1. (6 pts) Suppose a monkey sits at a keyboard with the letters A to Z on it and randomly hits keys until it has typed 10,000 letters. Use the normal approximation to the binomial distribution to estimate the probability that the monkey typed between 1,950 and 2,000 vowels (express the probability as a percent, rounded to the nearest tenth of a percent).

   Solution: This is a binomial distribution with probability $p = 5/26$; it is approximated by a normal curve with mean

   $$\mu = np = (10000)(5/26) = 1923.07$$

   and standard deviation

   $$\sigma = \sqrt{np(1-p)} = \sqrt{(10000)(5/26)(21/26)} = \sqrt{1553.25} = 39.411.$$

   So the probability of the monkey typing between 1950 and 2000 vowels is

   $$\texttt{normalcdf}(1950, 2000, 1923.07, 39.411) = .2217 = 22.2\%.$$

2. (5 pts) A gambler plays the slot machines at a casino every day for the month of April. He records his winnings and losses for each day as follows (negative numbers are losses, positive numbers are winnings):

   $$-1, -2, -1, 0, 2, -1, -1, 0, -4, -2, 250, -2, -4, -1, 0, 0, -3$$

   Which is more indicative of the gambler's daily performance, the *mean* or the *median*? Why?

   Solution: The mean will be skewed off by the outlier 250, so the median will be more indicative of the daily performance of the gambler.

3. (7 pts) The test results for a class are as follows (there are 22 students):

   $$61, 61, 63, 63, 68, 71, 75, 77, 77, 77, 81, 84, 85, 88, 88, 91, 92, 93, 95, 95, 98, 100$$

   What percent of the class scored within one standard deviation of the mean (round to the nearest tenth of a percent)?

   Solution: From the calculator, we see that the mean test score is 81.045 and the standard deviation is 12.304. So students who scored within one standard deviation of the mean scored between $81.045 - 12.304 = 68.741$ and $81.045 + 12.034 = 93.079$. There are 13 such scores, so 13 out of the 22 scored within one standard deviation of the mean. Writing this as a percent, that is

   $$\frac{13}{22} = .59090909 = 59.1\%.$$

4. The number of people living in a town who voted in the last few national elections is given in the following table:

Year:	1994	1996	1998	2000	2002	2004	2006
Population:	47,300	48,650	51,225	54,260	53,792	57,602	60,521

   (a) (3 pts) Find the least-squares line for this data (x should be the year, y the population).

   Solution: From the calculator, this is $y = 1073.821x - 2094307.143$.

   (b) (3 pts) How accurate is the least-squares line? Explain.

   Solution: The correlation coefficient is $r = .9827$ which is close to 1, so the line is very accurate.

   (c) (3 pts) Use the line you found in part (a) to estimate the population of the town in 2020.
   Plug in $x = 2020$ into the line from (a) to get

$$y = 1073.821(2020) - 2094307.143 = 74811.$$

5. (10 pts) Consider the two matrices

$$A = \begin{pmatrix} 1 & -5 & 3 \\ -1 & -2 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}.$$

Find the matrix products AB and BA (if defined). If either is undefined, say so and why.

Solution: AB is undefined because the number of columns of A is not equal to the number of rows of B.

$$\begin{aligned} BA &= \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -5 & 3 \\ -1 & -2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot 1 + 2 \cdot (-1) & 1 \cdot (-5) + 2 \cdot (-2) & 1 \cdot 3 + 2 \cdot 0 \\ -1 \cdot 1 + 0 \cdot (-1) & -1 \cdot (-5) + 0 \cdot (-2) & -1 \cdot 3 + 0 \cdot 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -9 & 3 \\ -1 & 5 & -3 \end{pmatrix}. \end{aligned}$$

6. Suppose

$$\begin{pmatrix} 1 & 1 & 3 & 4 \\ -1 & 2 & -1 & -1 \\ 2 & -3 & 3 & 2 \\ -3 & -1 & 2 & 3 \\ 0 & 1 & 2 & -5 \end{pmatrix}$$

is the augmented matrix corresponding to some system of linear equations.

(a) (3 pts) How many variables are in this system?

Solution: There are 3 variables (one less than the number of columns of the matrix).

(b) (3 pts) What are the possible number of solutions to this solution?

Solution: There are more equations than variables, so there could be 0, 1, or an infinite number of solutions.

(c) (3 pts) Without doing any calculations, what is the most likely number of solutions to this system? Why?

Solution: Since there are more equations than variables, one would expect that this system has no solution.

7. For the following systems of equations:

- Give the number of solutions to the system.
- If the system has one solution, solve the system and write your answer appropriately.
- If the system has more than one solution, write all the solutions to the equation in terms of a parameter a and give three explicit solutions to the system.

(a) (8 pts) $\begin{cases} x + y + z = 7 \\ x + 2y + 2z = 10 \\ 2x + 3y - 4z = 3 \end{cases}$

Solution: The augmented matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 7 \\ 1 & 2 & 2 & 10 \\ 2 & 3 & -4 & 3 \end{pmatrix}$$

which has row-echelon form (use the calculator to get this):

$$\begin{pmatrix} 1 & 1.5 & -2 & 1.5 \\ 0 & 1 & 8 & 17 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Writing the row-echelon form as equations, we have

$$\begin{cases} x + 1.5y - 2z = 1.5 \\ y + 8z = 17 \\ z = 2 \end{cases}$$

which by back-substitution, has one solution: $(4, 1, 2)$.

(b) (8 pts) $\begin{cases} x - y + 3z = 4 \\ 4x - 2y + z = 3 \\ 6x - 2y - 4z = -2 \end{cases}$

Solution:

$$\begin{pmatrix} 1 & -1 & 3 & 4 \\ 4 & -2 & 1 & 3 \\ 6 & -2 & -4 & -2 \end{pmatrix} \text{ has r-e form } \begin{pmatrix} 1 & -1/3 & -2/3 & -1/3 \\ 0 & 1 & -11/2 & -13/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

There are two nontrivial rows, so this system has an infinite number of solutions. Let $z = a$ and solve for the other two variables in terms of a:

$$y - (11/2)a = (-13/2) \Rightarrow y = (11/2)a - (13/2)$$

$$x - (1/3)[(11/2)a - (13/2)] - (2/3)a = (-1/3) \Rightarrow x = (5/2)a - (5/2)$$

So all solutions to the system are of the form $\left(\frac{5}{2}a - \frac{5}{2}, \frac{11}{2}a - \frac{13}{2}, a\right)$. Three solutions can be found when $a = 0, 1, 2$ (other answers possible):

$$\begin{aligned} a = 0 &\Rightarrow (-5/2, -13/2, 0) \\ a = 1 &\Rightarrow (0, -1, 1) \\ a = 2 &\Rightarrow (5/2, 9/2, 2) \end{aligned}$$

(c) (8 pts) $\begin{cases} x + 2y + 2z = 1 \\ 2x + 4y + 5z = 4 \end{cases}$

Solution:

$$\begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 5 & 4 \end{pmatrix} \text{ has r-e form } \begin{pmatrix} 1 & 2 & 2.5 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

There are two nontrivial rows, so this system has an infinite number of solutions. This time you let $y = a$ because z has to be equal to 2. Solving for x in terms of a and $z = 2$, we get

$$x + 2(a) + 2.5(2) = 2 \Rightarrow x = -2a - 3$$

So all solutions to the system are of the form $(-2a - 3, a, 2)$. Three solutions can be found when $a = 0, 1, 2$ (other answers possible):

$$\begin{aligned} a = 0 &\Rightarrow (-3, 0, 2) \\ a = 1 &\Rightarrow (-5, 1, 2) \\ a = 2 &\Rightarrow (-7, 2, 2) \end{aligned}$$

8. (12 pts) A lake is stocked each spring with three species of fish: trout, bass, and salmon. Three kinds of food (call them type I, II, and III) are available in the lake. Each trout requires an average of 1.32 units of food I, 2.9 units of food II, and 1.75 units of food III each day. Each bass requires 2.1 units of food I, .95 unit of food II, and .6 unit of food III daily. Each salmon requires .86 unit of food I, 1.52 units of food II, and 2.01 units of food III each day. There are 487.42 units of food I, 894.32 units of food II, and 651.06 units of food III available daily. How many trout, bass, and salmon should be put in the lake if the food is to be completely eaten by the fish and if the fish get the food they need?

   Solution:

   Let t be the number of trout in the lake, b be the number of bass in the lake, and s be the number of salmon in the lake. We have the three equations:

$$\begin{array}{c} \text{Food I} \Rightarrow \\ \text{Food II} \Rightarrow \\ \text{Food III} \Rightarrow \end{array} \quad \begin{cases} 1.32t + 2.1b + .86s = 487.42 \\ 2.9t + .95b + 1.52s = 894.32 \\ 1.75t + .6b + 2.01s = 651.06 \end{cases}$$

   Solving this system as usual, you get $t = 243, b = 38, s = 101$.

9. Consider the system of equations

$$\begin{cases} x - 3y = c \\ 2x + dy = 7 \end{cases}$$

   where c and d are constants.

   (a) (4 pts) Give specific values for c and d which make the above system have no solution. Justify your answer.

   Solution: Suppose you tried to solve this system using row operations. First, you would write the augmented matrix:

$$\begin{pmatrix} 1 & -3 & c \\ 2 & d & 7 \end{pmatrix}.$$

   Next, you would take -2 times Row 1 and add it to Row 2. If you do this here, you get

$$\begin{pmatrix} 1 & -3 & c \\ 0 & d+6 & 7-2c \end{pmatrix}.$$

   If the equation has no solution, you want the last row to be false, i.e. you want $d + 6 = 0$ and $7 - 2c \neq 0$. So $d = -6, c = 1$ works (there are lots of choices for c but only one choice for d.

(b) (4 pts) Give specific values for c and d which make the above system have more than one solution. Justify your answer.

Solution: Use the work in part (a). If there is more than one solution, you want the last row to be trivial. So you need $d + 6 = 0$ and $7 - 2c = 0$, in other words $d = -6$ and $c = 7/2$.

10. (10 pts) Find the maximum and minimum values (if they exist) of $z = y - 4x$ subject to the constraints $x + y \leq 10, 5x + 2y \geq 20, -x + 2y \geq 0, x \geq 0, y \geq 0$. If the maximum and/or minimum values do not exist, explain why not.

Solution: First, graph the constraints and find corner points. There are three corner points:

$$\begin{aligned} A = (0, 10) &\leftrightarrow \text{the intersection of } x + y = 10 \text{ and } 5x + 2y = 20 \\ B = (20/3, 10/3) &\leftrightarrow \text{the intersection of } x + y = 10 \text{ and } -x + 2y = 0 \\ C = (10/3, 5/3) &\leftrightarrow \text{the intersection of } -x + 2y = 0 \text{ and } 5x + 2y = 20. \end{aligned}$$

The feasible region is bounded so there is a maximum and minimum. Plug in the corner points into the objective function $z = y - 4x$ to get

$$\begin{aligned} A &: z = 10 \text{ (maximum)} \\ B &: z = -70/3 \text{ (minimum)} \\ C &: z = -35/3 \end{aligned}$$