Name:

Directions: This exam is to be done at home and returned to me by 12 PM on Friday, May 13. You may use books, your notes, etc. as a reference and I may give you some hints if you ask me questions. You can (and should) also use *Mathematica* as a resource to check your answers, etc. That said, you may not discuss the problems with others (unless there is a group of you in my office) and for all problems other than 1(b), to receive full credit your answers cannot rely on a *Mathematica* calculation.

Grading:

Problem	Points Possible	Points Earned
1	20	
2	10	
3	45	
4	35	
5	30	
6	40	
7	30	
8	40	
Total	250	

1. For each function $f : \mathbb{R} \to \mathbb{R}$, find all the periodic points of the function and sketch a phase portrait of $f(x)$:

   (a) (10 pts) $f(x) = -x^3$

   (b) (10 pts) $f(x) = e^{x-2}$ (decimal approximations are sufficient here)

2. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(1) = 2$, $f(2) = 3$, $f(3) = 4$, $f(4) = 5$, and $f(5) = 1$. Prove that f has a periodic point of prime period 3.

3. Let (X, T) be a dynamical system. We say that a point $p \in X$ is *non-wandering* (under T) if for every open set U containing p, there is a point $x \in U$ such that $T^n(x) \in U$ for some $n > 0$. Define $NW(T)$ to be the set of non-wandering points under T.

   (a) (10 pts) Prove that $NW(T)$ is always a closed set.

   (b) (5 pts) Let p be a periodic point for T. Is p non-wandering? Explain.

   (c) (10 pts) Suppose p is not periodic, but belongs to the stable set of some periodic point y. Is p non-wandering? Explain.

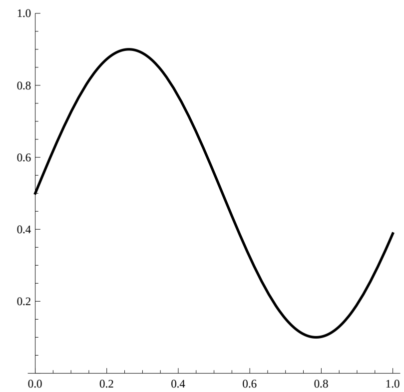
   (d) (10 pts) Let $f_r : [0, 1] \to [0, 1]$ be defined by $f_r(x) = rx(1 - x)$. Describe $NW(f_r)$ when $0 \leq r \leq 3$. (There may be different answers depending on the value of r).

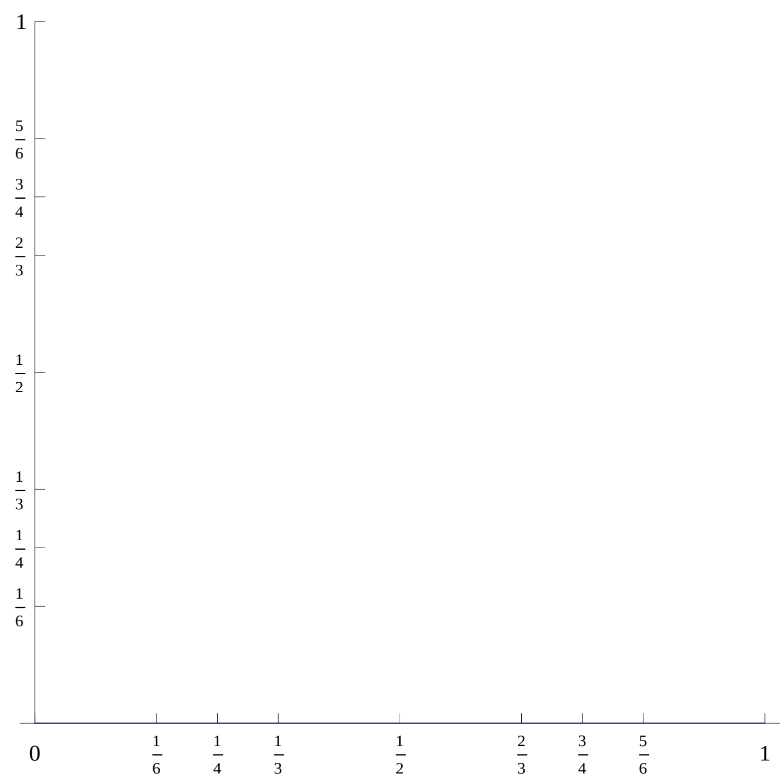
   (e) (10 pts) Suppose T is an arbitrary chaotic system. Describe $NW(T)$.

4. Given a continuous function $f : [0,1] \to [0,1]$, define the *double* of f to be the function $F : [0,1] \to [0,1]$ satisfying the following four properties:

- F is continuous;
- $F(2/3) = 0$ and $F(1) = 1/3$;
- F is linear on the interval $[1/3, 2/3]$ and also linear on the interval $[2/3, 1]$;
- $F(x) = \frac{1}{3}f(3x) + \frac{2}{3}$ if $x \in [0, 1/3]$

(a) (5 pts) Given the following graph of $f : [0,1] \to [0,1]$, sketch the graph of the double of f on the axes below:





(b) (This is a continuation of the problem on the previous page.) Suppose F is the double of some unknown continuous function f.

i. (10 pts) How many fixed points does F have? Classify the fixed points as attracting, repelling or neutral (prove your answer).

ii. (10 pts) Suppose p is a periodic point for F. Prove that the prime period of p must be a power of 2.

iii. (10 pts) Suppose q is a periodic point for f of prime period d. Show $q/3$ is a periodic point for F. What is its prime period?

5. For each family of functions given below, sketch a bifurcation diagram. Give all values of the parameter for which bifurcations occur, and classify each bifurcation.

   (a) (15 pts) $f_r(x) = x^2 + rx$; $r \in (-\infty, \infty)$

   (b) (15 pts) $f_r(x) = x^3 + r$; $r \in (-\infty, \infty)$

6. Let D be the subset of $\{A, B, C\}^{\mathbb{N}}$ consisting of all sequences obeying the following three rules: first, no two consecutive As are allowed; second, every B must be followed by an A; third, C cannot be followed by A.

   (a) (20 pts) Prove that (D, σ) is chaotic, where σ is the shift map.

   (b) (10 pts) Find the topological entropy of (D, σ). A decimal answer here is not sufficient.

   (c) (10 pts) Prove that (D, σ) is topologically conjugate to the golden mean shift (the *golden mean shift* is the shift on the subset of $\{0, 1\}^{\mathbb{N}}$ consisting of sequences with no two consecutive zeros).

7. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the rational function $f(z) = \frac{1}{z^2}$.

   (a) (10 pts) Find the fixed point(s) of f and classify them as attracting, repelling, or neutral.

   (b) (10 pts) Find the periodic cycle of prime period 2 and classify it as attracting, repelling or neutral.

   (c) (10 pts) Describe the Julia set of f (briefly explain your reasoning).

8. Let $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the rational function $g(z) = z^3 - 3z$; let I be the interval of real numbers $[-2, 2]$, thought of as a subset of $\widehat{\mathbb{C}}$:
   (a) (10 pts) Show I is completely invariant under both forward and backward iteration of g.
   (b) (10 pts) Show that if $|z| > 2$, then $z \in W^s(\infty)$.
   (c) (10 pts) Explain why g has no attracting periodic point, other than the fixed point at ∞.
   (d) (10 pts) "Guess" the topological entropy of $([-2, 2], g)$. Explain the logic behind your conjecture. (You don't necessarily have to guess correctly to receive credit here.)