1. Consider a strategic game where two players each choose a nonnegative number (call the numbers the players choose $x$ and $y$ respectively). Suppose the utility functions for this game are

$$u_1(x, y) = 100 + 20x + 2xy - x^2 - 50y \text{ and}$$

$$u_2(x, y) = 200 + 40y - 50x - 2y^2.$$

   (a) Find the best response function for each player.

   (b) Graph the best response functions on the same axis (placing the $x-$axis horizontally and the $y-$axis vertically). Be sure to label which function is $B_1$ and which one is $B_2$.

   (c) Find all (pure strategy) Nash equilibria of this game.

2. *Chicken* is a game played occasionally by irresponsible American teenagers. It works like this: two car drivers start at opposite ends of a road and drive toward each other at high speed. Both drivers have the same choices: keep going straight ($K$) or swerve off the road ($S$). If both drivers keep going straight, they crash their cars into one another. If one driver goes straight and the other swerves to avoid the crash, then the driver who kept going straight is the "victor" of the game and the driver who swerved is "chicken" ("chicken" is American slang for "coward"). Assume that if both drivers swerve, then they are both "chicken". This can be formulated as a strategic game represented by the following matrix:

$$\begin{array}{c} \\ K \\ S \end{array} \begin{array}{c} \begin{array}{cc} K & S \end{array} \\ \begin{pmatrix} (-w, -w) & (v, -s) \\ (-s, v) & (-s, -s) \end{pmatrix} \end{array}$$

where $s$, $v$ and $w$ are positive constants ($-s$ represents the amount of shame one feels by being "chicken", $v$ represents the value of being the "victor", and $-w$ represents the cost of wrecking a car).

   (a) Suppose $w > s > 0$ (this means each player prefers being a chicken to wrecking his car). Find all pure and mixed strategy Nash equilibria.

   (b) Suppose $s > w > 0$ (this means each player would rather wreck his car then be a chicken). Find all pure and mixed strategy Nash equilibria.

3. Consider a two-player symmetric strategic game with matrix

$$\begin{pmatrix} (0,0) & (4,-3) & (-2,-1) \\ (-3,4) & (0,0) & (5,3) \\ (-1,-2) & (3,5) & (0,0) \end{pmatrix}.$$

   (a) Find $E_1(\text{Row } 2, \overrightarrow{\beta})$ if $\overrightarrow{\beta} = (1/5, 2/5, 2/5)$.

(b) Find $U_1(\overrightarrow{\alpha}, \overrightarrow{\beta})$ if $\overrightarrow{\alpha} = (1/4, 0, 3/4)$ and $\overrightarrow{\beta} = (1/2, 1/2, 0)$.

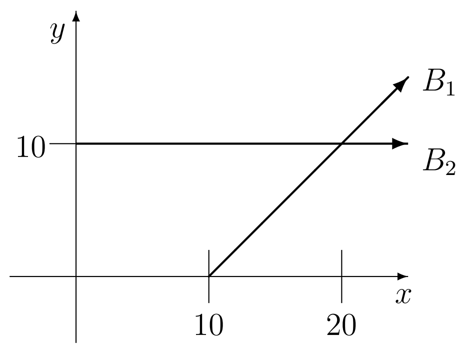
(c) This game has two Nash equilibria: a pure Nash equilibrium where player 1 always chooses the first row and player 2 always chooses the first columnn, and a mixed strategy Nash equilibrium where both players choose their actions according to the probability distribution $(1/3, 1/3, 1/3)$. Determine which one or ones (if any) of these two Nash equilibria are evolutionarily stable strategies; explain your answer.

4. (a) Give an example of a two-player strategic game where each player has two possible actions such that the game has exactly two Nash equilibria, neither of which are strict. It is sufficient to write a matrix which represents this game.

(b) Give an example of a two-player strategic game, where each player's set of possible actions is $[0, \infty)$, which has no pure strategy Nash equilibria. It is sufficient here to give the utility functions for each player.

1. (a) The best response function for player 1, $B_1(y)$, is the number $x$ which maximizes the function $u_1(x, y)$. To maximize this function, take a partial derivative and set equal to 0:
$$\frac{\partial u_1}{\partial x} = 20 + 2y - 2x = 0;$$
solving this for $x$ in terms of $y$ we see $B_1(y) = y + 10$.
(This is in fact a maximum since $\frac{\partial^2 u_1}{\partial x^2} = -2 < 0$.)
The best response function for player 2, $B_2(x)$, is the number $y$ which maximizes the function $u_2(x, y)$. To maximize this function, take a partial derivative and set equal to 0:
$$\frac{\partial u_1}{\partial y} = 40 - 4y = 0;$$
solving this for $y$ we see $B_2(x) = 10$.
(This is in fact a maximum since $\frac{\partial^2 u_2}{\partial y^2} = -4 < 0$.)

(b) Here is the graph:




(c) It is clear from the graph that the only intersection of the best response functions is $(20, 10)$; this is the only Nash equilibrium.

2. (a) First, by inspection there are two pure strategy Nash equilibria: $(K, S)$ and $(S, K)$. Next, we find interior mixed strategy Nash equilibria; suppose player 2's strategy is $(q, 1-q)$ (call this strategy "$q$"). Next we calculate expected payoffs for player 1 against strategy $q$:
$$E_1(\text{Row } 1, q) = -w(q) + v(1-q) = v - (v+w)q;$$
$$E_2(\text{Row } 2, q) = -s(q) - s(1-q) = -s.$$
At an interior mixed strategy Nash equilibrium these are equal; set them equal to get $v - (v+w)q = -s$; solve for $q$ to get
$$q = \frac{v+s}{v+w}.$$

Since $w > s$, this number is less than 1 so it is a valid solution $q$.

Perform the same calculation for player 2 (since this game is symmetric, you will get the same answer so you don't really need to do this); if player 1's strategy is $(p, 1-p)$, we see

$$E_2(\text{Col } 1, p) = -w(p) + v(1-p) = v - (v+w)p;$$

$$E_2(\text{Col } 2, p) = -w(p) + v(1-p) = v - (v+w)p;$$

set thse equal to get $v - (v+w)p = -s$; solve for $p$ to get

$$p = \frac{v+s}{v+w}.$$

We have the symmetric interior mixed strategy Nash equilibrium

$$\left(\frac{v+s}{v+w}, 1 - \frac{v+s}{v+w}\right)$$

(i.e. both players implement this mixed strategy).

(b) First, by inspection there is one pure strategy Nash equilibria: $(K, K)$. To find interior mixed strategy Nash equilibria, we repeat the same calculation as in part (a) to obtain

$$p = q = \frac{v+s}{v+w}.$$

But since $s > w$, here both $p$ and $q$ are larger than 1 so they are not valid strategies. So there is no mixed strategy Nash equilibrium here (other than the pure strategy one described earlier).

*Alternate solution:* If $s > w$, then Row 1 strictly dominates Row 2 and Column 1 strictly dominates Column 2. Therefore, since no mixed strategy Nash equilibrium can assign positive probability to any strictly dominated row or column, the only Nash equilibrium is the pure one $(K, K)$.

3. (a) $E_1(\text{Row } 2, \overrightarrow{\beta}) = (-3)(1/5) + (0)(2/5) + (5)(2/5) = 7/5$.

(b) $U_1(\overrightarrow{\alpha}, \overrightarrow{\beta}) = \sum_{j=1}^{m} \sum_{k=1}^{m} \alpha_j \beta_k u_1(a_j, b_k)$; calculating this explicitly and ignoring terms where $\alpha_j$ or $\beta_k$ is zero, we obtain $(1/4)(1/2)(0) + (1/4)(1/2)(4) + (3/4)(1/2)(-1)$ $+ (3/4)(1/2)(3) = 4/8 - 3/8 + 9/8 = 10/8 = 5/4$.

(c) The second Nash equilibrium is an interior mixed strategy Nash equilibrium; since there is a second Nash equilibrium, this cannot be an ESS by part (4) of the big theorem discussed in class last week.

*Alternate solution:* Let $\overrightarrow{\alpha} = (1/3, 1/3, 1/3)$. We need to show $U(\overrightarrow{\alpha}, \overrightarrow{\beta}) \leq U(\overrightarrow{\beta}, \overrightarrow{\beta})$ for some $\overrightarrow{\beta} \neq \overrightarrow{\alpha}$ which is a best response to $\overrightarrow{\alpha}$. But since $\overrightarrow{\alpha}$ is an interior, symmetric, mixed strategy Nash equilibrium, it must be the case that

$B_1(\overrightarrow{\alpha}) = P(A_1)$, that is, that all distributions are best responses to $\overrightarrow{\alpha}$. So therefore if we can show $U(\overrightarrow{\alpha}, \overrightarrow{\beta}) \leq U(\overrightarrow{\beta}, \overrightarrow{\beta})$ for any mixed strategy $\overrightarrow{\beta}$, then we will know $\overrightarrow{\alpha}$ is not an ESS. Try $\overrightarrow{\beta} = (0, 1/2, 1/2)$;

$$U(\overrightarrow{\beta}, \overrightarrow{\beta}) = \frac{1}{4}(5+3) = 2;$$

$$U(\overrightarrow{\alpha}, \overrightarrow{\beta}) = \frac{1}{6}(4 + (-2) + 5 + 3) = 5/3 < 2$$

so for this $\overrightarrow{\beta}$ we have $U(\overrightarrow{\alpha}, \overrightarrow{\beta}) \leq U(\overrightarrow{\beta}, \overrightarrow{\beta})$ and therefore $\overrightarrow{\alpha}$ is not an ESS.

For the first Nash equilibrium, set $\overrightarrow{\alpha} = (1, 0, 0)$ to be this strategy; we see $U(\overrightarrow{\alpha}, \overrightarrow{\alpha}) = 0$. Now let $\overrightarrow{\beta} = (\beta_1, \beta_2, \beta_3)$ be an arbitrary mixed strategy. Observe $U(\overrightarrow{\beta}, \overrightarrow{\alpha}) = -3\beta_2 - \beta_3$ which is less than 0 whenever $\overrightarrow{\beta} \neq \overrightarrow{\alpha}$, so this Nash equilibrium is strict and consequently this is an ESS.

4. (a) A simple example is

$$\begin{pmatrix} (1,1) & (0,0) \\ (1,1) & (0,0) \end{pmatrix};$$

the upper left and lower left action profiles are non-strict Nash equilibria and the other two action profiles are not Nash equilibria.

(b) Set $u_1(x, y) = x$ and $u_2(x, y) = y$; no $(x, y)$ can be a Nash equilibria because both players always have incentive to pick bigger and bigger numbers, no matter what numbers they pick.