MATH 130
Exam 5 Study Guide

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Chapter 1

Exam 5 Information

1.1 Exam 5 content

Exam 5 covers Chapter 5 in the 2024 version of my MATH 130 lecture notes.

1.2 Tasks for Exam 5

NOTE: This guide is not meant to be an exact representation of exam material. I always reserve the right to ask some questions that use the course material in a creative way.

1. Compute the inverse of a function
2. Solve equations, including:
    - any equation where the variable appears once;
    - any equation with only one type of like term;
    - any quadratic or quadratic-type equation;
    - any equation that can be easily factored;
    - rational equations;
    - proportions;
    - and equations involving tricks (log rules, trig identites, equations with complicated square root expressions, etc.)

Chapter 2

Old MATH 130 Exam 5s

1. (5.2) Match each function in the left-hand column to its inverse:

$\arctan \sqrt[3]{x+1}$	A. $\tan^3 x - 1$
$\arctan \sqrt[3]{x} + 1$	B. $\tan^3(x-1)$
$\sqrt[3]{\arctan x} + 1$	C. $\tan(x-1)^3$
$\sqrt[3]{\arctan x + 1}$	D. $\tan(x^3 - 1)$

In Problems 2-8, solve each equation. Answers should be reasonably simplified.

2. (5.2) $\cos \dfrac{x}{2} = \dfrac{\sqrt{3}}{2}$
3. (5.2) $3|x-7| = 6$
4. (5.3) $e^x - 6 - 16e^{-x} = 0$
5. (5.4) $x^4 - 13x^3 + 42x^2 = 0$
6. (5.2) $1 - \sqrt[3]{x^2+2} = -2$
7. (5.3) $3x^4 + 1 = 4x^2$
8. (5.2) $\ln(2 + 3\ln x) = -3$

Solutions

1. To compute their inverses, diagram each function and work backwards, starting with x. In order:

$$x \xrightarrow{+1} \quad \xrightarrow{\sqrt[3]{\ }} \quad \xrightarrow{\arctan} f(x)$$
$$\tan^3 x - 1 \xleftarrow[-1]{} \tan^3 x \xleftarrow[\wedge 3]{} \tan x \xleftarrow[\tan]{} x$$

This is choice $\boxed{\textbf{A}}$.

$$x \xrightarrow{\sqrt[3]{\ }} \quad \xrightarrow{\arctan} \quad \xrightarrow{+1} f(x)$$
$$\tan^3(x-1) \xleftarrow[\wedge 3]{} \tan(x-1) \xleftarrow[\tan]{} x-1 \xleftarrow[-1]{} x$$

This is choice $\boxed{\textbf{B}}$.

$$x \xrightarrow{\arctan} \quad \xrightarrow{\sqrt[3]{\ }} \quad \xrightarrow{+1} f(x)$$
$$\tan(x-1)^3 \xleftarrow[\tan]{} (x-1)^3 \xleftarrow[\wedge 3]{} x-1 \xleftarrow[-1]{} x$$

This is choice $\boxed{\textbf{C}}$.

$$x \xrightarrow{\arctan} \quad \xrightarrow{+1} \quad \xrightarrow{\sqrt[3]{\ }} f(x)$$
$$\tan(x^3-1) \xleftarrow[\tan]{} x^3 - 1 \xleftarrow[-1]{} x^3 \xleftarrow[\wedge 3]{} x$$

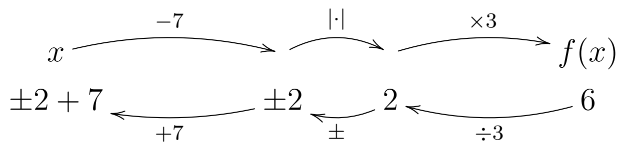
This is choice $\boxed{\textbf{D}}$.

2. x only appears once in this equation, so we treat the equation as $f(x) = b$, diagram the left-hand side and work backwards:

$$x \xrightarrow{\div 2} \quad \xrightarrow{\cos} f(x)$$
$$2\left(2\pi n \pm \tfrac{\pi}{6}\right) \xleftarrow[\times 2]{} 2\pi n \pm \tfrac{\pi}{6} \xleftarrow[2\pi n \pm \arccos]{} \tfrac{\sqrt{3}}{2}$$

Therefore $x = 2\left(2\pi n \pm \dfrac{\pi}{6}\right)$, i.e. $\boxed{x = 4\pi n \pm \dfrac{\pi}{3}}$.

3. x only appears once in this equation, so we treat the equation as $f(x) = b$, diagram the left-hand side and work backwards:



Therefore $x = \pm 2 + 7$, i.e. $x = -2 + 7 = 5$ or $x = 2 + 7 = 9$, meaning $\boxed{x = 5, 9}$.

4. Rewrite this as $e^x - 6 - \frac{16}{e^x} = 0$; then since this contains an e^x term and an $\frac{1}{e^x}$ term, multiply through by e^x to identify the equation as quadratic-type in e^x:

$$\begin{aligned} e^x - 6 - \frac{16}{e^x} &= 0 \\ e^x\left(e^x - 6 - \frac{16}{e^x}\right) &= e^x(0) \\ e^{2x} - 6e^x - 16 &= 0 \\ (e^x - 8)(e^x + 2) &= 0 \end{aligned}$$

$$\swarrow \qquad \searrow$$

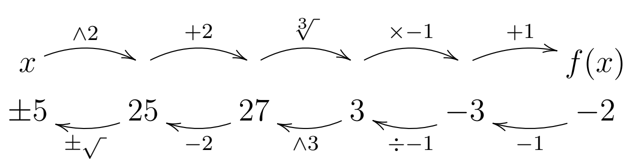
$$\begin{aligned} e^x - 8 &= 0 & \qquad e^x + 2 &= 0 \\ e^x &= 8 & e^x &= -2 \\ x &= \ln 8 & x &= \ln(-2) \text{ which DNE} \\ \boxed{x = \ln 8} \end{aligned}$$

5. Factor the left-hand side completely:

$$\begin{aligned} x^4 - 13x^3 + 42x^2 &= 0 \\ x^2(x^2 - 13x + 42) &= 0 \\ x^2(x-7)(x-6) &= 0 \end{aligned}$$

From $x^2 = 0$, we get $x = 0$; from $x - 7 = 0$, we get $x = 7$; from $x - 6 = 0$ we get $x = 6$. All together, the solutions are $\boxed{x = 0, 6, 7}$.

6. x only appears once in this equation, so we treat the equation as $f(x) = b$, diagram the left-hand side and work backwards:



Therefore $\boxed{x = \pm 5}$.

7. This is quadratic-type in x^2:

$$\begin{aligned} 3x^4 + 1 &= 4x^2 \\ 3x^4 - 4x^2 + 1 &= 0 \\ (3x^2 - 1)(x^2 - 1) &= 0 \end{aligned}$$

   From the left-hand factor, we get $3x^2 = 1$, i.e. $x^2 = \dfrac{1}{3}$, i.e. $x = \pm\sqrt{\dfrac{1}{3}}$. From the right-hand factor, we get $x^2 = 1$, i.e, $x = \pm\sqrt{1} = \pm 1$. So there are four solutions: $\boxed{x = \pm 1, \pm\sqrt{\dfrac{1}{3}}}$.

8. x only appears once in this equation, so we treat the equation as $f(x) = b$, diagram the left-hand side and work backwards:

$$x \xrightarrow{\ln} \quad \xrightarrow{\times 3} \quad \xrightarrow{+2} \quad \xrightarrow{\ln} f(x)$$

$$\exp\left(\tfrac{e^{-3}-2}{3}\right) \xleftarrow[\exp]{} \tfrac{e^{-3}-2}{3} \xleftarrow[\div 3]{} e^{-3} - 2 \xleftarrow[-2]{} e^{-3} \xleftarrow[\exp]{} -3$$

   Therefore $\boxed{x = \exp\left(\dfrac{e^{-3} - 2}{3}\right)}$.

2.1 Relevant exam questions from Spring 2018

1. Solve each equation:

   a) $e^{2x-1} = 17$ b) $\dfrac{3}{x+2} + 1 = \dfrac{2}{x-1}$ c) $6\sin^2 x = 6$

2. Solve each equation:

   a) $\ln(e^{x/3} + 1) = 4$ b) $2\sqrt{2x^2 - 5} - 7 = -3$ c) $e^x + 30e^{-x} = 13$

3. Choose any two of the following five equations and solve them.

   a) $\sin^2 x = 2\cos x + 1$
   b) $5^{x-2} = 2^x$
   c) $x^{5/2} - 7x^2 = 18x^{3/2}$
   d) $\log_3(8x + 3) + \log_3 x = 4$
   e) $x + 4 = \sqrt{8 - x}$

4. Solve each of the following equations:

   a) $e^{3-x} = 6$
   b) $|x + 3| - 2 = 0$
   c) $\sqrt{x^2 + 4} = 3$
   d) $2\sin 4x = 1$

5. Solve each of the following equations:

   a) $e^{2x} - e^x - 6 = 0$
   b) $x^5 + 4x^4 = 32x^3$
   c) $\sin x - \cos x = 0$

Solutions

1. a) x only appears once, so we can use arrow diagrams:

$$\begin{array}{ccccccc} x & \xrightarrow{2x} & & \xrightarrow{x-1} & & \xrightarrow{e^x} & 17 \\ \frac{1}{2}(\ln 17 + 1) & \xleftarrow[\frac{1}{2}x]{} & \ln 17 + 1 & \xleftarrow[x+1]{} & \ln 17 & \xleftarrow[\ln x]{} & \end{array}$$

Therefore $\boxed{x = \dfrac{1}{2}(\ln 17 + 1)}$.

b) This is a rational equation with least common denominator $(x+2)(x-1)$. Multiply through by this common denominator to get

$$(x+2)(x-1)\frac{3}{x+2} + (x+2)(x-1)(1) = (x+2)(x-1)\frac{2}{x-1};$$

cancel to get

$$\begin{aligned} 3(x-1) + (x+2)(x-1) &= 2(x+2) \\ 3x - 3 + x^2 + x - 2 &= 2x + 4 \\ x^2 + 4x - 5 &= 2x + 4 \\ x^2 + 2x - 9 &= 0 \end{aligned}$$

Last, solve this with the quadratic formula to get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4(1)(-9)}}{2} = \boxed{\frac{-2 \pm \sqrt{40}}{2}}.$$

(This answer simplifies to $x = -1 \pm \sqrt{10}$, but you need not simplify it.)

c) x only occurs once, so we can use arrow diagrams:

$$\begin{array}{ccccccc} x & \xrightarrow{\sin x} & & \xrightarrow{x^2} & & \xrightarrow{6x} & 6 \\ \frac{\pi}{2} + 2\pi n, \frac{-\pi}{2} + 2\pi n & \xleftarrow[\substack{\arcsin x + 2\pi n \\ \pi - \arcsin x + 2\pi n}]{} & \pm 1 & \xleftarrow[\pm\sqrt{x}]{} & 1 & \xleftarrow[\frac{1}{6}x]{} & \end{array}$$

Therefore $\boxed{x = \pm\frac{\pi}{2} + 2\pi n}$.

2. a) x only occurs once, so we can use arrow diagrams:

$$\begin{array}{ccccccccc} x & \xrightarrow{x/3} & & \xrightarrow{e^x} & & \xrightarrow{x+1} & & \xrightarrow{\ln x} & 4 \\ 3\ln(e^4 - 1) & \xleftarrow[3x]{} & \ln(e^4 - 1) & \xleftarrow[\ln x]{} & e^4 - 1 & \xleftarrow[x-1]{} & e^4 & \xleftarrow[e^x]{} & \end{array}$$

Therefore $\boxed{x = 3\ln(e^4 - 1)}$.

b) x only occurs once, so we can use arrow diagrams:

$$\begin{array}{ccccccccccccc} x & \xrightarrow{x^2} & & \xrightarrow{2x} & & \xrightarrow{x-5} & & \xrightarrow{\sqrt{x}} & & \xrightarrow{2x} & & \xrightarrow{x-7} & -3 \\ \pm\sqrt{\frac{9}{2}} & \xleftarrow[\pm\sqrt{x}]{} & \frac{9}{2} & \xleftarrow[\frac{1}{2}x]{} & 9 & \xleftarrow[x+5]{} & 4 & \xleftarrow[x^2]{} & 2 & \xleftarrow[\frac{1}{2}x]{} & 4 & \xleftarrow[x+7]{} & \end{array}$$

Therefore $\boxed{x = \pm\sqrt{\frac{9}{2}}}$.

c) Rewrite this as $e^x + \dfrac{30}{e^x} = 13$; since this equation contains "something" and "1 over something", multiply though by the "something" (in this case e^x) to get a quadratic-type equation:

$$\begin{aligned} e^x\left(e^x + \frac{30}{e^x}\right) &= e^x(13) \\ e^{2x} + 30 &= 13e^x \\ e^{2x} - 13e^x + 30 &= 0 \\ (e^x - 10)(e^x - 3) &= 0 \end{aligned}$$

Therefore $e^x - 10 = 0$ or $e^x - 3 = 0$, i.e. $\boxed{x = \ln 10, \ln 3}$.

3. a) For the equation $\sin^2 x = 2\cos x + 1$, use the Pythagorean identity to rewrite $\sin^2 x$ as $1 - \cos^2 x$:

$$\begin{aligned} 1 - \cos^2 x &= 2\cos x + 1 \\ 0 &= \cos^2 x - 2\cos x \\ 0 &= \cos x(\cos x - 2) \end{aligned}$$

Therefore $\cos x = 0$ (i.e. $x = \pm\frac{\pi}{2} + 2\pi n$) or $\cos x = 2$ (no solutions). Altogether, we have $\boxed{x = \pm\frac{\pi}{2} + 2\pi n}$.

b) For the equation $5^{x-2} = 2^x$, take the natural logarithm of both sides:

$$\begin{aligned} \ln 5^{x-2} &= \ln 2^x \\ (x-2)\ln 5 &= x\ln 2 \\ x\ln 5 - 2\ln 5 &= x\ln 2 \\ x\ln 5 - x\ln 2 &= 2\ln 5 \\ x(\ln 5 - \ln 2) &= 2\ln 5 \\ x &= \boxed{\frac{2\ln 5}{\ln 5 - \ln 2}}. \end{aligned}$$

(This could be rewritten as $x = \dfrac{\ln 25}{\ln \frac{5}{2}} = \log_{5/2} 25$.)

c) For the equation $x^{5/2} - 7x^2 = 18x^{3/2}$, set one side equal to zero, and factor:

$$\begin{aligned} x^{5/2} - 7x^2 &= 18x^{3/2} \\ x^{5/2} - 7x^2 - 18x^{3/2} &= 0 \\ x^{3/2}(x - 7x^{1/2} - 18) &= 0 \\ x^{3/2}(x^{1/2} - 9)(x^{1/2} + 2) &= 0 \end{aligned}$$

From the first factor, we get $x^{3/2} = 0$, i.e. $x = 0$. From the second factor, we get $x^{1/2} = 9$, i.e. $\sqrt{x} = 9$, i.e. $x = 9^2 = 81$. From the last factor, we get $x^{1/2} = \sqrt{x} = -2$ which has no solution. Altogether, we have $\boxed{x = 0, 81}$.

d) For the equation $\log_3(8x+3) + \log_3 x = 4$, combine using a log rule:

$$\begin{aligned} \log_3(8x+3) + \log_3 x &= 4 \\ \log_3[x(8x+3)] &= 4 \\ \log_3(8x^2+3x) &= 4 \end{aligned}$$

Thinking of the arrow diagram

$$\begin{array}{ccc} 8x^2+3x & \xrightarrow{\log_3 x} & 4 \\ 3^4 = 81 & \xleftarrow[3^x]{} & \end{array}$$

we get $8x^2 + 3x = 81$, i.e.

$$\begin{aligned} 8x^2 + 3x - 81 &= 0 \\ (x-3)(8x+27) &= 0 \end{aligned}$$

From the first factor, we get $x = 3$. From the second factor, we get $x = \dfrac{-27}{8}$. But the second solution doesn't work when plugged into the equation (you would have to take the log of a negative number), so we get only $\boxed{x = 3}$.

e) For the equation $x + 4 = \sqrt{8-x}$, square both sides to get a quadratic:

$$\begin{aligned} x + 4 &= \sqrt{8-x} \\ (x+4)^2 &= 8 - x \\ x^2 + 8x + 16 &= 8 - x \\ x^2 + 9x + 8 &= 0 \\ (x+8)(x+1) &= 0 \end{aligned}$$

Therefore we get two solutions $x = -8, -1$. However, when plugging in $x = -8$ in the original equation, we get $-4 = \sqrt{16}$, which is false. This leaves only the valid solution $\boxed{x = -1}$.

4. In each of these equations, x only appears once, so we can solve each equations with arrows:

a) $e^{3-x} = 6$:

$$\begin{array}{ccccccc} x & \xrightarrow{-x} & & \xrightarrow{x+3} & & \xrightarrow{e^x} & 6 \\ -\ln 6 + 3 & \xleftarrow[-x]{} & \ln 6 - 3 & \xleftarrow[x-3]{} & \ln 6 & \xleftarrow[\ln x]{} & \end{array}$$

Therefore $\boxed{x = -\ln 6 + 3}$.

b) $|x+3|-2=0$:

$$\begin{array}{ccccccc} x & \xrightarrow{x+3} & & \xrightarrow{|x|} & & \xrightarrow{x-2} & 0 \\ -1,-5 & \xleftarrow[x-3]{} & \pm 2 & \xleftarrow[\pm x]{} & 2 & \xleftarrow[x+2]{} & \end{array}$$

Therefore $\boxed{x=-1,-5}$.

c) $\sqrt{x^2+4}=3$:

$$\begin{array}{ccccccc} x & \xrightarrow{x^2} & & \xrightarrow{x+4} & & \xrightarrow{\sqrt{x}} & 3 \\ \pm\sqrt{5} & \xleftarrow[\pm\sqrt{x}]{} & 5 & \xleftarrow[x-4]{} & 9 & \xleftarrow[x^2]{} & \end{array}$$

Therefore $\boxed{x=\pm\sqrt{5}}$.

d) $2\sin 4x=1$:

$$\begin{array}{ccccccc} x & \xrightarrow{4x} & & \xrightarrow{\sin x} & & \xrightarrow{2x} & 1 \\ \frac{\pi}{24}+\frac{\pi}{2n}, \frac{5\pi}{24}+\frac{\pi}{2n} & \xleftarrow[\frac{1}{4}x]{} & \frac{\pi}{6}+2\pi n, \frac{5\pi}{6}+2\pi n & \xleftarrow[\substack{\arcsin x+2\pi n \\ \pi-\arcsin x+2\pi n}]{} & \frac{1}{2} & \xleftarrow[\frac{1}{2}x]{} & \end{array}$$

Therefore $\boxed{x=\dfrac{\pi}{24}+\dfrac{\pi}{2n}, \dfrac{5\pi}{24}+\dfrac{\pi}{2n}}$.

5. a) This equation is quadratic-type, so factor it and set each factor equal to zero:

$$\begin{aligned} e^{2x}-e^x-6&=0 \\ (e^x-3)(e^x+2)&=0 \end{aligned}$$

From the first term, $e^x=3$ so $x=\ln 3$; from the second term we get $e^x=-2$ which has no solution. Thus the only solution is $\boxed{x=\ln 3}$.

b) Move all terms to the left-hand side, then factor and set each factor equal to zero:

$$\begin{aligned} x^5+4x^4&=32x^3 \\ x^5+4x^4-32x^3&=0 \\ x^3(x^2+4x-32)&=0 \\ x^3(x+8)(x-4)&=0 \end{aligned}$$

Therefore $\boxed{x=0, x=-8, x=4}$ are the three solutions.

c) Add $\cos x$ to both sides, divide through by $\cos x$:

$$\begin{aligned} \sin x-\cos x&=0 \\ \sin x&=\cos x \\ \frac{\sin x}{\cos x}&=1 \\ \tan x&=1 \end{aligned}$$

Therefore $\boxed{x = \frac{\pi}{4} + \pi n}$.