition 5.2.** *A function  $f : \mathbb{C} \rightarrow \mathbb{C}$  is called differentiable at  $z_0 \in \mathbb{C}$  if*

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

*exists, in which we call the limit  $f'(z_0)$ , the derivative of  $f$  at  $z_0$ . A function which is differentiable at every complex number is called analytic on  $\mathbb{C}$ .*

A word of caution: in the above limit,  $h$  is a complex number, i.e. is a vector  $(x, y)$  so complex limits (and complex derivatives) are more like multivariable limits than they are limits of one variable. In particular, in order for a derivative to exist the above limit has to give the same value if  $h$  approaches 0 from any direction.

In fact, the theory of derivatives of complex functions is deep and very interesting (but beyond the scope of these notes).

That said, all the usual rules of differentiation still hold (power rule, product rule, chain rule, trig rules, etc.) As with functions of real numbers, differentiable functions are necessarily continuous.

**5.1. Functions on the Riemann sphere.** Given a function defined on  $\mathbb{C}$ , we set  $f(\infty) = \lim_{z \rightarrow \infty} f(z)$  if this limit exists. If we can do this, then  $f$  becomes a function from  $\widehat{\mathbb{C}}$  to itself, i.e.  $f$  has been “extended” to a function of the Riemann sphere. For example, the function  $f(z) = 1/z$  is extended to the sphere by setting  $f(0) = \infty$  and  $f(\infty) = 0$ .

There is also a method of determining whether  $f$  is differentiable at  $\infty$  (not the usual definition of derivative). This is beyond the scope of these notes. If a function  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  has a derivative at every complex number and is differentiable at  $\infty$ , we call  $f$  an *meromorphic* function, or a function *analytic on  $\widehat{\mathbb{C}}$* . It turns out that there aren’t that many meromorphic functions; here is a fact typically discussed in a complex analysis course:

**Theorem 5.3.** *Every meromorphic function  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is a rational function, i.e.  $f(z) = \frac{P(z)}{Q(z)}$  where  $P$  and  $Q$  are polynomials with no common factors.*

Therefore the functions we are most interested in studying (that is, differentiable functions from the Riemann sphere to itself) are rational functions.

**Definition 5.4.** *Given a rational function  $f(z) = \frac{P(z)}{Q(z)}$ , we define the degree of  $f$  to be the maximum of the degrees of  $P$  and  $Q$ .*

Furthermore, for a rational function  $f(z) = \frac{P(z)}{Q(z)}$ ,  $f(\infty) = \lim_{z \rightarrow \infty} \frac{P(z)}{Q(z)}$  which is an easy limit to evaluate using pre-calculus and calculus rules (it is zero if the degree of  $P$  is less than the degree of  $Q$ , etc.).

Finally for a rational function  $f$ , we can set  $f'(\infty) = \lim_{z \rightarrow 0} f'(\frac{1}{z})$ . (This makes sense because  $z$  approaching 0 is equivalent to  $1/z$  approaching  $\infty$ .) It turns out that for a rational function  $f$ ,  $f'(\infty)$  is always equal to  $\lim_{z \rightarrow \infty} \frac{f(z)}{z}$ .