1. (a) Prove that the functions $f(x) = x^2$ and $g(x) = x^2 - 6x + 12$ are topologically conjugate.

   (b) Prove that the functions $T(x) = x^2$ and $S(x) = x^2 - 1$ are not topologically conjugate.

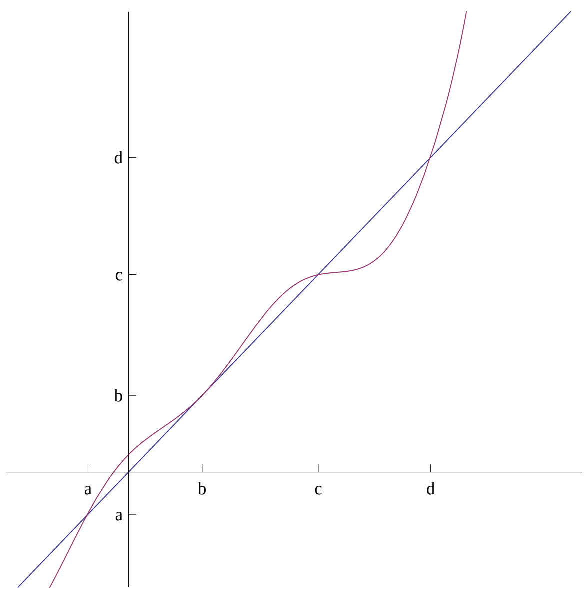
2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2x & x \leq 1/2 \\ 2 - 2x & x > 1/2 \end{cases}$$

   Prove that $f(x)$ has periodic points of all prime periods.

3. Describe a function (either by writing its formula or sketching its graph) $f : [0,1] \to [0,1]$ such that $f$ is $1-1$ and onto, but $f$ has no fixed point in $[0,1]$.

4. Here is the graph of some $C^1$ function $f(x)$ (plotted together with $y = x$). From the graph you see that the function has four fixed points: $a$,$b$,$c$ and $d$.




   (a) Classify each of the fixed points as attracting, repelling or neutral.

   (b) Find the stable set of each fixed point.

   (c) Find the stable set of infinity.

5. Let $f(x) = x - \frac{1}{x}$.

   (a) Let $I = [1,2]$. Find $f(I)$ and $f^{-1}(f(I))$.

   (b) Find all periodic points of $f$ which have prime period 2.

   (c) Calculate the Schwarzian derivative of $f$.

   (d) Prove that $f$ has a periodic point of prime period 3.

(e) Prove that all the periodic points of $f$ are repelling.

6. Let $\{0,1\}^{\mathbb{N}}$ be the set of all one-sided sequences of zeros and ones. Let $E$ be the subset of $\{0,1\}^{\mathbb{N}}$ consisting of all sequences which do not contain two consecutive 0s.

   (a) Prove that $E$ is a closed subset of $\Sigma_2$.

   (b) Explain why $E$ is invariant under the shift map $\sigma$ (i.e. show $\sigma(E) \subseteq E$).

   (c) How many fixed points does the dynamical system $(E, \sigma)$ have? List all the fixed points.

   (d) How many points of prime period 2 does the dynamical system $(E, \sigma)$ have? List all these points.

   (e) How many points of prime period 3 does the dynamical system $(E, \sigma)$ have? List all these points.

   (f) Show that there is a point $x \in E$ whose forward orbit under $\sigma$ is dense in $E$.

1. (a) Let $\phi(x) = x+3$. We see $(\phi \circ f)(x) = x^2+3$ and $(g \circ \phi)(x) = (x+3)^2 - 6(x+3) + 12 = x^2 + 6x + 9 - 6x - 18 + 12 = x^2 + 3$ so $\phi$ intertwines $f$ and $g$. But since $\phi$ is linear, it is $1-1$, onto and continuous and has continuous inverse, so $\phi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism. Therefore $\phi$ is a topological conjugacy and $f \cong g$.

   (b) Observe that $x = 0$ is an attracting fixed point of $T(x)$. Therefore if $\phi$ is a topological conjugacy from $(\mathbb{R}, T)$ to $(\mathbb{R}, S)$, $\phi(0)$ must be an attracting fixed point of $S(x)$. But the fixed points of $S$ are $\frac{1}{2}\left(1 \pm \sqrt{5}\right)$ and the derivative of $S$ at these fixed points is $1 \pm \sqrt{5}$. We see $1 + \sqrt{5} > 1$ and $1 - \sqrt{5} < -1$, so both fixed points of $S$ are repelling. Therefore $T$ and $S$ cannot be topologically conjugate.

2. $f$ is continuous so by Sarkovskii's Theorem it is sufficient to show that $f$ has a periodic point of prime period 3. Let $x = 2/9$; then $f(x) = 4/9$; $f^2(x) = 8/9$ and $f^3(x) = 2/9$ so $2/9$ is a periodic point of prime period 3.

3. If $f$ is continuous, $f$ must have a fixed point in $[0,1]$. Therefore any correct answer must necessarily be discontinuous somewhere in $[0,1]$. One correct answer is

$$f(x) = \begin{cases} x + \frac{1}{2} & x < 1/2 \\ x - \frac{1}{2} & x \geq 1/2 \end{cases}.$$

4. (a) $a$ and $d$ are repelling because the slope of $f$ at the fixed point is greater than 1; $b$ is neutral because the graph of $f$ is tangent to $y = x$ at $b$ (hence $f'(b) = 1$); $c$ is attracting because the graph clearly has slope between $-1$ and $0$ at $c$.

   (b) By considering cobweb diagrams together with the answer to (a), we see $W^s(a) = \{a\}$; $W^s(b) = (a, b]$; $W^s(c) = (b, d)$; $W^s(d) = \{d\}$.

   (c) $W^s(\infty) = (-\infty, a) \cup (d, \infty)$.

5. Let $f(x) = x - \frac{1}{x}$. Observe that $f$ has no fixed points, for if $f(x) = x$ then $1/x = 0$ which is impossible.

   (a) Note $f'(x) = 1 + \frac{1}{x^2} > 0$ so $f$ is everywhere increasing. Therefore $f(I) = [f(1), f(2)] = \left[0, \frac{3}{2}\right]$. Now $f^{-1}(f(I)) = f^{-1}([0, \frac{3}{2}])$. Notice that $f^{-1}(0) = \{1, -1\}$ and $f^{-1}(\frac{3}{2}) = \{2, -1/2\}$. So

$$f^{-1}([0, \frac{3}{2}]) = [-1, -1/2] \cup [1, 2].$$

   (b) $f^2(x) = f(f(x)) = x - \frac{1}{x} - \frac{1}{x - \frac{1}{x}} = \frac{x^4 - 3x^2 + 1}{x(x^2-1)}$. Setting $f^2(x) = x$ and solving for $x$, we see $x = \pm\frac{1}{\sqrt{2}}$. Since $f$ has no fixed points, these must be periodic with prime period 2.

   (c) By direct calculation $f'(x) = 1 + \frac{1}{x^2} = \frac{x^2+1}{x^2}$; $f''(x) = \frac{-2}{x^3}$; $f'''(x) = \frac{6}{x^4}$. Then by direct calculation,

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2 = \frac{6}{(x^2+1)^2}.$$

(d) First, $f$ is continuous everywhere except $x = 0$ so $f^3$ is continuous on any interval that does not contain 0 and avoids 0 for two iterates, i.e. any interval $J$ such that $0 \notin J$, $0 \notin f(J)$ and $0 \notin f^2(J)$.

Consider the interval $J = [\frac{4}{3}, \frac{3}{2}]$. Since $f$ is increasing, $f(J) = [f(\frac{4}{3}), f(\frac{3}{2})] = [\frac{7}{12}, \frac{5}{6}]$ and $f^2(J) = f([\frac{7}{12}, \frac{5}{6}]) = [\frac{-95}{84}, \frac{-11}{30}]$. Then $f^3(J) = f([\frac{-95}{84}, \frac{-11}{30}]) = [\frac{-1969}{7980}, \frac{779}{330}]$. Notice that $f^3(J) \supseteq J$. Since $f^3$ is continuous on $J$, by a theorem from class (essentially this is the Intermediate Value Theorem), $f^3$ has a fixed point $p$ in $J$. Since $f$ has no fixed points, $p$ must be a periodic point of prime period 3 for $f$.

(e) $f'(x) = 1 + \frac{1}{x^2} > 1$ for all $x$. Suppose $p$ is periodic for $f$ with prime period $k$. Then $(f^k)'(p) = f'(p)f'(f(p))f'(f^2(p)) \cdots f'(f^{k-1}(p)) > 1$ since all the numbers being multiplied are $> 1$. Therefore $p$ must be repelling.

6. (a) Let $F$ be the complement of $E$. A sequence belongs to $F$ if it contains two consecutive 0s. But for any fixed index $k$, the set of sequences which have two consecutive 0s starting at that index is a cylinder set $C(0,0;k,k+1)$. Thus $F = \bigcup_{k=0}^{\infty} C(0,0;k,k+1)$ is a countable union of cylinder sets and is therefore open. Since $E$ is the complement of $F$, $E$ is closed.

(b) This is somewhat obvious; if $x \in E$ then $x$ has no two consecutive 0s. Clearly then $\sigma(x)$ also has no two consecutive 0s because $\sigma$ does not rearrange or insert characters into the sequence $x$; it just "deletes" the first symbol.

(c) One: 1111111.....

(d) Two: 10101010101.... and 01010101010101....

(e) Three: 110110110110... and 101101101101101101... and 011011011011...

(f) Let $x \in E$ be a "Morse-like" sequence; we build $x$ by starting with all blocks of length 1, then all possible blocks of zeros and ones of length 2, then all blocks of length 3, etc. However, at each stage we do not include any blocks which have two consecutive zeros in them, and between every block we put a 1 (in case the last digit of one block was a zero and the first digit of the next block was a zero). In other words $x$ looks like

$$x = 0\ 1\ 1 \quad 1 \quad 01\ 1\ 10\ 1\ 11 \quad 1 \quad 010\ 1\ 011\ 1\ 101\ 1\ 110\ \ldots$$

Notice $x$ has no two consecutive zeros, so $x \in E$. Let $y \in E$ be arbitrary and let $\epsilon > 0$. Choose $k$ large enough so that $\frac{1}{2^k} < \epsilon$; then consider the block of the first $k+1$ symbols in $y$ (namely $y_0 y_1 y_2 \ldots y_k$). This block appears somewhere in $x$ because of the way $x$ was constructed, so for some $n \geq 0$, $\sigma^n(x) = y_0 y_1 y_2 \ldots y_k \ldots$ and since $\sigma^n(x)$ and $y$ agree on the first $k+1$ symbols, $d(\sigma^n(x), y) < \frac{1}{2^k} < \epsilon$. Therefore the forward orbit of $x$ comes arbitrarily close to every $y \in E$ and hence this orbit is dense.