Precalculus Lecture Notes

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Contents

Chapter 1

Preliminaries

1.1 Basic mathematics terminology

Expressions

Definition 1.1 *An* **expression** *(or* **mathematical expression***) is a combination of numbers, variables and/or symbols that represents a single object.*

EXAMPLES OF EXPRESSIONS

- 8
- t
- $4\sin\theta\cos 2\theta$
- $17.35 - 12.82 + 3.51$
- $\dfrac{(x+2)(y^2 - 3\cdot 2^y)}{xy^5 - \sqrt{3x^4\cos 3x}} + \left(\dfrac{16+7tx}{3-\frac{1}{4-7y}}\right)^{-3/5}$
- $\Gamma(\mathfrak{G})/\widetilde{H}_2(\mathbb{R},\mathbb{Z})$

Terms

Definition 1.2 *The* **terms** *of an expression are the individual parts of that expression that are added (i.e. separated by + (or −) signs). Reading from left to right, we call the terms the* **first term (of the expression)**, *the* **second term**, *etc.*

A **constant term** *is a term of an expression that has no variables in it.*

The **coefficient** *of a term is the constant being multiplied by variables in the term.*

Two terms are called **like (terms)** *if, other than their coefficients, they are the same. Otherwise the terms are called* **unlike (terms)**.

EXAMPLE 1

Consider this expression:

$$3x + 5 - 7x^2 + 9 - x$$

1. What is the first term of this expression?
2. What is the third term of this expression?
3. Identify the constant term(s) of this expression.
4. What is the coefficient on the first term?
5. What is the coefficient on the last term?
6. Are the second and last terms of this expression like?
7. Identify all the like terms (if any) in this expression.

EXAMPLE 2

Consider this expression:

$$4xy + 7x + 10xy - 2yx.$$

1. What is the coefficient on the third term?
2. Identify all the like terms (if any) in this expression.

Some general rules for legally manipulating terms and/or expressions

Seven things you can always legally do to any expression at any time

1. Write terms in a different order
2. Combine like terms
3. Replace equals with equals
4. Add 0 creatively
5. Multiply by 1 creatively
6. FOIL, distribute and/or factor
7. Apply other legal math rules

Let's go through these seven things, one-by-one:

1. **Write terms in a different order:** you can write the terms of an expression in any order you want. For instance,

   $$7x - 5 + 2x^2 \text{ is the same thing as } 2x^2 + 7x - 5,$$

   which we write as

   $$7x - 5 + 2x^2 = 2x^2 + 7x - 5.$$

2. **Combine like terms** using addition and subtraction:

   $$1 + 2x + 3x^2 + 7x - 5 + 2x^2 = 5x^2 + 9x - 4.$$

   WARNING: Be careful not to combine unlike terms:

   $$5 + 8\cos x \neq 13\cos x$$

3. **Replace equals with equals:** this means that you can substitute/replace any term or expression with an equal quantity. Here are some examples:

You can replace this...	...with this:
$2^2x - 3t$	$4x - 3t$
$7x - 3t$	$4x + 3x - 3t$

   Another example: if you know $s = 8$, then you can replace $s^2 - 4$ with $8^2 - 4$.

We often write these replacements using = signs. For example, the first line of the chart above would usually be written

$$2^2 x - 3t = 4x - 3t.$$

Often, we chain together several steps where, in each step, we replace equals with equals:

$$\begin{aligned} 4(2 \cdot 3^2 - 5) + 7 &= 4(2 \cdot 9 - 5) + 7 \\ &= 4(18 - 5) + 7 \\ &= 4(13) + 7 \\ &= 52 + 7 \\ &= 59. \end{aligned}$$

Make sure when doing this that you follow universally agreed upon rules for order of operations (see Section 1.3).

4. You can **creatively add** 0 to a term or expression at any time.

   An uncreative addition of 0: $\quad 7x = 7x + 0$

   More creative: $\quad x^2 + 8x + 7 = x^2 + 8x + 7 + 16 - 16$

   Why might you do this?

5. You can **creatively multiply by** 1 to a term or expression at any time.

$$\frac{3x+2}{\sqrt{t-4}} = \frac{3x+2}{\sqrt{t-4}} \cdot 1$$

$$\frac{\sin\theta}{\tan\theta} = \frac{\sin\theta}{\tan\theta} \cdot \frac{\cos\theta}{\cos\theta}$$

$$x^2 + 5 = (x^2 + 5)\left(3^2 - 8\right)$$

$$\frac{3x+2}{\sqrt{t-4}} + 7x^2 = \frac{3x+2}{\sqrt{t-4}} \cdot \frac{\sqrt{t-4}}{\sqrt{t-4}} + 7x^2$$

Why might you do this? One reason is to add/subtract fractions:

$$\frac{2}{3} + \frac{3}{5} =$$

6. You can **FOIL, distribute and/or factor** terms or expressions (more on this in Section 1.4).

7. You can **apply other legal rules** of arithmetic and algebra to a term or expression (more on these later, at various points in the course).

Equations

Definition 1.3 *An* **equation** *is a statement that asserts that two expressions are equal. In this situation, we put an = between the two expressions.*

In an equation, the expression before the = is called the **left-hand side (LHS)** *and the expression after the = is called the* **right-hand side (RHS)**.

EXAMPLES OF EQUATIONS

- $3 + 5 = 8$
- $8x + 3xy - 4y^2 = 17$
- $3 \sin 2\theta + 4 \cos^2 \theta = 5 \tan 3\theta - 1$
- $\dfrac{(x+2)(y^2 - 3 \cdot 2^y)}{xy^5 - \sqrt{3x^4 \cos 3x}} + \left(\dfrac{16 + 7tx}{3 - \frac{1}{4-7y}} \right)^{-3/5} = 3x^6 + y^3 \tan(x - 3y^2) - \dfrac{4\sqrt[3]{y+5}}{7t^3 - 2t^2 + 5e^x}$

Equals signs and arrows

An = in math has a very specific meaning. It means that what precedes the = is an expression (i.e. a quantity), and that what follows the = is also an expression, and that those expressions are the same (i.e. equal).

EXAMPLES OF HORRIBLE MISUSE OF EQUALS SIGNS

$$x + 8 = 15 = \boxed{7} \quad \text{or} \quad x + 8 = 15 = x = \boxed{7}.$$

WHAT THAT PERSON ACTUALLY MEANT

$$\begin{aligned} x + 8 &= 15 \\ x &= \boxed{7}. \end{aligned}$$

Arrows in mathematical sentences also have specific meaning. A double arrow $\Rightarrow$ is short for the word "therefore". The arrow $\Rightarrow$ means that whatever comes after the $\Rightarrow$ follows as a logical consequence of what comes before the $\Rightarrow$.

A single arrow $\to$ is used for functions (more on this later) or limits (more in calculus), not logical implication or to indicate steps in a problem.

EXAMPLES OF HORRIBLE MISUSE OF ARROWS

$$5(x+4) \Rightarrow 5x+20 \qquad 5(x+4) \to 5x+20$$

EXAMPLES OF CORRECT USAGE OF $\Rightarrow$

$$x = 7 \;\Rightarrow\; x^2 = 49.$$

$$x + 8 = 15 \;\Rightarrow\; x = 7.$$

WARNING: I penalize the misuse of arrows and equals signs. This is a pet peeve of mine.

EXAMPLE 3

Consider this equation:

$$5x^2 + 3x - 7 = 4x^2 - 2x^2 + 9$$

1. What is the second term on the left-hand side?

   Answer: $3x$

2. What is the constant term on the right-hand side?

   Answer: 9

3. Which side of the equation contains like terms?

   Answer: the RHS ($4x^2$ and $-2x^2$ are like)

4. What is the x term on the left-hand side?

   Answer: $3x$

5. What is the coefficient on the first term of the left-hand side?

   Answer: 5

General rules for manipulating equations

Equations are like balanced scales. Whenever you change the value of one side of an equation, you must change the value of the other side in the same way (to keep the scale balanced).

EXAMPLE 4

Consider this equation:

$$3x^2 + 4x - 2 = 10x^2 - 9x + 12$$

1. Add $3x$ to both sides of this equation.

2. If this equation is manipulated so that the left-hand side becomes $4x - 2$, what must the right-hand side be?

3. If this equation is manipulated so that the right-hand side becomes $10x^2 - 5x$, what must the left-hand side be?

4. Add or subtract something from both sides of this equation so that the right-hand side becomes 12.

Note: Combining like terms, creatively adding 0 and creatively multiplying by 1 do not change the value of an expression.

So you can do these types of things on one side of an equation without doing them on the other side.

Here is an example of valid math:

$$3\sin 2\theta + 4\cos^2\theta = 5\tan 3\theta - 1$$
$$3\sin 2\theta + 4\cos^2\theta \cdot \frac{\tan^2\theta}{\tan^2\theta} = 5\tan 3\theta - 1 + \cos\theta - \cos\theta$$

But this next "algebra" isn't valid, because the LHS changed in a way the RHS didn't:

$$3\sin 2\theta + 4\cos^2\theta = 5\tan 3\theta - 1$$
$$3\sin 2\theta + 4\cos^2\theta \cdot \tan^2\theta = 5\tan 3\theta - 1$$

Inequalities

Definition 1.4 *An* **inequality** *is a statement that asserts one expression is more than (and/or equal to) another. In this situation, we put $<$, $\leq$, $>$ or $\geq$ between the expressions.*

An inequality is called **strong** *or* **strict** *if it has $<$ or $>$, and is called* **weak** *if it has $\leq$ or $\geq$.*

EXAMPLES OF INEQUALITIES

a) $3 < 5 + 6$ is a strong inequality.

b) $5x + 7 \geq 3 - 4y$ is a weak inequality.

General rules for manipulating inequalities

You manipulate inequalities the same as you would equations, with one exception: *when you multiply or divide by a negative number, the direction of the inequality changes*.

$$5x < 25 \qquad\qquad -\frac{x}{3} \geq -4$$

1.2 Quick review of arithmetic

Multiplication of fractions

To multiply fractions, multiply both the numerators and denominators:

§1.2 EXAMPLE 1

a) $\frac{3}{8} \cdot \frac{5}{7}$

b) $\frac{-8}{9} \cdot \frac{3}{20}$

Negative signs in fractions

A negative sign in a fraction can be in the top, bottom, or out in front; it doesn't change the fraction:

$$-1 \cdot \frac{a}{b} = -\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$$

Of course, negatives in both the numerator and denominator cancel:

$$\frac{-a}{-b} = \frac{a}{b}$$

Division of fractions

The **reciprocal** of fraction $\frac{a}{b}$ is $\frac{1}{\frac{a}{b}} = \frac{b}{a}$ (i.e. flip the fraction over). To divide one fraction by another, multiply the first by the reciprocal of the second:

§1.2 EXAMPLE 2

a) $\frac{3}{8} \div \frac{5}{7}$

b) $\frac{\frac{7}{6}}{\frac{-5}{3}}$

Solution: $\frac{\frac{7}{6}}{\frac{-5}{3}} = \frac{7}{6} \div \frac{-5}{3} = \frac{7}{6} \cdot \frac{3}{-5} = \frac{21}{-30} = \boxed{-\frac{7}{10}}$.

c) $4 \div \left(\frac{3}{8}\right)$

Solution: $4 \div \left(\frac{3}{8}\right) = 4 \cdot \frac{8}{3} = \frac{4}{1} \cdot \frac{8}{3} = \boxed{\frac{32}{3}}$.

d) $\dfrac{\frac{3}{8}}{4}$

e) $\dfrac{3}{\frac{8}{4}}$

WARNING: The answers to (d) and (e) above are different. In general,

$$\frac{\frac{a}{b}}{c} \neq \frac{a}{\frac{b}{c}},$$

so you should avoid writing this:

Addition and subtraction of fractions

Adding/subtracting fractions is harder than multiplying/dividing them, because to add fractions you need to find a ____________________________ .

§1.2 Example 3

a) $\frac{2}{7} + \frac{3}{4}$

b) $\frac{3}{5} - \frac{1}{2} + \frac{7}{8}$

Solution:

$$\frac{3}{5} - \frac{1}{2} + \frac{7}{8} = \frac{3(8)}{5(8)} - \frac{1(20)}{2(20)} + \frac{7(5)}{8(5)} = \frac{24}{40} - \frac{20}{40} + \frac{35}{40} = \frac{24 - 20 + 35}{40} = \boxed{\frac{39}{40}}.$$

c) $\frac{5}{12} + \frac{1}{8}$

Solution: $\frac{5}{12} + \frac{1}{8} = \frac{5(2)}{12(2)} + \frac{1(3)}{8(3)} = \frac{10}{24} + \frac{3}{24} = \boxed{\frac{13}{24}}.$

Whole number exponents

First, a **whole number** is a number like $1, 2, 3, 4.....$

> **Definition 1.5** *If n is a whole number, then the expression x^n is short for the product of n copies of x:*
> $$x^n = x \cdot x \cdot x \cdots x$$
> *In this context, x^n is called the n^{th}* **power** *of x; x is called the* **base***; and n is called the* **exponent***.*

QUICK EXAMPLES

a) $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = \boxed{32}$

We pronounce this "2 to the 5^{th} power is 32".

In this expression, 2 is the base and 5 is the exponent.

b) $\left(\dfrac{5}{3}\right)^3 = \dfrac{5}{3} \cdot \dfrac{5}{3} \cdot \dfrac{5}{3} = \boxed{\dfrac{125}{9}}$

c) $-3^4 = -(3 \cdot 3 \cdot 3 \cdot 3) = \boxed{-81}$

d) $(-3)^4 = (-3)(-3)(-3)(-3) = \boxed{81}$.

Exponent rules

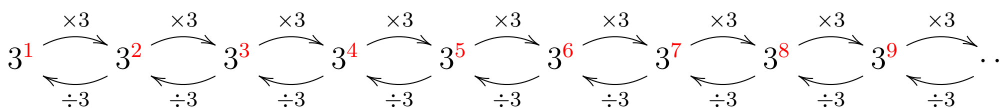
In calculus, it is vital to be able to manipulate expressions involving exponents. This manipulation involves appliction of what are called *rules (or laws) of exponents*. To understand exponent rules, let's start by considering a picture which lists the powers of 3:

$$3 \qquad 9 \qquad 27 \qquad 81 \qquad 243 \qquad 729 \qquad \cdots$$

Writing these numbers as exponents, the same picture is

$$3^{1} \underset{\div 3}{\overset{\times 3}{\rightleftarrows}} 3^{2} \underset{\div 3}{\overset{\times 3}{\rightleftarrows}} 3^{3} \underset{\div 3}{\overset{\times 3}{\rightleftarrows}} 3^{4} \underset{\div 3}{\overset{\times 3}{\rightleftarrows}} 3^{5} \underset{\div 3}{\overset{\times 3}{\rightleftarrows}} 3^{6} \underset{\div 3}{\overset{\times 3}{\rightleftarrows}} \cdots$$

The exponents (shown in red) make a number line (of sorts), where multiplying by 3 moves you one unit to the right and dividing by 3 moves you one unit to the left.



OBSERVATION # 1

Multiplying by 3^b corresponds to moving b places to the right on this diagram. So if you start at 3^a and multiply by 3^b, you go from position a to position ________ . Writing this idea as an exponent rule, this is

OBSERVATION # 2

Suppose you multiply by 3^a b times. Each time you multiply by 3^a, you move to the right a units, so if you do this b times, you have moved to the right a total of $\square$ units, which corresponds to multiplying by $\square$. Writing this as an exponent rule, we have

OBSERVATION # 3

If I wanted to know what 3^0 is, for the diagram to make sense I would have to get to 3^0 by starting at ______ and ________________ by 3. This makes

$$3^0 = 3^1 \div 3 = \square.$$

OBSERVATION # 4

Dividing by 3^b is equivalent to moving b units to the left. So if you start at 3^a and divide by 3^b, you go from position a to position ________ . Writing this idea as an exponent rule, we have

There's nothing special about the 3 in these observations. They hold for any base x, meaning that we've derived these rules:

Theorem 1.6 (Laws of exponents)

Anything[a] to zero power is 1: $x^0 = 1$

Anything to first power is itself: $x^1 = x$

To multiply powers, add the exponents: $x^a x^b = x^{a+b}$.

Iterated exponents multiply: $(x^a)^b = x^{ab}$

To divide powers, subtract the exponents: $\dfrac{x^a}{x^b} = x^{a-b}$

[a]One caveat: if $x = 0$, then $x^0 = 0^0$ which is tricky... you need calculus to evaluate 0^0.

Theorem 1.7 (Exponents respect multiplication and division)

$$(xy)^a = x^a y^a \qquad \left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

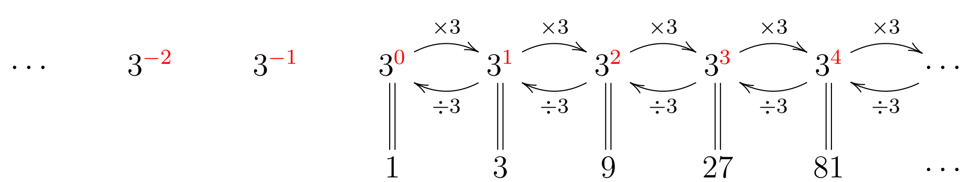
WARNING: Exponents do not respect addition and subtraction: in general,

$$(x+y)^a \neq x^a + y^a \qquad (x-y)^a \neq x^a - y^a$$

Integer exponents

First, an **integer** is a positive or negative whole number, or zero. Integers are numbers like ..., $-3, -2, -1, 0, 1, 2, 3$, ...

We know what happens when an exponent is either a positive whole number or zero. What if the exponent is negative?



$$
\begin{array}{ccccccccccccc}
\cdots & 3^{-a} & \cdots & 3^{-3} & 3^{-2} & 3^{-1} & 3^{0} & 3^{1} & 3^{2} & 3^{3} & \cdots & 3^{a} & \cdots \\
 & \| & & \| & \| & \| & \| & \| & \| & \| & & \| & \\
\cdots & \frac{1}{3^{a}} & \cdots & \frac{1}{27} & \frac{1}{9} & \frac{1}{3} & 1 & 3 & 9 & 27 & \cdots & 3^{a} & \cdots
\end{array}
$$

Again, there's nothing special about 3 (we could use any base). This leads to the following rule:

Theorem 1.8 (Negative exponents)

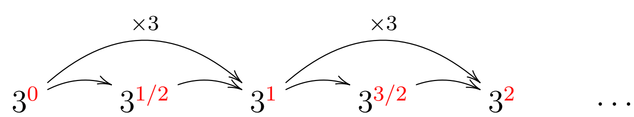
$$x^{-a} = \frac{1}{x^a} = \left(\frac{1}{x}\right)^a .$$

Rational exponents, square roots and other radicals

First, a **rational number** is any number that can be written as the quotient of two integers. These include

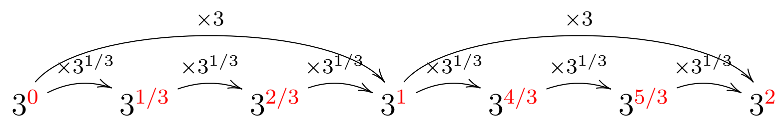
$$\frac{5}{3}; \quad \frac{-1985445}{979438}; \quad 19 \qquad\qquad 8.27$$

Continuing with our diagram with powers of 3, let's try to think about what 3^a would be if a was a rational number that wasn't an integer. Let's start with $a = \frac{1}{2}$:



So $3^{1/2}$ must be a number which, when multiplied by itself, gives 3. This is what we call the **square root** of 3 and denote by $\sqrt{3}$. We conclude:

Now, what about $3^{1/3}$? Here, we think of the diagram



We can see from this picture that $3^{1/3} = \boxed{\quad\quad}$ and that $3^{4/3} = \boxed{\quad\quad\quad\quad\quad\quad}$.

Similarly, $3^{1/n} = \boxed{\quad\quad}$ and that $3^{m/n} = \boxed{\quad\quad\quad\quad\quad\quad}$.

More generally, there is nothing special about using 3 as a base. We have these rules for rational (fractional) exponents:

Theorem 1.9 (Rational exponents)

$$\sqrt{x} = x^{1/2} \qquad \sqrt[n]{x} = x^{1/n} \qquad x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$$

Expressions involving a $\sqrt{\ }$, $\sqrt[3]{\ }$ or $\sqrt[n]{\ }$ are called **radicals** (these signs are called **radical signs**). Since radicals are effectively rewritten exponents, they respect multiplication and division but not addition and subtraction:

Theorem 1.10 (Radicals respect multiplication and division)

$$\sqrt{xy} = \sqrt{x}\,\sqrt{y} \qquad \sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y} \qquad \sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}} \qquad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

WARNING: Radicals do not respect addition and subtraction: in general,

$$\sqrt{x \pm y} \neq \sqrt{x} \pm \sqrt{y} \qquad \sqrt[n]{x \pm y} \neq \sqrt[n]{x} \pm \sqrt[n]{y}$$

One more thing about square roots:. There is a difference between the statements

$$x^2 = c \qquad \text{and} \qquad x = \sqrt{c}.$$

Square roots (and other even roots) are never allowed to be negative. So

$$\sqrt{25} = 5 \text{ (not } \pm 5\text{).}$$

However, if we have the equation $x^2 = 25$, then x might be negative, so $x^2 = 25$ leads to $x = \pm\sqrt{25} = \pm 5$ (not just 5).

Examples

§1.2 EXAMPLE 1

Evaluate each expression:

1. 3^0
2. 0^3
3. 0^{-3}
4. $\sqrt{24}\sqrt{6}$
5. $\dfrac{4^8}{4^5}$
6. $32^{2/5}$
7. $27^{-2/3}$
8. $7^{4/5}\sqrt[5]{7}$
9. $\sqrt{(-8)^3}$

§1.2 EXAMPLE 2

Simplify each expression (by combining the exponents as much as possible):

1. $\sqrt[4]{b^3}\sqrt{b}\sqrt[4]{b^7}$

2. $(3x^2y^3)y^2$

   Solution: $(3x^2y^3)y^2 = 3x^2y^{3+2} = \boxed{3x^2y^5}$.

3. $\left(\dfrac{2x^2}{3\sqrt{y}}\right)^4 \left(\dfrac{4y^2}{12x}\right)$

§1.2 EXAMPLE 3

Rewrite each expression using radical signs, so that it contains no fractional or negative exponents:

1. $x^{1/3}$

   Solution:. $x^{1/3} = \boxed{\sqrt[3]{x}}$.

2. $x^{-2/3}$

   Solution:. $x^{-2/3} = \dfrac{1}{x^{2/3}} = \boxed{\dfrac{1}{\sqrt[3]{x^2}}}$. $\left(\boxed{\dfrac{1}{(\sqrt[3]{x})^2}}\right.$ is also correct.)

3. $19^{1/2}$

   Solution:. $19^{1/2} = \boxed{\sqrt{19}}$.

4. $17^{2/3}50^{3/4}x^{1/8}$

   Solution:. $17^{2/3}50^{3/4}x^{1/8} = \boxed{\sqrt[3]{17^2}\sqrt[4]{50^3}\sqrt[8]{x}}$.

§1.2 EXAMPLE 4 (VERY IMPORTANT IN CALCULUS)

Rewrite each given expression in the form $\square x^{\square}$, where the boxes are numbers:

1. $\dfrac{8x^7}{4x^2}$
2. $2x^2(3x^3)^2$
3. $\dfrac{1}{5\sqrt{x}}$
4. $\dfrac{3}{x}$
5. $\sqrt[5]{x}$
6. $\dfrac{4}{x^5}$
7. $\dfrac{3}{12x^{-2}}$
8. $\dfrac{1}{\sqrt{x^5}}$
9. $\sqrt{2x^5}$
10. $\left(\dfrac{3x}{5}\right)^{-1}$
11. $x^3\sqrt{x}$
12. $\sqrt[3]{64x^3}$
13. $\dfrac{3x^{-3}}{5x^{-2}}$
14. 4

1.3 Order of operations

MOTIVATION

Consider these two expressions:

$$4 + 3 \cdot 5 \qquad\qquad 18 \div 3 \times 2$$

A reasonable person might interpret either of these expressions in two ways:

	$4 + 3 \cdot 5$	$18 \div 3 \times 2$
Reasonable person # 1:		
Reasonable person # 2:		

If we don't decree one of these people to be wrong, this is a big problem!

We need universally agreed-upon procedures for evaluating complicated expressions like these.

This means we have to know what to do first, and in what order to do things.

In other words, we have to have a standard **order of operations** that everyone agrees (has agreed) to follow.

Problems with PEMDAS

You may be aware of the expression "PEMDAS" which is frequently used to teach order of operations. This is not a bad thing to know, but it is dangerous.

- There is some batshit crazy wrong stuff related to PEMDAS / order of operations floating around on places like YouTube. Be very careful looking at this stuff.
- PEMDAS doesn't mention functions like $\sin$, $\log$, $\cos$, etc. How do functions fit into order of operations? PEMDAS doesn't say (we'll talk about this in Chapter 2).
- PEMDAS makes it seem like M comes before D. But in reality, M and D are "tied", in that you work out all multiplication and division from left to right (you don't do all the multiplication before all the division). A and S are similar.

In other words, PEMDAS should really be thought of as

$$\text{P} - \text{E} - \binom{\text{M}}{\text{D}} - \binom{\text{A}}{\text{S}}.$$

Put another way, here is our order of operations (for now):

1. **P**arentheses (and other grouping symbols like brackets) supersede everything else.

   Be careful! There are often invisible parentheses that are present even if they aren't written. Here is an example:

   $$8 + \frac{3+7}{9-4} =$$

   When writing a horizontal fraction bar, there are always invisible parentheses around the top and bottom of that fraction.

2. **E**xponents, from left-to-right
3. **M**ultiplication and **D**ivision, from left-to-right
4. **A**ddition and **S**ubtraction, from left-to-right

Quick Examples

a) $8 - 12 + 2 = -4 + 2 = \boxed{-2}$.

b) $8 - (12 + 2) = 8 - 14 = \boxed{-6}$.

c) $24 \div 4 \times 2 = 6 \times 2 = \boxed{12}$.

d) $3 + \dfrac{7-3}{2} = 3 + \dfrac{4}{2} = 3 + 2 = \boxed{5}$.

e) $5 \cdot 2^3 - (3^2 + 2) = 5 \cdot 8 - (9 + 2) = 40 - 11 = \boxed{29}$.

f) $(5-1)^2 - 4 \cdot 2 = 4^2 - 8 = 16 - 8 = \boxed{8}$.

1.4 Distributing, FOILing and factoring

Distributing

By **distributing**, we mean applying what is called the **distributive law**, which says that for numbers a, x and y,

$$a(x + y) = ax + ay.$$

This law is based on an area calculation, where we figure the area of a rectangle two different ways:

§1.4 EXAMPLE 1

Distribute, and then simplify by combining like terms:

a) $5(x + 3)$

Solution: $5(x + 3) = \boxed{5x + 15}$.

b) $2x^{3/2}(5x^2 + 8\sqrt{x})$

c) $5(x + 3) + x(x - 2)$

Solution: $5(x + 3) + x(x - 2) = 5x + 15 + x^2 - 2x = \boxed{x^2 + 3x + 15}$.

d) $3(x^2 + 4) - 5(x + 1) + 4(2x^2 + x)$

WARNING:. Distributing is special to multiplication. Operations other than multiplication **do not** distribute!

$$
\begin{aligned}
(x+y)^2 &\neq x^2+y^2 \\
\sin(x+y) &\neq \sin x + \sin y \\
\cos(x+y) &\neq \cos x + \cos y \\
\sqrt{x+y} &\neq \sqrt{x} + \sqrt{y} \\
\frac{1}{x+y} &\neq \frac{1}{x} + \frac{1}{y} \\
|x+y| &\neq |x| + |b| \\
\log(x+y) &\neq \log x + \log y
\end{aligned}
$$

The **only** thing that distributes is multiplication:

$$5(x+y) = 5x+5y \qquad a(x+y) = ax+ay$$

In general, if you see $+$ or $-$ inside parentheses or otherwise grouped, you should think "Aw, crap - I probably cannot do anything simple with this". The presence of addition and/or subtraction makes algebraic manipulation harder, generally speaking.

On the other hand, $\times$ or $\div$ inside parentheses is often not so bad:

$$
\begin{aligned}
(xy)^2 &= x^2y^2 & \left(\frac{x}{y}\right)^a &= \frac{x^a}{y^a} \\
|xy| &= |x|\,|y| & \frac{1}{xy} &= \frac{1}{x}\cdot\frac{1}{y} \\
\sqrt{xy} &= \sqrt{x}\,\sqrt{y} & \sqrt{\frac{x}{y}} &= \frac{\sqrt{x}}{\sqrt{y}} \\
\log(xy) &= \log x + \log y & \log\left(\frac{x}{y}\right) &= \log x - \log y
\end{aligned}
$$

etc.

FOILing

By **FOILing**, we mean applying this law:

$$(a+b)(x+y) = ax + ay + bx + by.$$

We call this **FOILing** because four terms you get come from the **F**irst, **O**utside, **I**nside and **L**ast terms of those being rewritten.

Like distributing, FOILING is also justified by an area calculation:

FOILing a sum and difference

Something special happens when you FOIL the sum and difference of the same terms:

$$(a+b)(a-b) =$$

It is useful to remember this fact:

> **Theorem 1.11 (FOILing a sum and difference)** *For any terms a and b,*
>
> $$(a+b)(a-b) = a^2 - b^2.$$

§1.4 EXAMPLE 2

FOIL (and/or distribute), and then simplify by combining like terms:

a) $(2x+1)(x-3)$

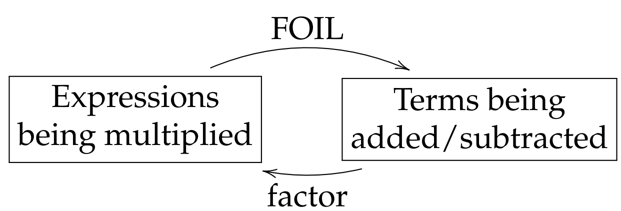
b) $(2x+5)(2x-5)$

c) $(x + 7)(3x - 1) + 2x(x - 5)$

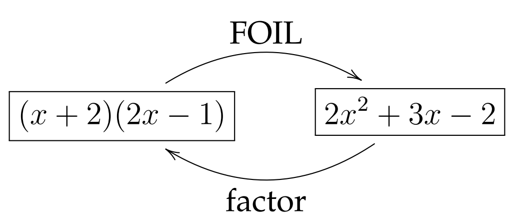
d) $(\sqrt{x} + 2)\left(3x^{3/2} - 5x^{-1/2}\right)$

Factoring

Factoring the "opposite" or "inverse" procedure of FOILING/distributing. Here's the idea:



Here's a specific example of the concept:



To factor an expression, pull out common factors, look for a difference of squares, and then try to factor what's left.

Pulling out common factors

§1.4 EXAMPLE 3

Completely factor each expression:

a) $3x - 12$

b) $5x^3 + 10x^2$

c) $x^2 + 5x$

Factoring a difference of squares

If we see any expression of the form $a^2 - b^2$, we can undo the FOILING rule we saw earlier to write

$$a^2 - b^2 = (a + b)(a - b).$$

This is called **factoring a difference of squares**.

§1.4 EXAMPLE 4

Completely factor each expression:

a) $x^2 - 9$

b) $x^2 - 14$

c) $25t^2 - 81$

d) $x^4 - x^2$

Factoring $x^2 \pm bx \pm c$

Expressions like this should be factored as $(x \pm \Box)(x \pm \triangle)$. I tend to factor these by trial and error, with the object of finding two numbers $\Box$ and $\triangle$ that multiply to the constant term c and add or subtract to give the coefficient b on the x term.

§1.4 EXAMPLE 5

Completely factor each expression:

a) $x^2 - 7x + 12$

b) $x^2 + 8x - 33$

c) $x^5 - 12x^4 + 36x^3$

d) $3x^2 + 9x - 30$

e) $x^2 + x + \frac{1}{4}$

Factoring $ax^2 \pm bx \pm c$

These tend to factor into something like $(\diamondsuit x \pm \square)(\heartsuit x \pm \triangle)$, where the shapes are numbers. If the a isn't a common factor that can be pulled out, then you try to factor these by trial and error, or use the "box method".

§1.4 EXAMPLE 6

Completely factor each expression:

a) $2x^2 + 3x - 20$

b) $18x^2 + 21x - 4$

$18x^2$	
	-4

c) $12x^3 + 4x^2 - 16x$

1.5 The coordinate plane

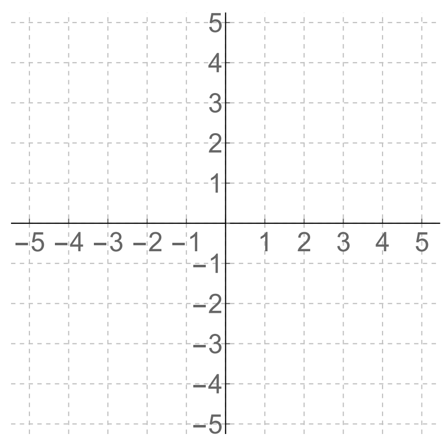
Ordered pairs

Just as real numbers can be thought of as points on a number line, **ordered pairs** (x, y) can be thought of as points in a plane. The first number in an ordered pair is called the **x-coordinate** and measures the horizontal distance the point (x, y) is from the origin; the second number in the pair is called the **y-coordinate** and measures the vertical distance that (x, y) is from the origin.

§1.5 EXAMPLE 1

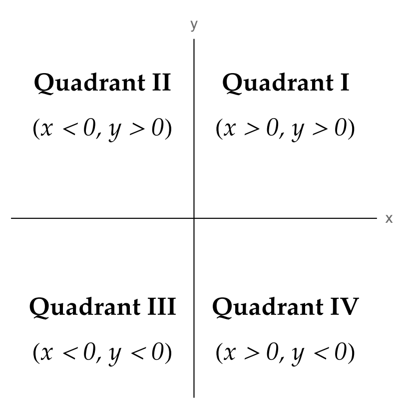
Graph the following points on the provided axes:

$$(0, -3) \qquad (-2, 0) \qquad (5, -1) \qquad (-3, 4)$$



Quadrants

The x- and y-axes divide the coordinate plane into four **quadrants**, numbered by Roman numerals:

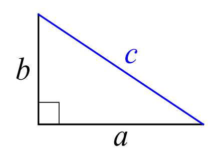


Distance formula & Pythagorean theorem

Recall the Pythagorean Theorem, which relates the lengths of the three sides of a right triangle:

Theorem 1.12 (Pythagorean Theorem) *If $\triangle ABC$ is a right triangle with legs a and b and hypotenuse c, then*

$$a^2 + b^2 = c^2.$$

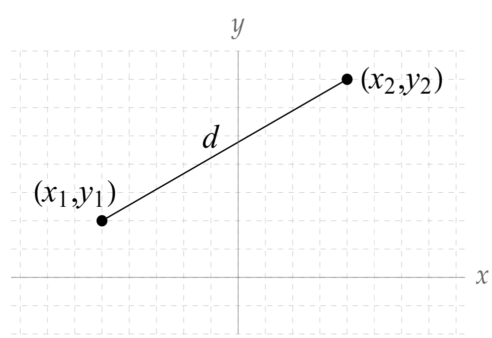


The Pythagorean Theorem can be used to find a formula for the distance between two points in the xy-plane:

Theorem 1.13 (Distance formula) *The distance between points (x_1, y_1) and (x_2, y_2) is*

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

WHY THIS FORMULA IS TRUE



§1.5 EXAMPLE 2

Compute the distance between each given pair of points:

a) $(3, 7)$ and $(-5, 10)$

b) $(2,-3)$ and $(9,1)$

Solution: by the distance formula, this is

$$\begin{aligned} d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} &= \sqrt{(1-(-3))^2+(9-2)^2} \\ &= \sqrt{4^2+7^2} \\ &= \sqrt{16+49} = \boxed{\sqrt{65}}. \end{aligned}$$

c) $(4,0)$ and (x,x^2)

Note: horizontal and vertical distances do not require the distance formula:

§1.5 EXAMPLE 3

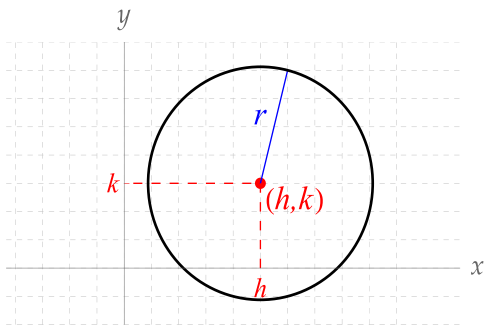
Compute the distance between each given pair of points:

a) $(5,8)$ and $(5,-3)$

b) $(-2,-6)$ and $(-2,-1)$

Equations of circles

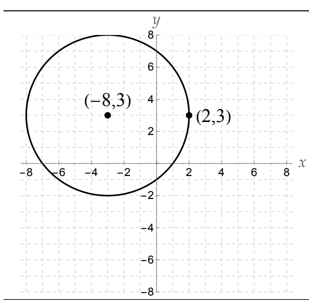
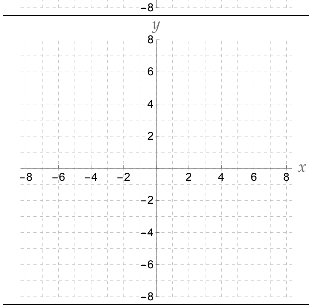
The distance formula can be used to describe the set of points in the xy-plane that lie on any circle. Suppose we have a circle centered at (h, k) with radius r:

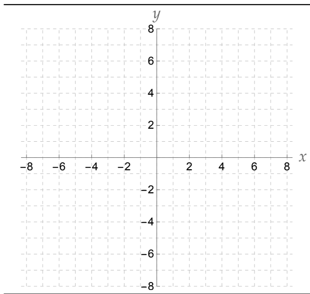


> **Theorem 1.14 (Equations of circles)** *The equation of the circle of radius r centered at (h, k) is*
> $$(x-h)^2 + (y-k)^2 = r^2.$$

§1.5 Example 4

Complete the following table:

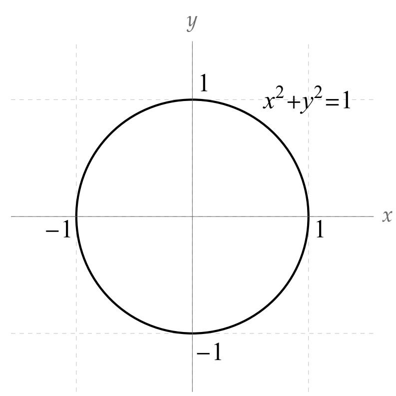
Graph of circle	Equation of circle	Radius Diameter Center
		
	$(x-3)^2 + (y-2)^2 = 4$	

Graph of circle	Equation of circle	Radius Diameter Center
		$r = 3$ $(-3, 0)$

The unit circle

Definition 1.15 (Unit circle) *The circle of radius* 1 *centered at* $(0, 0)$ *is called the* **unit circle**. *Its equation is*

$$x^2 + y^2 = 1.$$



1.6 Quick review of unit circle trigonometry

RECALL

The trigonometric functions sine, cosine and tangent are used to convert *rotational measurements* into *distance measurements*:

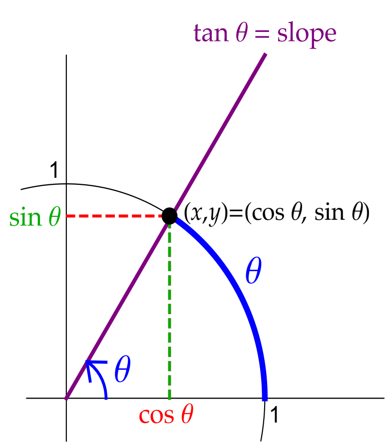
Definition 1.16 (Unit circle definition of sine, cosine and tangent) *Take a number θ. Starting at the point $(1, 0)$, mark off an arc of length θ on the unit circle (go counterclockwise if $\theta > 0$ and clockwise if $\theta < 0$). In other words, mark off an angle of θ radians.*

Call the point on the unit circle where the arc ends (x, y).

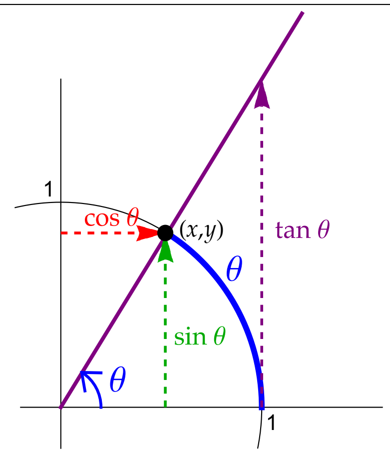
Then define the **sine**, **cosine** *and* **tangent** *of θ to be:*

$$\begin{aligned} \sin\theta = \sin(\theta) &= y \\ \cos\theta = \cos(\theta) &= x \\ \tan\theta = \tan(\theta) &= \frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \text{ slope of the terminal side of } \theta \end{aligned}$$

Trig functions as coordinates



Trig functions as directed distances



In the second picture, we call these "directed" distances because they could be negative (if the arrows point left or down, for example).

§1.5 EXAMPLE 1

Evaluate each quantity:

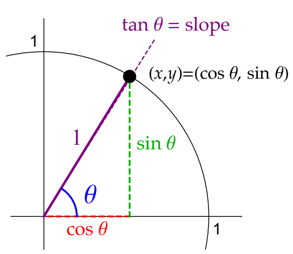
a) $\cos\pi$

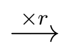
b) $\sin\frac{3\pi}{2}$

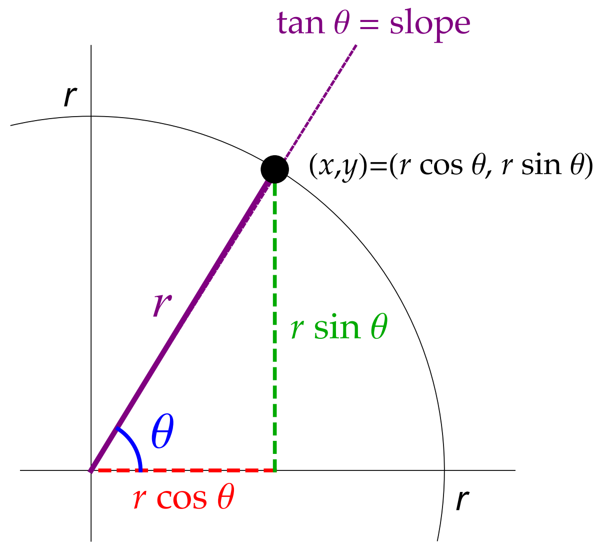
c) $\tan 2\pi$

d) $\cos\frac{-\pi}{2}$

If we are applying trig to a problem where our circle doesn't have radius 1, we think of "blowing up" the previous pictures by a factor of r, where r is the radius of the circle we're considering:



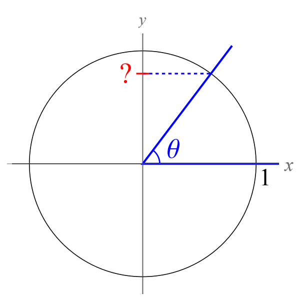




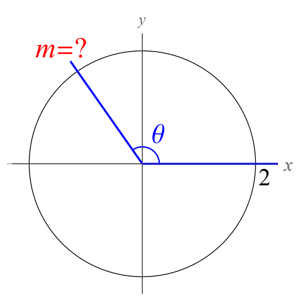
§1.5 EXAMPLE 2

In each picture, you are given a length or coordinate marked with a "?". Write a formula for the "?" in terms of the other numbers and variables given in the picture:

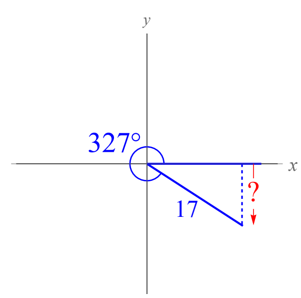
a)



b)



c)



Secant, cosecant and cotangent

There are three other trig functions, which are the reciprocals of the functions we defined above:

Definition 1.17 *Let θ be a number or an angle. Define the* **secant**, **cosecant** *and* **cotangent** *of θ to be, respectively,*

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

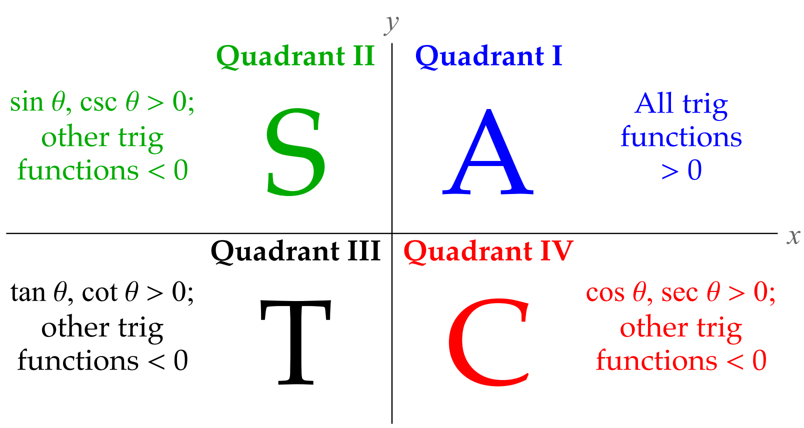
Generally speaking, secant, cosecant and cotangent aren't that useful, but they streamline some computations in calculus, so we'll cover a couple of important things about these functions later.

Signs of the trig functions

Whether or not a trig function of θ is positive or negative depends on the quadrant in which the angle θ lies. We use the phrase

All Scholars Take Calculus

to help remember the signs of the trig functions:



Definition 1.18 *An angle is called* **quadrantal** *if it is a multiple of $\frac{\pi}{2}$ (i.e. is a multiple of 90°). Such angles include*

$$0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \pm 2\pi, \pm\frac{5\pi}{2}, \pm 3\pi, \pm\frac{7\pi}{2}, \ldots$$

These angles do not belong to any of the four quadrants I-IV.

Trig functions of special angles

A "special angle" is any multiple of $30°$ or any multiple of $45°$, i.e. angles like

$$0°, 30°, 60°, 90°, 120°, 150°, -30°, -60°, -90°, -120°, -150°, \dots$$

$$45°, 90°, 135°, 180°, 225°, -45°, -90°, -135°, -180°, -225°, \dots$$

In radians, these special angles are any multiple of $\frac{\pi}{6}$, any multiple of $\frac{\pi}{4}$, any multiple of $\frac{\pi}{3}$, any multiple of $\frac{\pi}{2}$ or any multiple of π:

$$\frac{\pi}{6}, \frac{5\pi}{6}, \frac{-\pi}{6}, \frac{-5\pi}{6}, \frac{7\pi}{6}, \frac{-7\pi}{6}, \frac{\pi}{3}, \frac{-\pi}{3}, \frac{3\pi}{4}, \frac{-5\pi}{4}, \frac{3\pi}{2}, \frac{-5\pi}{2}, 0, \pi, -2\pi, 3\pi, \dots$$

These are the angles you get when you divide a right angle into halves or thirds, and are the most commonly used angles in math courses and in real-world situations.

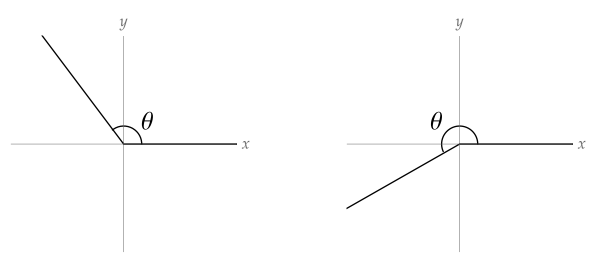
> **DISCLAIMER:** You will pretty much never see degrees used in any math class again. All problems in MATH 130 will be asked in radians, and any problem asking for an angle in MATH 130 is asking for an answer in radians.

That said, most folks are more used to thinking in degrees. If you prefer to think in degrees, there are certain angles (in radians) that you should just "know" how many degrees they correspond to:

$$\pi = 180° \qquad \frac{\pi}{2} = 90° \qquad \frac{\pi}{3} = 60° \qquad \frac{\pi}{4} = 45° \qquad \frac{\pi}{6} = 30°$$

Reference angles

Given angle θ, a reference angle of θ is an angle $\widehat{\theta}$ in Quadrant I which can be obtained from θ by reflecting θ across the x- and/or y-axes:



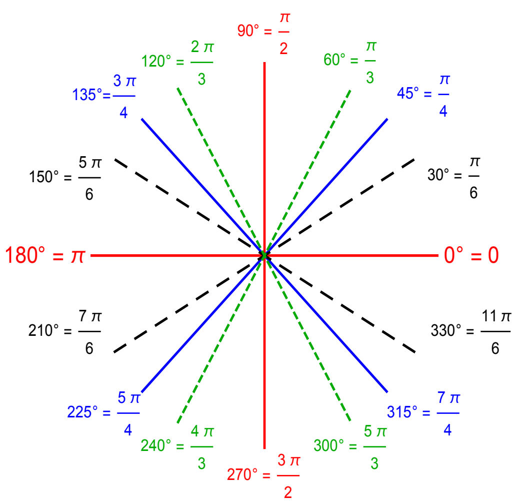
That means that if (x, y) is on the terminal side of $\widehat{\theta}$, then $(\pm x, \pm y)$ is on the terminal side of θ.

Therefore, since the trig functions are computed from such a point, we know:

Theorem 1.19 *If $\widehat{\theta}$ is the reference angle of θ, then*

$$\sin\theta = \pm\sin\widehat{\theta}; \qquad \cos\theta = \pm\cos\widehat{\theta}; \qquad \tan\theta = \pm\tan\widehat{\theta}.$$

It is fairly easy to compute the reference angle of a special angle given in radians:



IF IN LOWEST TERMS:

(whole #) $\pi \longleftrightarrow$

$\dfrac{(\text{odd \#})\,\pi}{2} \longleftrightarrow$

$\dfrac{(\text{whole \#})\,\pi}{4} \longleftrightarrow$

$\dfrac{(\text{whole \#})\,\pi}{3} \longleftrightarrow$

$\dfrac{(\text{whole \#})\,\pi}{6} \longleftrightarrow$

Theorem 1.20 *If θ is in radians, expressed as a fraction $\dfrac{a\pi}{b}$ where $\dfrac{a}{b}$* ***is in lowest terms*** *and $b = 2, 3, 4$, or 6, then the reference angle of θ is $\widehat{\theta} = \dfrac{\pi}{b}$.*

How to compute trig functions of special angles in any quadrant

1. Determine whether or not the angle θ is quadrantal (i.e. whether or not θ is a multiple of $\frac{\pi}{2}$ or π).
2. **If θ is quadrantal**, determine the point on the unit circle at angle θ. This point will be $(\pm 1, 0)$ or $(0, \pm 1)$. Then (like Example 1 earlier):

$$\begin{aligned} \cos\theta &= x & &\longrightarrow \text{ flip over } \cos\theta \text{ to get } \sec\theta \\ \sin\theta &= y & &\longrightarrow \text{ flip over } \sin\theta \text{ to get } \csc\theta \\ \tan\theta &= \text{ slope} & &\longrightarrow \text{ flip over } \cot\theta \text{ to get } \cot\theta \end{aligned}$$

3. **If θ is not quadrantal**,
   a) Determine the reference angle $\widehat{\theta}$ of θ. The ref. angle will be $30°$, $45°$ or $60°$.
   b) Compute $\sin\widehat{\theta}$ or $\cos\widehat{\theta}$ or $\tan\widehat{\theta}$ by remembering the following table (or using a "finger-counting trick"):

Theorem 1.21 (Trig functions of special angles in Quadrant I)

θ *in degrees*	θ *in radians*	$\sin\theta$	$\cos\theta$	$\tan\theta$
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

   c) If asked to compute $\sec\theta$, $\csc\theta$ or $\cot\theta$, flip over the trig function you found in the previous step as needed.
   d) Determine the quadrant θ is in; this will tell you the sign of your final answer based on the "All Scholars Take Calculus" rules.
   e) Your final answer is the $+/-$ sign from step (d) together with the number from steps (b) and (c).

§1.5 EXAMPLE 3

Compute the exact value of each quantity:

a) $\sin \frac{\pi}{6}$

b) $\cos \frac{5\pi}{3}$

c) $\tan \frac{3\pi}{4}$

d) $\cos \frac{-\pi}{3}$

e) $\sin \frac{\pi}{2}$

f) $\sec \frac{2\pi}{3}$

g) $\sin \pi$

h) $\cos 4\pi$

i) $\cos \frac{15\pi}{4}$

j) $\csc \frac{7\pi}{6}$

k) $\tan \frac{4\pi}{3}$

Solution:. $\frac{4\pi}{3} = 4 \cdot 60°$ is in Quadrant III (making the answer positive);

the reference angle is $\widehat{\theta} = \frac{\pi}{3} = 60°$; $\tan \widehat{\theta} = \sqrt{3}$; so $\tan \frac{4\pi}{3} = \boxed{\sqrt{3}}$.

l) $\sin \frac{-5\pi}{3}$

Solution: $\frac{-5\pi}{3} = -5 \cdot 60°$ is in Quadrant I (making the answer positive);

the reference angle is $\widehat{\theta} = \frac{\pi}{3} = 60°$; $\sin \widehat{\theta} = \frac{\sqrt{3}}{2}$; so $\sin \frac{-5\pi}{3} = \boxed{\frac{\sqrt{3}}{2}}$.

m) $\cot \frac{\pi}{6}$

Solution: $\frac{\pi}{6} = 30°$ is in Quadrant I (making the answer positive);

$\tan 30° = \sqrt{3}$; so $\cot 30° = \boxed{\frac{1}{\sqrt{3}}}$.

n) $\cos \frac{3\pi}{2}$

Solution: $\frac{3\pi}{2} = 3 \cdot 90° = 270°$ is quadrantal;

the point on the unit circle is $(0, -1)$; so $\cos \frac{3\pi}{2} = \boxed{0}$.

o) $\tan \frac{3\pi}{2}$

Solution: $\frac{\pi}{2} = 90°$ is quadrantal;

this angle is vertical so its slope is undefined; so $\tan \frac{\pi}{2} \boxed{\text{DNE}}$.

p) $\sin \frac{5\pi}{4}$

Solution:. $\frac{5\pi}{4} = 5 \cdot 45°$ is in Quadrant III (making the answer negative);

the reference angle is $\widehat{\theta} = \frac{\pi}{4} = 45°$;

$\sin \widehat{\theta} = \sin 45° = \frac{\sqrt{2}}{2}$; so $\sin \frac{5\pi}{4} = \boxed{-\frac{\sqrt{2}}{2}}$.

1.7 Chapter 1 Homework

Exercises from Section 1.1

1. Classify each mathematical sentence below as an *expression*, an *equation*, an *inequality*, or *none of the above*:

   a) $4\sin^2\theta = 2\sin\theta + 3$
   b) $x \longleftrightarrow y$
   c) $3x^4 - 5y^2t + 17 = 12xy - 9t^2x$
   d) $4(3^{2y-5}) + 3z$
   e) $3 + 4 \longrightarrow 7$
   f) $8z + 3z^2 - 5 \leq 13z + 2$

2. Consider this equation:

$$2 + \frac{1}{2}x^2 = \frac{1}{2} - \frac{3}{4}x + \frac{1}{4}x^2$$

   a) What is the last term on the right-hand side?
   b) What is the x term on the right-hand side?
   c) What is the x term on the left-hand side?
   d) Manipulate this equation (by adding and subtracting terms from both sides) so that the right-hand side is 3.
   e) Manipulate this equation so that the left-hand side is 0.

3. Classify each inequality as *strict* or *weak*:

   a) $4 < 7$
   b) $x + 3y \geq 9$

4. Combine like terms:

   a) $3 + 4\cos\theta - 1$
   b) $7x + 3x^2 - 4 + 5x + 2x - 8x^2 + 3 + x^2$
   c) $3xy - 2yx + 4x^2y + 3xy^2$

5. Consider this equation:

$$y^2 - 3y + 7 - 4y + 8y^2 - 5 = 4$$

   a) Subtract $2y^2$ from both sides of this equation.
   b) Combine all the like terms on the left-hand side.
   c) Is it *legal* or *illegal* to rewrite this equation as

$$y^2 - 3y + 7 - 4y + 8y^2 - 5 + 9 = 4 + 9?$$

d) Is it *legal* or *illegal* to rewrite this equation as

$$9 + y^2 - 3y + 7 - 4y + 8y^2 - 5 = 4 + 9?$$

e) Is it *legal* or *illegal* to rewrite this equation as

$$2(y^2 - 3y + 7 - 4y + 8y^2 - 5) = 2(4)?$$

f) Is it *legal* or *illegal* to rewrite this equation as

$$2(y^2) - 3y + 7 - 4y + 8y^2 - 5 = 2(4)?$$

g) Is it *legal* or *illegal* to rewrite this equation as

$$(2 - 5 + 4)(y^2 - 3y + 7 - 4y + 8y^2 - 5) = 4?$$

h) Is it *legal* or *illegal* to rewrite this equation as

$$(2 - 5 + 4)y^2 - 3y + 7 - 4y + 8y^2 - 5 = 4?$$

i) Is it *legal* or *illegal* to rewrite this equation as

$$y^2 - 3y + 7 - 4y + 8y^2 - 5 = 4 + 8z - 8z?$$

6. Solve for x in each inequality:

a) $3x + 7 \leq 25$

b) $4 - 9x < 16 - 3x$

Answers

1.
a) equation
b) none of the above
c) equation
d) expression
e) none of the above
f) inequality

2.
a) $\frac{1}{4}x^2$
b) $-\frac{3}{4}x$
c) 0
d) $\frac{1}{4}x^2 + \frac{3}{4}x + \frac{9}{2} = 3$
e) $0 = -\frac{3}{2} - \frac{3}{4}x - \frac{1}{4}x^2$

3.
a) strict
b) weak

4. a) $2 + 4\cos\theta$
   b) $14x - 4x^2 - 1$
   c) $xy + 4x^2y + 3xy^2$

5. a) $y^2 - 3y + 7 - 4y + 8y^2 - 2y^2 = 4 - 2y^2$
   b) $9y^2 - 7y + 2 = 4$
   c) Legal (we added 9 to both sides)
   d) Legal (we added 9 to both sides)
   e) Legal (we multiplied both sides by 2)
   f) Illegal (we didn't multiply through the entire left-hand side by 2)
   g) Legal (this is creative multiplication by 1)
   h) Legal (this is creative multiplication by 1)
   i) Legal (this is creative addition of 0)

6. a) $x \leq 6$
   b) $x > -2$

Exercises from Section 1.2

1. Compute each quantity:

   a) $\frac{2}{7} + \frac{5}{2}$
   b) $\frac{3}{8} \cdot \frac{4}{11}$
   c) $\frac{-3}{5} \div \frac{12}{7}$
   d) $\frac{2}{\frac{-3}{5}}$
   e) $-\frac{1}{3}\left(\frac{2}{3} - \frac{3}{4}\right)$
   f) $-\frac{11}{8} \div 3$
   g) $\frac{4}{3} - \frac{5}{2} + \frac{11}{6}$
   h) $5 - \frac{17}{8}$
   i) $\frac{\frac{1}{2} + \frac{2}{3}}{3 - \frac{7}{4}}$
   j) $\left(\frac{2}{5} - \frac{1}{3}\right)\left(\frac{5}{4} + \frac{1}{2}\right)$

2. Compute each quantity:

   a) 3^4
   b) -2^3
   c) 1^{15}
   d) $0^7 + (-7)^2$
   e) $\left(\frac{2}{3}\right)^5$
   f) $\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^3$
   g) $(-5)^2 + (-3)^3$
   h) $\left(\frac{4}{5}\right)^1 - \left(\frac{2}{5}\right)^2$

3. Simplify each expression using exponent rules:

a) x^5x^2
b) $\dfrac{x^3x^6}{x^4}$
c) $x\dfrac{x^8}{x^3x^2}$
d) $(x^3)^3$
e) $\dfrac{(x^4)^5}{(x^3)^2x^3}$
f) $\left(\left(x^2\right)^4\right)^3$
g) $(x^3)^5x^2(x^4)^2$
h) $\dfrac{x^3(x^2)^4}{x^5(x^3)^2}$

4. Compute each quantity:

a) $\sqrt{121}$
b) $\sqrt{-16}$
c) $\sqrt[3]{27}$
d) $\sqrt[3]{-8}$
e) $\sqrt{16\cdot 25}$
f) $\sqrt[3]{8\cdot 27\cdot 1000}$
g) $\sqrt[5]{32}$
h) $\sqrt[17]{1}$
i) $\sqrt{0}$
j) $\sqrt[4]{625}-\sqrt[4]{1}$
k) $\sqrt[4]{-16}$
l) $\sqrt{49}+2\sqrt{64}$

5. Compute each quantity:

a) $4^{3/2}$
b) $16^{-1/4}$
c) $3^{2/3}\sqrt[3]{3}$
d) $(-9)^{5/2}$
e) $-9^{5/2}$
f) 2^{-5}
g) 7^0
h) $27^{2/3}$
i) 0^50^3

6. Compute each quantity:

a) $\dfrac{80^{1/2}}{20^{1/2}}$
b) $0^{2/3}$
c) $\dfrac{3^{13}3^5}{(3^7)^2}$
d) 5^{-3}
e) $5^{-2}5^4$
f) 1^{-8}
g) $\left(\dfrac{3}{4}\right)^{-2}$
h) $3^{-2}+4^{-1}$
i) $0^{-1/3}$
j) $8^{-3}8^{5/3}$
k) 2^{1-4}
l) $\left(\dfrac{1}{8}\right)^{-1/3}$

7. Simplify each expression, by combining the exponents as much as possible:

a) $(2xy)^2x^3$
b) $(x^7)^{-3}x^5$
c) $\sqrt{t}\sqrt{t}\sqrt{t}$
d) $\sqrt[5]{x^2}\sqrt{x}$
e) $\left(\dfrac{-x^2y}{4}\right)^3$
f) $\left(\dfrac{2x}{y}\right)^{2/3}\left(\dfrac{2}{x}\right)^{-4/3}$
g) $\dfrac{r}{s^{3/4}}\cdot\dfrac{r^2}{\sqrt{s}}\cdot\dfrac{r^{-3/2}}{\sqrt[4]{s}}$
h) $\dfrac{x^{-3}}{\sqrt[3]{y}}\cdot\dfrac{y^3}{x^2}\cdot\sqrt[3]{\dfrac{x^4}{y}}$

8. Rewrite each expression using radical signs, so that it contains no fractional or negative exponents:

a) $t^{2/5}$
b) y^{-5}
c) $x^{1/4}$
d) $x^{3/2}y^{-1}$
e) $23^{2/5}$
f) $\left(\frac{1}{x}\right)^{-4}$
g) $19^{-2/3}24^{-5/3}$
h) $x^{-2}z^{1/2}w^{-3/4}$
i) $(2x)^{-7/3}$
j) $y^0t^3x^{-3}t^{-1/6}$

9. Rewrite each expression in the form $\square x^{\square}$, where the boxes are numbers:

a) $\sqrt[7]{x}$
b) $\sqrt{x^5}$
c) $\frac{1}{x^8}$
d) $\frac{1}{\sqrt[3]{x}}$
e) $\frac{1}{\sqrt[4]{x^3}}$
f) $\sqrt{3x}$
g) $\sqrt{3x^7}$
h) $\frac{7}{-x}$
i) $\left(\frac{2}{x}\right)^3$
j) $\frac{3}{(2x)^2}$

10. Rewrite each expression in the form $\square x^{\square}$, where the boxes are numbers:

a) $\frac{x^{-3}x^2}{x^{-4}x^7}$
b) $(2x^3)^2$
c) $x^5\left(\frac{x^2}{2}\right)^2$
d) 11
e) $\sqrt[3]{x^2}\sqrt[3]{x^4}\sqrt[3]{x}$
f) $\sqrt{x}\sqrt[3]{x}$
g) $\frac{24x^5}{6x^3}$
h) $\sqrt[9]{15x^4}$
i) $\left(\frac{x}{-2}\right)^{-2}$
j) $\left(\frac{3}{x^2}\right)^{-1}$

11. Rewrite each expression in the form $\square x^{\square}$, where the boxes are numbers:

a) $x\sqrt[7]{x}$
b) $x\sqrt{x}\sqrt{3x}$
c) $x^3\sqrt{x}$
d) $(x^2)^3(2x^3)^4x$
e) $(4x^3)^{-2}(2x^2)^3$
f) $\left(\frac{4}{x^3}\right)^{-2}(2x)^4$
g) $\left(\frac{(2x)^2}{x}\right)^3(4x)^{-2}$

Answers

1. a) $\frac{39}{14}$ b) $\frac{3}{22}$ c) $-\frac{7}{20}$ d) $-\frac{10}{3}$ e) $\frac{1}{36}$ f) $-\frac{11}{24}$ g) $\frac{2}{3}$ h) $\frac{23}{8}$ i) $\frac{14}{15}$ j) $\frac{7}{60}$

2. a) 81 b) -8 c) 1 d) 49 e) $\frac{32}{243}$ f) $\frac{3}{16}$ g) -2 h) $\frac{16}{25}$

3. a) x^7 b) x^5 c) x^4 d) x^9 e) x^{11} f) x^{24} g) x^{25} h) 1

4. a) 11 b) DNE c) 3 d) -2 e) 20 f) 60 g) 2 h) 1 i) 0 j) 4 k) DNE l) 23

5. a) 8 b) $\frac{1}{2}$ c) 3 d) DNE e) -243 f) $\frac{1}{32}$ g) 1 h) 9 i) 0

6. a) 2 b) 0 c) 81 d) $\frac{1}{125}$ e) 25 f) 1 g) $\frac{16}{9}$ h) $\frac{13}{36}$ i) DNE j) $\frac{1}{16}$ k) $\frac{1}{8}$ l) 2

7. a) $4x^5y^2$ b) x^{-16} c) t d) $x^{9/10}$ e) $\frac{-x^6y^3}{64}$ f) $\frac{2^{-2/3}x^2}{y^{2/3}}$ g) $\frac{r^{3/2}}{s^{3/2}}$ h) $x^{-11/3}y^{7/3}$

8. a) $\sqrt[5]{t^2}$ b) $\frac{1}{y^5}$ c) $\sqrt[4]{x}$ d) $\frac{\sqrt{x^3}}{y}$ e) $\sqrt[5]{23^2}$ f) x^4 g) $\frac{1}{\sqrt[3]{19^2 \cdot 24^5}}$ h) $\frac{\sqrt{z}}{x^2\sqrt[4]{w^3}}$ i) $\frac{1}{\sqrt[3]{(2x)^7}}$ j) $\frac{\sqrt[6]{t^{17}}}{x^3}$

9. a) $x^{1/7}$ b) $x^{5/2}$ c) x^{-8} d) $x^{-1/3}$ e) $x^{-3/4}$ f) $\sqrt{3}x^{1/2}$ g) $\sqrt{3}x^{7/2}$ h) $-7x^{-1}$ i) $8x^{-3}$ j) $\frac{3}{4}x^{-2}$

10. a) x^{-4} b) $4x^6$ c) $\frac{1}{4}x^9$ d) $11x^0$ e) $x^{7/3}$ f) $x^{4/3}$ g) $4x^2$ h) $\sqrt[9]{15}\,x^{4/9}$ i) $\frac{1}{4}x^{-2}$ j) $\frac{1}{3}x^{-2}$

11. a) $x^{8/7}$ b) $\sqrt{3}x^2$ c) $x^{7/2}$ d) $16x^{19}$ e) $\frac{1}{2}x^0$ (a.k.a. $\frac{1}{2}$) f) x^{10} g) $4x$

Exercises from Section 1.3

In Exercises 1-6, evaluate each expression:

1. $6^2 - 3$
2. $-2(4-2)^3$
3. $12 \div 2 \cdot 8$
4. $5 + 3 \cdot 2^2$
5. $-3^2 + 4$
6. $2 \cdot 4^2 - 4 \cdot 3^2$

Answers

1. 33
2. -16
3. 48
4. 17
5. -5
6. -4

Exercises from Section 1.4

In Exercises 1-15, expand ("expand" is a synonym for "distribute" or "FOIL") each expression and then simplify by combining like terms:

1. $4x(x-2)$
2. $\sqrt[3]{x}\left(x^{5/3} + 3\sqrt{x}\right)$
3. $(x-3)(2x+1)$
4. $(x+3)(x-3)$
5. $(x-3)^2$
6. $(x+3)(y-3)$
7. $\left(x - \frac{1}{4}\right)\left(x + \frac{1}{2}\right)$
8. $4(x^3 - 3x) + 2(3x^2 + x) - 3x^2(x+1)$
9. $(x - \sqrt{7})(x + \sqrt{7})$
10. $(3x+4)^2$
11. $(x-3)(2x+1) - (x-2)(3x-4)$
12. $\sqrt{x}\,(x\sqrt{x} + 3)$
13. $\left(w^{2/5} + 1\right)\left(w^{1/3} - 1\right)$
14. $\left(x^{-5/2} + 4x^{-1/2}\right)\left(x^2 + \frac{1}{2}\right)$
15. $(5-3t)(5+3t)$

In Exercises 16-35, factor each expression completely:

16. $3x^5 - 18x^3$
17. $18x^5 + 9x^3 + 36$
18. $x^2 + 5x - 50$
19. $t^2 - 25$
20. $x^5 - 4x^4 - 12x^3$
21. $x^2 - \frac{4}{9}$
22. $x^3 - x^4$
23. $2x^2 - 7x - 4$
24. $r^2 + 3r + 2$

25. $2x^2 - 4x - 70$
26. $49 - 4x^2$
27. $4x^2 - 4x - 15$
28. $24x^2 + 20x - 24$
29. $w^2 - \frac{1}{3}$
30. $2t^3 - 16t^2 + 32t$
31. $3x^2 - 17$
32. $w^2 + \frac{4}{3}w + \frac{1}{3}$
33. $x^2 + 2x + \frac{3}{4}$
34. $5x^3 - 11x^2 - 12x$
35. $x^4 - 121x^2$

Answers

1. $4x^2 - 8x$
2. $x^2 + 3x^{5/6}$
3. $2x^2 - 5x - 3$
4. $x^2 - 9$
5. $x^2 - 6x + 9$
6. $xy - 3x + 3y - 9$
7. $x^2 + \frac{1}{4}x - \frac{1}{8}$
8. $x^3 + 3x^2 - 10x$
9. $x^2 - 7$
10. $9x^2 + 24x + 16$
11. $-x^2 + 5x - 11$
12. $x^2 + 3\sqrt{x}$
13. $w^{11/15} - w^{2/5} + w^{1/3} - 1$
14. $x^{-1/2} + \frac{1}{2}x^{-5/2} + 4x^{3/2} + 2x^{-1/2}$
15. $25 - 9t^2$
16. $3x^3(x - \sqrt{6})(x + \sqrt{6})$
17. $9(2x^5 + x^3 + 4)$
18. $(x + 10)(x - 5)$
19. $(t - 5)(t + 5)$
20. $x^3(x - 6)(x + 2)$
21. $\left(x - \frac{2}{3}\right)\left(x + \frac{2}{3}\right)$
22. $x^3(1 - x)$
23. $(2x + 1)(x - 4)$
24. $(r + 2)(r + 1)$
25. $2(x + 5)(x - 7)$
26. $(7 - 2x)(7 + 2x)$
27. $(2x + 3)(2x - 5)$
28. $4(3x - 2)(2x + 3)$
29. $\left(w - \sqrt{\frac{1}{3}}\right)\left(w + \sqrt{\frac{1}{3}}\right)$
30. $2t(t - 4)(t - 4)$ a.k.a. $2t(t - 4)^2$
31. $\left(\sqrt{3}x - \sqrt{17}\right)\left(\sqrt{3}x + \sqrt{17}\right)$
32. $\left(w + \frac{1}{3}\right)(w + 1)$
33. $\left(x + \frac{1}{2}\right)\left(x + \frac{3}{2}\right)$
34. $x(5x + 4)(x - 3)$
35. $x^2(x - 11)(x + 11)$

Exercises from Section 1.5

1. Graph the following points on a single set of coordinate axes:

   a) $(0, 3)$
   b) $(-3, 2)$
   c) $(5, -1)$
   d) $(-4, 0)$
   e) $(-3, -7)$
   f) $\left(-\sqrt{16}, 8^{1/3}\right)$

2. To which quadrant do each of these points belong?

   a) $\left(-\frac{1}{4}, \frac{7}{2}\right)$
   b) $(3, 8)$
   c) $(4, 0)$
   d) $\left(-\sqrt{7}, \sqrt[3]{13}\right)$

3. Compute the distance between each given pair of points:

   a) $(3, -6)$ and $(3, -2)$
   b) $(0, 0)$ and $(5, 12)$
   c) $(-3, 7)$ and $(4, 2)$
   d) (y, y^2) and $(0, 5)$
   e) $\left(\frac{1}{2}, \frac{2}{3}\right)$ and $\left(-\frac{3}{2}, \frac{8}{3}\right)$
   f) $(x, \sqrt{x})$ and $(3, 0)$

4. Sketch the graph of each circle:

   a) The circle with center $(-5, 3)$ that has radius 2
   b) The circle with equation $(x + 2)^2 + y^2 = 16$
   c) The circle with diameter going from $(8, -3)$ to $(8, 7)$
   d) The circle with equation $(x - 3)^2 + (y - 5)^2 = 7^2$

5. a) What is the lowest point on the circle described in part (d) of the previous exercise?
   b) What is the right-most point on the circle described in part (d) of the previous exercise?
   c) What is the left-most point on the circle whose equation is $(x - 11)^2 + (y - 6)^2 = 49$?
   d) What is the diameter of the circle described in part (c) of this exercise?

6. Write the equation of each circle described:

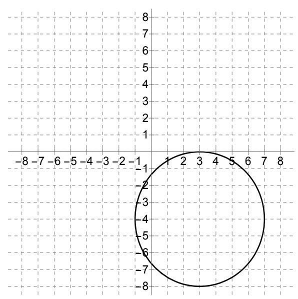
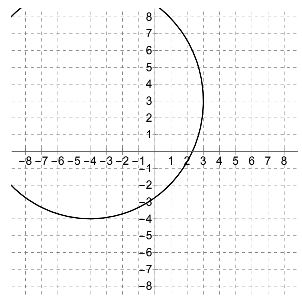
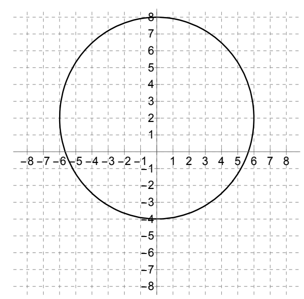
   a) The circle is centered at the origin and has radius 10.
   b) The circle is centered at $(-5, 0)$ and has radius 4.
   c) The circle is centered at $(3, -2)$ and has diameter 9.
   d) The circle is centered at $(-4, 11)$ and passes through $(-4, 3)$.
   e) The circle has a diameter that goes from $(3, 6)$ to $(3, 10)$.

f) The highest point on the circle is $(-8, 7)$ and the left-most point on the circle is $(-11, 4)$.

g) The circle is graphed below, at left.

h) (Part of) the circle is graphed below, in the middle.

i) The circle is graphed below, at right.

7. Find the radius and center of each circle:

a) $x^2 + (y-2)^2 = 9$

b) $(x-3)^2 + (y-5)^2 = 17$

c) $x^2 + y^2 = 40$

d) $\left(x + \frac{2}{3}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{4}{9}$

e) $\left(x - \sqrt{5}\right)^2 + \left(y - \sqrt{7}\right)^2 = 1$

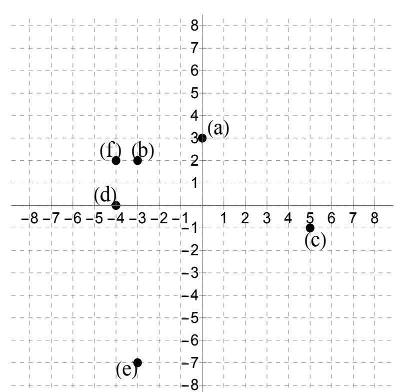
8. a) If you were to graph the circle in part (b) of the previous exercise, would its graph go below the x-axis at all?

b) If you were to graph the circle in part (b) of the previous exercise, would its graph go to the left of the y-axis at all?

c) In how many points does the circle in part (d) of the previous exercise intersect the x-axis?

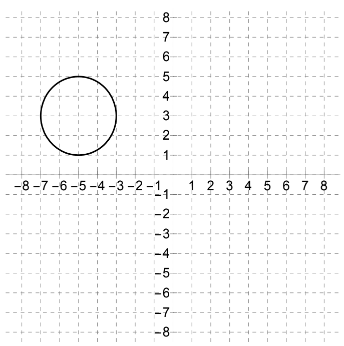
d) In how many points does the circle in part (d) of the previous exercise intersect the y-axis?

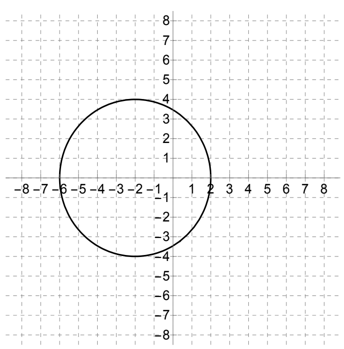
Answers

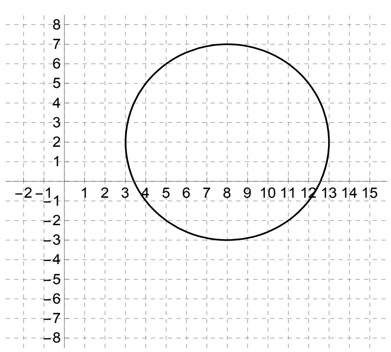
1. 

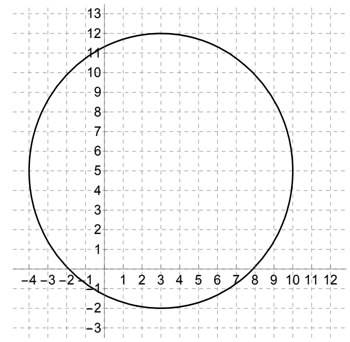
2. a) II
   b) I
   c) $(4,0)$ does not belong to any quadrant.
   d) II

3. a) 4
   b) 13
   c) $\sqrt{74}$
   d) $\sqrt{y^2+(y^2-5)^2}$
   e) $\sqrt{8}$
   f) $\sqrt{(x-3)^2+x}$

4. a) 

   b) 

   c) 

   d) 

5. a) $(3,-2)$
   b) $(10,5)$
   c) $(4,6)$
   d) 14

6. a) $x^2+y^2=100$
   b) $(x+5)^2+y^2=16$
   c) $(x-3)^2+(y+2)^2=\dfrac{81}{4}$
   d) $(x+4)^2+(y-11)^2=64$
   e) $(x-3)^2+(y-8)^2=4$
   f) $(x+8)^2+(y-4)^2=9$
   g) $(x-3)^2+(y+4)^2=16$
   h) $(x+4)^2+(y-3)^2=49$
   i) $x^2+(y-2)^2=36$

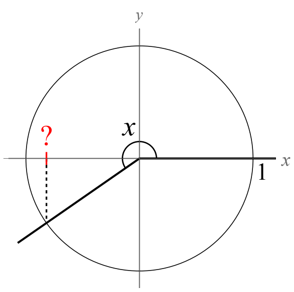
7. a) center $(0,2)$; radius 3
   b) center $(3,5)$; radius $\sqrt{17}$
   c) center $(0,0)$; radius $\sqrt{40}$
   d) center $\left(-\dfrac{2}{3},\dfrac{1}{2}\right)$; radius $\dfrac{2}{3}$
   e) center $(\sqrt{5},\sqrt{7})$; radius 1

8. a) No
   b) Yes
   c) 2
   d) 1

Exercises from Section 1.6

1. Suppose $\tan t = 3$. What trig function of t must equal $\dfrac{1}{3}$?
2. Suppose $\sec x = \dfrac{7}{2}$. What trig function of x must equal $\dfrac{2}{7}$?
3. Suppose $\sin x = -\dfrac{2}{5}$. What trig function of x must equal $-\dfrac{5}{2}$?
4. Suppose $\tan x < 0$ and $\cos x < 0$. What quadrant must the angle x belong to?
5. Suppose $\sin\theta < 0$ and $\cos\theta > 0$. What quadrant must the angle θ belong to?
6. Suppose $\sec\theta > 0$ and $\cot\theta < 0$. What quadrant must the angle θ belong to?
7. Suppose $\tan x > 0$ and $\csc x < 0$. What quadrant must the angle x belong to?

In Exercises 8-13, write a formula for the "?" indicated in the picture, in terms of the other numbers and/or variables given in the picture:

8.


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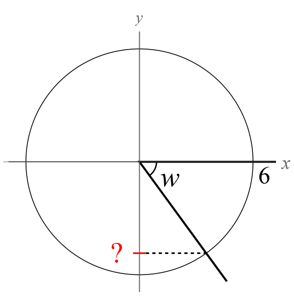

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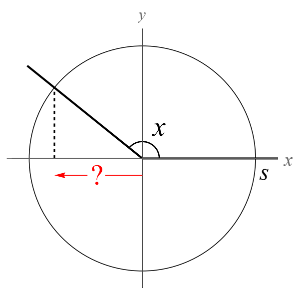
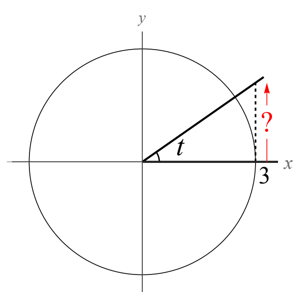


11.


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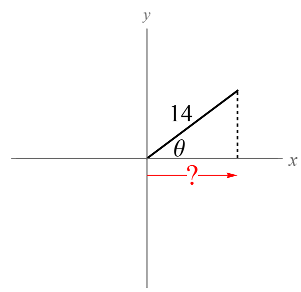
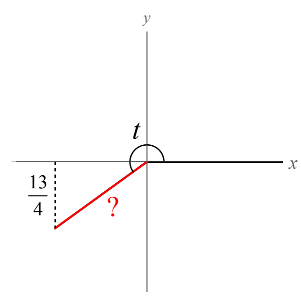


13.


In Exercises 14-67, compute each quantity:

14. $\tan -5\pi$
15. $\sin \dfrac{11\pi}{2}$
16. $\cos -\pi$
17. $\tan \dfrac{-13\pi}{6}$
18. $\sin \dfrac{-7\pi}{4}$
19. $\cos -9\pi$
20. $\tan \pi$
21. $\cos \dfrac{5\pi}{4}$
22. $\tan \dfrac{-\pi}{6}$

23. $\tan -\pi$
24. $\sec -\frac{\pi}{6}$
25. $\tan \frac{3\pi}{4}$
26. $\cot \frac{\pi}{3}$
27. $\csc \pi$
28. $\sin 0$
29. $\tan -\frac{2\pi}{3}$
30. $\cot 0$
31. $\cot \frac{\pi}{2}$
32. $\sec \frac{2\pi}{3}$
33. $\sin \frac{11\pi}{6}$
34. $\tan \frac{-4\pi}{3}$
35. $\tan 0$
36. $\cos \frac{-\pi}{3}$
37. $\sin 4\pi$
38. $\cos \frac{-7\pi}{2}$
39. $\tan \frac{-5\pi}{6}$
40. $\sin \frac{-\pi}{3}$
41. $\cos -5\pi$
42. $\sin \frac{-5\pi}{2}$
43. $\cos \frac{9\pi}{4}$
44. $\sin -\frac{3\pi}{4}$
45. $\cot -\frac{5\pi}{6}$
46. $\tan \frac{\pi}{2}$
47. $\sec \frac{23\pi}{2}$
48. $\tan \pi$
49. $\sin \frac{-11\pi}{6}$
50. $\sin \frac{27\pi}{2}$
51. $\csc \frac{33\pi}{2}$
52. $\cos \frac{\pi}{3}$
53. $\tan \frac{-2\pi}{3}$
54. $\cos \frac{7\pi}{6}$
55. $\cos \frac{5\pi}{6}$
56. $\sin -6\pi$
57. $\cot \frac{15\pi}{4}$
58. $\sin \frac{29\pi}{6}$
59. $\sin -\pi$
60. $\sec \frac{11\pi}{3}$
61. $\sin \frac{11\pi}{6}$
62. $\tan \frac{5\pi}{2}$
63. $\csc \frac{\pi}{2}$
64. $\cos \pi$
65. $\cos \frac{2\pi}{3}$
66. $\tan \frac{-\pi}{4}$
67. $\sin -12\pi$

Answers

1. $\cot t$
2. $\cos x$
3. $\csc x$
4. II
5. IV
6. IV
7. III
8. $? = \cos x$
9. $? = 6 \sin w$
10. $? = s \cos x$
11. $? = 3 \tan t$
12. $? = 14 \cos \theta$
13. $? = \frac{13}{4} \csc \theta$
14. $\tan -5\pi = 0$
15. $\sin \frac{11\pi}{2} = -1$
16. $\cos -\pi = -1$
17. $\tan \frac{-13\pi}{6} = -\frac{1}{\sqrt{3}}$

18. $\sin\frac{-7\pi}{4} = \frac{\sqrt{2}}{2}$
19. $\cos -9\pi = -1$
20. $\tan\pi = 0$
21. $\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$
22. $\tan\frac{-\pi}{6} = -\frac{1}{\sqrt{3}}$
23. $\tan -\pi = 0$
24. $\sec -\frac{\pi}{6} = -\frac{2}{\sqrt{3}}$
25. $\tan\frac{3\pi}{4} = -1$
26. $\cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$
27. $\csc\pi$ DNE
28. $\sin 0 = 0$
29. $\tan -\frac{2\pi}{3} = \sqrt{3}$
30. $\cot 0$ DNE
31. $\cot\frac{\pi}{2} = 0$
32. $\sec\frac{2\pi}{3} = -2$
33. $\sin\frac{11\pi}{6} = -\frac{1}{2}$
34. $\tan\frac{-4\pi}{3} = -\sqrt{3}$
35. $\tan 0 = 0$
36. $\cos\frac{-\pi}{3} = \frac{1}{2}$
37. $\sin 4\pi = 0$
38. $\cos\frac{-7\pi}{2} = 0$
39. $\tan\frac{-5\pi}{6} = \frac{1}{\sqrt{3}}$
40. $\sin\frac{-\pi}{3} = -\frac{\sqrt{3}}{2}$
41. $\cos -5\pi = -1$
42. $\sin\frac{-5\pi}{2} = -1$
43. $\cos\frac{9\pi}{4} = \frac{\sqrt{2}}{2}$
44. $\sin -\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$
45. $\cot -\frac{5\pi}{6} = \sqrt{3}$
46. $\tan\frac{\pi}{2}$ DNE
47. $\sec\frac{23\pi}{2}$ DNE
48. $\tan\pi = 0$
49. $\sin\frac{-11\pi}{6} = \frac{1}{2}$
50. $\sin\frac{27\pi}{2} = -1$
51. $\csc\frac{33\pi}{2} = 1$
52. $\cos\frac{\pi}{3} = \frac{1}{2}$
53. $\tan\frac{-2\pi}{3} = \sqrt{3}$
54. $\cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$
55. $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$
56. $\sin -6\pi = 0$
57. $\cot\frac{15\pi}{4} = -1$
58. $\sin\frac{29\pi}{6} = \frac{1}{2}$
59. $\sin -\pi = 0$
60. $\sec\frac{11\pi}{3} = 2$
61. $\sin\frac{11\pi}{6} = -\frac{1}{2}$
62. $\tan\frac{5\pi}{2}$ DNE
63. $\csc\frac{\pi}{2} = 1$
64. $\cos\pi = -1$
65. $\cos\frac{2\pi}{3} = -\frac{1}{2}$
66. $\tan\frac{-\pi}{4} = -1$
67. $\sin -12\pi = 0$

Chapter 2

Functions

2.1 Introducing functions

§2.1 MOTIVATING EXAMPLE A

Let's create a "function" called "A" that can be used to compute the area of a square, in terms of the side length of the square.

§2.1 MOTIVATING EXAMPLE B

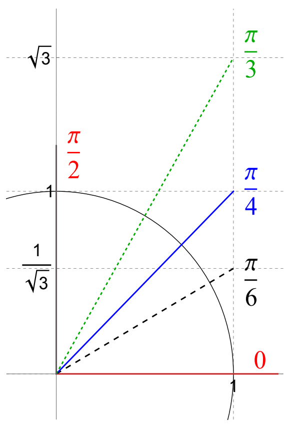
Suppose you buy frozen pizzas at Meijer. Each frozen pizza costs $\$5.00$, with the catch that frozen pizzas are "buy one, get one free". Create a function called "price" that models this situation:

§2.1 MOTIVATING EXAMPLE C

Suppose you make \$20 per hour (and are paid time-and-a-half for each hour over 40 you work each week). Let's analyze your weekly pay (before taxes and other deductions) in terms of the number of hours you work and create a function called "pay" modeling this:

§2.1 Motivating Example D

Compute the slope of an angle, in terms of the measure (in radians, of course) of the angle. The function that models this is called ________ .



angle (radians)	slope
0	
$\frac{\pi}{6}$	
$\frac{\pi}{4}$	
$\frac{\pi}{3}$	
$\frac{\pi}{2}$	
$\frac{3\pi}{4}$	-1
π	0
2π	0
x	
$\vdots$	$\vdots$

$0 \xrightarrow{\tan}$ $\qquad \tan 0$

$\frac{\pi}{6} \xrightarrow{\tan}$ $\qquad \tan \frac{\pi}{6}$

$\frac{\pi}{4} \xrightarrow{\tan}$ $\qquad \tan \frac{\pi}{4}$

$\frac{\pi}{3} \xrightarrow{\tan}$ $\qquad \tan \frac{\pi}{3}$

$\frac{\pi}{2} \xrightarrow{\tan}$ $\qquad \tan \frac{\pi}{2}$

$\frac{3\pi}{4} \xrightarrow{\tan} -1$ $\qquad \tan \frac{3\pi}{4} = -1$

$\pi \xrightarrow{\tan} 0$ $\qquad \tan \pi = 0$

$2\pi \xrightarrow{\tan} 0$ $\qquad \tan 2\pi = 0$

$x \xrightarrow{\tan}$ $\qquad \vdots$

$\vdots \qquad \vdots$

§2.1 Motivating Example E

Given a human being, determine the year in which they were born; call this process "y":

person	birth year
George Washington	1732
Denzel Washington	1954
Taylor Swift	1989
Babe Ruth	1895
Caligula	12
Woodbridge Ferris	1853
$\vdots$	$\vdots$

G.Washington $\xrightarrow{y}$ 1732 $\qquad y$(G.Washington) $= 1732$

D. Washington $\xrightarrow{y}$ 1954 $\qquad y$(D. Washington) $= 1954$

Taylor Swift $\xrightarrow{y}$ 1989 $\qquad y$(Taylor Swift) $= 1989$

Babe Ruth $\xrightarrow{y}$ 1895 $\qquad y$(Babe Ruth) $= 1895$

Caligula $\xrightarrow{y}$ 12 $\qquad y$(Caligula) $= 12$

W. Ferris $\xrightarrow{y}$ 1853 $\qquad y$(W. Ferris) $= 1853$

$\vdots \qquad \vdots \qquad \vdots$

Examples A-E above are examples of *functions*. A function is as a *procedure* or a *rule of assignment* that produces outputs from inputs (i.e. assigns outputs to inputs):

- This procedure might be simple, and it might be very complicated.
- The procedure might follow a mathematical rule that is easy to write down (like Motivating Example A);
- the procedure might be impossible to describe without inventing a new name for it (like Motivating Example D);
- the procedure might require different cases, depending on what the input is (like Motivating Examples B or C);
- the procedure might take rather complicated mathematical notation to write (we haven't really seen this in the examples); and
- the procedure might not be "mathematical" at all (like Motivating Example E).

What makes each of these procedures a function is this important principle:

IMPORTANT: for a procedure to be a function, it must be the case that if you perform the procedure twice with the same input each time, you must get the same output each time. In other words,

each input of a function leads to at most one output.

More precisely:

Definition 2.1 *A* **function** f *is a procedure which generates outputs from inputs, in such a way that each input leads to* ***one and only one*** *output.*

The output produced from input x *is denoted* $f(x)$ *(this is pronounced "*f *of* x*").*

Functions are named by lots of different things:

- most commonly, by a single lowercase letter (f, g, k, etc.);
- capital letters, Greek or Hebrew letters (F, A, α, ψ, $\aleph$, etc.);
- words or phrases ($\sin$, $\cos$, $\sec$, $\log$, $\ln$, etc.);
- symbols ($\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[n]{\ }$, etc.); or
- using "grouping-type" symbols that enclose the input, like $|\ |$ (absolute value) or $\lfloor\ \rfloor$.

Don't misinterpret the parentheses in $f(x)$!

WARNING: All your life you have been told that parenthesis means multiplication, i.e.

$$3(2) = 6 \quad \text{or} \quad a(b + c) = ab + ac.$$

If f is a function, the parenthesis in "$f(x)$" do not mean multiplication. In particular, $f(x)$ does not mean f times x, and $f(a + b)$ is not the same thing as $f(a) + f(b)$ (in general).

$f(x)$ means, literally, this:

"the output of function f when x is the input".

and is better understood through the diagram

$$x \xrightarrow{f} f(x).$$

By the way, this diagram also shows the difference between "f" and "$f(x)$":

- $f(x)$ is the **output** of function f when the input is x; but
- f is the **function** itself. f is neither the input nor the output–it is the name of the **procedure** that produces outputs from inputs.

A useful analogy: at the grocery store, x is an item you buy, $f(x)$ is the price of that item, and f is the cash register (which scans your item and tells you how much you have to pay).

Sometimes the parenthesis in the $f(x)$ is omitted and we just write fx, especially if the function is named after a word or phrase, rather than a single letter.

We usually write $\sin x$ instead of $\sin(x)$, for instance.

But $\sin x$ does NOT mean something called $\sin$ times x.

$\sin x$ refers to the output of the function $\sin$ (which computes y-coordinates from angles), when the input is x.

2.2 The rule of a function

Motivating examples A-E discussed earlier were described in words. Here is another example of a function described in words:

§2.2 EXAMPLE 1

Let g be a function which takes its input, multiplies the input by two less than the input, then takes the cosine of that product, then multiplies the square of that by 2, and finally subtracts 4 times the input cubed.

Questions:

- Was this description of g easy to read?
- Was this description easy to interpret?
- Do you think you could easily perform mathematical operations on this g (combining it with other functions, etc.)?
- Would you enjoy writing descriptions like this of functions you encounter or compute?

We need a way of describing a function that has two attributes:

1. it is **efficient** (meaning it can be written quickly, with a minimal number of symbols and words), and
2. it is **effective** (meaning that the description is useful for performing mathematical operations that solve important applied problems).

Here's how we do this:

We compute the output of the function if the input is generic and arbitrary.

In math, generic and arbitrary things are represented by variables like x.

So we compute the output of the function if the input is a generic x.
(You can use a letter or symbol other than x. The choice of variable often has to do with what the variable means. For instance, if the input to a function is *time*, we probably will use _____ instead of x.)

This gives us some formula with x in it.

It is valid to say that $f(x)$ **equals** this formula (assuming the function is named f), since the formula gives the output associated to x.

(If we used t as the input, our formula would be $f(t) = ...$, not $f(x) = ...$)

This leads to an equation called a *rule* for the function:

Definition 2.2 *Let f be a function. A* **rule** *for f is an equation of the form*

$$f(x) = \textit{something}$$

where the "something" is the output that comes from input x.

As mentioned, we don't have to use x for our generic input; we can use t or y or θ or some other letter or symbol. These would produce rules that look like

$$f(t) = \text{something of } t \qquad f(y) = \text{something of } y \qquad f(\theta) = \text{something of } \theta$$

Main concept: To describe a function, it is sufficient to write down its rule.

Reason: Think of the x as a placeholder which represents where the input goes. Given a rule for f, to find any output you take whatever input you are given and replace all the xs in the rule with the appropriate input.

§2.2 EXAMPLE 2

Write down a rule for each function:

a) Let h be the function which takes the cube root of its input and then adds 4 to produce the output.

b) Let k be the function that takes the product of the sine of its input and the tangent of twice its input (to produce its output).

c) Let F be the function that squares its input, then takes the cosine of that.

d) Let f be the function that adds 3 to twice the input, takes the seventh power of that, subtracts twice the input and then takes a cube root of that to produce the output.

BACK TO EXAMPLE 1

Let g be a function which takes its input, multiplies the input by two less than the input, then takes the cosine of that product, then multiplies the square of that by 2, and finally subtracts 4 times the input cubed.

What is a rule for g?

§2.2 EXAMPLE 3

<table>
<tr><th>Rule of f</th><th>Arrow diagram</th><th>Table of values</th><th>Description of f in words</th></tr>
<tr><td></td><td></td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>-1</td><td></td></tr><tr><td>0</td><td></td></tr><tr><td>2</td><td></td></tr><tr><td>4</td><td></td></tr></table></td><td>f takes the input, multiplies it by itself, then adds 4 times the input</td></tr>
<tr><td>$f(x) = 4\cos x$</td><td></td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>0</td><td></td></tr><tr><td>$\frac{\pi}{3}$</td><td></td></tr><tr><td>$\frac{\pi}{2}$</td><td></td></tr><tr><td>π</td><td></td></tr></table></td><td></td></tr>
<tr><td></td><td>$x \xrightarrow{f} x+2$
$-3 \xrightarrow{f}$
$0 \xrightarrow{f}$
$3 \xrightarrow{f}$
$4 \xrightarrow{f}$
$\vdots$</td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>0</td><td></td></tr><tr><td>2</td><td></td></tr><tr><td>4</td><td></td></tr><tr><td>10</td><td></td></tr></table></td><td></td></tr>
</table>

Why rules of functions are effective

§2.2 EXAMPLE 4

Let $f(x) = x^2 - 3x$. Compute and simplify each quantity:

1. $f(3)$
2. $f(-2)$
3. $f(4) + f(1)$
4. $f(t)$
5. $f(\heartsuit)$
6. $f(\text{peanut})$
7. $f(x-1)$
8. $f(3x)$
9. $3f(x)$
10. $\dfrac{f(x+h) - f(x)}{h}$

Applying a function to a set

Definition 2.3 *Suppose f is a function and $\{x_1, x_2, x_3, ...\}$ is some set/list of inputs. By*

$$f(\{x_1, x_2, x_3, ...\}),$$

we mean the set/list of outputs

$$\{f(x_1), f(x_2), f(x_3), ...\}.$$

In other words, to apply a function f to a set, apply it to each member of the set.

§2.2 EXAMPLE 5

Let $f(x) = x^2 - 3x$. Compute $f(\{1, 2, 3\})$.

Applying functions to both sides of an equation

In Chapter 1 we talked about equations as being like balanced scales. Since functions only have one output, if you apply a function to two equal inputs you must get equal outputs. In symbols, this means

If $a = b$, then $f(a) = f(b)$ for any function f.

and restated, this is:

> **General principle of manipulating equations:** If you start with any equation and apply the same function to both sides, what you get is still an equation.

§2.2 EXAMPLE 6

In each part of this example, you are given an equation and a function. Apply the given function to both sides of the equation, simplifying both sides of what you get:

a) $3x + 2 = 9$ $\qquad f(x) = x - 2$

b) $2x - 1 = 5x + 3$ $\qquad f(x) = \frac{1}{2}(x + 1)$

c) $4\cos x = 3x - 7$ $\qquad f(x) = x^2$

Solution:

$$\begin{aligned} 4\cos x &= 3x - 7 \\ (4\cos x)^2 &= (3x - 7)^2 \\ 16\cos^2 x &= (3x - 7)^2 \end{aligned}$$

d) $2x^3 + 5 = 8x^3$ $\qquad f(x) = \sqrt[3]{x}$

2.3 Domain, codomain and range

We can associate to each function three sets, which describe the kinds of inputs and outputs that the function has:

Definition 2.4
The set of inputs to a function f is called the **domain** *of f. This set is denoted $Dom(f)$.*

Any set which contains all the outputs of f is called a **codomain** *of f.*

The set of actual outputs of f is called the **range** *of f. This set is denoted $Range(f)$.*

If the domain of a function f is a subset of some set A and if B is a codomain of f, then we write

$$f : A \to B$$

and say "f is a function from A to B".

§2.3 EXAMPLE 1

In Motivating Example A (area of a square), we described the function $A(x) = x^2$.

$$Dom(A) =$$

$$Range(A) =$$

$$\text{codomain of } A =$$

so it is valid to write

§2.3 EXAMPLE 2

In Motivating Example B (the pizza example), we described a function called price.

$$Dom(\text{price}) =$$

$$Range(\text{price}) =$$

$$\text{codomain of price} =$$

so it is valid to write

Most of the time, the domains and codomains of functions we study in MATH 120 (and in calculus) are subsets of $\mathbb{R}$, the set of real numbers. In other words, we study functions f for which we can write

So if you see "Let $f : \mathbb{R} \to \mathbb{R}$" in some statement or exercise, that just means that f is a function which has as its inputs some set of real numbers, and produces real numbers as outputs.

Interval notation

Notice that the range of the function A in Example 1 was described with an inequality. It is convenient to have some shorthand notation for sets described by inequalities, so that we can write the domains and ranges of functions more quickly.

§2.3 EXAMPLE 3

a) Sketch a picture of the set of real numbers x satisfying $1 < x \leq 5$ (this means $1 < x$ and $x \leq 5$).

b) Sketch a picture of the set of real numbers x satisfying $x \geq -4$.

We adapt the ideas of these examples to describe certain sets with two numbers, where the set goes from the first number to the second.

We use [and] are for weak inequalities $\leq$ and $\geq$ where the endpoint(s) is/are included in the set; we use (and) for strict inequalities $<$ and $>$ where the endpoint(s) isn't/aren't in the set.

We use ∞ (**infinity**) to represent the right edge of a set with no biggest number, and $-\infty$ (**negative infinity**) for the left edge of a set with no smallest number.

Definition 2.5 *Let a and b be numbers.*

This notation ↓	*describes the set of real numbers x satisfying this inequality:*
$[a, b]$	$a \leq x \leq b$
(a, b)	$a < x < b$
$[a, b)$	$a \leq x < b$
$(a, b]$	$a < x \leq b$
$[a, \infty)$	$a \leq x$
(a, ∞)	$a < x$
$(-\infty, a]$	$x \leq a$
$(-\infty, a)$	$x < a$

Any set of any of these eight types is called an **interval**.

NOTE: ∞ and $-\infty$ never have square brackets on them. That's because these sets don't actually include endpoints called "∞" or "$-\infty$".

One ambiguity

In a vacuum, something like $(3, 5)$ might mean one of two different things: it is either the ordered pair $(x, y) = (3, 5)$, or it is the interval of real numbers running from 3 to 5 but including neither endpoint.

Whether $(3, 5)$ is an ordered pair or an interval depends on the context in which it is written.

Equality of functions

Definition 2.6 *Two functions f and g are called* **equal** *if the functions have the same domain, and if they produce the same output for every input in their common domain. In this situation we write $f = g$.*

The difference between "$f = g$" and "$f(x) = g(x)$"

- Writing $f(x) = g(x)$ means the outputs of f and g are the same *for one particular input x* (which you might be trying to solve for).
- Writing $f = g$ means the outputs of f and g are the same *for every input x*.

2.4 Piecewise-defined functions

A common class of functions $f : \mathbb{R} \to \mathbb{R}$ can be described by a procedure like this:

- Take your input x.
- If x is one type of input, do one thing to produce $f(x)$.
- If x is a second type of input, do something else to produce $f(x)$.

A function f that works like this is called a **piecewise-defined function**.

§2.4 EXAMPLE 1

Let $f : \mathbb{R} \to \mathbb{R}$ be the absolute value function $f(x) = |x|$.

a) Give a description of this function f in words.

b) Write a rule for the description given in part (a).

Remark: We don't want to have to write this rule out over and over. This is why we invent the symbol $|x|$ for absolute value.

c) Compute $f(7)$ and $f\left(-\dfrac{2}{3}\right)$.

§2.4 EXAMPLE 2

Let $g(x) = \begin{cases} -2x+1 & x \leq 3 \\ x-2 & 3 < x < 5 \\ 7 & x \geq 5 \end{cases}$. Compute each quantity:

a) $g(-2)$

b) $g(5)$

c) $g(8)$

Who cares about piecewise-defined functions? Everyone should. Here's why:

§2.4 EXAMPLE 3

Here is a piecewise-defined function that determines the amount $T(x)$ of federal income tax (not counting deductions) owed by an individual who has income x in the year 2023:

$$T(x) = \begin{cases} .1x & x < 11000 \\ .12x - 220 & 11000 \leq x < 44725 \\ .22x - 4692.50 & 44725 \leq x < 95375 \\ .24x - 6600 & 95375 \leq x < 182100 \\ .32x - 21168 & 182100 \leq x < 231250 \\ .35x - 28105.50 & 231250 \leq x < 578125 \\ .37x - 39667.30 & x \geq 578125 \end{cases}$$

a) If you make $\$35000$ in 2023, how much do you owe in tax?

b) How much tax would a citizen owe on lottery winnings of $\$250000$?

2.5 Multifunctions

RECALL: MOTIVATING EXAMPLE B FROM §2.1

(This was the function "price" where price(x) is the price of x frozen pizzas purchased at $\$5$ per pizza, where every other pizza was free.)

x	price(x)
0	0
1	5
2	5
3	10
4	10
5	15
⋮	⋮

$0 \xrightarrow{\text{price}} 0$

$1 \xrightarrow{\text{price}} 5$

$2 \xrightarrow{\text{price}} 5$

$3 \xrightarrow{\text{price}} 10$

⋮ ⋮

Question: Can you go backwards? More precisely, can you tell how many pizzas you bought, based on the amount you spent?

A procedure that produces a list of (maybe more than one) possible output(s) from each single input is called a *multifunction*:

Definition 2.7 *A* **multifunction** *f is a procedure which takes each input x (in the domain) and produces outputs (in the codomain) from that input, where* ***each input might lead to more than one output****.*

If f is a multifunction and x is an input with outputs $y_1, y_2, y_3, \ldots$ then we list the associated outputs as a set by writing

$$f(x) = \{y_1, y_2, y_3, \ldots\};$$

Equality of multifunctions is the same as equality of functions:

Definition 2.8 *Let f and g be multifunctions. We say f and g are* **equal** *if f and g have the same domain, and if $f(x) = g(x)$ for every input x in their common domain.*

§2.5 EXAMPLE 1

Consider the "plus-minus" multifunction

$$f(x) = \pm x.$$

Compute each quantity:

1. $f(6)$

2. $f(0)$

§2.5 EXAMPLE 2

Let g be the multifunction $g(x) = \pm\sqrt{x}$. Compute each quantity:

1. $g(16)$

2. $g(144)$

3. $g(0)$

4. $g(-9)$

2.6 Inverting a function

In the previous section, we were motivated by trying to *undo* or *reverse* the price function of Motivating Example B (the pizza example).

Undoing/reversing a function is called *inverting* the function. Based on our motivating example, we see that in general, to invert a function, we have to use a multifunction.

Definition 2.9 *Let $f : \mathbb{R} \to \mathbb{R}$. Then the* **inverse** *of f, denoted f^{-1} and pronounced "f inverse", is the multifunction with all these properties:*

- *Outputs of f are inputs of f^{-1} and vice versa (meaning that the domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f).*
- *$f^{-1}(y)$ is defined to be the set of all the inputs of f that produces output y.*

WARNING: the -1 in this notation is **not** an exponent.

$$f^{-1}(x) \text{ does not mean } (f(x))^{-1} = \frac{1}{f(x)}.$$

If you have to write $\frac{1}{f(x)}$ as an exponent, write it as $(f(x))^{-1}$.

It is useful to visualize inverses with the following arrow diagram:

$$x \underset{f^{-1}}{\overset{f}{\rightleftarrows}} y$$

Another way of looking at this: $f^{-1}(y)$ is a set/list of all the things that would work as "?" in this diagram:

$$? \xrightarrow{f} y$$

Contrast this with $f(x)$, which is at most one item which works as the "?" in this diagram:

$$x \xrightarrow{f} ?$$

§2.6 EXAMPLE 1

Let price be the function described in Motivating Example B. If you don't remember, price is the function with this table of values:

x	0	1	2	3	4	5	$\cdots$
price(x)	0	5	5	10	10	15	$\cdots$

Compute each quantity:

a) price 3

b) price$^{-1}(3)$

P.S. The expression in (b) is pronounced "price inverse of 3".

c) price$^{-1}(10)$

d) price$^{-1}\, 0$

e) price^{-1}(price(1))

f) price(price$^{-1}\, 5$)

Parts (e) and (f) of Example 1 illustrate the major concept concept behind inverses: the inverse f^{-1} is a multifunction which "undoes" whatever f does (and vice versa):

Theorem 2.10 *If f is a function with inverse f^{-1}, then*

$$f^{-1}(f(x)) \text{ includes } x \quad \text{and} \quad f(f^{-1}(y)) = y.$$

§2.6 EXAMPLE 2

Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2$. In parts (a)-(e), write an arrow diagram that explains what is being asked for, and then compute the indicated quantity:

a) $f(4)$

b) $f^{-1}(4)$

c) $f^{-1}(-4)$

d) $f^{-1}(0)$

e) $f^{-1}(y)$

f) Is f^{-1} a function $\mathbb{R} \to \mathbb{R}$?

Remark: The work in part (e) can be thought of as finding a rule for f^{-1}.

§2.6 EXAMPLE 3

Let $g : \mathbb{R} \to \mathbb{R}$ be $g(x) = 5 - 3x$. In parts (a)-(d), write an arrow diagram that explains what is being asked for, and then compute the indicated quantity:

a) $g(2)$

   Solution: $\boxed{2 \xrightarrow{g} ?}$; $\quad g(2) = 5 - 3(2) = \boxed{-1}$.

b) $g^{-1}(2)$

   Solution: $\boxed{? \xrightarrow{g} 2}$;

c) $g^{-1}(y)$

d) $g^{-1}(x)$

e) Is g^{-1} a function $\mathbb{R} \to \mathbb{R}$?

§2.6 EXAMPLE 4

Suppose f and g are functions, each with domain $\{-3, -2, -1, 0, 1, 2, 3\}$, described completely by the table of values given below:

x	-3	-2	-1	0	1	2	3
$f(x)$	2	1	-2	0	0	-3	2
$g(x)$	1	3	4	2	0	-3	-2

Compute each quantity:

a) $f(2)$

b) $f^{-1}(2)$

c) $g^{-1}(-3)$

d) $g^{-1}(-1)$

One-to-one functions

In general, the inverse of a function is a multifunction. But sometimes, the inverse of a function is itself a function (like Example 3 of this section)! This happens if...

Definition 2.11 *A function f is called* **one-to-one**, *a.k.a.* **injective**, *a.k.a.* $1-1$ *if* different inputs to f must always lead to different outputs. *Algebraically, this means*

$$x_1 \neq x_2 \text{ implies } f(x_1) \neq f(x_2).$$

§2.2 EXAMPLE 5

Determine whether or not each given function is one-to-one:

1. The price function coming from Motivating Example B (pizzas $\$5$ each, every other pizza free)

2. A function modeling the price you pay for bags of chips, if bags of chips are $\$3.29$ each

3. The function in Motivating Example E (which gives the birth year as a function of the person in question)

4. The function f described by the table in Example 4 (one page ago)

5. The function g described by the table in Example 4

6. The function $h(x) = |x|$

7. The function $k(x) = \sin x$

8. The function $F(x) = x^3$

Here's why we care whether or not a function is one-to-one:

Theorem 2.12 *The inverse of a one-to-one function is itself a function.*

Using inverses to solve equations

A major reason we care about inverses is that an understanding of inverses is important in solving equations. Here are some basic examples:

§2.2 EXAMPLE 6

a) Consider the equation $x - 13 = 17$.

The reason we add 13 to both sides has to do with functions and inverses.

b) Consider the equation $8x = 48$. To solve this, we divide both sides by 8:

$$\begin{aligned} 8x &= 48 \\ \frac{8x}{8} &= \frac{48}{8} \\ x &= 6 \end{aligned}$$

Why do we divide both sides by 8? Well, if we let $f(x) = 8x$, we can think of the equation $8x = 48$ as

$$f(x) = 48 \qquad \text{i.e.} \qquad x \xrightarrow{f} 48 \ \text{ or } \ x \xrightarrow{\times 8} 48\ .$$

The inverse of the function $f(x) = 8x$ is $f^{-1}(x) = \boxed{}$, so we get the solution as $f^{-1}(48) = \dfrac{48}{8} = \boxed{6}$. As an arrow diagram, this is

$$6 \underset{f^{-1}}{\overset{f}{\rightleftarrows}} 48 \qquad \text{or} \qquad 6 \underset{\div 8}{\overset{\times 8}{\rightleftarrows}} 48\ .$$

c) Consider the equation $x^2 = 81$.

More on these ideas later, both in Section 2.8 and when we study equations in more detail.

2.7 Order of operations with functions

§2.7 EXAMPLE 1

Let $f(x) = x^2 + 3x$. Compute and simplify the following expressions:

a) $f(2-1)$

b) $f(2) - f(1)$

c) $f(2) - 1$

§2.7 EXAMPLE 2

Let $h(x) = 2\sqrt{x} + 1$. Compute and simplify the following expressions:

a) $h(4 \cdot 9)$

b) $4h(9)$

c) $9h(4)$

d) $h(4)h(9)$

e) $h(4h(9))$

§2.7 EXAMPLE 3

Let $g(t) = 3t^2$. Compute and simplify the following expressions:

1. $g(t+1)$

2. $g(t) + 1$

3. $g(t) + g(1)$

One of the most important things to master in MATH 130 is how to interpret expressions with functions that have "invisible parentheses" in them. In particular, this means learning order of operations with functions.

> **A new general rule:** when reading an expression, whenever you see a function, immediately after the function there is an invisible (.
>
> Everything after the invisible (is grouped until you get to one of three things: addition, subtraction or the name of another function.

§2.3 EXAMPLE 2

Let job $x = 4x - 1$. Compute and simplify each expression:

1. job(x)
2. job x
3. job$(2 \cdot 3)$
4. job $2 \cdot 3$
5. 2 job 3
6. job $(1 + 3)$
7. job $1 + 3$
8. job 1 + job 3
9. job $2 \cdot 4 - 1$
10. job $(3$ job $1)$
11. job 3 job 1

Order of operations with functions and exponents

Definition 2.13 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For any $n \neq -1$, the notation $f^n(x)$ (a.k.a. $f^n x$) means $(f(x))^n$.*

REMINDER: $f^{-1}(x)$ is the inverse of f, not $f(x)$ to the -1 power.

One instance where this notation is used is in the following trig identity:

Theorem 2.14 (Pythagorean Identity) *For any real number x,*

$$\cos^2 x + \sin^2 x = 1.$$

§2.7 EXAMPLE 3

Suppose car $x = 3x - 1$. Compute each quantity:

a) car 2^2

b) car^2 2

c) car^3 1

d) car $3^2 \cdot 2$

Solution: car $3^2 \cdot 2 =$ car $(3^2 \cdot 2) =$ car $(9 \cdot 2) =$ car $18 = 3(18) - 1 = \boxed{53}$.

e) 2 car 3^2

Solution: 2 car $3^2 = 2$ car $(3^2) = 2$ car $9 = 2\,[3(9) - 1] = 2\,[26] = \boxed{52}$.

f) 3 car^3 $(2 - 1)^2 + 5$

g) car^{-1} 1

Substitutions in functional expressions

§2.7 EXAMPLE 4

a) If you know $x = 7$, how does the expression buzz $x + 3$ simplify?
(Put another way, how do you substitute "$x = 7$" into "buzz $x + 3$"?)

b) If you know buzz $x = 7$, how does the expression buzz $x + 3$ simplify?

c) If you know $x = 7$, how does the expression buzz $(x + 1)$ simplify?

d) If you know buzz $x = 7$, how does the expression buzz $(x + 1)$ simplify?

e) If you know $x = 7$, how does the expression buzz ${}^2x$ simplify?

f) If you know buzz $x = 3$, how does the expression buzz ${}^2x$ simplify?

§2.7 EXAMPLE 5

a) Substitute $x = 2$ into moon ${}^2x + 3$ sun ${}^2x = 17$.
Solution: $\boxed{\text{moon } {}^2 2 + 3 \text{ sun } {}^2 2 = 17}$.

b) Substitute moon $x = 2$ into moon ${}^2x + 3$ sun ${}^2x = 17$. Then solve for sun x.

c) Substitute moon ${}^2x = 2$ into moon ${}^2x + 3$ sun${}^2x = 17$. Then solve for sun x.

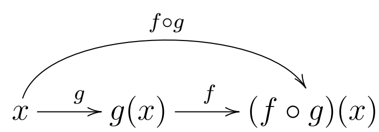
2.8 Composition of functions

Definition 2.15 *Let f and g be functions. The* **composition** *of f and g, denoted $f \circ g$, is the function defined by the rule*

$$(f \circ g)(x) = f(g(x)).$$

Composing functions enables us to build more complicated functions out of easier ones. In particular, composition corresponds to doing functions in sequence, **from right to left**: to evaluate $f \circ g$, you do the procedure g first, then do the procedure f.

An arrow diagram to explain:



§2.8 Example 1

Let $f(x) = x^2$, $g(x) = 3x - 4$ and $h(x) = 2x^2 + 1$. Compute and simplify the rule for each of these functions:

a) $f \circ g$

b) $g \circ f$

c) $g \circ g$

d) $g \circ h \circ f$

WARNING: composition of functions is not commutative (the order matters). In general,

$$(f \circ g)(x) \neq (g \circ f)(x).$$

§2.8 EXAMPLE 2

Continuing with the functions $f(x) = x^2$, $g(x) = 3x - 4$ and $h(x) = 2x^2 + 1$ described in Example 1, first draw an arrow diagram indicating what is asked for; then compute each quantity:

a) $(f \circ g)(3)$

b) $(g \circ h)(-2)$

Solution: $\boxed{-2 \xrightarrow{h} \quad \xrightarrow{g} ?}$;

$(g \circ h)(-2) = g(h(-2)) = g\left(2(-2)^2 + 1\right) = g\left(2(4) + 1\right) = g(9) = 3(9) - 4 = \boxed{23}$.

c) $(g \circ g \circ f)(3)$

§2.8 EXAMPLE 3

Suppose f and g are one-to-one functions, each with domain $\{-3, -2, -1, 0, 1, 2, 3\}$, described by the table of values given below:

x	-3	-2	-1	0	1	2	3
$f(x)$	2	1	-2	0	3	-3	-1
$g(x)$	1	3	-1	2	0	-3	-2

Compute each quantity:

a) $(f \circ g)(3)$

b) $(f^{-1} \circ g)(-2)$

c) $(g^{-1} \circ f^{-1})(-1)$

d) $(f \circ g \circ f^{-1})(0)$

Composing multifunctions

We compose multifunctions in the same way that we compose functions:

§2.8 EXAMPLE 5

Suppose f and g are multifunctions with $g(1) = \{4, 5, 6\}$, $f(4) = \{2, 3\}$, $f(5) = \{3, 4\}$ and $f(6) = 10$. Compute $(f \circ g)(1)$.

Inverting a composition

Let g be the "function" representing the procedure of putting your socks on.

Then g^{-1} is the "function" ______________________________ .

Also, let f be the "function" of putting your shoes on.

Then f^{-1} is the "function" ______________________________ .

When you get ready to go outside, usually you perform the function that is written

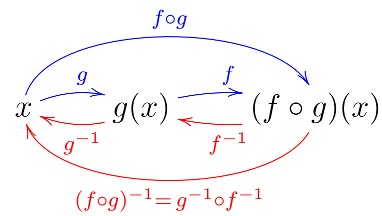
symbolically as ____________ .

Question: When you come back inside, what do you usually do? (There are two ways of writing the answer.)

Theorem 2.16 *Let f and g be functions. Then*

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}.$$

An arrow diagram to explain this:



Elementary functions

Every chemical compound is made up of molecules, where each molecule is itself built from atomic *elements*. As an example, water is built from 2 hydrogen atoms and an oxygen atom (water is H_2O).

Suppose you have a (reasonable) mathematical expression with a single variable that only appears once, like

$$\cos\sqrt[5]{3x^2+5} \qquad \sin^2 4x - 8 \qquad \ln(5\arctan(2x^2-3)) \qquad \text{etc.}$$

Every expression like this is the rule for a function, and that function is a composition of functions from a list of functions called **elementary functions**, in the same way that every molecule is made up of atoms from the elements in the periodic table.

On the next page, I give a list of names and rules of the elementary functions, together with how they are represented in arrow diagrams:

List of elementary functions

Class	Examples	Labelled arrows
Absolute value function	$f(x) = \|x\|$	$\xrightarrow{\|\cdot\|}$ or $\xrightarrow{\text{abs}}$
Addition / subtraction by a constant	$f(x) = x + 3$ $f(x) = x - 1.752$ $f(x) = x + \frac{\sqrt{7}}{4}$ etc.	$\xrightarrow{+3}$ $\xrightarrow{-1.752}$ $\xrightarrow{+\sqrt{7}/4}$
Multiplication / division by a constant	$f(x) = 2x$ $f(x) = \frac{2}{5}x$ $f(x) = \pi x$ $f(x) = \frac{x}{\sqrt{3}}$ etc.	$\xrightarrow{\cdot 2}$ or $\xrightarrow{\times 2}$ $\xrightarrow{\cdot 2/5}$ or $\xrightarrow{\times 2/5}$ $\xrightarrow{\cdot \pi}$ or $\xrightarrow{\times \pi}$ $\xrightarrow{\div\sqrt{3}}$
Whole number power functions	$f(x) = x^2$ $f(x) = x^3$ $f(x) = x^n$ etc.	$\xrightarrow{\wedge 2}$ $\xrightarrow{\wedge 3}$ $\xrightarrow{\wedge n}$
Reciprocal function	$f(x) = x^{-1} = \dfrac{1}{x}$	$\xrightarrow{\frac{1}{\cdot}}$ or $\xrightarrow{1/\cdot}$ or $\xrightarrow{\wedge -1}$
Root functions	$f(x) = \sqrt{x}$ $f(x) = \sqrt[3]{x}$ $f(x) = \sqrt[n]{x}$ etc.	$\xrightarrow{\sqrt{\ }}$ $\xrightarrow{\sqrt[3]{\ }}$ $\xrightarrow{\sqrt[n]{\ }}$
Elementary trig functions	$f(x) = \sin x$ $f(x) = \cos x$ $f(x) = \tan x$	$\xrightarrow{\sin}$ $\xrightarrow{\cos}$ $\xrightarrow{\tan}$
Inverse trig functions	$f(x) = \arcsin x$ $f(x) = \arccos x$ $f(x) = \arctan x$	$\xrightarrow{\arcsin}$ $\xrightarrow{\arccos}$ $\xrightarrow{\arctan}$
Exponential function	$f(x) = e^x$	$\xrightarrow{\exp}$
Natural logarithm function	$f(x) = \ln x$	$\xrightarrow{\ln}$

Functions you may have heard of that are <u>not</u> elementary:

$$5 - x \qquad x^{-2} \qquad x^{3/4} \qquad \csc x \qquad \sec x \qquad \cot x$$

$$\cosh x \qquad 2^x \qquad \log_6 x \qquad \log x$$

Diagramming functions

In this section, we are going to examine the connection between composition of functions and order of operations.

§2.8 EXAMPLE 5

In each part of this problem, you are given a function which is a composition of one or more elementary functions. **Diagram the function**. This means you are to write an arrow diagram which indicates which elementary functions are composed (and in what order) to produce the output of the given function.

a) $f(x) = (x+1)^3$

b) $f(x) = x^3 + 1$

c) $f(x) = x^{-3}$

d) $f(x) = 5x^{2/3}$

e) $f(x) = 1 + \dfrac{7}{|x|}$

f) $f(x) = 4 - \sqrt{x+1}$

g) $f(x) = \sec x$

§2.8 EXAMPLE 6

Diagram these functions:

a) $f(x) = \sin^3 2x$

b) $f(x) = 2\sin^3 x$

c) $f(x) = 2\sin x^3$

d) $f(x) = \sin 2x^3$

Solution: $\boxed{x \xrightarrow{\wedge 3}\xrightarrow{\times 2}\xrightarrow{\sin} f(x)}$

e) $f(x) = \sin(2x)^3$

Solution: $\boxed{x \xrightarrow{\times 2}\xrightarrow{\wedge 3}\xrightarrow{\sin} f(x)}$

§2.8 EXAMPLE 7

In each part, you are given an arrow diagram with elementary functions that indicates how $f(x)$ is produced from x. Write the rule for $f(x)$, simplifying your answer as much as possible. (This is called **reverse diagramming** a function.)

a) $x \xrightarrow{\sin} \xrightarrow{-3} \xrightarrow{\wedge 4} \xrightarrow{1/\cdot} f(x)$

b) $x \xrightarrow{-\pi} \xrightarrow{\sin} \xrightarrow{+4} f(x)$

c) $x \xrightarrow{|\cdot|} \xrightarrow{\cos} \xrightarrow{-1} \xrightarrow{\tan} f(x)$

d) $x \xrightarrow{\times 2} \xrightarrow{\wedge 3} \xrightarrow{1/\cdot} \xrightarrow{\sqrt{\ }} \xrightarrow{\div 4} f(x)$

2.9 Arithmetic operations on functions

In the last section, we talked about combining elementary functions to make more complicated functions using the operation of *composition*.

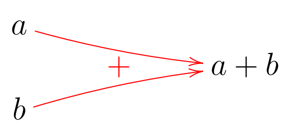
We can also build complicated functions out of elementary ones using other operations that are motivated by basic arithmetic operations we do with numbers.

Adding functions

The process of adding two numbers is a *binary operation*: that is, you add two numbers (i.e. have two inputs) and get one sum (i.e. you have one output). To think of addition as an arrow diagram, consider an example like this:

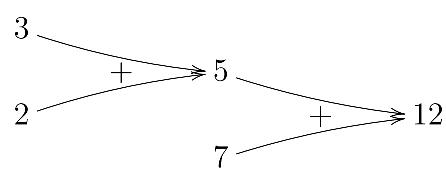
EXAMPLE: Suppose you have 8 red apples and 3 green apples. How many total apples do you have?

More generally, adding two numbers a and b looks like this:



Of course, you can add more than two numbers together, but you have to add them *two at a time*:

$3 + 2 + 7 = ?$



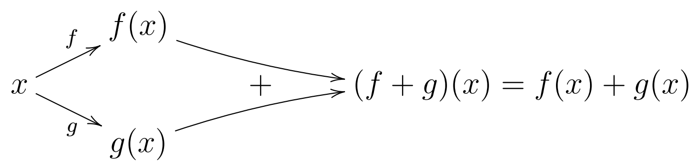
We can also add two functions f and g to get another function called "$f + g$".

Here's how we do this: to define $f + g$, we need to say what its output, given a generic input x. To get the output of $f + g$, we first compute $f(x)$, then compute $g(x)$, and add those together to get the output $(f + g)(x)$. That idea leads to this definition:

Definition 2.17 *Given functions f and g, the* **sum** *of f and g is the function $f + g$ whose rule is*

$$(f + g)(x) = f(x) + g(x).$$

Put another way, $f + g$ is the function with this arrow diagram:

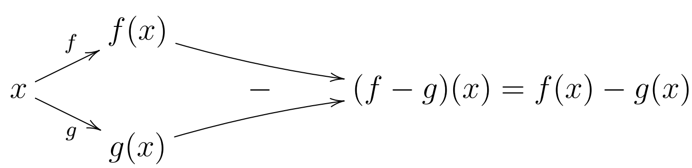


Subtracting functions

Subtracting two functions is similar to adding them: to define function $f - g$ from f and g, we decree that from input x, we first compute $f(x)$ and $g(x)$, then subtract those to get the output of the newly defined $f - g$.

Definition 2.18 *Given functions f and g, the* **difference** *of f and g is the function $f - g$ whose rule is*

$$(f - g)(x) = f(x) - g(x).$$



§2.9 EXAMPLE 1

Suppose f and g are functions described by the table of values given below:

x	-3	-2	-1	0	1	2	3
$f(x)$	-3	0	-2	4	4	-2	1
$g(x)$	5	-3	2	12	0	-7	-3

In each problem, compute the given quantity and in parts (a) and (c), draw an arrow diagram indicating what is being asked for.

a) $(f + g)(-1)$

b) $(f + g^{-1})(0)$

c) $(f - g)(3)$

d) $(f + g \circ f)(2)$

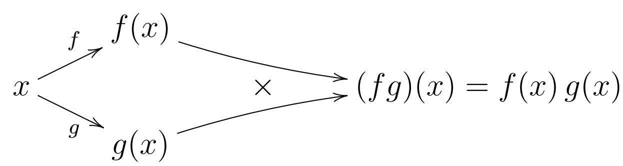
Multiplying and dividing functions

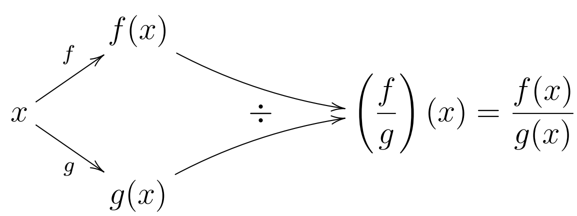
Definition 2.19 *Given functions f and g, the* **product** *of f and g is the function fg whose rule is*

$$(fg)(x) = f(x)g(x).$$

The **quotient** *of f and g is the function $\frac{f}{g}$ whose rule is*

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$





§2.9 Example 2

Suppose f and g are functions described by the table of values given below:

x	-3	-2	-1	0	1	2	3
$f(x)$	-3	0	-2	4	4	-2	1
$g(x)$	5	-3	2	12	0	-7	-3

Compute each quantity:

a) $(fg)(2)$

b) $\left(\frac{f}{g}\right)(-1)$

c) $\left(\frac{g}{f}\right)(-2)$

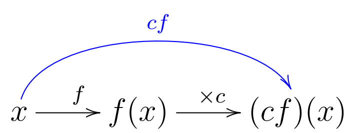
d) $\left(\dfrac{fg}{f+g^{-1}}\right)(3-1)$

Scalar multiples of functions

There is one other operation we can do on a function. We can take a function and multiply it by a number to produce a new function:

> **Definition 2.20** *Let f be a function and let c be a number. The* **scalar multiple**, *a.k.a.* **constant multiple** *of f is the function cf whose rule is*
>
> $$(cf)(x) = c\,f(x).$$

Here's the arrow diagram for this, which indicates that cf is a function which first does f, then multiplies the answer by the scalar/constant c:



§2.9 Example 3

Suppose f and g are functions described by the table of values given below:

x	-3	-2	-1	0	1	2	3
$f(x)$	-3	0	-2	4	4	-2	1
$g(x)$	5	-3	2	12	0	-7	-3

Compute each quantity:

a) $(4f)(2)$

b) $(2g)(-2\cdot -1)$

c) $\left(\dfrac{g}{3f+g}\right)(-3)$

d) $(4f \circ g)(-1)$

§2.9 EXAMPLE 4

Let $f(x) = 2x - 3$, $g(x) = x^2 + 2$ and $h(x) = 5 - 3x$. Compute the rule for each function:

1. $f - 2g$

2. gh

3. h^2

4. $\dfrac{f}{g - 4h}$

5. $f + f \circ h$

6. $hg \circ f$

Theorem 2.21 (Order of function operations) *When building complicated functions from easier ones, the order of operations is:*

1. *obey any grouping symbols like parentheses or brackets;*
2. *exponents;*
3. *multiplication/division (from left to right);*
4. *composition (from left to right);*
5. *addition/subtraction (from left to right).*

You could shorthand the content of this theorem as "PEMDCAS", which is really

$$\text{P} - \text{E} - \begin{pmatrix} \text{M} \\ \text{D} \end{pmatrix} - \text{C} - \begin{pmatrix} \text{A} \\ \text{S} \end{pmatrix}.$$

More diagramming of functions

§2.9 EXAMPLE 5

Diagram each function:

a) $f(x) = x^4 + \cos x$

b) $f(x) = x^3 \sin \sqrt{x}$

c) $f(x) = \dfrac{8x}{x^{3/4} - 1} + 3x \tan x$

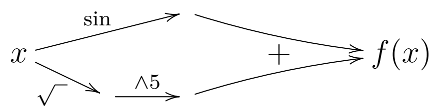
d) $f(x) = (x^5 - 3)^3 (7 \tan x - 3)^4 - |\arctan x|$

e) $f(x) = (x + 3x^2 \cos x)^4 - \log x$

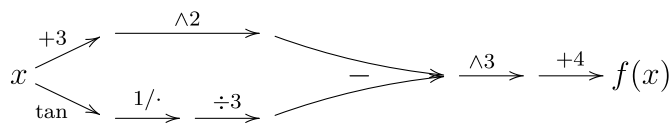
§2.9 EXAMPLE 6

Reverse-diagram each of these pictures (and simplify your answers):

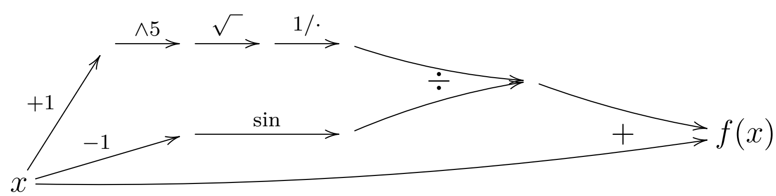
a)



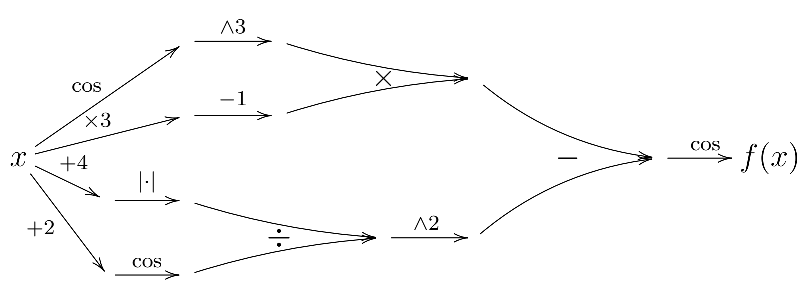
b)



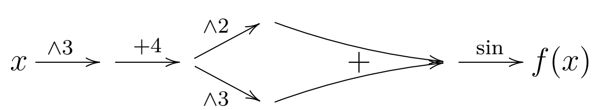
c)



d)



e)

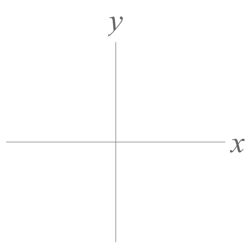


2.10 Graphs of functions

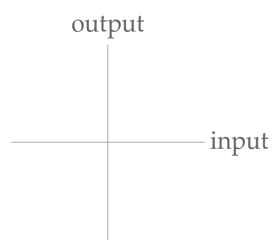
It is often useful to draw a picture representing a function. Such a picture is called a *graph*:

> **Definition 2.22** *The* **graph** *of a function (or multifunction) $f : \mathbb{R} \to \mathbb{R}$ is (a picture of) the set of all points (x, y) such that $y = f(x)$.*

Put another way, this means that we think of the horizontal coordinate of any point on the graph as being an input to the function, and the vertical coordinate is the corresponding output:



should really be thought of as

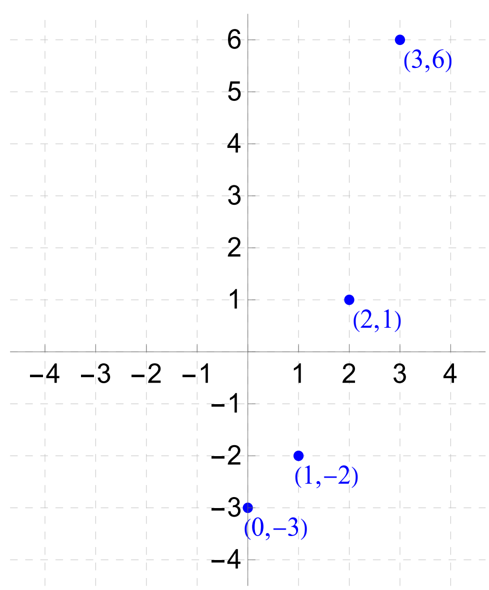


We can make a reasonable guess as to what the graph of a function looks like by plotting some points coming from a table of values:

§2.10 Example 1

Let $f(x) = x^2 - 3$. To produce a picture of the graph of f, let's pick some inputs and find the corresponding outputs:

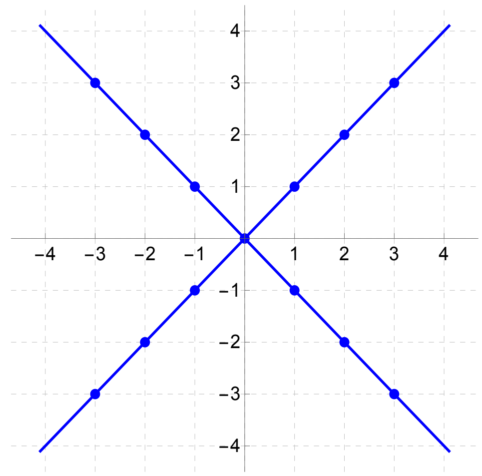
input x	output $f(x)$	ordered pair
-3		
-2		
-1		
0		
1		
2		
3		



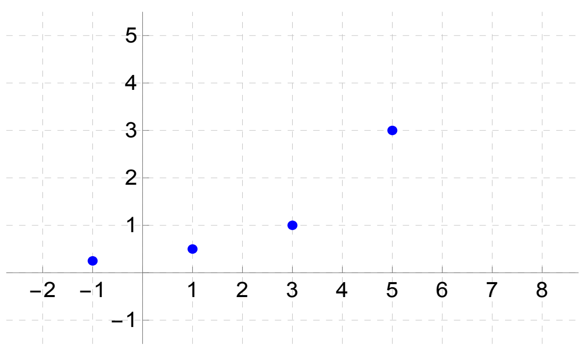
§2.10 EXAMPLE 2

Sketch a graph of the multifunction $f(x) = \pm x$.

input x	output(s) $f(x)$	ordered pair(s)
-3	$\pm(-3) = \{-3, 3\}$	$(-3, -3)$ and $(-3, 3)$
-2	$\pm(-2) = \{-2, 2\}$	$(-2, -2)$ and $(-2, 2)$
-1	$\pm(-1) = \{-1, 1\}$	$(-1, -1)$ and $(-1, 1)$
0	0	$(0, 0)$
1	$\pm 1 = \{-1, 1\}$	$(1, -1)$ and $(1, 1)$
2	$\pm 2 = \{-2, 2\}$	$(2, -2)$ and $(2, 2)$
3	$\pm 3 = \{-3, 3\}$	$(3, -3)$ and $(3, 3)$



Potential problem(s) with producing a graph from table of values:



Ways to get around this:

1. **Use technology:** http://www.desmos.com or a graphing calculator (or software like *Mathematica* that you might learn how to use in a future course).

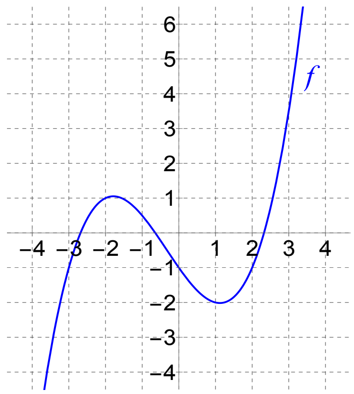
2. **Use stuff we'll learn in Chapters 3 and 4:** we'll learn some general theory about classes of functions (closely related to the classes of elementary functions). This will tell us what a lot of graphs basically look like without having to construct tables.

3. **Take calculus:** where you learn some more powerful theory and computational techniques that tell you basically what any graph looks like without having to construct tables.

Strengths and weaknesses of graphs

Graphs are good for:

- allowing you to *estimate* values (synonym: outputs) of the function:

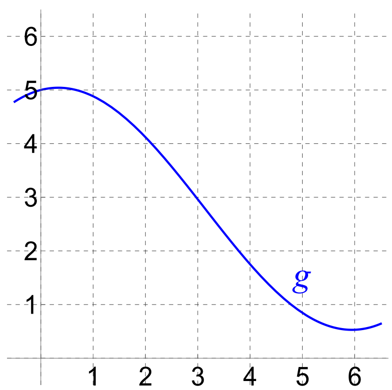


$f(3) \approx$

$f(-2) \approx$

$f^{-1}(4) \approx$

- identifying *relationships* between outputs and inputs, i.e. showing how the output *changes* in response to a change in input:

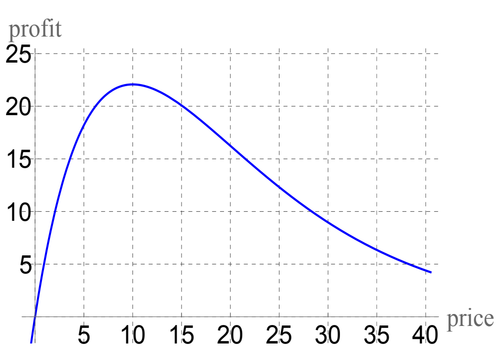


As x increases from 2 to 4,

$g(x)$ ________________

from _____ to _____ .

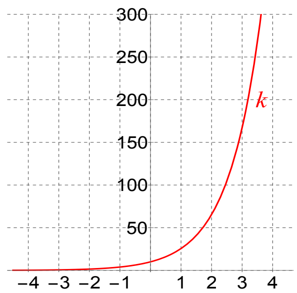
- identifying *optimal* situations related to the function:



The maximum possible profit is _____, which occurs when the product is priced at _____ .

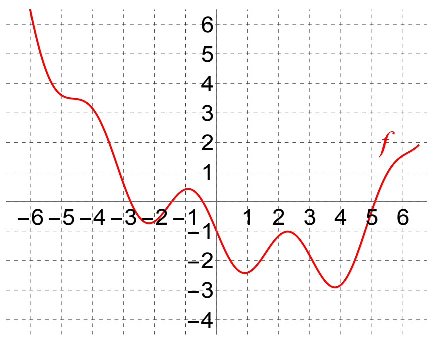
Graphs are not good for:

- determining *exact* values of functions:



$k(3) = ?$

- performing *algebra* on functions:



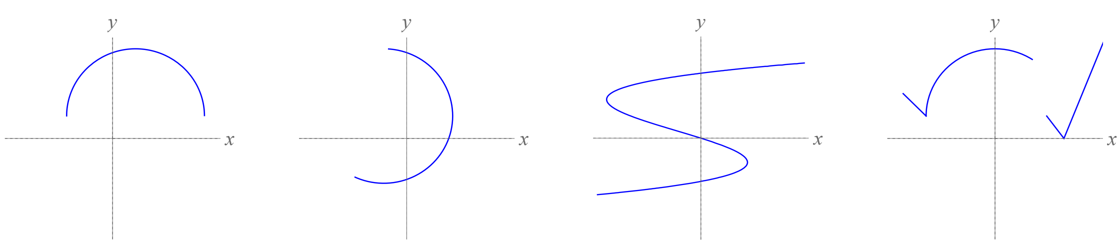
What is the graph of

$f \circ (f^2 + 3f)$?

Vertical and horizontal line tests

§2.10 EXAMPLE 3

The graphs of several multifunctions are given below. Determine whether each graph is the graph of a function $y = f(x)$:



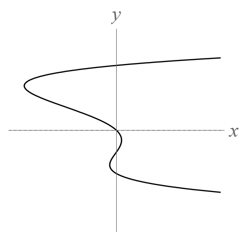
Theorem 2.23 (Vertical Line Test) *A graph in the (x, y)-plane is the graph of a function $y = f(x)$ if and only if the graph intersects every vertical line in at most one point.*

Enrichment: A graph that fails the vertical line test isn't a function $y = f(x)$, BUT it can be the graph of a different kind of function (not $y = f(x)$). More on this in Calculus 3 (MATH 320).

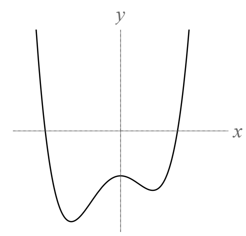
There is also a Horizontal Line Test, which tells us from a graph whether or not a function is one-to-one:

Theorem 2.24 (Horizontal Line Test) *A function $f : \mathbb{R} \to \mathbb{R}$ is one-to-one (a.k.a. injective) if and only if every horizontal line strikes the graph of f in at most one point.*

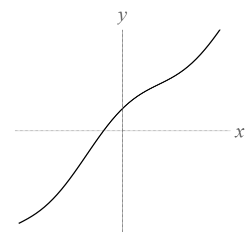
A graph of a multifunction that is **not** a function $y = f(x)$



A graph of a function $y = f(x)$ that is **not** one-to-one



A graph of a one-to-one function $y = f(x)$

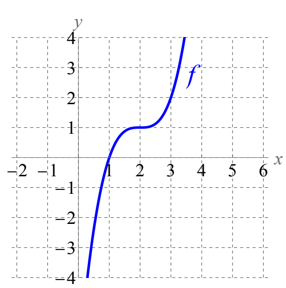


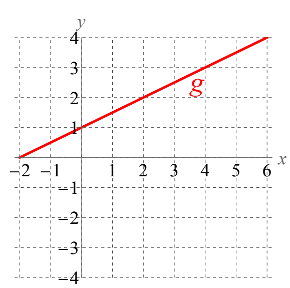
Graphs of piecewise-defined functions

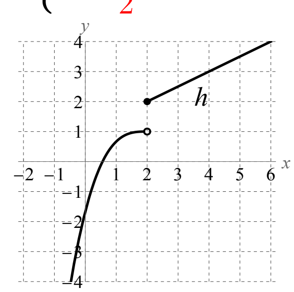
§2.10 Example 4

Consider the graphs of these three functions:

$$f(x) = \frac{1}{3}(x-2)^3 + 1 \qquad g(x) = \frac{1}{2}x + 1 \qquad h(x) = \begin{cases} \frac{1}{3}(x-2)^3 + 1 & x < 2 \\ \frac{1}{2}x + 1 & x \geq 2 \end{cases}$$

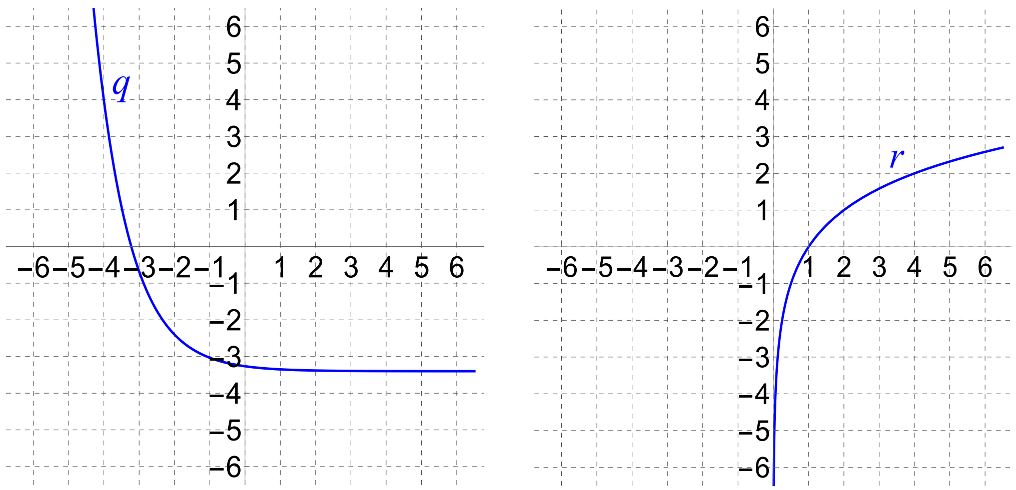






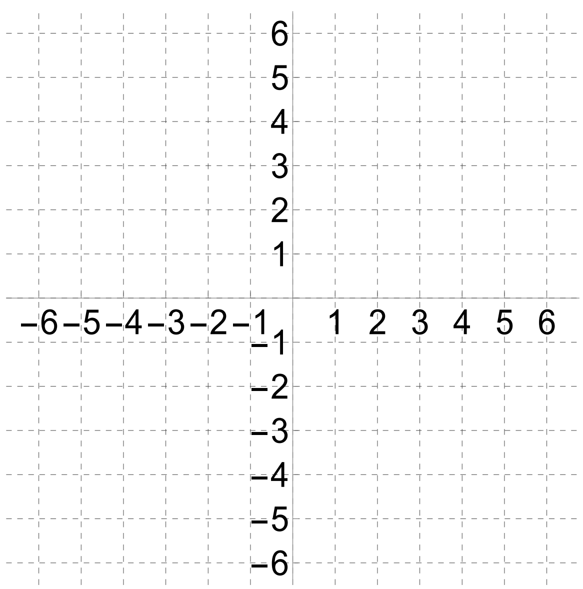
§2.10 EXAMPLE 5

The graphs of functions $q : \mathbb{R} \to \mathbb{R}$ and $r : \mathbb{R} \to \mathbb{R}$ are shown below:



Use these graphs to sketch the graph of

$$f(x) = \begin{cases} q(x) & x < -1 \\ 3 & -1 \le x \le 2 \\ 4 & x = 2 \\ r(x) & x > 2 \end{cases} .$$



x- and y-intercepts

Definition 2.25 *Let $f : \mathbb{R} \to \mathbb{R}$.*

- *The **x-intercept(s)** of a function are points on the graph of f that have y-coordinate 0. These are the points where the graph of f intercepts the x-axis.*
- *The **y-intercept** of a function is the point on the graph of f that has x-coordinate 0. These is the point where the graph of f intercepts the y-axis.*

Note: a function cannot have more than one y-intercept (otherwise the VLT would be violated), but it can have any number of x-intercepts.

Note: some functions may not have x- and/or y-intercepts.

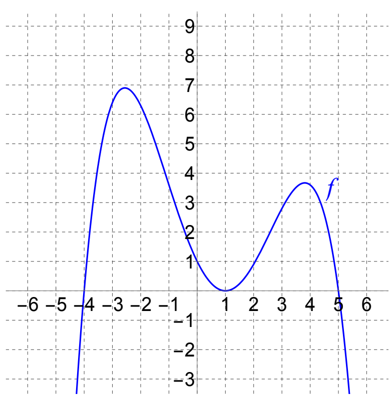
To find x-intercepts, set $f(x) = 0$ (i.e. $y = 0$) and solve for x.

To find the y-intercept, compute $f(0)$ (i.e. set $x = 0$).

Keep in mind: x- and y-intercepts are *points* (ordered pairs), not numbers.

§2.10 EXAMPLE 6

Use the graph of f given below to estimate all x- and y-intercepts of f.



§2.10 EXAMPLE 7

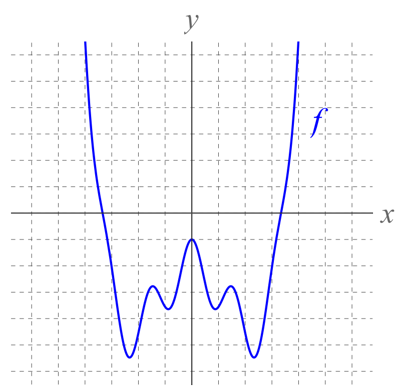
Compute the x- and y-intercepts of the function $g(x) = \dfrac{2}{3}x - \dfrac{8}{3}$.

Symmetry

It is useful to take note of functions whose graphs have *symmetry*, because certain symmetries in a graph can be interpreted in terms of algebra.

§2.10 EXAMPLE 8

Consider the function f that has this graph:



a) Describe the symmetry of this graph.

b) Write down an algebraic formula that corresponds to this symmetry.

Definition 2.26 *A function $f : \mathbb{R} \to \mathbb{R}$ is called* **even** *if its graph is symmetric across the x-axis. Algebraically, this means $f(-x) = f(x)$.*

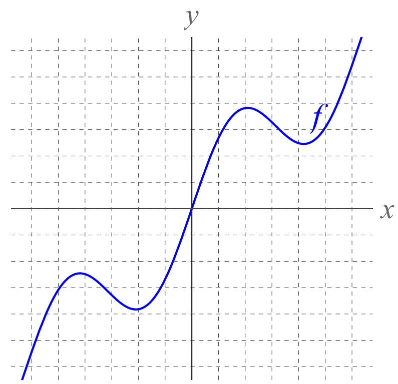
We use the word "even" because any function made up only of even powers of x is even, including things like:

$$f(x) = 3x^2 \qquad g(x) = 5x^8 - 3x^{-2} \qquad h(x) = \sqrt{5}\,x^{100} + 3x^{46} - 2x^2 + 3$$

In trigonometry, we encounter two functions that are even: __________ and __________ .

§2.10 EXAMPLE 9

Consider the function f that has this graph:



a) Describe the symmetry of this graph.

b) Write down an algebraic formula that corresponds to this symmetry.

> **Definition 2.27** *A function* $f : \mathbb{R} \to \mathbb{R}$ *is called* **odd** *if its graph is unchanged if you rotate the entire plane by* $180°$ *about the origin. Algebraically, this means* $f(-x) = -f(x)$.

We use the word "odd" because any function whose terms are odd powers of x will be odd, including things like:

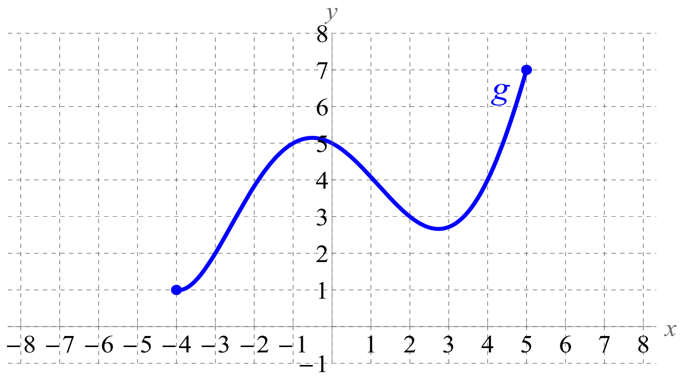
$$f(x) = 3x^49 \qquad g(x) = 7x - 8x^{-3} \qquad h(x) = \frac{2}{3}x^{107} - 4.873x^5 + \frac{5}{\sqrt{7}x^5}$$

In trigonometry, we encounter four functions that are odd: __________ ,
__________ , __________ and __________ .

2.11 Reading graphs

§2.11 Example 1

The graph of some unknown function g is shown below.



Use the graph of g to answer these questions/estimate these quantities:

a) What is the domain of g?

b) What is the range of g?

c) $g(2)$

d) $g(-1) + 1$

e) $g(2 \cdot 2)$

f) $4g(1) - \frac{1}{2}g(-3)$

g) $g^{-1}(3)$

h) $g^{-1}(2)$

i) Near $x = 2$, would $g(x)$ increase or decrease as x increases?

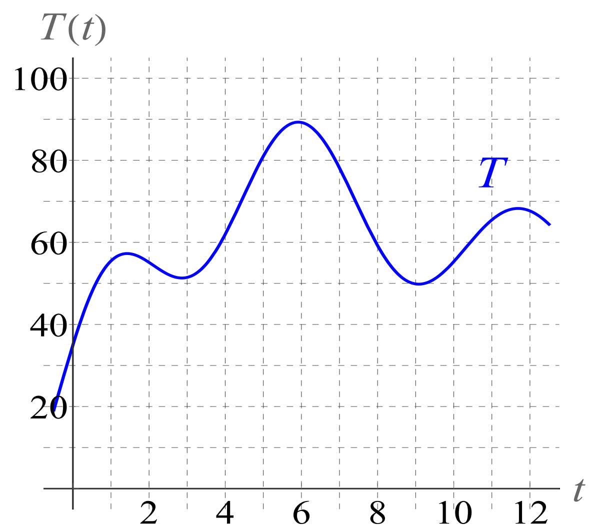
j) Near $x = -3$, would $g(x)$ increase or decrease as x increases?

k) Which is larger, $g(-1)$ or $g(3)$?

l) What is the maximum value achieved by g?

§2.11 EXAMPLE 2

Suppose that the temperature T, in $°C$, of a filament at time t (in minutes) is given by a function whose graph is shown below:



Use this graph to estimate answers to the following questions:

a) What is the temperature of the filament at time 5?

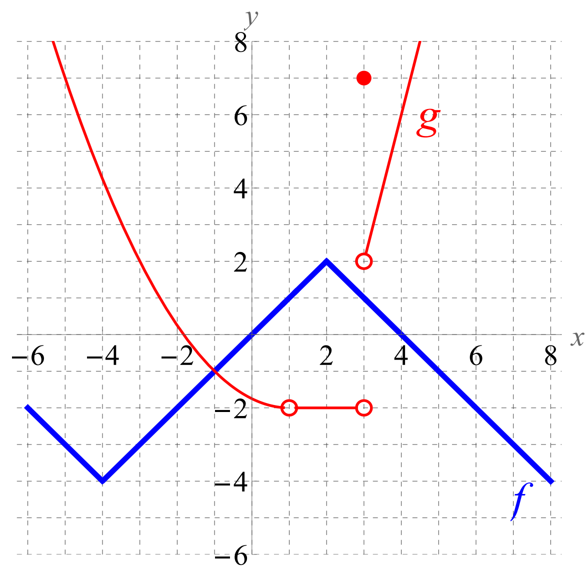
b) At what time(s) is the temperature of the filament equal to $70°$?

c) At time 10.5, is the filament getting hotter or colder?

d) At what time is the temperature of the filament greatest?

§2.11 EXAMPLE 3

The graphs of f and g are given in the picture below (f is the thick graph; g is the thin graph).



Use these graphs to estimate each quantity:

a) $g(3)$

b) $g(1)$

c) $(f+g)(5)$

d) $(fg)(4)+1$

e) $(g \circ f)(0)$

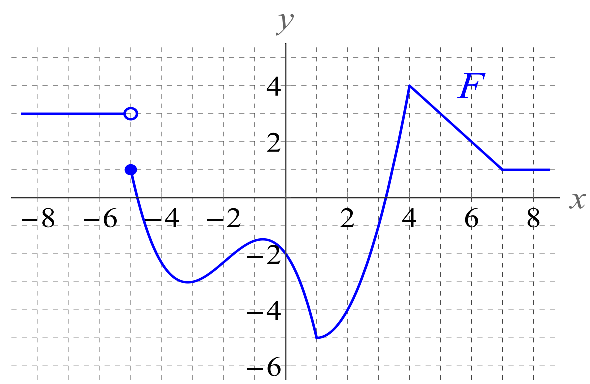
f) $f^3(-1)$

g) $\left(\frac{f}{g} \circ f\right)(2)$

h) $\dfrac{f(4)-2}{4-g(2)}$

§2.11 EXAMPLE 4

Let F be the function whose graph is given here:



Also, suppose G is the function given by the table of values

x	-3	-2	-1	0	1	2	3
$G(x)$	4	8	3	-5	3	1	7

and last, suppose $H : \mathbb{R} \to \mathbb{R}$ is the function $H(x) = \begin{cases} 3 - x & x < 1 \\ x^2 + 2 & x \geq 1 \end{cases}$.

Use all this information to compute (or at least estimate) each quantity:

a) $2 + F(-6)$

e) $(3F - G + 2H)(1)$

b) $(FG)(0)$

f) $G(H(0) - F(5))$

c) $(H \circ F)(3 + 1)$

g) $(F + H \circ G)(1)$

d) $(FH^2)(2)$

h) $3(2F \circ G^{-1})(1)$

2.12 Graphs, equations and inequalities

Every equation in one variable has a left-hand side and a right-hand side. If we set $f(x)$ equal to the left-hand side and $g(x)$ equal to the right-hand side, we see that

Every equation in one variable can be thought of as $f(x) = g(x)$.

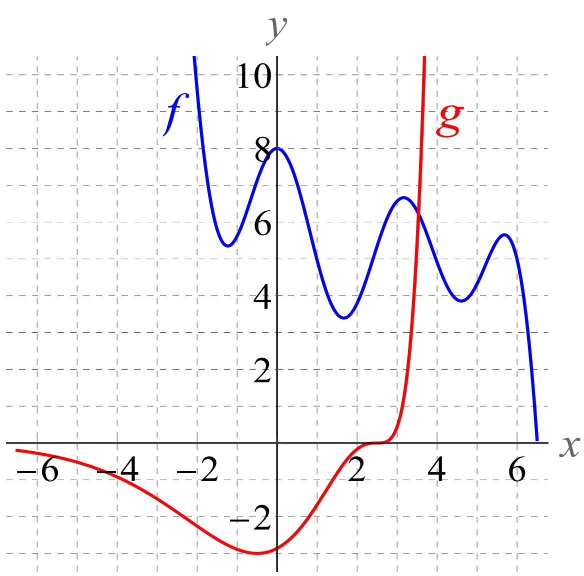
This allows us to use graphs to approach equations. Suppose, for example, we have the equation

$$2\cos 2x - \frac{x^3(x-5)^2}{80} + 6 = 10\left(x - \frac{5}{2}\right)^3 \arctan(e^{x-4}). \tag{2.1}$$

We can let $f(x) = 2\cos 2x - \frac{x^3(x-5)^2}{80} + 6$ and $g(x) = 10\left(x - \frac{5}{2}\right)^3 \arctan(e^{x-4})$; then this equation becomes

$$f(x) = g(x).$$

To see how graphs help us understand this equation, let's look at the graphs of f and g on the same axes;



Based on these graphs:

1. How many solutions does equation (2.1) above have?

2. What is/are the solution(s) of equation (2.1)?

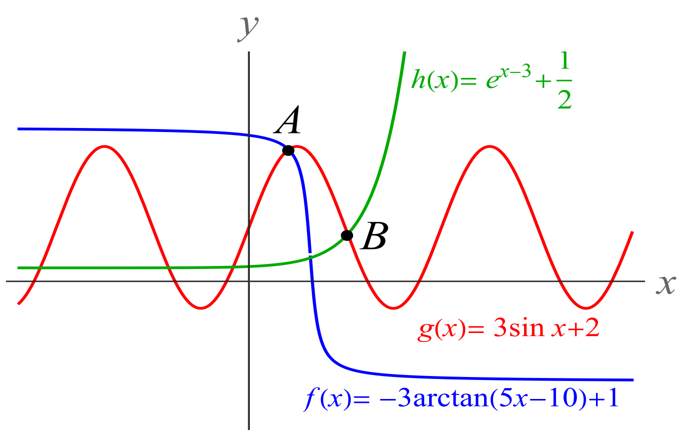
Theorem 2.28 *The solution(s) of the equation $f(x) = g(x)$ are the x-coordinates of the point(s) where the graphs of f and g intersect.*

§2.12 EXAMPLE 1

Find the intersection point of the graphs of $f(x) = 3 - 4x$ and $g(x) = 5x + 8$.

§2.12 EXAMPLE 2

Consider these graphs:



a) What equation would you solve to find point A?

b) What equation would you solve to find point B?

c) When you solve the equation you wrote down in part (b), how many solutions would you get?

d) How would you know which of those solutions is the x-coordinate of point B?

§2.12 EXAMPLE 3

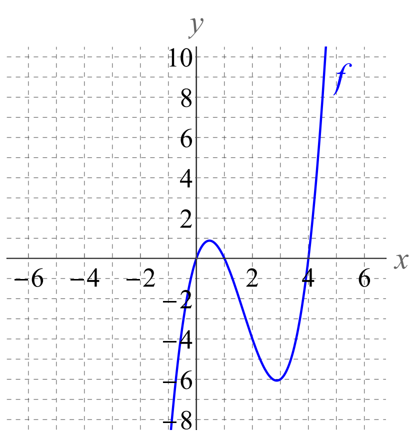
If a company produces x units of its product, then the company incurs costs $C(x) = 30x^{2/3} + 25$ and has revenue $R(x) = xe^x$. Describe a method to determine the number of units the company needs to produce in order to break even.

An important special case

We've seen that every equation in one variable is $f(x) = g(x)$. However, if we move all the non-constant terms of one side of this equation to the other side, we can rewrite the equation as $f(x) = b$ (**not** the same f we started with in $f(x) = g(x)$), where b is a constant. For example, given the equation

$$x^3 + 3x - 5 = 5x^2 - x \tag{2.2}$$

Here's a graph of $f(x) = x^3 - 5x^2 + 4x$:



Based on this graph:

a) How many solutions does equation (2.2) above have?

b) What is/are the solution(s) of equation (2.2)?

Theorem 2.29 *The solutions of the equation $f(x) = b$ are the x-coordinates of points where the graph of f has height b.*

What about inequalities?

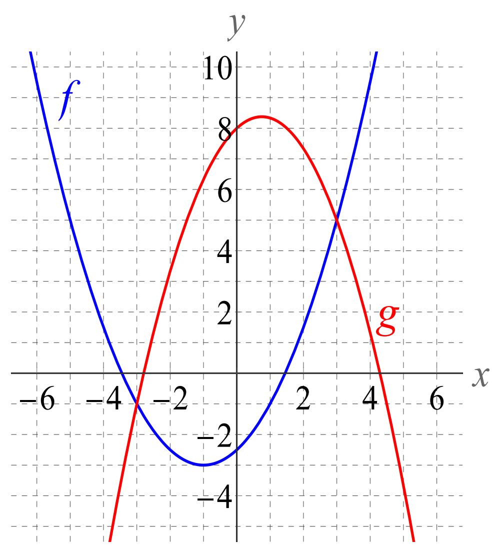
In the same vein that every equation in one variable is $f(x) = g(x)$, every inequality in one variable is either

$$f(x) > g(x) \qquad \text{(a.k.a. } g(x) < f(x))$$

or

$$f(x) \geq g(x) \qquad \text{(a.k.a. } g(x) \leq f(x)).$$

Suppose we have functions f and g with the following graphs:



Based on these graphs,

1. What is the solution set of the inequality $g(x) \geq f(x)$?

2. What is the solution set of the inequality $g(x) > f(x)$?

Theorem 2.30 *The solution set of $f(x) > g(x)$ is the set of x-coordinates where the graph of f lies above the graph of g.*

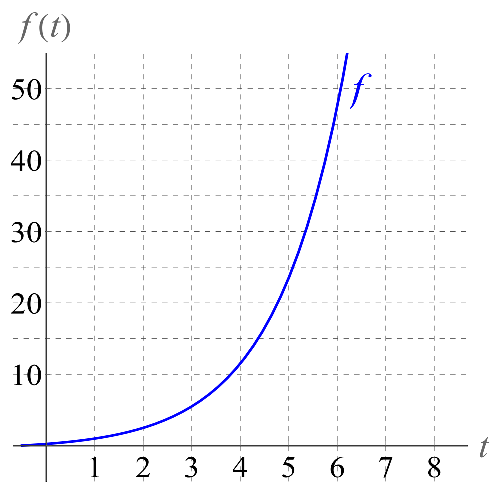
The solution set of $f(x) > b$ is the set of x-coordinates for which the graph of f lies above height b.

The solution set of $f(x) < b$ is the set of x-coordinates for which the graph of f lies below height b.

2.13 Average and instantaneous rate of change

§2.13 EXAMPLE 1

Here is the graph of a function f, where $f(t)$ represents the distance (in meters) an object has travelled after t seconds:



a) How far has the object travelled 1 second after its departure?

b) How far has the object travelled 2 seconds after its departure?

c) How far did the object travel between times 1 and 2?

d) How much time elapsed between times 1 and 2?

e) What is the object's average velocity between times 1 and 2?

f) How far has the object travelled 5 seconds after its departure?

g) What is the object's average velocity between times 1 and 5?

h) What is the object's average velocity between times 3 and 3.0001?

i) What is the object's instantaneous velocity at time 3?

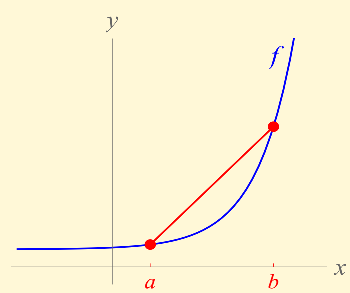
Definition 2.31 *Let $f : \mathbb{R} \to \mathbb{R}$, and suppose $a < b$.*

- *The* **net change** *in f from $x = a$ to $x = b$ is the change in the output, i.e.*

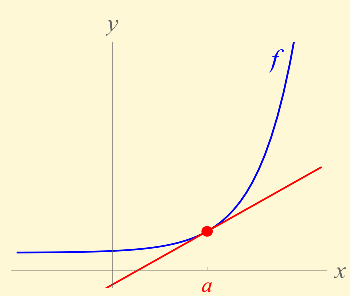
$$\text{net change } = \triangle y = f(b) - f(a).$$

- *The* **average rate of change** *of f from $x = a$ to $x = b$ is the change in the output per unit change in the input, i.e.*

$$\begin{array}{c}\text{average rate}\\ \text{of change}\end{array} = \frac{f(b)-f(a)}{b-a} = \frac{\triangle \text{ outputs}}{\triangle \text{ inputs}} = \frac{\text{rise}}{\text{run}} = \begin{array}{c}\text{slope of red secant}\\ \text{line shown below}\end{array}$$



- *The* **instantaneous rate of change** *of f at $x = a$ is the slope of the line passing through $(a, f(a))$ which best[a] approximates the curve f. This line is called the* **tangent line to f at a**.



[a]Exactly which line "best" approximates f is discussed in calculus.

§2.13 EXAMPLE 2

Let $f(x) = \frac{1}{2}x(x+1)+3$.

a) Compute the net change of f from $x=2$ to $x=5$.

b) Compute the average rate of change of f from $x=2$ to $x=5$.

§2.13 EXAMPLE 3

The temperature of my grill is initially 70° F when I turn it on. Three minutes later, its temperature is 330°, and seven minutes after I turn it on its temperature is 490°. Determine the average rate of change in my grill's temperature between times $t=3$ and $t=7$.

2.14 Chapter 2 Homework

Exercises from Section 2.2

In Exercises 1-5, translate each description of a function into a formula:

1. Let f be the function which takes the cosine of its input, then adds 4 to produce the output.
2. Let F be the function which subtracts 7 from half the input, then takes the cube root of that to produce the output.
3. Let g be the function which doubles its input, then subtracts 7, then multiplies that by 3, then adds four times the input, then squares the result to produce the output.
4. Let G be the function that takes the square root of the sum of the input and the square root of the input to produce the output.
5. Let h be the function that takes twice the input, subtracts 9, squares that, then adds 3 times the two-thirds power of the input, and finally divides by the difference of 13 and the sine of twice the input to produce the output.

In Exercises 6-13, complete the table of values for each given function:

6. $f(x) = 2x^2 - 5x + 3$

x	-2	-1	$-\frac{1}{2}$	0	$\frac{2}{3}$	1	2	$\sqrt{5}$
$f(x)$								

7. $f'(x) = \sin x + \cos x$

   Note: Don't get scared by the $'$ after the f here. f' can be the name of a function, just like f or $\sin$ or $\cos$. The name of this function f' is pronounced "f prime".

x	0	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{5\pi}{4}$
$f'(x)$							

8. $g(x) = \tan 2x$

x	$-\pi$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$g(x)$										

9. $\widehat{g}(x) = 10x^{-1/3}$

   Note: the name of this function is pronounced "g hat".

x	-8	-1	0	1	4	8	15	27
$\widehat{g}(x)$								

10. snow $x = 3x^{2/3} + 2$

x	-8	-4	-1	0	$\frac{1}{8}$	1	5	8	27
snow x									

11. $k : \mathbb{R} \to \mathbb{R}$ is the function takes the reciprocal of the input, then adds 3, then takes the reciprocal of that, and then multiplies that by 4 to get the output.

x	-3	-1	0	1	4
$k(x)$					

12. $M : \mathbb{R} \to \mathbb{R}$ is the function which gives as its output the minimum of four times the input and ten minus the input.

t	-2	-1	0	1	2	3	4
$M(t)$							

13. yes : $\mathbb{R} \to \mathbb{R}$ is the function that produces its output by multiplying the input by the maximum of zero and the cosine of the input.

x	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$
yes x							

In Exercises 14-27, let $f(x) = 2x^2 - 3x + 2$. Complete and simplify each quantity:

14. $f(3)$
15. $f(-2)$
16. $f(\#)$
17. $f\left(\frac{3}{2}\right)$
18. $f(3) + f(-1)$
19. $f(\text{train})$
20. $f(\{1, 2, 4\})$
21. $f(\heartsuit) - f(\diamondsuit)$

22. $f(x+1)$
23. $f(x+t)$
24. $f(4x)$
25. $4f(x)$
26. $f(x+3)-f(x-2)$
27. $\dfrac{f(x+h)-f(x)}{h}$
28. Translate the computation you did in Exercise 15 into an arrow diagram.
29. Translate the computation you did in Exercise 22 into an arrow diagram.

In Exercises 30-38, let $g(t)=\cos\dfrac{t}{2}$. Complete and simplify each quantity:

30. $g(z)$
31. $g(\pi)$
32. $g(3\pi)$
33. $g\left(\dfrac{2\pi}{3}\right)$
34. $g(-4\pi)+g(0)$
35. $g\left(\dfrac{\pi}{3}\right)$
36. $g(3\triangle)$
37. $g(2x)$
38. $g\left(\dfrac{3\pi}{2}\right)+g\left(\dfrac{5\pi}{3}\right)$

In Exercises 39-50, let $h(x)=x^{-3/2}$. Complete and simplify each quantity:

39. $h(4)$
40. $h(7)$
41. $h(-2)$
42. $h(\square)$
43. $h\left(\dfrac{1}{4}\right)$
44. $h(\text{banana})$
45. $h(x^2)$
46. $h(4x)$
47. $h(x+3)$
48. $h(9)+h(16)$
49. $h\left(\dfrac{25}{9}\right)+h\left(\dfrac{1}{9}\right)$
50. $5h(t)$

In Exercises 51-60, you are given an equation and a function. Apply the function to both sides of the equation, simplifying both sides of what you get:

51. $\dfrac{x}{3}+\dfrac{4}{9}=\dfrac{7}{6}$; $\quad f(x)=x-\dfrac{1}{3}$
52. $4x+3=22$; $\quad f(x)=\dfrac{1}{2}x$
53. $8x^3=19$; $\quad f(x)=\sqrt[3]{x}$
54. $8x^3+1=19$; $\quad f(x)=\sqrt[3]{x}$
55. $x^6=19$; $\quad f(x)=\sqrt{x}$
56. $x^{2/3}=5$; $\quad f(x)=x^{3/5}$
57. $x^{-3/4}=\dfrac{3}{2}$; $\quad f(x)=x^4$
58. $\cos 2x=-1$; $\quad f(x)=4x$
59. $\sin 2x=\dfrac{1}{2}$; $\quad f(x)=x+2$
60. $x^{-2}=8$; $\quad f(x)=x^{-1/2}$

61. In each part of this exercise, you are given a function. Apply this function to both sides of the equation

$$5x+7=29,$$

simplifying both sides of what you get:

a) $a(x) = \frac{x}{5}$

b) $b(x) = x - 7$

c) $c(x) = \frac{x}{5} - 7$

d) $d(x) = \frac{x-5}{7}$

62. In the previous question, for which function (a, b, c or d) would it make the most sense to apply that function to both sides of $5x + 7 = 29$? Explain your answer.

63. In each part of this exercise, you are given an equation. Write down a function f so that, if you applied f to both sides of the equation, you would isolate x on one side of the equation.

    a) $x + 9 = 23$

    b) $8x = 14$

    c) $x - 6 = 13$

    d) $\frac{2}{3}x = -4$

    e) $3x + 7 = -5$

    f) $-\frac{x}{2} + \frac{1}{4} = \frac{3}{8}$

64. In each part of this problem, you are given a function. Apply this function to both sides of the equation
$$x^{-1/4} = 3,$$
simplifying both sides of what you get:

    a) $a(x) = x^{-1}$

    b) $b(x) = x^4$

    c) $c(x) = \frac{1}{x}$

    d) $d(x) = x^{-4}$

65. In the previous question, for which function (a, b, c or d) would it make the most sense to apply that function to both sides of $x^{-1/4} = 3$? Explain your answer.

66. In general, if you have the equation $x^a = b$, where a and b are constants, what function would you need to apply to both sides of the equation in order to isolate x on the left-hand side?

67. In each part of this exercise, you are given an equation. Write down a function f that, if you applied f to both sides of the equation, you would isolate x on the left-hand side.

    Hint: If necessary, rewrite the left-hand side as $x^{\square}$, then use the result of the previous exercise.

    a) $x^5 = 22$

    b) $x^{-1} = 22$

    c) $x^{1/3} = 22$

    d) $x^{-4} = 22$

    e) $x^{-5/2} = 22$

    f) $x^{4/3} = 22$

    g) $\sqrt{x} = 22$

    h) $x^3 = 22$

    i) $\sqrt[3]{x} = 22$

68. Same directions as previous exercise:

a) $\sqrt[3]{x^4} = 22$

b) $(\sqrt[3]{x})^4 = 22$

c) $x^8 = 22$

d) $\dfrac{1}{x^7} = 22$

e) $\sqrt{x^5} = 22$

f) $x^{1/9} = 22$

g) $\dfrac{1}{\sqrt{x}} = 22$

h) $\dfrac{1}{\sqrt[4]{x^7}} = 22$

i) $x^{-1/3} = 22$

j) $x^{77} = 22$

Answers

1. $f(x) = \cos x + 4$

2. $F(x) = \sqrt[3]{\dfrac{1}{2}x - 7}$

3. $g(x) = (3(2x-7) + 4x)^2$

4. $G(x) = \sqrt{x + \sqrt{x}}$

5. $h(x) = \dfrac{(2x-9)^2 + 3x^{2/3}}{13 - \sin 2x}$

6.

x	-2	-1	$-\frac{1}{2}$	0	$\frac{2}{3}$	1	2	$\sqrt{5}$
$f(x)$	21	10	6	3	$\frac{5}{9}$	0	1	$13 - 5\sqrt{5}$

7.

x	0	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{5\pi}{4}$
$f'(x)$	1	$\frac{\sqrt{3}}{2} + \frac{1}{2}$	$\sqrt{2}$	1	$\frac{\sqrt{3}}{2} - \frac{1}{2}$	-1	$-\sqrt{2}$

8.

x	$-\pi$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$g(x)$	0	0	$\sqrt{3}$	0	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	DNE	$-\sqrt{3}$	0	0

9.

x	-8	-1	0	1	4	8	15	27
$\widehat{g}(x)$	-5	-10	DNE	10	$\frac{10}{\sqrt[3]{4}}$	5	$\frac{10}{\sqrt[3]{15}}$	$\frac{10}{3}$

10.

x	-8	-4	-1	0	$\frac{1}{8}$	1	5	8	27
snow x	14	$3\sqrt[3]{16} + 2$	5	2	$\frac{11}{4}$	5	$3\sqrt[3]{25} + 2$	14	29

11.

x	-3	-1	0	1	4
$k(x)$	$\frac{3}{2}$	2	0	1	$\frac{16}{13}$

12.

t	-2	-1	0	1	2	3	4
$M(t)$	-8	-4	0	4	8	7	6

13.

x	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$
yes x	0	$-\frac{\pi\sqrt{2}}{8}$	0	$\frac{\pi\sqrt{2}}{8}$	0	0	0

14. 11
15. 16
16. $2\#^2 - 3\# + 2$
17. 2
18. 18
19. $2\text{train}^2 - 3\text{train} + 2$
20. $\{1, 4, 22\}$
21. $2\heartsuit^2 - 3\heartsuit - 2\diamondsuit^2 + 3\diamondsuit$
22. $2x^2 + x + 1$
23. $2(x+t)^2 - 3(x+t) + 2$
24. $32x^2 - 12x + 2$
25. $8x^2 - 12x + 8$
26. $20x - 5$
27. $4x + 2h - 3$
28. $-2 \xrightarrow{f} 16$
29. $x + 1 \xrightarrow{f} 2x^2 + x + 1$
30. $\cos\frac{z}{2}$
31. 0
32. 0
33. $\frac{1}{2}$
34. 2
35. $\frac{\sqrt{3}}{2}$
36. $\cos\frac{3\triangle}{2}$
37. $\cos x$
38. $-\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}$
39. $\frac{1}{8}$
40. $7^{-3/2}$
41. DNE
42. $\square^{-3/2}$
43. 8
44. $\text{banana}^{-3/2}$
45. x^{-3}
46. $\frac{1}{8}x^{-3/2}$
47. $(x+3)^{-3/2}$
48. $\frac{1}{27} + \frac{1}{64} = \frac{91}{1728}$
49. $\frac{27}{125} + 27 = \frac{3402}{125}$

50. $5t^{-3/2}$

51. $\dfrac{x}{3}+\dfrac{1}{9}=\dfrac{5}{6}$

52. $2x+\dfrac{3}{2}=11$

53. $2x=\sqrt[3]{19}$

54. $\sqrt[3]{16x^2+1}=\sqrt[3]{19}$

55. $x^3=\sqrt{19}$

56. $x^{2/5}=5^{3/5}$

57. $x^{-3}=\dfrac{81}{16}$

58. $4\cos 2x=-4$

59. $\sin 2x+2=\dfrac{5}{2}$

60. $x=\dfrac{1}{\sqrt{8}}$

61.
   a) $x+\dfrac{7}{5}=\dfrac{29}{5}$
   b) $5x=22$
   c) $x-\dfrac{28}{5}=-\dfrac{6}{5}$
   d) $x=\dfrac{22}{5}$

62. d is best, because it isolates x by itself (solving the equation for x).

63.
   a) $f(x)=x-9$
   b) $f(x)=\dfrac{x}{8}$
   c) $f(x)=x+6$
   d) $f(x)=\dfrac{3}{2}x$
   e) $f(x)=\dfrac{x-7}{3}$
   f) $f(x)=-2\left(x-\dfrac{1}{4}\right)$

64. In each part of this problem, you are given a function. Apply this function to both sides of the equation
$$x^{-1/4}=3,$$
simplifying both sides of what you get:
   a) $x^{1/4}=3$ a.k.a. $\sqrt[4]{x}=3$
   b) $x^{-1}=81$ a.k.a. $\dfrac{1}{x}=81$
   c) $x^4=\dfrac{1}{3}$
   d) $x=\frac{1}{81}$

65. d is best, because it isolates x by itself (solving the equation for x).

66. You should apply the function $f(x)=x^{1/a}$ to both sides in order to isolate x.

67.
   a) $f(x)=x^{1/5}$ a.k.a. $f(x)=\sqrt[5]{x}$
   b) $f(x)=x^{-1}$ a.k.a. $f(x)=\dfrac{1}{x}$
   c) $f(x)=x^3$
   d) $f(x)=x^{-1/4}$ a.k.a. $f(x)=\dfrac{1}{\sqrt[4]{x}}$
   e) $f(x)=x^{-2/5}$ a.k.a. $f(x)=\dfrac{1}{(\sqrt{x})^5}$
   f) $f(x)=x^{3/4}$ a.k.a. $f(x)=\sqrt[4]{x^3}$
   g) $f(x)=x^2$
   h) $f(x)=x^{1/3}$ a.k.a. $f(x)=\sqrt[3]{x}$
   i) $f(x)=x^3$

68.
a) $f(x) = x^{3/4}$ a.k.a. $f(x) = (\sqrt[3]{x})^4$
b) $f(x) = x^{3/4}$ a.k.a. $f(x) = (\sqrt[3]{x})^4$
c) $f(x) = x^{1/8}$ a.k.a. $f(x) = \sqrt[8]{x}$
d) $f(x) = x^{-1/7}$ a.k.a. $f(x) = \frac{1}{\sqrt[7]{x}}$
e) $f(x) = x^{2/5}$ a.k.a. $f(x) = \sqrt[5]{x^2}$
f) $f(x) = x^9$
g) $f(x) = x^{-2}$ a.k.a. $f(x) = \dfrac{1}{x^2}$
h) $f(x) = x^{-4/7}$ a.k.a. $f(x) = \frac{1}{\sqrt[7]{x^4}}$
i) $f(x) = x^{-3}$ a.k.a. $f(x) = \dfrac{1}{x^3}$
j) $f(x) = x^{1/77}$ a.k.a. $f(x) = \sqrt[77]{x}$

Exercises from Section 2.3

1. Complete this table:

INTERVAL NOTATION	INEQUALITY	PICTURE OF INTERVAL
$[5, 9)$		
	$3 < x$	
$(-\infty, -4]$		
		
	$2 \le x \le 7$	
$\left(\frac{2}{3}, \frac{5}{4}\right)$		
		
	$x \ge 0$	

2. In this problem, let $f(x) = 14 - 5x$ and let $g(x) = 3 + 2x^2 + 4x$. Classify these statements as true or false:

   a) $f(1) = g(1)$ b) $f(2) = g(2)$ c) $f = g$

3. In this problem, let $h(x) = (x + 2)^2 - (x + 1)^2$ and let $k(x) = 2x + 3$. Classify these statements as true or false:

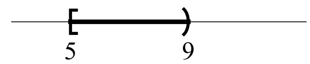
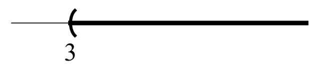
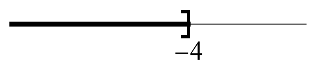
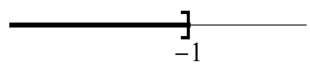
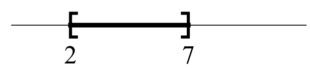
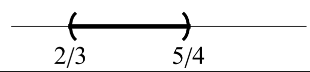
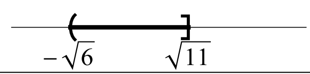
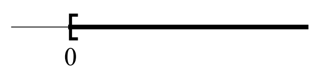
a) $h(4) = k(4)$ b) $h = k$

4. In this problem, let $m(x) = \left(x + \frac{1}{2}\right)^2 + 2\left(x - \frac{1}{4}\right)^2$ and let $n(x) = 4x^2 - \frac{5}{8}$. Classify these statements as true or false:

a) $m(-1) = n(-1)$ b) $m = n$

Answers

1.

$[5, 9)$	$5 \leq x < 9$	
$(3, \infty)$	$3 < x$	
$(-\infty, -4]$	$x \leq -4$	
$(-\infty, -1]$	$x \leq -1$	
$[2, 7]$	$2 \leq x \leq 7$	
$\left(\frac{2}{3}, \frac{5}{4}\right)$	$\frac{2}{3} < x < \frac{5}{4}$	
$(-\sqrt{6}, \sqrt{11}]$	$-\sqrt{6} < x \leq \sqrt{11}$	
$[0, \infty)$	$x \geq 0$	

2. a) True b) False c) False

3. a) True b) True

4. a) True b) False

Exercises from Section 2.4

1. Write a (piecewise-defined) rule/formula for the function f described as follows:

   If the input is positive, then the output is the cube root of the tangent of the input; if the input is negative or zero, the output is 1 plus the sine of the input.

2. Write a (piecewise-defined) rule for the function g described as follows:

   If the input x is greater than 11, then the output is $x^{-2/5}$; if the input is greater than 2 but less than 11 then the output is the cosine of twice the input; if the input is at most 2 the output is 7; $g(11) = 0$.

3. Look at Motivating Example C on page 46 of the lecture notes. Write a rule for the function "pay".

In Exercises 4-9, suppose $F : \mathbb{R} \to \mathbb{R}$ is the function

$$F(x) = \begin{cases} 3x-4 & x < -4 \\ 2x+1 & -4 \le x < 5 \\ x-3 & x \ge 5 \end{cases}.$$

Compute and simplify each quantity:

4. $F(-5)$
5. $F(-2)$
6. $F(5)$
7. $F(0) + F(8)$
8. $F(0+8)$
9. $F(11)$

In Exercises 10-19, suppose

$$\widehat{k}(x) = \begin{cases} \cos 4x & x < 4 \\ 13 & x = 4 \\ 10x^{-1/3} & 4 \le x < 9 \\ x^{2/3} & x \ge 9 \end{cases}.$$

Compute and simplify each quantity:

10. $\widehat{k}\left(-\frac{\pi}{16}\right)$
11. $\widehat{k}(7-3)$
12. $\widehat{k}(64)$
13. $\widehat{k}(\pi)$
14. $3\widehat{k}(8)$
15. $\widehat{k}(3\pi)$
16. $\widehat{k}(27)$
17. $\widehat{k}(125) + \widehat{k}(8)$
18. $\widehat{k}(77) - \widehat{k}(77)$
19. $\widehat{k}\left(-\frac{\pi}{8}\right)\widehat{k}(24)$

20. Look up (on page 60 of the lecture notes) the function T which determines the amount of federal income tax owed as a function of an individual's salary x. For each of these questions, write an expression using function notation that gives the answer to the question, and then compute the answer (in this problem, it is OK to use a calculator):

    a) In the year 2023, how much tax would a person with a salary of 215000 owe?

    b) How much would the person in part (a) have left after paying taxes?

    c) If two individuals, one making 20000 and the other making 70000, each file tax returns independently, what is the total amount they will pay in tax in the year 2023?

Answers

1. $f(x) = \begin{cases} \sqrt[3]{\tan x} & x > 0 \\ 1 + \sin x & x \leq 0 \end{cases}$

2. $g(x) = \begin{cases} 7 & x \leq 2 \\ \cos 2x & 2 < x < 11 \\ 0 & x = 11 \\ x^{-2/5} & x > 11 \end{cases}$

3. pay $x = \begin{cases} 20x & 0 \leq x \leq 40 \\ 20(40) + 30(x - 40) & x > 40 \end{cases}$

   (The second line of this could be simplified to $30x - 400$.)

4. -19
5. -3
6. 2
7. 6
8. 5
9. 8
10. $\dfrac{\sqrt{2}}{2}$
11. 13
12. 16
13. 1
14. 15
15. $(3\pi)^{2/3}$
16. 9
17. 30
18. 0
19. 0
20. a) quantity: $T(215000)$; answer: 47632
    b) quantity: $215000 - T(215000)$; answer 167368
    c) quantity: $T(20000) + T(70000)$; answer 12887.50

Exercises from Section 2.5

1. Throughout this exercise, let $p(x) = \pm(3x + 2)$. Compute and simplify each quantity:

   a) $p(5)$ b) $p(-3)$ c) $p\left(\dfrac{-3}{4}\right)$

2. Throughout this exercise, let mean be the multifunction mean $x = \pm\sqrt{6x^3}$. Compute and simplify each quantity:

   a) mean 1
   b) mean 6
   c) mean (6)
   d) mean 0
   e) mean $\sqrt[3]{6}$
   f) mean -6
   g) mean $\dfrac{1}{6}$
   h) mean $\sqrt[3]{24}$

3. Throughout this exercise, suppose c is a multifunction which takes its input and produces as its output a list of all the integers which are at least 0 but less than or equal to the input. Compute each quantity:

   a) $c(7)$ b) $c\left(\dfrac{2}{3}\right)$ c) $c(2\pi)$ d) $c\left(\dfrac{28}{5}\right)$

Answers

1.
   a) $\{-17, 17\}$
   b) $\{-7, 7\}$
   c) $\{-\frac{1}{4}, \frac{1}{4}\}$

2.
   a) $\{-\sqrt{6}, \sqrt{6}\}$
   b) $\{-36, 36\}$
   c) $\{-36, 36\}$
   d) 0
   e) $\{-6, 6\}$
   f) DNE
   g) $\{-\frac{1}{6}, \frac{1}{6}\}$
   h) $\{-12, 12\}$

3.
   a) $\{0, 1, 2, 3, 4, 5, 6, 7\}$
   b) 0
   c) $\{0, 1, 2, 3, 4, 5, 6\}$
   d) $\{0, 1, 2, 3, 4, 5\}$

Exercises from Section 2.6

1. Let f be the function described completely by this table of values:

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$	2	2	-3	-3	0	0	-6	6	1	4	-1

In each part, write an arrow diagram that explains what is being asked for, and then compute the indicated quantity:

   a) $f(2)$
   b) $f^{-1}(2)$
   c) $f(0)$
   d) $f^{-1}(0)$
   e) $f^{-1}(-2)$
   f) $f(6)$
   g) $f^{-1}(6)$
   h) $f(4)$

2. Using the same function f as in Exercise 1, compute each indicated quantity:

   a) $f(f^{-1}(2))$
   b) $f^{-1}(f(1))$
   c) $f^{-1}(1)$
   d) $f(f^{-1}(-3)$
   e) $f^{-1}(f(-5))$
   f) $f(f(3))$
   g) $f^{-1}(f^{-1}(3))$
   h) $f(-2)$

3. Let wolf : $\mathbb{R} \to \mathbb{R}$ be wolf $x = 3x+4$. In each part, write an arrow diagram that explains what is being asked for, and then compute the indicated quantity.

   a) wolf t
   b) wolf 3
   c) wolf$^{-1}19$
   d) wolf $\frac{1}{2}$
   e) wolf$^{-1} - 10$
   f) wolf^{-1}q

4. Let $g(x) = x^2 + 3$. Compute each indicated quantity:

   a) $g(1)$
   b) $g(-4)$
   c) $g^{-1}(3)$
   d) $g^{-1}(1)$
   e) $g^{-1}(19)$
   f) $g^{-1}(g(2))$
   g) $g(g^{-1}(4))$
   h) $g(g^{-1}(286^{2/3}))$
   i) $g^{-1}(y)$
   j) $g^{-1}(x)$

5. Suppose F, G and H are functions, each with domain $\{-2, -1, 0, 1, 2, 3, 4, 5\}$ described completely by this table of values:

x	-2	-1	0	1	2	3	4	5
$F(x)$	5	7	-2	3	0	-5	6	1
$G(x)$	-1	5	2	11	17	-3	-4	6
$H(x)$	$\sqrt{17}$	$\sqrt{5}$	$\sqrt{11}$	$\sqrt{19}$	$\sqrt{5}$	$\sqrt{13}$	$\sqrt{26}$	1

   a) Is F^{-1} a function?
   b) Is G^{-1} a function?
   c) Is H^{-1} a function?

6. In each part of this question you are given a function f. Determine whether or not its inverse f^{-1} is also a function:

   a) $f(x) = 2x$
   b) $f(x) = \frac{x}{2} + \frac{2}{3}$
   c) $f(x) = x^4$
   d) $f(x) = x^5$
   e) $f(x) = \sqrt{x}$
   f) $f(x) = \frac{x}{9}$
   g) $f(x) = x - 5$
   h) $f(x) = x^{2/3}$
   i) $f(x) = |x|$
   j) $f(x) = 6x - 5$
   k) $f(x) = \sin x$
   l) $f(x) = 12x^{17}$

7. Determine whether or not each function is one-to-one.

   a) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x$. (In other words, this is just the function $f(x) = 3x$.)
   b) The function $g(x) = 2 - 7x$.
   c) The function $h(x) = \cos x$
   d) The function $k(x) = x^2 - 3x + 7$
   e) The function that assigns to each date its day of the week (for example, if the input is January 29, 2024, the output would be Monday).
   f) The function that assigns to each US state its capital city (for instance, if the input is Michigan, the output would be Lansing).
   g) The function that assigns to every living US citizen his/her Social Security number.

h) The function that assigns to every NFL player the number of touchdowns he has scored in his career.

i) The function that assigns to every key on a piano the pitch of the note that is played when that key is pressed.

j) The function in Motivating Example C on page 46 of the lecture notes (that computes the amount you earn in a week if you make $\$20$ per hour plus time-and-a-half for overtime).

Answers

1. a) $f(2) = 6$
   b) $f^{-1}(2) = \{-5, -4\}$
   c) $f(0) = 0$
   d) $f^{-1}(0) = \{-1, 0\}$
   e) $f^{-1}(-2)$ DNE
   f) $f(6)$ DNE
   g) $f^{-1}(6) = 2$
   h) $f(4) = 4$

2. a) $f(f^{-1}(2)) = 2$
   b) $f^{-1}(f(1)) = 1$
   c) $f^{-1}(1) = 3$
   d) $f(f^{-1}(-3) = -3$
   e) $f^{-1}(f(-5)) = \{-5, -4\}$
   f) $f(f(3)) = f(1) = -6$
   g) $f^{-1}(f^{-1}(3))$ DNE
   h) $f(-2) = -3$

3. a) wolf $t = 3t + 4$
   b) wolf $3 = 13$
   c) wolf$^{-1} 19 = 5$
   d) wolf $\frac{1}{2} = \frac{11}{2}$
   e) wolf$^{-1} - 10 = -\frac{14}{3}$
   f) wolf$^{-1} q = \frac{1}{3}(q - 4)$

4. a) $g(1) = 4$
   b) $g(-4) = 19$
   c) $g^{-1}(3) = 0$
   d) $g^{-1}(1)$ DNE
   e) $g^{-1}(19) = \{-4, 4\}$
   f) $g^{-1}(g(2)) = g^{-1}(7) = \{-2, 2\}$
   g) $g(g^{-1}(4)) = 4$
   h) $g(g^{-1}(286^{2/3})) = 286^{2/3}$
   i) $g^{-1}(y) = \pm\sqrt{y - 3}$
   j) $g^{-1}(x) = \pm\sqrt{x - 3}$

5. a) Yes
   b) Yes
   c) No

6. a) Yes ($f^{-1}(x) = \frac{1}{2}x$).
   b) Yes ($f^{-1}(x) = 2x - \frac{4}{3}$).
   c) No ($f^{-1}(x) = \pm\sqrt[4]{x}$).
   d) Yes ($f^{-1}(x) = \sqrt[5]{x}$).
   e) Yes ($f^{-1}(x) = x^2$).
   f) Yes ($f^{-1}(x) = 9x$).
   g) Yes ($f^{-1}(x) = x + 5$).
   h) No ($f^{-1}(x) = \pm x^{3/2}$).
   i) No ($f^{-1}(x) = \pm x$).

j) Yes ($f^{-1}(x) = \frac{1}{6}(x+5)$).

k) No ($f^{-1}(0)$ includes 0, π, 2π and other stuff).

l) Yes ($f^{-1}(x) = \sqrt[17]{\frac{x}{12}}$).

7. a) Yes (since $f^{-1}(x) = \frac{x}{3}$, which is a function).
   b) Yes (since $g^{-1}(x) = \frac{1}{7}(x+2)$ is a function).
   c) No (because $\cos 0 = \cos 2\pi = 1$).
   d) No (because $k(0) = k(3) = 7$).
   e) No (many different dates occur on the same day of the week).
   f) Yes (every state has a different capital city).
   g) Yes (no two different people have the same SS number).
   h) No (there are different football players with the same number of touchdowns).
   i) Yes (unless the piano is out of tune).
   j) Yes (each possible pay amount corresponds only to one amount of time worked).

Exercises from Section 2.7

1. Throughout this exercise, let hand $x = \frac{1}{2}x + \frac{3}{2}$. Compute and simplify each quantity:

a) hand 9
b) hand $7 \cdot 3$
c) hand $(7 \cdot 3)$
d) hand $(7) \cdot 3$
e) 3 hand 7
f) 7 hand 3
g) $3+$hand 7
h) hand $3+7$
i) hand $7+3$
j) hand $(3+7)$
k) hand $(7+3)$
l) hand $3+$ hand 7
m) hand $(3+$ hand $7)$
n) hand 3 hand 7
o) hand $(3$ hand $7)$
p) hand 3^3
q) hand$^3 3$
r) 2 hand 5^2

2. As in the previous exercise, let hand $x = \frac{1}{2}x + \frac{3}{2}$. Compute and simplify each quantity:

a) hand $2x$
b) hand x^2
c) hand$^2 x$
d) 2 hand x
e) 2 hand x^2
f) hand $^2 x^2$
g) hand $^3(-5)$
h) hand $^2 2$
i) 2 hand$^2 5$
j) $(2$ hand $5)^2$
k) 2 hand$^2 5^2$
l) hand $^{-1} 3$
m) hand $^{-1}(-1)$
n) $-$hand $^{-1} 4$

3. Throughout this exercise, let foot $x = x^{-1/3} + 1$. Compute and simplify each quantity:

a) foot $x + 1$
b) foot $(x + 1)$
c) foot $x+$ foot 1
d) $1+$ foot x
e) $x+$ foot 1
f) foot $4 \cdot 2$
g) foot $(4 \cdot 2)$
h) foot $(4) \cdot 2$
i) 8 foot 8
j) foot $8 \cdot 8$
k) foot foot 8
l) foot 8 foot 8
m) foot -1
n) $-$ foot 1
o) foot $-1-$ foot 1
p) foot $-1 - 1$
q) foot $(-1 - 1)$
r) $-\text{foot}^2(-1)$
s) $\text{foot}^2 1$
t) $\text{foot}^{-1} 2$

4. Throughout this exercise, let $g(x) = \begin{cases} 1 - 2x & x < -3 \\ x + 5 & -3 \leq x < 2 \\ 3x + 4 & x \geq 2 \end{cases}$. Compute and simplify each quantity:

a) $g(5 - 7)$
b) $g(5) - g(7)$
c) $g(5) - 7$
d) $2g(-2 \cdot 2)$
e) $g(1)g(3 - 4)$
f) $g(-1 \cdot 4)$
g) $-g(4)$
h) $4g(-1)$
i) $g(-1) \cdot 4$
j) $g^2(-2)$
k) $3g(0)g^2(-4)$
l) $3g^2(1)$

5. Throughout this exercise, let $h(x) = \cos 2x$. Compute and simplify each quantity:

a) $h\left(\frac{\pi}{8}\right) h\left(\frac{\pi}{3}\right)$
b) $h\left(\frac{\pi}{8}\right) + h\left(\frac{\pi}{3}\right)$
c) $h\left(\frac{\pi}{8} + \frac{\pi}{4}\right)$
d) $h\left(\frac{\pi}{8}\right) + \frac{\pi}{4}$
e) $h^2\left(\frac{\pi}{12}\right)$
f) $h\left(-\frac{\pi}{6}\right)$
g) $h^2\left(-\frac{\pi}{6}\right)$
h) $-4h^2\left(-\frac{\pi}{6}\right)$
i) $h(2x)$
j) $h^2(x)h(4x)$
k) $h(x + 1)^2$
l) $h^2(x + 1)$
m) $h(2x) - h\left(\frac{x}{2}\right)$

6. a) If you know string $x = -3$, how does the expression string $(x + 2)$ simplify?
b) If you know string $x = -3$, how does the expression string $x + 2$ simplify?
c) If you know $x = -3$, how does the expression string $x + 2$ simplify?
d) If you know $x = -3$, how does the expression string $(x + 2)$ simplify?

e) If you know string $x = -3$, how does the expression string$^2(x+2)$ simplify?

f) If you know string $x = -3$, how does the expression string$^2 x + 2$ simplify?

g) If you know $x = -3$, how does the expression string$^2(x+2)$ simplify?

h) If you know $x = -3$, how does the expression string$^2 x + 2$ simplify?

i) If you know string $x = -3$, how does the expression 2 string $(x+2)$ simplify?

j) If you know string $x = -3$, how does the expression 2 string $x + 2$ simplify?

k) If you know string$^3 x = -3$, how does the expression string$^3 x + 2$ simplify?

l) If you know string$^3 x = -3$, how does the expression string$^3(x+2)$ simplify?

7. a) Substitute cat $x = 4$ into the equation 3 cat$^2 x -$ hat $^2 x = 1$.

   b) Substitute $x = 4$ into the equation 3 cat$^2 x -$ hat $^2 x = 1$.

   c) Substitute cat$^2 x = 4$ into the equation 3 cat$^2 x -$ hat $^2 x = 1$.

   d) Substitute 3cat$^2 x = 4$ into the equation 3 cat$^2 x -$ hat $^2 x = 1$.

   e) In which part(s) of (a)-(d) is it possible to solve for hat x after performing the substitution?

   f) In which part(s) of (a)-(d) is is possible to solve for x after performing the substitution?

8. Recall from trigonometry that for any x, the Pythagorean Identity $\cos^2 x + \sin^2 x = 1$ holds.

   a) Suppose $\cos x = \frac{4}{7}$. Substitute this into the Pythagorean Identity and solve for $\sin x$.

   b) Suppose $\sin x = -\frac{2}{3}$. Substitute this into the Pythagorean Identity and solve for $\cos x$.

   c) Suppose $\sin^2 x = \frac{8}{11}$. Substitute this into the Pythagorean Identity and solve for $\cos x$.

   d) Suppose $\cos^2 x = \frac{2}{5}$. Substitute this into the Pythagorean Identity and solve for $\sin x$.

Answers

1. a) 6 b) 12 c) 12 d) 15 e) 15 f) 21 g) 8 h) 10 i) 8 j) $\frac{13}{2}$ k) $\frac{13}{2}$ l) 8 m) $\frac{11}{2}$ n) 15 o) 9 p) 15 q) 27 r) 28

2. a) $x+\frac{3}{2}$ b) $\frac{1}{2}x^2+\frac{3}{2}$ c) $\left(\frac{1}{2}x+\frac{3}{2}\right)^2$ d) $x+3$ e) x^2+3 f) $(x^2+3)^2$ g) -1 h) $\frac{25}{4}$ i) 32 j) 64 k) 392 l) 3 m) -5 n) -5

3. a) $x^{-1/3}+2$ b) $(x+1)^{-1/3}+1$ c) $x^{-1/3}+3$ d) $x^{-1/3}+2$ e) $x+2$ f) $\frac{3}{2}$ g) $\frac{3}{2}$ h) $2\left(4^{-1/3}+1\right)$ i) 12 j) $\frac{5}{4}$ k) $\left(\frac{2}{3}\right)^{-1/3}+1$ l) $\frac{9}{4}$ m) 0 n) -2 o) 2 p) -1 q) $-2^{-1/3}+1$ r) 0 s) 4 t) 1

4. a) 3 b) -6 c) 12 d) -18 e) 24 f) 9 g) -16 h) 16 i) 16 j) 9 k) 1215 l) 108

5. a) $-\frac{\sqrt{2}}{4}$ b) $\frac{\sqrt{2}}{2}-\frac{1}{2}$ c) $-\frac{\sqrt{2}}{2}$ d) $\frac{\sqrt{2}}{2}+\frac{\pi}{4}$ e) $\frac{3}{4}$ f) $\frac{1}{2}$ g) $\frac{1}{4}$ h) -1 i) $\cos 4x$ j) $\cos^2 2x\cos 8x$ k) $\cos 2(x+1)^2$ l) $\cos^2 2(x+1)$ m) $\cos 4x-\cos x$

6. a) It doesn't. b) -1 c) string $(-3)+2$ d) string -1 e) It doesn't. f) 11 g) string$^2(-1)$ h) string$^2(-3)+2$ i) It doesn't. j) -4 k) -1 l) It doesn't.

7. a) $48-$ hat $^2x=1$ b) 3 cat$^2 4-$ hat $^2 4=1$ c) $12-$ hat $^2x=1$ d) $4-$ hat $^2x=1$ e) (a), (c) and (d). f) None

8. a) $\sin x = \pm\frac{\sqrt{33}}{7}$.

b) $\cos x = \pm\frac{\sqrt{5}}{3}$.

c) $\cos x = \pm\sqrt{\frac{3}{11}}$.

d) $\sin x = \pm\sqrt{\frac{3}{5}}$.

Exercises from Section 2.8

In Exercises 1-16, let f, g, h, k, m and n be these functions:

$$f(x) = 8x \qquad g(x) = \frac{x}{2} \qquad h(x) = x+7 \qquad k(x) = x-3 \qquad m(x) = x^2 \qquad n(x) = \sqrt[3]{x}$$

Compute and simplify the rule for each of these functions:

1. $f \circ f$
2. $g \circ f$
3. $f \circ h$
4. $h \circ f$
5. $m \circ n$
6. $m \circ f$
7. $m \circ k$
8. $f \circ h \circ g$
9. $n \circ f$
10. $f \circ n$
11. $n \circ n$
12. $m \circ h \circ m$
13. $m \circ m \circ m \circ m \circ m$
14. $h \circ h \circ h \circ h \circ h$
15. $g \circ g \circ g \circ g$
16. $k \circ k \circ k \circ k$

In Exercises 17-19, let $F(x) = x^2 - 2x$ and let $G(x) = 3x^2 + x$. Compute the rule for each function. Simplify by expanding these rules out and combine any like terms:

17. $F \circ G$
18. $G \circ F$
19. $F \circ F$

In Exercises 20-27, let $f(x) = x + \pi$, $g(x) = 4x$ and $h(x) = \tan x$. In each part of the problem, draw an arrow diagram indicating what is being asked for, and then compute the quantity:

20. $(f \circ g)(\pi)$
21. $(g \circ f)(\pi)$
22. $(h \circ g)\left(\frac{\pi}{12}\right)$
23. $(f \circ h)\left(\frac{3\pi}{4}\right)$
24. $(h \circ f)\left(\frac{3\pi}{4}\right)$
25. $(f \circ g \circ h)\left(\frac{\pi}{6}\right)$
26. $(f \circ h \circ g)\left(\frac{\pi}{6}\right)$
27. $(g \circ h \circ g)\left(\frac{\pi}{4}\right)$

In Exercises 28-39, let F and F' be the one-to-one functions described completely by this table of values:

x	-3	-2	-1	0	1	2	3
$F(x)$	2	1	-1	3	-2	4	-3
$F'(x)$	1	3	-2	0	2	-1	5

Use this table to compute each quantity:

28. $F' \circ F(-2)$
29. $F \circ F'(-2)$
30. $F \circ F(1)$
31. $F' \circ F'(-1)$
32. $F \circ (F')^{-1}(3)$
33. $F' \circ F^{-1}(4)$
34. $F' \circ (F')^{-1}(4)$
35. $F^{-1} \circ F'(-1)$
36. $F^{-1} \circ F^{-1}(2)$
37. $(F' \circ F' \circ F')(2)$
38. $(F' \circ F \circ F)^{-1}(1)$
39. $(F' \circ F^{-1} \circ F' \circ F)(-2)$

In Exercises 40-63, diagram each function (similar to Examples 5 and 6 in Section 2.8 of the notes):

40. $f(x) = (x-3)^8$
41. $f(x) = \sqrt{x} + 2$
42. $f(x) = 4x - 2$
43. $f(x) = \dfrac{2}{3} \cos \dfrac{5x}{3}$
44. $f(x) = \csc x$
45. $f(x) = \dfrac{1}{8} - \dfrac{x}{2}$
46. $f(x) = (5x)^{-2}$
47. $f(x) = \cot 3x$
48. $f(x) = 4 \sec(x-2)$
49. $f(x) = x^{-7}$
50. $f(x) = \tan x + \pi$
51. $f(x) = \sqrt[3]{x-4}$
52. $f(x) = (5|x| - 2)^3$
53. $f(x) = (5x+7)^{-2}$
54. $f(x) = \sin \cos x$
55. $f(x) = \sin^4 3x$
56. $f(x) = \dfrac{7}{\sqrt{x^3+1}}$
57. $f(x) = x^{2/3} - 7$
58. $f(x) = 3 + \sin \sqrt{x}$
59. $f(x) = \sin 3x^{-2}$
60. $f(x) = \dfrac{1}{2x-3}$
61. $f(x) = 4 \tan^2 x^3$
62. $f(x) = \dfrac{7}{x^9 - 8}$
63. $f(x) = \dfrac{2}{7x}$

In Exercises 64-71, diagram each function:

64. $f(x) = \dfrac{1}{6} \cos^4 x$
65. $f(x) = \left(\dfrac{\cos x}{6}\right)^4$
66. $f(x) = \dfrac{1}{6} \cos x^4$
67. $f(x) = \cos\left(\dfrac{x}{6}\right)^4$

68. $f(x) = \dfrac{\cos^4 x}{6}$

69. $f(x) = \cos^4\left(\dfrac{x}{6}\right)$

70. $f(x) = \cos\dfrac{x^4}{6}$

71. $f(x) = \dfrac{\cos x^4}{6}$

In Exercises 72-100, reverse diagram each of these arrow pictures (similar to Example 7 in Section 2.8 of the notes). If possible, simplify the rule for the function:

72. $x \xrightarrow{+3} \xrightarrow{\wedge 2} \xrightarrow{\times 4} f(x)$

73. $x \xrightarrow{\times 4} \xrightarrow{\wedge 2} \xrightarrow{+3} f(x)$

74. $x \xrightarrow{\times 4} \xrightarrow{+3} \xrightarrow{\wedge 2} f(x)$

75. $x \xrightarrow{\wedge 2} \xrightarrow{+3} \xrightarrow{\times 4} f(x)$

76. $x \xrightarrow{\wedge 2} \xrightarrow{\times 4} \xrightarrow{+3} f(x)$

77. $x \xrightarrow{\wedge 2} \xrightarrow{\sqrt[3]{}} f(x)$

78. $x \xrightarrow{\sin} \xrightarrow{1/\cdot} f(x)$

79. $x \xrightarrow{\wedge 2} \xrightarrow{\wedge 5} f(x)$

80. $x \xrightarrow{\times 6} \xrightarrow{\tan} f(x)$

81. $x \xrightarrow{\tan} \xrightarrow{\times 6} f(x)$

82. $x \xrightarrow{\tan} \xrightarrow{\tan} f(x)$

83. $x \xrightarrow{\times 3} \xrightarrow{\tan} \xrightarrow{1/\cdot} f(x)$

84. $x \xrightarrow{-5} \xrightarrow{\wedge 3} \xrightarrow{1/\cdot} f(x)$

85. $x \xrightarrow{+3} \xrightarrow{|\cdot|} f(x)$

86. $x \xrightarrow{\wedge 2} \xrightarrow{\cos} \xrightarrow{\wedge 2} f(x)$

87. $x \xrightarrow{\cos} \xrightarrow{\wedge 2} \xrightarrow{\wedge 2} f(x)$

88. $x \xrightarrow{\wedge 2} \xrightarrow{\wedge 2} \xrightarrow{\cos} f(x)$

89. $x \xrightarrow{-2} \xrightarrow{\wedge 2} \xrightarrow{\sqrt[5]{\ }} \xrightarrow{1/\cdot} f(x)$

90. $x \xrightarrow{-2} \xrightarrow{\wedge 2} \xrightarrow{1/\cdot} \xrightarrow{\sqrt[5]{\ }} f(x)$

91. $x \xrightarrow{-2} \xrightarrow{1/\cdot} \xrightarrow{\sqrt[5]{\ }} \xrightarrow{\wedge 2} f(x)$

92. $x \xrightarrow{\cos} \xrightarrow{\sin} \xrightarrow{\times 3} f(x)$

93. $x \xrightarrow{\sqrt{\ }} \xrightarrow{\sin} \xrightarrow{\wedge 2} f(x)$

94. $x \xrightarrow{\times -2} \xrightarrow{+3} \xrightarrow{\sqrt{\ }} \xrightarrow{\wedge 5} \xrightarrow{-6} \xrightarrow{\cos} f(x)$

95. $x \xrightarrow{\div 8} \xrightarrow{+3} \xrightarrow{\sin} \xrightarrow{\wedge 5} \xrightarrow{-6} \xrightarrow{\times 4} f(x)$

96. $x \xrightarrow{\cos} \xrightarrow{\times 2} \xrightarrow{+9} \xrightarrow{\sqrt{\ }} \xrightarrow{\sin} \xrightarrow{\wedge 3} f(x)$

97. $x \xrightarrow{\times -1} \xrightarrow{+3} \xrightarrow{\cos} \xrightarrow{\cos} \xrightarrow{+8} f(x)$

98. $x \xrightarrow{-4} \xrightarrow{1/\cdot} \xrightarrow{|\cdot|} \xrightarrow{\div 3} \xrightarrow{\wedge 2} f(x)$

99. $x \xrightarrow{\times -2} \xrightarrow{+3} \xrightarrow{\sqrt{\ }} \xrightarrow{\wedge 5} \xrightarrow{-6} \xrightarrow{\cos} f(x)$

100. $x \xrightarrow{+2} \xrightarrow{\wedge 9} \xrightarrow{-3} \xrightarrow{\sqrt[5]{\ }} \xrightarrow{+3} \xrightarrow{\wedge 4} \xrightarrow{\cos} \xrightarrow{1/\cdot} f(x)$

Answers

1. $f \circ f$
2. $(g \circ f)(x) = 4x$
3. $(f \circ h)(x) = 8x + 56$
4. $(h \circ f)(x) = 8x + 7$
5. $(m \circ n)(x) = x^{2/3}$
6. $(m \circ f)(x) = 64x^2$
7. $(m \circ k)(x) = (x - 3)^2$
8. $(f \circ h \circ g)(x) = 4x + 56$
9. $(n \circ f)(x) = 2\sqrt[3]{x}$
10. $(f \circ n)(x) = 8\sqrt[3]{x}$
11. $(n \circ n)(x) = x^{1/6} = \sqrt[6]{x}$
12. $(m \circ h \circ m)(x) = (x^2 + 7)^2$
13. $(m \circ m \circ m \circ m \circ m)(x) = x^{32}$
14. $(h \circ h \circ h \circ h \circ h)(x) = x + 35$

15. $(g \circ g \circ g \circ g)(x) = \dfrac{x}{16}$
16. $(k \circ k \circ k \circ k)(x) = x - 12$
17. $(F \circ G)(x) = 9x^4 + 6x^3 - 5x^2 - 2x$
18. $(G \circ F)(x) = 3x^4 - 12x^3 + 13x^2 - 2x$
19. $(F \circ F)(x) = x^4 - 4x^3 + 2x^2 + 4x$
20. $\pi \xrightarrow{g} \xrightarrow{f} ?$

    $(f \circ g)(\pi) = 5\pi$
21. $\pi \xrightarrow{f} \xrightarrow{g} ?$

    $(g \circ f)(\pi) = 8\pi$
22. $\dfrac{\pi}{12} \xrightarrow{g} \xrightarrow{h} ?$

    $(h \circ g)\left(\dfrac{\pi}{12}\right) = \sqrt{3}$
23. $\dfrac{3\pi}{4} \xrightarrow{h} \xrightarrow{f} ?$

    $(f \circ h)\left(\dfrac{3\pi}{4}\right) = -1 + \pi$
24. $\dfrac{3\pi}{4} \xrightarrow{f} \xrightarrow{h} ?$

    $(h \circ f)\left(\dfrac{3\pi}{4}\right) = -1$
25. $\dfrac{\pi}{6} \xrightarrow{h} \xrightarrow{g} \xrightarrow{f} ?$

    $(f \circ g \circ h)\left(\dfrac{\pi}{6}\right) = \dfrac{4}{\sqrt{3}} + \pi$
26. $\dfrac{\pi}{6} \xrightarrow{g} \xrightarrow{h} \xrightarrow{f} ?$

    $(f \circ h \circ g)\left(\dfrac{\pi}{6}\right) = -\sqrt{3} + \pi$
27. $\dfrac{\pi}{4} \xrightarrow{g} \xrightarrow{h} \xrightarrow{g} ?$

    $(g \circ h \circ g)\left(\dfrac{\pi}{4}\right) = 0$
28. $F' \circ F(-2) = 2$
29. $F \circ F'(-2) = -3$
30. $F \circ F(1) = 1$
31. $F' \circ F'(-1) = 3$
32. $F \circ (F')^{-1}(3) = 1$
33. $F' \circ F^{-1}(4) = -1$
34. $F' \circ (F')^{-1}(4)$ DNE
35. $F^{-1} \circ F'(-1) = 1$
36. $F^{-1} \circ F^{-1}(2) = 3$
37. $(F' \circ F' \circ F')(2) = 3$
38. $(F' \circ F \circ F)^{-1}(1) = -2$
39. $(F' \circ F^{-1} \circ F' \circ F)(-2) = 1$
40. $x \xrightarrow{-3} \xrightarrow{\wedge 8} f(x)$
41. $x \xrightarrow{\sqrt{\ }} \xrightarrow{+2} f(x)$
42. $x \xrightarrow{\times 4} \xrightarrow{-2} f(x)$
43. $x \xrightarrow{\times 5/3} \xrightarrow{\cos} \xrightarrow{\times 2/3} f(x)$
44. $x \xrightarrow{\sin} \xrightarrow{1/\cdot} f(x)$
45. $x \xrightarrow{\div -2} \xrightarrow{+1/8} f(x)$
46. $x \xrightarrow{\times 5} \xrightarrow{\wedge 2} \xrightarrow{1/\cdot} f(x)$
47. $x \xrightarrow{\times 3} \xrightarrow{\tan} \xrightarrow{1/\cdot} f(x)$
48. $x \xrightarrow{-2} \xrightarrow{\cos} \xrightarrow{1/\cdot} \xrightarrow{\times 4} f(x)$
49. $x \xrightarrow{\wedge 7} \xrightarrow{1/\cdot} f(x)$
50. $x \xrightarrow{\tan} \xrightarrow{+\pi} f(x)$
51. $x \xrightarrow{-4} \xrightarrow{\sqrt[3]{\ }} f(x)$
52. $x \xrightarrow{|\cdot|} \xrightarrow{\times 5} \xrightarrow{-2} \xrightarrow{\wedge 3} f(x)$
53. $x \xrightarrow{\times 5} \xrightarrow{+7} \xrightarrow{\wedge 2} \xrightarrow{1/\cdot} f(x)$
54. $x \xrightarrow{\cos} \xrightarrow{\sin} f(x)$
55. $x \xrightarrow{\times 3} \xrightarrow{\sin} \xrightarrow{\wedge 4} f(x)$

56. $x \xrightarrow{\wedge 3} \xrightarrow{+1} \xrightarrow{\sqrt{}} \xrightarrow{1/\cdot} \xrightarrow{\times 7} f(x)$
57. $x \xrightarrow{\wedge 2} \xrightarrow{\sqrt[3]{}} \xrightarrow{-7} f(x)$
58. $x \xrightarrow{\sqrt{}} \xrightarrow{\sin} \xrightarrow{+3} f(x)$
59. $x \xrightarrow{\wedge 2} \xrightarrow{1/\cdot} \xrightarrow{\times 3} \xrightarrow{\sin} f(x)$
60. $x \xrightarrow{\times 2} \xrightarrow{-3} \xrightarrow{1/\cdot} f(x)$
61. $x \xrightarrow{\wedge 3} \xrightarrow{\tan} \xrightarrow{\wedge 2} \xrightarrow{\times 4} f(x)$
62. $x \xrightarrow{\wedge 9} \xrightarrow{-8} \xrightarrow{1/\cdot} \xrightarrow{\times 7} f(x)$
63. $x \xrightarrow{1/\cdot} \xrightarrow{\times 2/7} f(x)$
64. $x \xrightarrow{\cos} \xrightarrow{\wedge 4} \xrightarrow{\div 6} f(x)$
65. $x \xrightarrow{\cos} \xrightarrow{\div 6} \xrightarrow{\wedge 4} f(x)$
66. $x \xrightarrow{\wedge 4} \xrightarrow{\cos} \xrightarrow{\div 6} f(x)$
67. $x \xrightarrow{\div 6} \xrightarrow{\wedge 4} \xrightarrow{\cos} f(x)$
68. $x \xrightarrow{\cos} \xrightarrow{\wedge 4} \xrightarrow{\div 6} f(x)$
69. $x \xrightarrow{\div 6} \xrightarrow{\cos} \xrightarrow{\wedge 4} f(x)$
70. $x \xrightarrow{\wedge 4} \xrightarrow{\div 6} \xrightarrow{\cos} f(x)$
71. $x \xrightarrow{\wedge 4} \xrightarrow{\cos} \xrightarrow{\div 6} f(x)$
72. $f(x) = 4(x+3)^2$
73. $f(x) = 16x^2 + 3$
74. $f(x) = (4x+3)^2$
75. $f(x) = 4x^2 + 12$
76. $f(x) = 4x^2 + 3$
77. $f(x) = x^{2/3}$
78. $f(x) = \csc x$
79. $f(x) = x^{10}$
80. $f(x) = \tan 6x$
81. $f(x) = 6 \tan x$
82. $f(x) = \tan \tan x$
83. $f(x) = \cot 3x$
84. $f(x) = (x-5)^{-3}$
85. $f(x) = |x+3|$
86. $f(x) = \cos^2 x^2$
87. $f(x) = \cos^4 x$
88. $f(x) = \cos x^4$
89. $f(x) = (x-2)^{-5/2}$
90. $f(x) = (x-2)^{-5/2}$
91. $f(x) = (x-2)^{-5/2}$
92. $f(x) = 3 \sin \cos x$
93. $f(x) = \sin^2 \sqrt{x}$
94. $f(x) = \cos((3x-6)^{5/2} - 6)$
95. $f(x) = 4 \sin^5 \left(\frac{x}{8} + 3\right) - 24$
96. $f(x) = \sin^3 \sqrt{2 \cos x + 9}$
97. $f(x) = \cos \cos(3x-3) + 8$
98. $f(x) = \dfrac{1}{9(x-4)^2}$
99. $f(x) = \cos((3-2x)^{5/2} - 6)$
100. $f(x) = \sec\left(\sqrt[5]{(x+2)^9 - 3} + 3\right)^4$

Exercises from Section 2.9

In Exercises 1-20, suppose f, g, f' and g' are functions described by this table of values:

x	0	1	2	3	4	5	6	7
$f(x)$	2	-3	1	4	-5	0	2	6
$g(x)$	4	2	2	9	-3	-2	1	5
$f'(x)$	2	7	10	-3	-4	1	0	8
$g'(x)$	2	3	-4	1	-1	-6	3	2

Use this table to compute each quantity:

1. $fg(3)$
2. $2g(4)$
3. $f(3) + g(5)$
4. $\dfrac{g'}{g}(3)$
5. $(g - g')(1)$
6. $f^2(4)$
7. $(3g' - f)(5)$
8. $(f + f \circ g)(2)$
9. $(f \circ g^{-1} + f'g')(1)$
10. $g \circ g'(3)(f + f')(4)$
11. $(f'g + g'f)(4)$
12. $\dfrac{f'g - g'f}{g^2}(2)$
13. $fg(6) - f' \circ fg(5)$
14. $(f \circ 3g')(0)$
15. $f^2 \circ f(5 - 2)$
16. $7 + f'g(2)$
17. $7 + f' \circ g(2)$
18. $(f-g)(7)+3f'\circ g(3\cdot 2)$
19. $f(2g'(g^2(2^2) - 6))$
20. $\left(\dfrac{f + g'}{g' \circ f}\right)(5)$

In Exercises 21-32, assume F, G, H and K are the functions

$$F(x) = \frac{1}{2} - \frac{x}{4} \qquad G(x) = 8x + 3 \qquad H(x) = 2x^2 \qquad K(x) = x^2 - 1$$

Compute the rule for each given function, simplifying where reasonable:

21. $F + G$
22. $12F$
23. GH
24. $\dfrac{1}{4}K^2$
25. $\dfrac{1}{6}G - F$
26. $F + F \circ G$
27. $\dfrac{3H}{K + 4F}$
28. $H \circ (H + K)$
29. $K \circ GH$
30. $G^2 + 4K$
31. $\dfrac{G}{H} \circ F$
32. $(K + F) \circ (H - G)$

In Exercises 33-44, diagram the given function:

33. $f(x) = x^4 - 5x^2$

34. $f(x) = \cos x \sin x$

35. $f(x) = \cos \sin x$

36. $f(x) = 3x + \dfrac{x-2}{x+8}$

37. $f(x) = \dfrac{x}{8x^2+2}$

38. $f(x) = x^3 \sin 4x$

39. $f(x) = \cos^2 x + 2\sin 3x$

40. $f(x) = \sqrt[3]{2x^4 + 9|x-2|}$

41. $f(x) = \dfrac{3}{x} - 4\tan x^5$

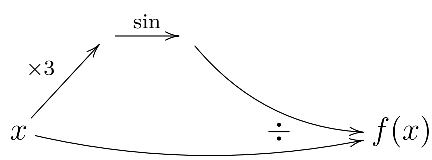
42. $f(x) = 2x^5 + \tan(x^{-7}\sin x)$

43. $f(x) = (\cos x + x^2)(4\sqrt{x} + 9\sin x)$

44. $g(x) = \dfrac{\sqrt{14x - \cos\frac{x}{3}}}{\cos(x^2 + 3\cos x)}$

In Exercises 45-55, reverse-diagram the given arrow picture, and simplify the rule for the function if appropriate:

45.



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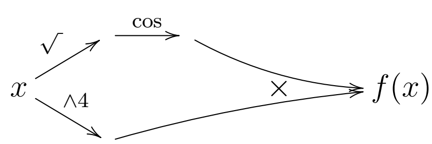

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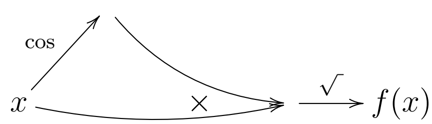

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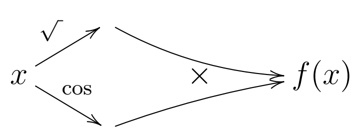

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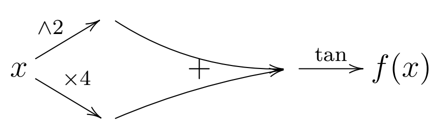

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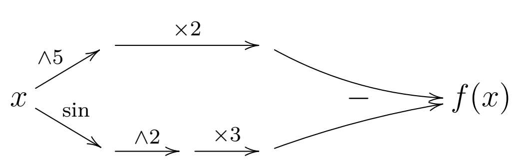

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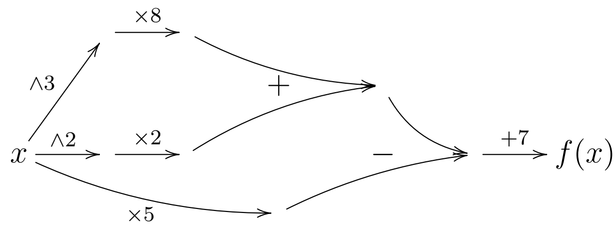
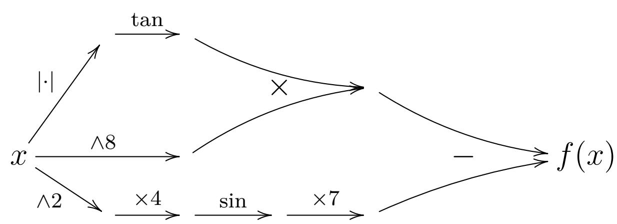
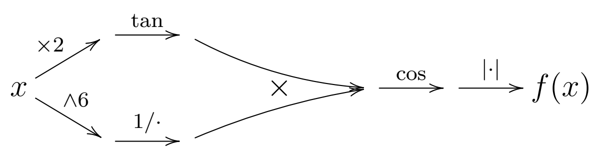

×8
∧3
+
x
∧2
×2
−
+7
f(x)
×5

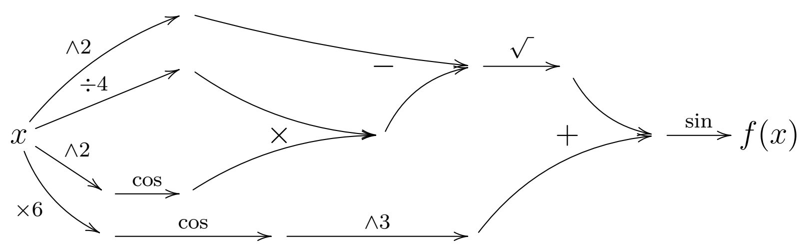
52.


tan
|·|
×
x
∧8
−
f(x)
∧2
×4
sin
×7

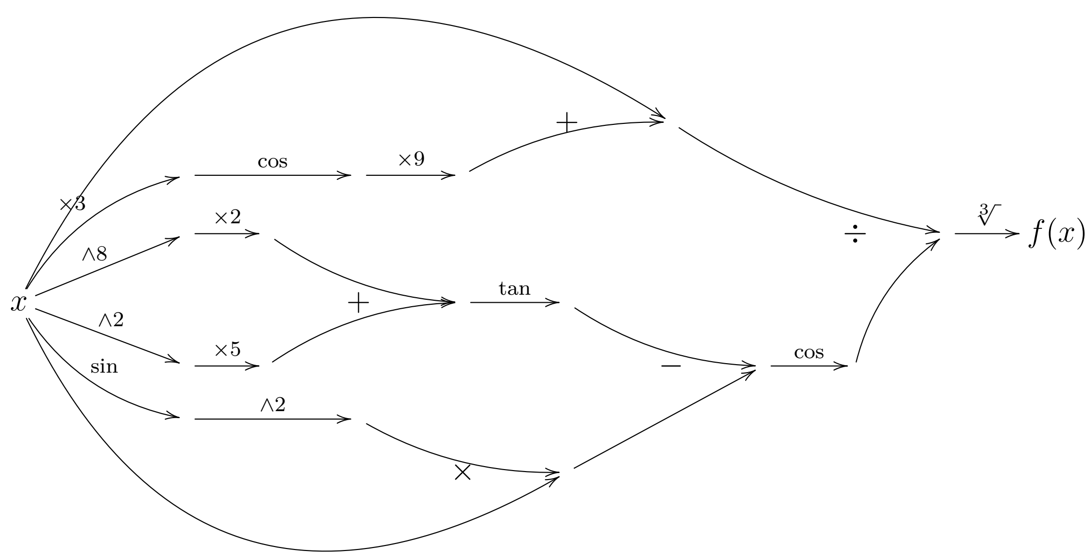
53.


tan
×2
x
∧6
×
cos
|·|
f(x)
1/·

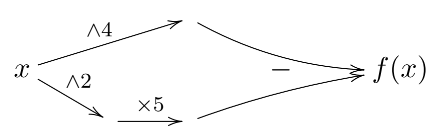
54.

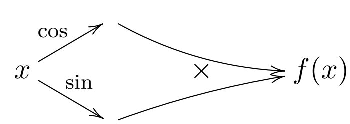

∧2
√
÷4
−
x
×
+
sin
f(x)
∧2
cos
×6
cos
∧3

55.


+
cos
×9
×3
×2
÷
∛
f(x)
∧8
x
+
tan
∧2
×5
sin
−
cos
∧2
×

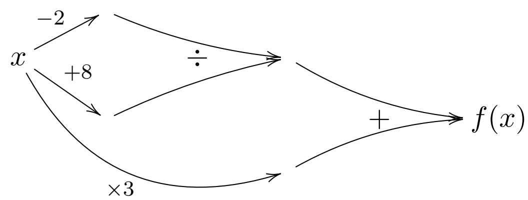
Answers

1. $fg(3) = 36$
2. $2g(4) = -6$
3. $f(3) + g(5) = 2$
4. $\dfrac{g'}{g}(3) = \dfrac{1}{9}$
5. $(g - g')(1) = -1$
6. $f^2(4) = 25$
7. $(3g' - f)(5) = -18$
8. $(f + f \circ g)(2) = 2$
9. $(f \circ g^{-1} + f'g')(1) = 23$
10. $g \circ g'(3)(f + f')(4) = -18$
11. $(f'g + g'f)(4) = 17$
12. $\dfrac{f'g - g'f}{g^2}(2) = 6$
13. $fg(6) - f' \circ fg(5) = 0$
14. $(f \circ 3g')(0) = 2$
15. $f^2 \circ f(5 - 2) = 25$
16. $7 + f'g(2) = 27$
17. $7 + f' \circ g(2) = 17$
18. $(f - g)(7) + 3f' \circ g(3 \cdot 2) = 22$
19. $f(2g'(g^2(2^2) - 6)) = 1$
20. $\left(\dfrac{f + g'}{g' \circ f}\right)(5) = -3$
21. $(F + G)(x) = \dfrac{31}{4}x + \dfrac{7}{2}$
22. $12F(x) = 6 - 3x$
23. $GH(x) = 16x^3 + 6x^2$
24. $\dfrac{1}{4}K^2(x) = \dfrac{1}{4}(x^2 - 1)^2$
25. $\left(\dfrac{1}{6}G - F\right)(x) = \dfrac{19}{12}x$
26. $F + F \circ G = \dfrac{1}{4} - \dfrac{9}{4}x$
27. $\dfrac{3H}{K + 4F} = \dfrac{6x^2}{x^2 - x + 1}$
28. $H \circ (H + K) = 2(3x^2 - 1)^2$
29. $(K \circ GH)(x) = (16x^3 + 6x^2)^2 - 1$
30. $(G^2 + 4K)(x) = 68x^2 + 48x + 5$
31. $\left(\dfrac{G}{H} \circ F\right)(x) = \dfrac{56 - 16x}{(x - 2)^2}$
32. $((K + F) \circ (H - G))(x) =$ $\dfrac{131x^2 - 524x - 180}{(x - 2)^2}$
33. 

34. 

35. $x \xrightarrow{\sin} \xrightarrow{\cos} f(x)$

36.



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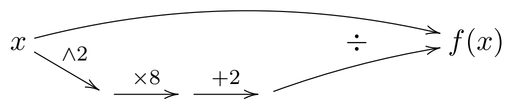

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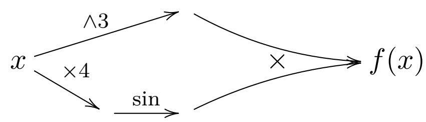

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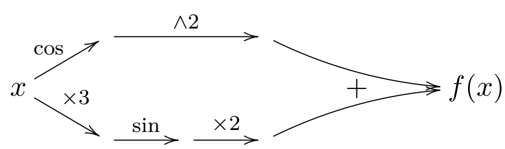

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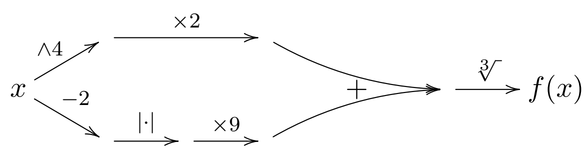

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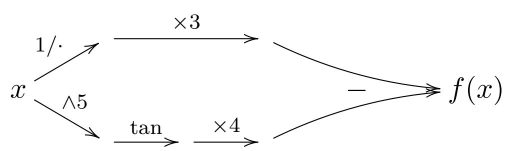

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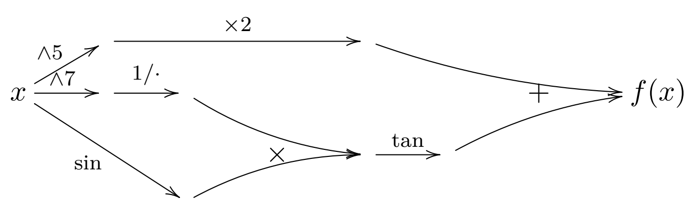

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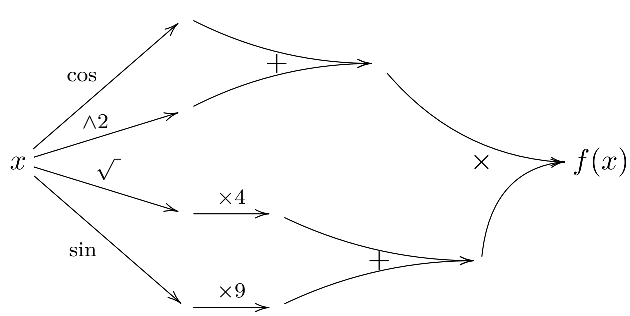

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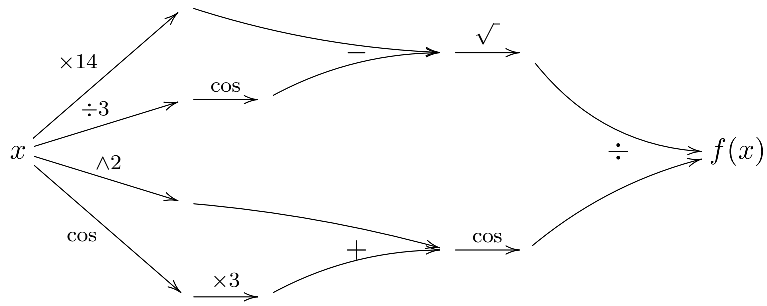


45. $f(x) = \dfrac{\sin 3x}{x}$

46. $f(x) = x^4 \cos \sqrt{x}$

47. $f(x) = \sqrt{x \cos x}$

48. $f(x) = \sqrt{x} \cos x$

49. $f(x) = \tan(x^2 + 4x)$

50. $f(x) = 2x^5 - 3\sin^2 x$

51. $f(x) = 8x^3 + 2x^2 - 5x + 7$

52. $f(x) = x^8 \tan |x| - 7 \sin 4x^2$

53. $f(x) = \left|\cos\left(x^{-6} \tan 2x\right)\right|$

54. $f(x) = \sin\left(\sqrt{x^2 - \dfrac{x}{4}\cos x^2} + \cos^3 6x\right)$

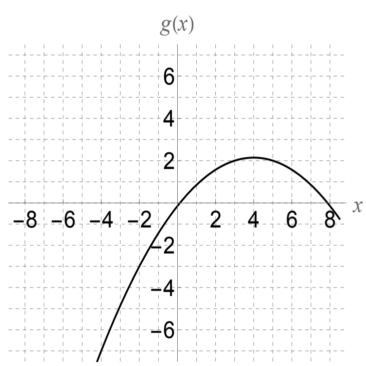
55. $f(x) = \sqrt[3]{\dfrac{x + 9\cos 3x}{\cos\left(\tan(2x^8 + 5x^2) - x\sin^2 x\right)}}$

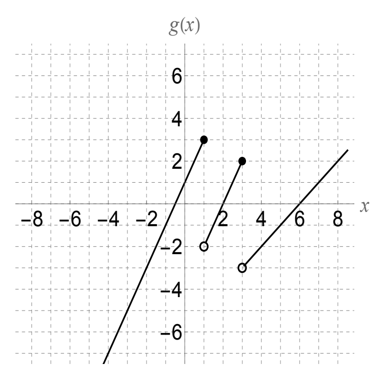
Exercises from Section 2.10

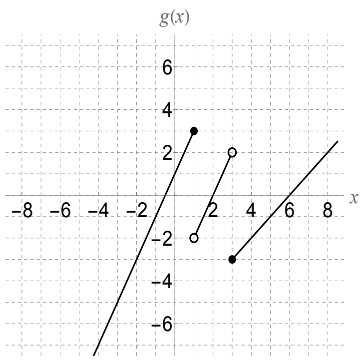
1. Suppose a function F has this (partial) table of values:

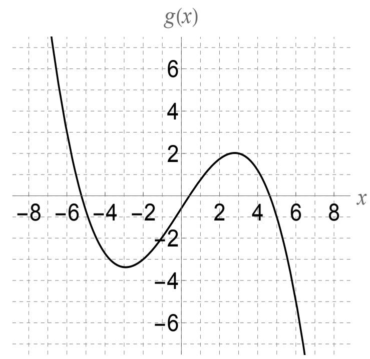
x	-2	3	5
$F(x)$	-3	2	-1

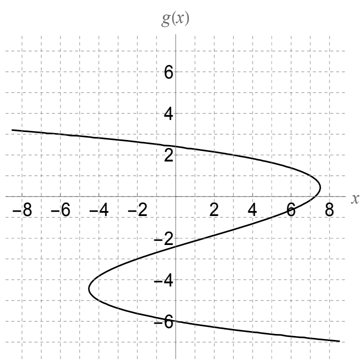
Based on this table, which one or ones of the graphs shown below could be the graph of F?

a) 

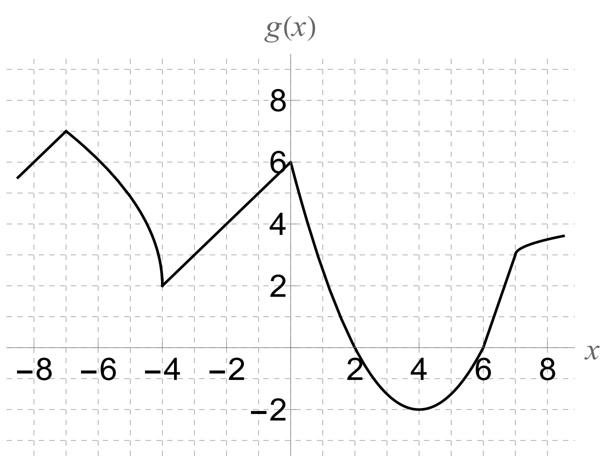
b) 

c) 

e) 

d) 

The graph of some unknown function g is shown here:



Use this graph to answer the questions in Exercises 2-16:

2. Estimate $g(-2)$.

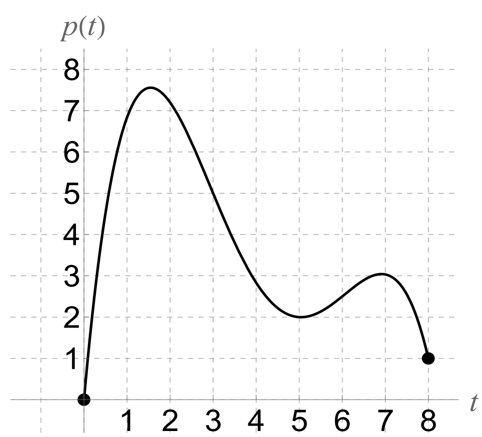
3. Estimate $g(3)$.

4. Estimate $g(7)$.

5. Estimate $g^{-1}(2)$.

6. Estimate $g^{-1}(9)$.
7. Estimate $g^{-1}(7)$.
8. As x increases from 1 to 4, how does g change?
9. As x increases from -3 to 0, how does g change?
10. How many x-intercepts does g have?
11. What are the x-intercepts of g?
12. At what value of x is $g(x)$ maximized?
13. What is the maximum value of $g(x)$?
14. At what value of x is $g(x)$ minimized?
15. What is the minimum value of $g(x)$?
16. Is g one-to-one?

Suppose that a worker's productivity t hours after he begins his shift is given by $p(t)$, where p is the function whose graph is shown here:



Use this graph to answer the questions posed in Exercises 17-26:

17. Estimate the productivity of the worker 3 hours after he begins his shift.
18. What is the mathematical language for the expression being asked for in Exercise 17?
19. Estimate the productivity of the worker 7 hours after he begins his shift.
20. At what time(s) is the worker's productivity equal to 5?

21. What is the mathematical language for the expression being asked for in Exercise 20?

22. How does the worker's productivity change between times 4 and 8?

23. What is the worker's maximum productivity during his shift?

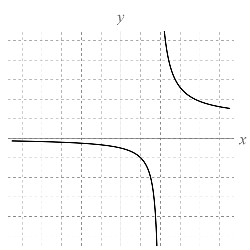
24. At what time is the worker's productivity greatest?

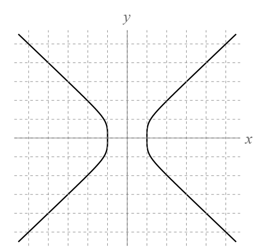
25. What is the maximum productivity of the worker between times 5 and 7?

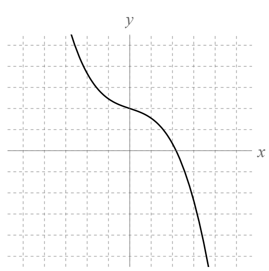
26. What is the minimum productivity of the worker between times 3 and 6?

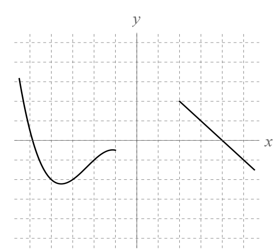
27. In each part of this problem, you are given the graph of a multifunction. For each graph:

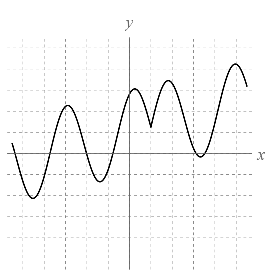
    - determine whether or not this is the graph of a function $\mathbb{R} \to \mathbb{R}$, and
    - if it is the graph of a function, determine whether or not the function is one-to-one.

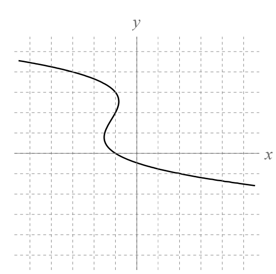
a) 

b) 

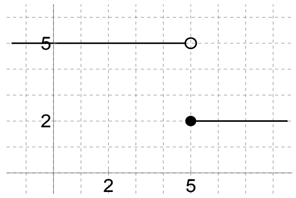
(c) 

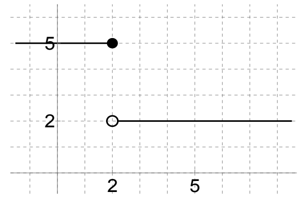
(e) 

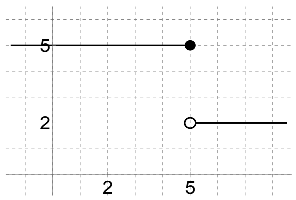
(d) 

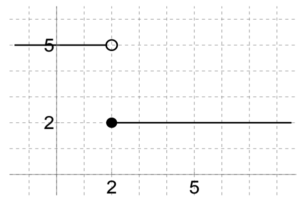
(f) 

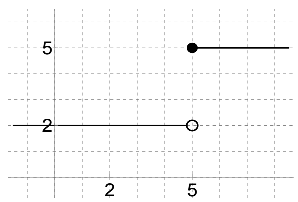
28. Here are the graphs of eight functions, labelled A through H:

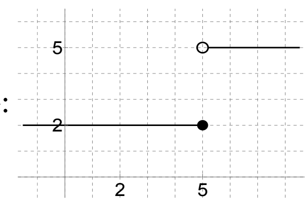
A:


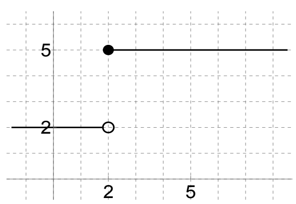
E:


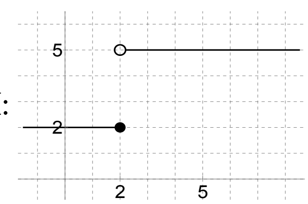
B:


F:


C:


G:


D:


H:


For each function whose rule is given below, pick out which picture above (A through H) is the graph of that function:

a) $f(x) = \begin{cases} 5 & x < 2 \\ 2 & x \geq 2 \end{cases}$

b) $f(x) = \begin{cases} 2 & x < 2 \\ 5 & x \geq 2 \end{cases}$

c) $f(x) = \begin{cases} 5 & x < 5 \\ 2 & x \geq 5 \end{cases}$

d) $f(x) = \begin{cases} 2 & x < 5 \\ 5 & x \geq 5 \end{cases}$

e) $f(x) = \begin{cases} 5 & x \leq 2 \\ 2 & x > 2 \end{cases}$

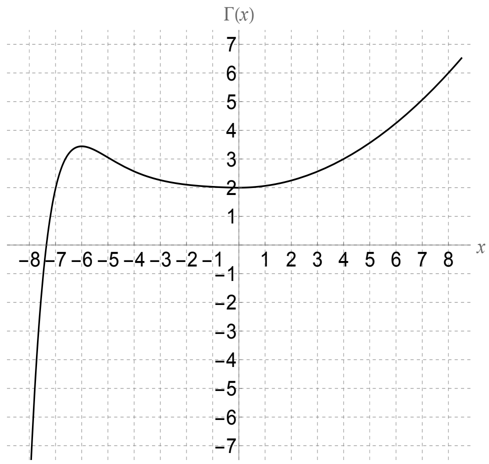
f) $f(x) = \begin{cases} 2 & x \leq 2 \\ 5 & x > 2 \end{cases}$

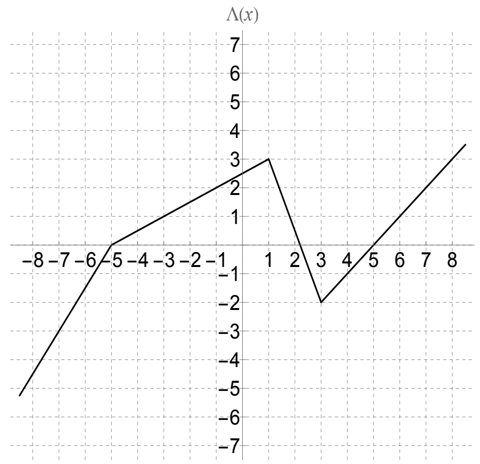
g) $f(x) = \begin{cases} 5 & x \leq 5 \\ 2 & x > 5 \end{cases}$

h) $f(x) = \begin{cases} 2 & x \leq 5 \\ 5 & x > 5 \end{cases}$

29. Sketch a graph of $g(x) = \begin{cases} -3 & x < -1 \\ 4 & x = -1 \\ 2 & x > -1 \end{cases}$.

30. The graphs of unknown functions Γ (Gamma) and Λ (Lambda) are shown below:





Use these graphs to sketch the graph of

$$f(x) = \begin{cases} \Gamma(x) & x < -1 \\ \Lambda(x) & x \geq -1 \end{cases}.$$

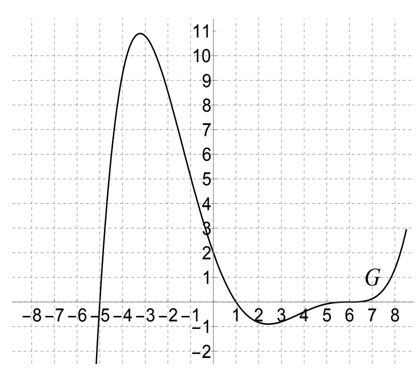
31. Use the graphs of Γ and Λ from Exercise 30 to sketch the graph of

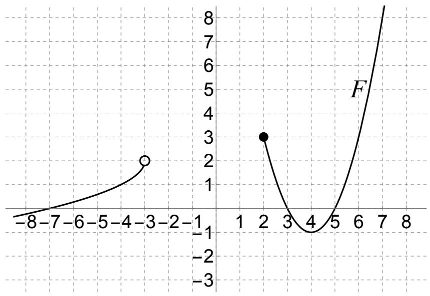
$$g(x) = \begin{cases} -2 & x < -3 \\ 5 & x = -3 \\ \Lambda(x) & x > -3 \end{cases}$$

32. Use the graphs of Γ and Λ from Exercise 30 to sketch the graph of

$$h(x) = \begin{cases} \Lambda(x) & x < 1 \\ -2 & 1 \leq x < 3 \\ \Gamma(x) & x \geq 3 \end{cases}$$

33. For each function, estimate its x- and y-intercepts from its graph:

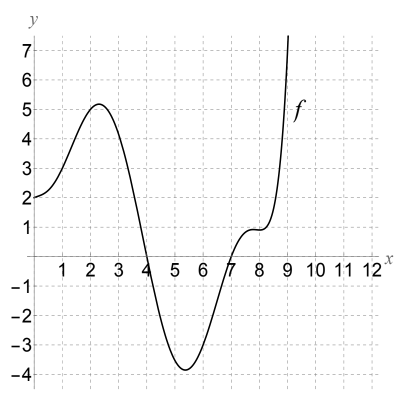
a) 

b) 

In Exercises 34-40, compute the x- and y-intercepts of the given function:

34. $f(x) = 3 - \frac{2}{3}x$
35. $g(x) = x^2 - 7x - 18$
36. $h(x) = 2x^3 - 9x^2 - 18x$
37. $k(x) = x^2 + 4$
38. $r(x) = 9x - 7$
39. $s(x) = 4x^2 + 8x - 140$
40. $t(x) = 16x^2 - 9$

In Exercises 41-46, assume that f is an even function, where the graph of f (for positive x) is shown below:

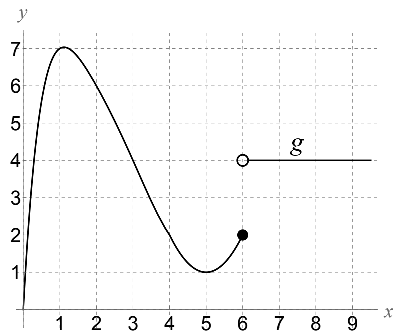


Use this graph to compute/estimate each indicated quantity.

Hint: Use the fact that f is even to tell you what the rest of the graph of f looks like.

41. $f(6)$
42. $f(-6)$
43. $f(-2)$
44. $5f(3) - 2f(-9)$
45. $f(13) - f(-13)$
46. $f^{-1}(0)$

In Exercises 47-59, assume that g is an odd function, where the graph of g (for positive x) is shown below:



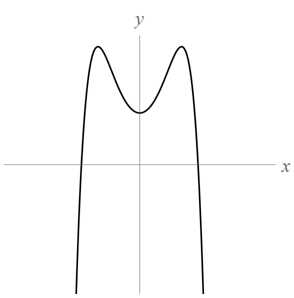
In Exercises 47-56, compute/estimate each indicated quantity.

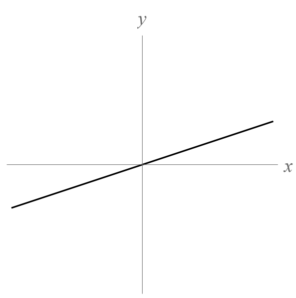
Hint: Use the fact that g is odd to tell you what the rest of the graph of g looks like.

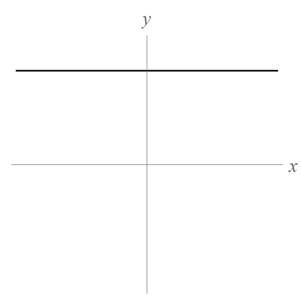
47. $g(2)$
48. $g(-2)$
49. $g(-4)$
50. $g(-6)$
51. $g^{-1}(7)$
52. $g^{-1}(-7)$
53. $g^{-1}(-3)$
54. $g(13) + g(-13)$
55. $g^2(-3)g(-5)$
56. $2g(3) + 3g(-8)$

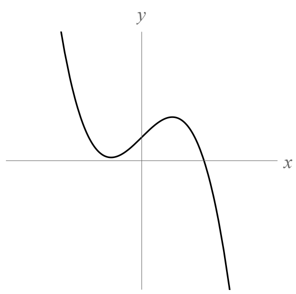
57. What is the maximum value of g?
58. What is the minimum value of g?
59. What input produces the minimum value of g?

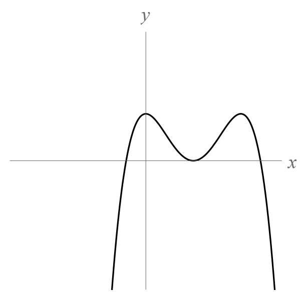
In Exercises 60-68, describe the symmetry in the given graph (your choices are "even", "odd", "both (even and odd)" or "neither (even nor odd)").

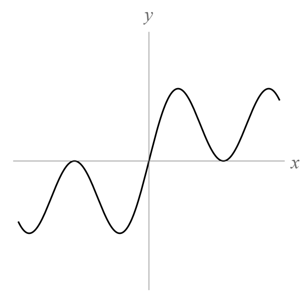
60. 

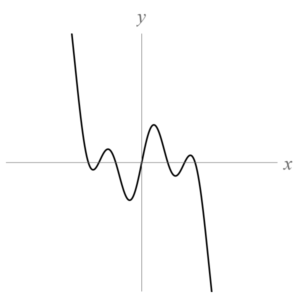
61. 

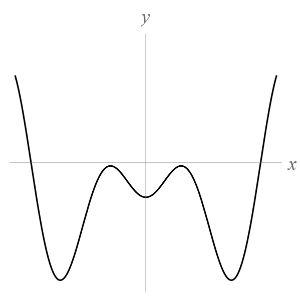
62. 

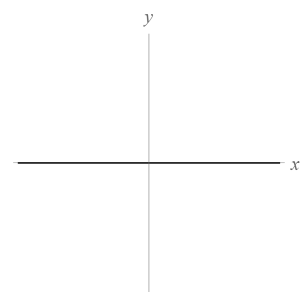
63. 

65. 

67. 

64. 

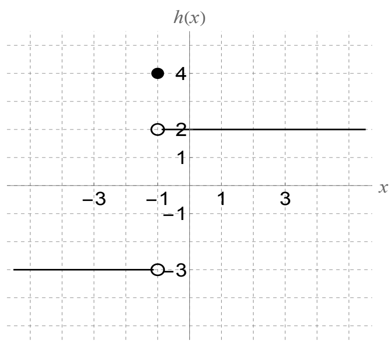
66. 

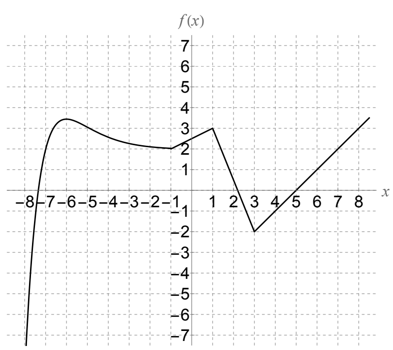
68. 

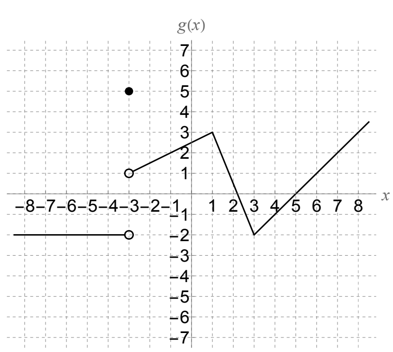
Answers

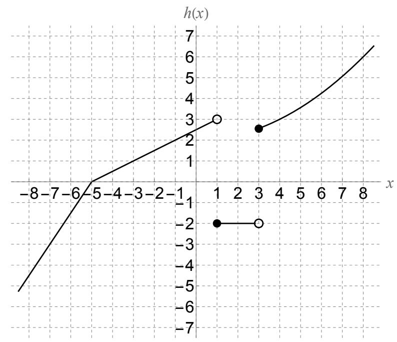
1. (b) and (e) could be the graph of F.
2. 4
3. -1.6
4. 3
5. $\{-4, 1.2, 6.7\}$
6. DNE
7. -7
8. g decreases from 2.5 to -1.75.
9. g increases from 3 to 6.
10. 2
11. $(2, 0)$ and $(6, 0)$.
12. -7
13. 7
14. 4
15. -2
16. No
17. 5
18. $p(3)$
19. 3
20. $t = .5$ and $t = 3$
21. $p^{-1}(5)$
22. His productivity decreases from 3 to 1.
23. 7.5
24. 1.5 hours after he begins his shift.
25. 3
26. 2
27. a) one-to-one function
    b) not a function $\mathbb{R} \to \mathbb{R}$
    c) one-to-one function
    d) function, but not one-to-one
    e) function, but not one-to-one
    f) not a function $\mathbb{R} \to \mathbb{R}$

28. a) F
    b) D
    c) A
    d) C
    e) E
    f) H
    g) B
    h) G

29. 

30. 

31. 

32. 

33. a) x-ints: $(-5,0)$, $(1,0)$, $(6,0)$; y-int: $(0,2)$.
    b) x-ints: $(-7,0)$, $(3,0)$, $(5,0)$; there is no y-intercept.

34. x-int: $\left(\frac{9}{2},0\right)$
    y-int: $(0,3)$

35. x-ints: $(9,0)$, $(-2,0)$
    y-int: $(0,-18)$

36. x-ints: $(0,0)$, $\left(-\frac{3}{2},0\right)$, $(6,0)$
    y-int: $(0,0)$

37. no x-intercepts;
    y-int: $(0,4)$.

38. x-int: $\left(\frac{7}{9},0\right)$
    y-int: $(0,-7)$

39. x-ints: $(-7,0)$, $(5,0)$;
    y-int: $(0,-140)$.

40. x-ints: $\left(-\frac{4}{3},0\right)$, $\left(\frac{4}{3},0\right)$;
    y-int: $(0,-9)$.

41. -3

42. -3

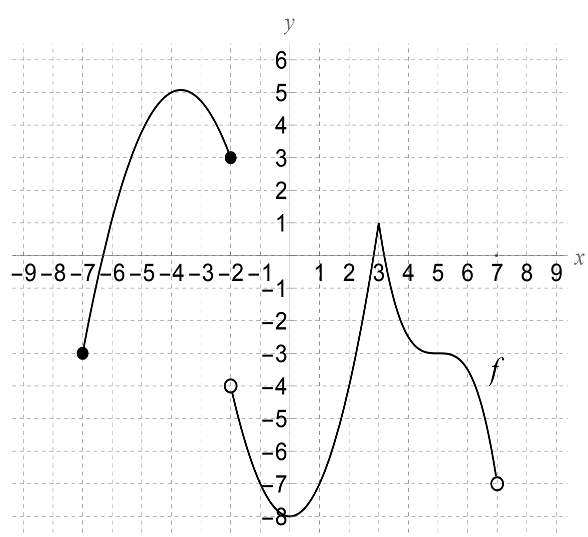
43. 5

44. 8

45. 0
46. $\{-7, -4, 4, 7\}$
47. 6
48. -6
49. -2
50. -2
51. 1
52. -1
53. $\{-.2, -3.2\}$
54. 0
55. -16
56. -4
57. 7
58. -7
59. -1
60. even
61. odd
62. even
63. neither
64. odd
65. neither
66. even
67. odd
68. both

Exercises from Section 2.11

In Exercises 1-23, suppose that f is the function whose graph is given here:

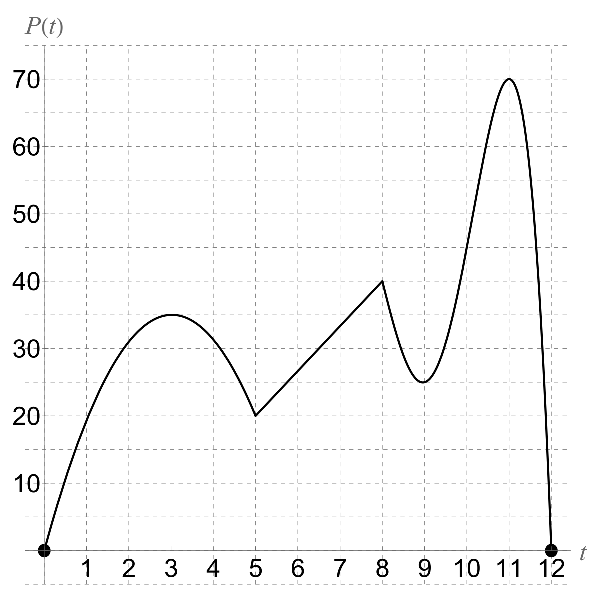


Use this graph to answer these questions/estimate these quantities:

1. What is the domain of f?
2. What is the range of f?

3. What is the maximum value of f?
4. Which is greater, $f(3)$ or $f(5)$?
5. How many x-intercepts does f have?
6. What is the y-intercept of f?
7. Near $x = 1.5$, is $f(x)$ increasing or decreasing as x increases?
8. Near $x = 6$, is $f(x)$ increasing or decreasing as x increases?
9. $f(3)$
10. $f(7)$
11. $-f(-5)$
12. $f(-2)$
13. $8f(5)$
14. $f(2) + 4$
15. $f(2 - 2)$
16. $f(2) - f(2)$
17. $2f(1) + 6f(-4)$
18. $2f^2(2)$
19. $f(-6) - 2f(-1)$
20. $f(-3 \cdot 2 - 1)$
21. $f^{-1}(1)$
22. $f^{-1}(6)$
23. $f^{-1}(-7)$

In Exercises 24-35, suppose that the power (in watts) output of an engine at time t (in seconds) is given by a function P whose graph is given here:

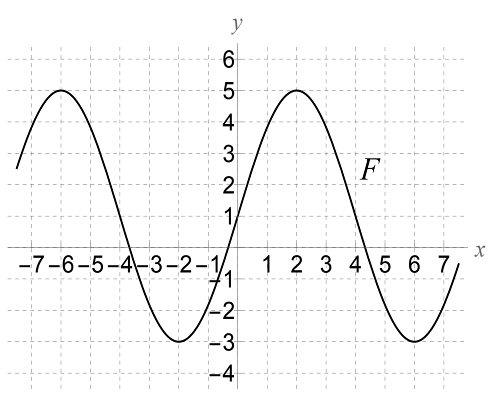


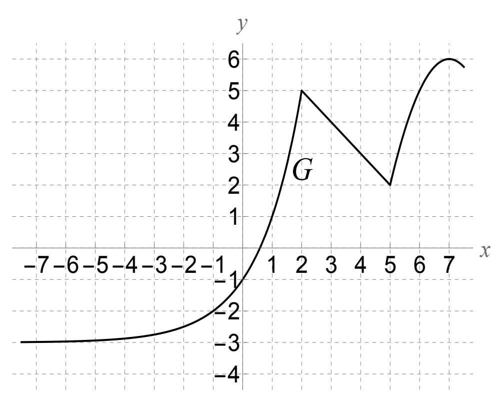
Use this graph to estimate the answers to these questions:

24. What is the power output of the engine at time 5 sec?
25. What is the power output of the engine at time 7 sec?

26. Which is greater, the power output of the engine at time 2 or the power output at time 9?
27. How much greater is the power output at time 3 than the power output at time 7?
28. From time 3 sec to time 5 sec, how does the power output of the engine change?
29. At what time(s) is the power output of the engine equal to 40 W?
30. At how many times is the power output of the engine equal to 30 W?
31. At how many times is the power output of the engine equal to 90 W?
32. What is the maximum power output of the engine?
33. At what time is the power output of the engine largest?
34. What is the maximum power output of the engine during its first 6 seconds of operation?
35. At what time between times 3 and 8 is the power output of the engine minimized?

In Exercises 36-55, assume that F and G are the functions whose graphs are given below:



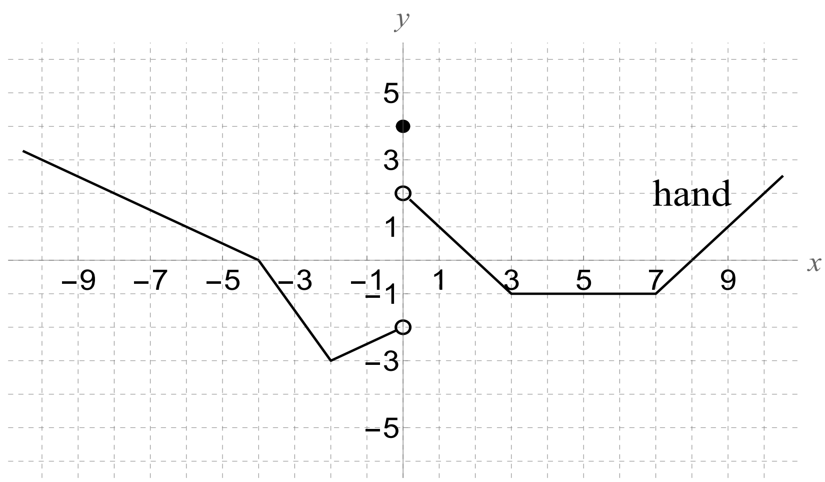


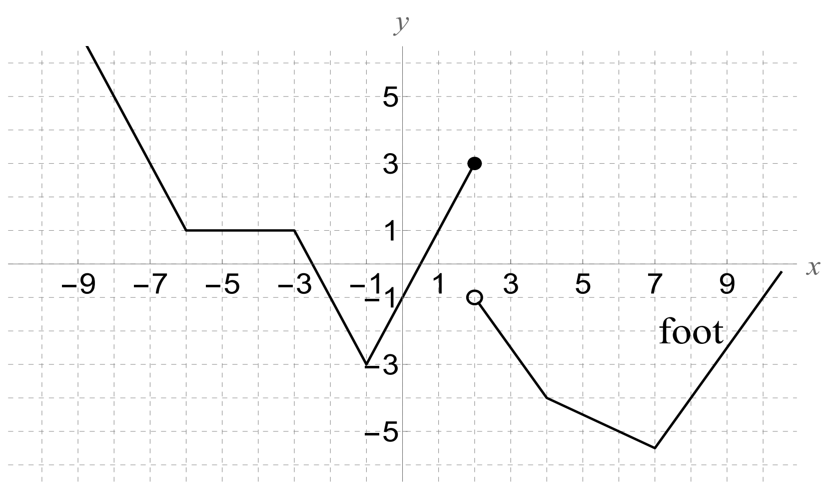
Use these graphs to compute (or at least estimate) each quantity:

36. $F(1)$
37. $G(3)$
38. $G^{-1}(6)$
39. $F^{-1}(6)$
40. $F(5)$
41. $(F+G)(5)$
42. $(FG)(-1)$
43. $F \circ G(3)$
44. $G \circ F(3)$

45. $F \circ F(-2)$
46. $10F(3)$
47. $G^3(2)$
48. $4F^2(-2)$
49. $F(-3) + F(6)$
50. $\left(\dfrac{G}{F}\right)(-1)$
51. $G(2) - 3$
52. $F \circ G \circ G(3)$
53. $(F + F \circ G)(0)$
54. $G(2 - 3)$
55. $4FG(-6)$

In Exercises 56-73, assume that hand and foot are the functions whose graphs are given below:



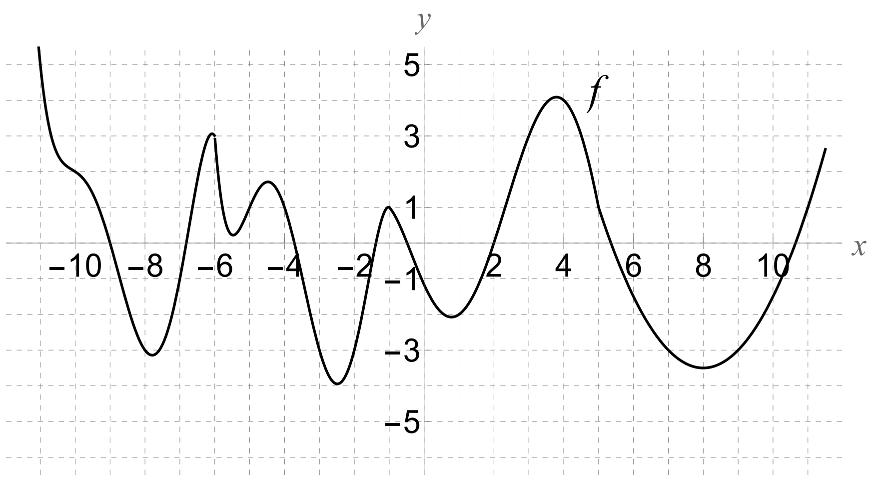


Use these graphs to compute (or at least estimate) each quantity:

56. foot -1
57. hand $3 \cdot 2$
58. 4 foot -3
59. foot $0 + 6$
60. hand $(2 + 7)$
61. 3 foot$^2(4)$

62. $\text{foot}^{-1}(5)$
63. foot $-4+9$
64. $7-$ $\text{hand}^{-1}(-3)$
65. foot hand 2
66. foot 2 hand $(7-5)$
67. hand 0 foot(hand $4+3$)
68. foot $4+3$ hand 2
69. foot $(2$ hand $2)$
70. $\text{foot}^2(-2)$ $\text{hand}^3 3$
71. (foot $+4$ hand)(-5)
72. $\dfrac{\text{hand } 0+3}{8-\text{foot } 2}$
73. $\dfrac{\text{foot}}{\text{hand}}(7)$

In Exercises 74-93, assume that f is the function whose graph is given here:



Also, assume g and g' are functions with this table of values:

x	-4	-3	-2	-1	0	1	2	3	4	5	6
$g(x)$	2	1	9	0	4	-1	5	-3	2	1	4
$g'(x)$	16	9	8	3	-2	-3	4	1	0	-1	7

Finally, assume

$$h(x) = 2\sqrt{x}+1 \qquad \text{and} \qquad h'(x) = \begin{cases} x^2-5 & x<3 \\ x+2 & x \geq 3 \end{cases}.$$

Use all this information to compute (or at least estimate) each quantity:

74. $g(2) + f(-7)$
75. $g(2) - 7$
76. $h'g(-5)$
77. $h' \circ g(-3)$
78. $g(2 + f(-7))$
79. $f \circ g(3)$
80. $g^2 \circ f(5)$
81. $(g + 2f)(5)$
82. $hf(4 \cdot 3 - 8)$
83. $(3h' - g)(2)$
84. $\left(\frac{f}{g'}\right)(-3)$
85. $h \circ f(-4)$
86. $f(h(9) - h'(5))$
87. $(g'h + gh')(4)$
88. $(g')^{-1}(5 - 1)$
89. $hh'(1)$
90. $(2f^2 - 3g + h)(4)$
91. $g(-2) + g' \circ f(7)$
92. $h'(4 + f \circ f^2(-8))$
93. $\frac{hh' - gg'}{gh}(4)$

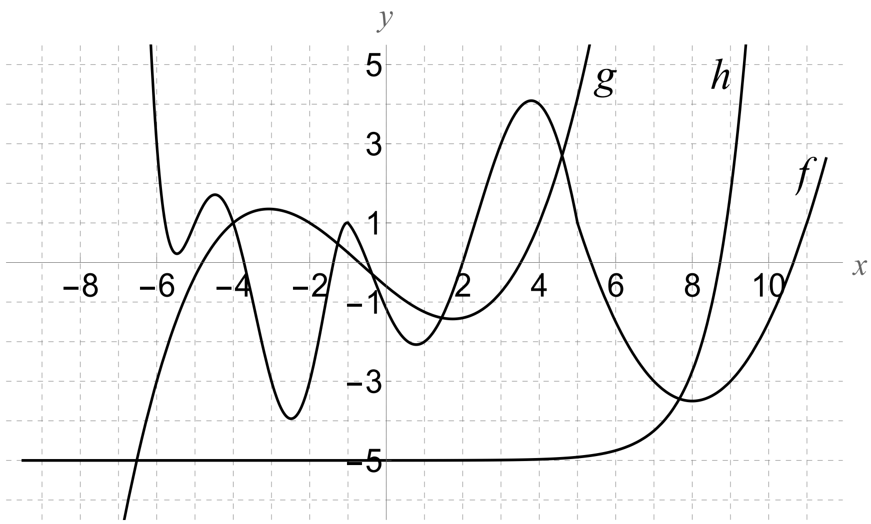
Answers

1. $[-7, 7)$
2. $[-8, 5]$
3. 5
4. $f(3)$
5. 3
6. $(0, -8)$
7. increasing
8. decreasing
9. 1
10. DNE
11. -4
12. 3
13. -24
14. 0
15. -8
16. 0
17. 16
18. 32
19. 15
20. -3
21. $\{-6, 3\}$
22. DNE
23. $\{-1, 1\}$
24. 20 W
25. 33 W
26. The power output of the engine is greater at time 2.
27. 2 W
28. The power output of the engine decreases from 35 W to 20 W.
29. At times 8 sec, 9.8 sec, and 11.7 sec.
30. 6
31. 0
32. 70 W
33. At time $t = 11$ sec.
34. 35 W
35. At time $t = 5$ sec.
36. 4
37. 4
38. 7
39. DNE
40. -2

41. 0
42. 2
43. 1
44. 3.25
45. -2
46. 37.5
47. 125
48. 36
49. -5
50. 1
51. 2
52. 3.75
53. $-.75$
54. -2
55. -60
56. -3
57. -1
58. 4
59. 5
60. 1
61. 48
62. -8
63. 10
64. 9
65. -1
66. 0
67. 12
68. -4
69. -1
70. -1
71. 3
72. $\frac{7}{5}$
73. 5.5
74. 4
75. -2
76. 4
77. -4
78. -1
79. -3
80. 1
81. 3
82. 20
83. -8
84. $-\frac{1}{3}$
85. 3
86. -1
87. 12
88. 2
89. -12
90. 31
91. 18
92. -4
93. 3

Exercises from Section 2.12

In Exercises 1-10, assume that f, g and h are the functions whose graphs are shown here:



Use these graphs to answer the following questions:

1. Assuming that the graphs of f, g and h do not touch when $x > 10$ or $x < -10$:

   a) How many solutions does the equation $f(x) = g(x)$ have?

   b) Of these solutions to the equation $f(x) = g(x)$, how many of them are negative?

   c) Of these solutions to the equation $f(x) = g(x)$, how many of them are bigger than 3?

   d) Which is larger, the solution of $f(x) = h(x)$ or the solution of $g(x) = h(x)$?

   e) What is the solution set of $f(x) \geq g(x)$?

   f) What is the solution set of $h(x) \geq f(x)$?

   g) What is the solution set of $g(x) < h(x)$?

2. Estimate the smallest positive solution of $f(x) = g(x)$.

3. If you did not assume that the graphs of f and g do not touch when $x > 10$ or $x < -10$, what would you know about the number of solutions of the equation $f(x) = g(x)$?

4. Estimate a solution of $g(x) = h(x)$.

5. Estimate a solution of $f(x) = h(x)$.

6. Estimate a solution of $h(x) = 1$.

7. Estimate all the solutions of $g(x) = -1$ that can be read from this graph.

8. Assuming that the minimum value of f is shown on this graph, how many solutions does $f(x) = -6$ have?

9. Assuming that h is a function that increases for all x, what is the solution set of $h(x) \geq -2$?

10. Estimate all the solutions of $f(x) = 3$ that can be read from this graph.

In each of Exercises 11-14, compute the point(s) (if any) where the graphs of f and g intersect:

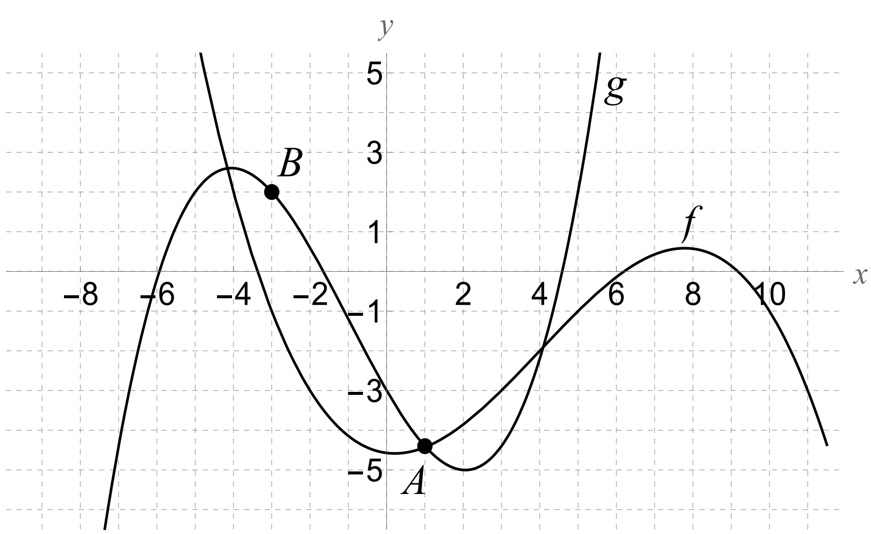
11. $f(x) = 4x + 9$ and $g(x) = -3x - 19$

12. $f(x) = \frac{2}{3}x - \frac{3}{4}$ and $g(x) = \frac{1}{2}x + \frac{7}{3}$

13. $f(x) = -5 - x$ and $g(x) = x^2 - x - 14$

14. $f(x) = 2x^2 + 3x - 7$ and $g(x) = 2x^2 - 8x + 1$

In Exercises 15-22, the graphs of $f(x) = \frac{1}{42}(-x^3 + 12x^2 - 5x - 192)$, and and $g(x) = \frac{1}{15}(x^3 + 3x^2 - 25x - 45)$ are shown here:



You may assume that f increases (from left to right) for all $x < -6$, that f decreases for all $x > 8$, that g decreases for all $x < -4$, and that g increases for all $x > 4$. Use these graphs to estimate answers to the following questions:

15. What is the solution set of $\frac{1}{42}(-x^3+12x^2-5x-192) > \frac{1}{15}(x^3+3x^2-25x-45)$?
16. How many solutions are there to $\frac{1}{15}(x^3+3x^2-25x-45) = 1$?
17. How many solutions to $\frac{1}{15}(x^3+3x^2-25x-45) = 1$ are negative?
18. What is the solution set of $\frac{1}{15}(x^3+3x^2-25x-45) \leq -1$?
19. What is the largest solution of $\frac{1}{42}(-x^3+12x^2-5x-192) = -1$?
20. What is the largest negative solution of $\frac{1}{42}(-x^3+12x^2-5x-192) = -1$?
21. Describe a method of obtaining the coordinates of the point marked A.
22. Describe a method of obtaining the x-coordinate of the point marked B.

Answers

1. a) 5
   b) 3
   c) 1
   d) The solution of $f(x) = h(x)$ is larger.
   e) $(-\infty, -4]$, $[-.12, -.5]$, $[1.5, 4.5]$
   f) $[7.7, \infty)$
   g) $(-\infty, -7.5)$
2. $x \approx 1.5$
3. The equation would have at least 5 solutions.
4. $x \approx -7.7$
5. $x \approx 7.7$
6. $x \approx 8.8$
7. $x \approx -5.2, .5, 2.8$
8. None
9. $[8.2,]infty)$
10. $x \approx -6, 3, 4.2$
11. $(-4, -7)$
12. $\left(\frac{37}{2}, \frac{139}{12}\right)$
13. $(3, -8)$ and $(-3, -2)$
14. $\left(\frac{8}{11}, -\frac{455}{121}\right)$
15. $(-\infty, -4.2), (1, 4)$
16. 3
17. 2
18. $(-\infty, -6.2], [-1, 4.2]$
19. $x = 10$
20. $x = -3$

21. Solve the equation $\frac{1}{42}(-x^3+12x^2-5x-192) = \frac{1}{15}(x^3+3x^2-25x-45)$ (i.e. $f(x) = g(x)$) for x, and take the smallest positive solution. This gives you the x-coordinate; then plug that x-coordinate into either f or g to obtain the y-coordinate.

22. Solve the equation $\frac{1}{15}(x^3+3x^2-25x-45) = 2$ (i.e. $g(x) = 2$) for x.

Exercises from Section 2.13

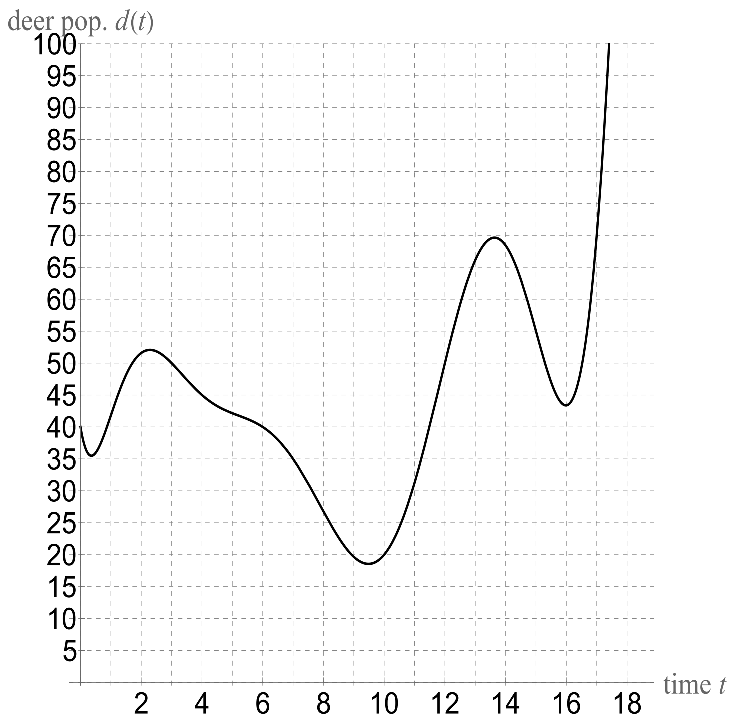
In Exercises 1-10, suppose that R and S are functions given by the following table of values:

x	0	1	2	3	4	5	6	7	8
$R(x)$	1	4	8	12	15	13	7	4	10
$S(x)$	7	11	5	5	2	6	10	13	15

1. Compute the net change of R from $x = 1$ to $x = 5$.
2. Compute the net change of R from $x = 2$ to $x = 7$.
3. Compute the net change of S from $x = 3$ to $x = 8$.
4. Compute the net change of $R + S$ from $x = 1$ to $x = 5$.
5. Compute the net change of $R \circ S$ from $x = 0$ to $x = 2$.
6. Compute the average rate of change of R from $x = 3$ to $x = 7$.
7. Compute the average rate of change of S from $x = 2$ to $x = 3$.
8. Compute the average rate of change of S from $x = 1$ to $x = 7$.
9. Compute the average rate of change of $2R - S$ from $x = 4$ to $x = 8$.
10. Compute the average rate of change of RS from $x = 2$ to $x = 6$.
11. Let $Q(t) = 2t^2 - 3t$.
    a) Compute the net change of Q from $t = -3$ to $t = 2$.
    b) Compute the average rate of change of Q from $t = 1$ to $t = 5$.
    c) The computation you did in part (b) represents the slope of the line connecting what two points?
12. Let $P(x) = 4x^{1/2} + x$.
    a) Compute the net change of P from $x = 4$ to $x = 9$.

b) Compute the average rate of change of P from $x = 12$ to $x = 16$.

In Exercises 13-19, assume that the population of deer in a forest at time t (in years after 2000) is given by the function d whose graph is given here:



Use this graphs to answer the following questions:

13. Estimate the net change in the deer population from 2006 to 2011.

14. Estimate the average rate of change in the deer population from 2010 to 2017.

15. Estimate the instantaneous rate of change of the deer population in 2004.

16. Estimate the instantaneous rate of change of the deer population in 2010.

17. Estimate the instantaneous rate of change of the deer population in 2016.

18. At what year between 2004 and 2015 is the deer population greatest? Explain your answer.

19. At what year between 2004 and 2015 is the deer population increasing most rapidly? Explain your answer.

Answers

1. 9
2. -4
3. 10
4. 4
5. 9
6. -2
7. 0
8. $\dfrac{1}{3}$
9. $-\dfrac{23}{4}$
10. $\dfrac{15}{2}$
11. a) -25
    b) 9
    c) $(1, -1)$ and $(5, 35)$.
12. Let $P(x) = 4x^{1/2} + x$.
    a) 9
    b) $5 - \sqrt{12}$
13. -10 deer
14. $\dfrac{50}{7}$ deer/year
15. -10 deer/year
16. 5 deer/year
17. 0 deer/year
18. Around year 2013.5, because the graph of d is highest at that point.
19. Around year 2012, because the graph of d is steepest at that point.

Chapter 3

A library of algebraic functions

BIG PICTURE

In the next two chapters, we are going to study several classes of functions. Collectively, these functions:

- model lots of real-world phenomena by themselves;
- are combined (with $+$, $-$, $\times$, $\div$ and $\circ$) to make pretty much every function that models anything;
- are needed to solve equations coming from typical applications; and/or
- are the easiest examples necessary to illustrate calculus concepts.

Each class of function we study has a handful of important properties you need to internalize. These properties are ultimately laid out in the tables at the end of Chapter 4.

The functions we study will break into two types: the first type, consisting of those coming from the operations $+$, $-$, $\times$, $\div$, powers and roots, are functions called **algebraic**. Algebraic functions are the subject matter of Chapter 3.

The second type of functions, which include trig functions, exponentials and logarithms, are called **transcendental** and will be discussed in Chapter 4.

3.1 Linear functions

Definition 3.1 *A* **linear function** *is a function* $f : \mathbb{R} \to \mathbb{R}$ *with a constant rate of change.*

In this definition, "rate of change" means both *average* rate of change and *instantaneous* rate of change: for a linear function, both of these are constant.

Remark: If you ever take MATH 322 (Linear Algebra), you will see a different definition of the word "linear" that doesn't match this one.

Linear functions are by far the most important class of functions to be familiar with. This is because they have three very important properties:

1. linear functions are **ubiquitous**, meaning that they occur in lots of different fields (biology, economics, physics, engineering, health, etc.);
2. linear functions are (relatively) **easy** to describe and analyze mathematically;
3. linear functions can be used to **approximate** harder functions (more on this in calculus).

§3.1 Example 1

a) Suppose the balance on your credit card statement is given in the table below. Is the balance a linear function of the number of months that have passed?

time (months)	0	1	2	3	4	5	6
balance ($)	800	750	700	650	500	200	150

b) The number of miles you have gone on a trip at various times is given in the table below. Is the distance travelled a linear function of the amount of time that has passed?

time (hrs)	0	1	2	3	4	5	6
distance (mi)	0	45	90	135	180	225	270

c) Suppose that the grade you will make on an exam depends on the number of hours you study, as given in the chart below. Is your grade a linear function of the amount of time you study?

time (hrs)	0	1	2	3	4	5	6
grade (pts)	25	50	75	85	90	92	93

d) In DNA molecules, the GC content (this is the % of guanine and cytosine in the molecule) and its melting point (the temperature at which the DNA molecule will denature) is given in the table below. Is the melting point a linear function of the GC content?

GC content (%)	0	10	20	30	40	50	60
melting point (°C)	80	85	90	95	100	105	110

Slope

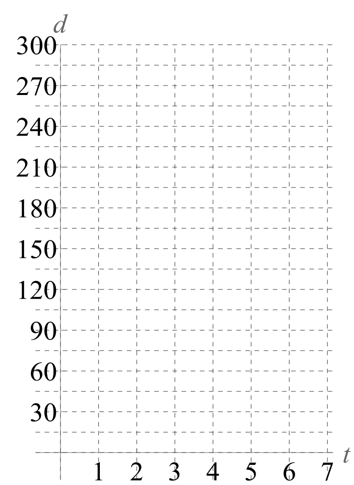
§3.1 Example 2

Below, you are given tables of values for linear functions. For each table of values,

i. graph the function,
ii. compute the rate of change of the function (since the function is linear, this rate will be constant), and
iii. identify what the rate of change of the function has to do with the graph.

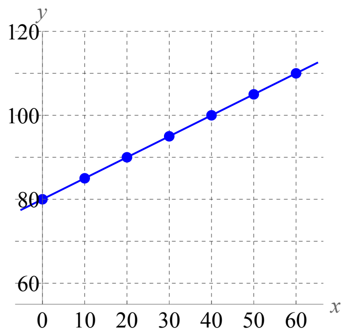
a) (Example 1 (b) earlier)

time t (hrs)	0	1	2	3	4	5	6
distance $d(t)$ (mi)	0	45	90	135	180	225	270



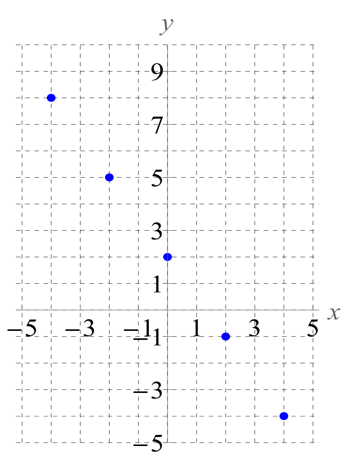
b) (Example 1 (d) earlier)

GC content x (%)	0	10	20	30	40	50	60
melting point $f(x)$ (°C)	80	85	90	95	100	105	110



c)

x	-4	-2	0	2	4
$f(x)$	8	5	2	-1	-4



Definition 3.2 *The* **slope** *of a linear function is its constant rate of change. This is a number, usually denoted by m, that represents how much the graph of the function goes up/down for each unit x moves to the right.*

§3.1 EXAMPLE 3

a) Suppose a linear function f has slope 4. If $f(5) = 2$, what is $f(6)$?

b) Suppose a linear function f has slope -2. If $f(1) = 14$, what is $f(7)$?

c) Suppose a linear function f has slope 3. If $f(3) = 11$, what is $f(-2)$?

§3.1 EXAMPLE 4

Complete the following table:

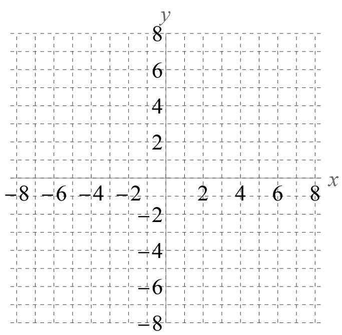
slope of linear function	amount the input changes	amount the output changes
7	4	
	12	2
$\frac{3}{5}$	-8	
$-\frac{2}{3}$		$-\frac{3}{4}$

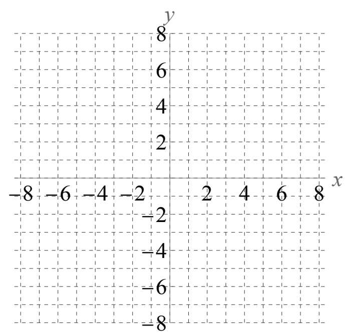
§3.1 Example 5

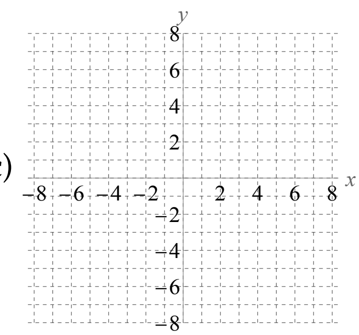
a) Graph the linear function g which passes through $(-4, 3)$ and has slope 2.

b) Graph the linear function h that passes through $(2, 0)$ with slope $\dfrac{1}{2}$.

c) Graph the linear function h that passes through $(-1, 5)$ with slope 0.

a) 

b) 

c) 

These graphs explain why we call linear functions *linear*:

Theorem 3.3 *The graph of a linear function is a line.*

P.S. All lines are straight; all straight things are lines.
If a graph isn't straight, it would be called a *curve* or a *path*, not a line.

Since the slope of a line is its rate of change, we can adapt the rate of change formula we learned in Chapter 2 to produce a formula for the slope:

Theorem 3.4 (Slope formula) *If f is a linear function so that (x_0, y_0) and (x_1, y_1) are on the graph of f, then the slope of f is*

$$m = \frac{rise}{run} = \frac{\triangle y}{\triangle x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\triangle\, output}{\triangle\, input}.$$

§3.1 Example 6

Compute the slope of the line that passes through each pair of points:

a) $(3, -2)$ and $(-5, -6)$

b) $(5, -3)$ and $(-2, 4)$

c) $(8,7)$ and $(-12,1)$

Solution: $m = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{1-7}{-12-8} = \dfrac{-6}{-20} = \boxed{\dfrac{3}{10}}$.

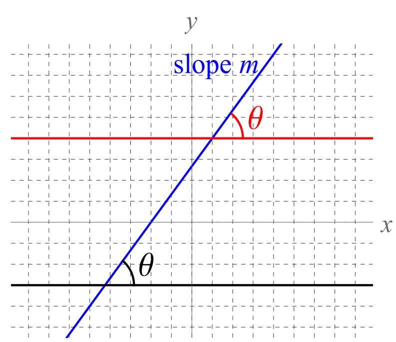
d) $(4,-6)$ and $(4,2)$

Solution: $m = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{2-(-6)}{4-4} = \dfrac{8}{0}$

Slope and angle

Slope is very strongly connected with the trig function *tangent*:

Theorem 3.5 *If f is a linear function with slope m and if θ is the angle the graph of f makes with the horizontal, then $m = \tan\theta$.*



§3.1 Example 7

a) Find the slope of a line which makes an angle of $\frac{\pi}{6}$ with the horizontal.

Solution: The slope is $m = \tan\frac{\pi}{6} = \boxed{\dfrac{1}{\sqrt{3}}}$.

b) Find the slope of any vertical line.

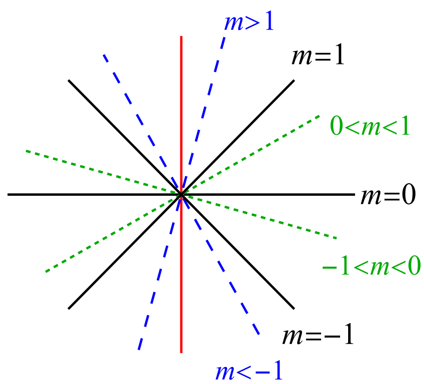
Solution: Vertical lines have an angle of _____ = _____ with the horizontal,

so their slope is $m = \tan \boxed{}$ which

Note: this computation reinforces Example 6 (d) above.

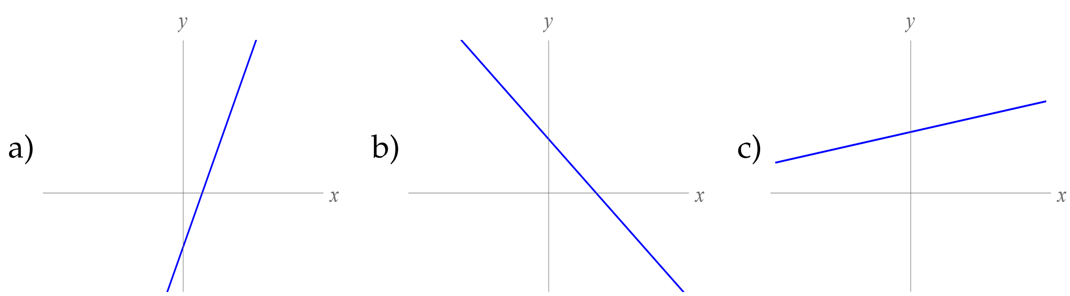
Estimating slope

Assuming the scales on the x- and y-axes are the same, we can estimate the slope of a line by looking at it:



§3.1 EXAMPLE 8

Estimate the slope of each linear function by looking at its graph (assume the scales on the x- and y-axes are the same):



3.2 More on linear functions

The rule of a linear function

§3.1 EXAMPLE 3(B) (EARLIER)

Suppose a linear function f has slope 3. If $f(1) = 4$, what is $f(7)$?

Solution: $f(7) = f(1) + m(7-1) = 4 + 3(7-1) = 4 + 3 \cdot 6 = \boxed{22}$.

We want to generalize this example, to give a *general* rule for all linear functions. Suppose a linear function f passes through the point (x_0, y_0) and has slope m. Then

We have derived this important fact:

Theorem 3.6 (Point-slope formula) *The linear function passing through the point* (x_0, y_0) *with slope* m *has rule*

$$f(x) = y_0 + m(x - x_0)$$

a.k.a.

$$y = y_0 + m(x - x_0).$$

§3.2 EXAMPLE 1

Write an equation (a.k.a. rule) of each line with the given properties:

a) the line has slope $\frac{-3}{4}$ and passes through $\left(\frac{1}{2}, \frac{-7}{2}\right)$

b) the line passes through $(-5, 3)$ and $(4, -15)$

c) the line passes through $(6, 2)$ and makes an angle of $\frac{2\pi}{3}$ with the horizontal

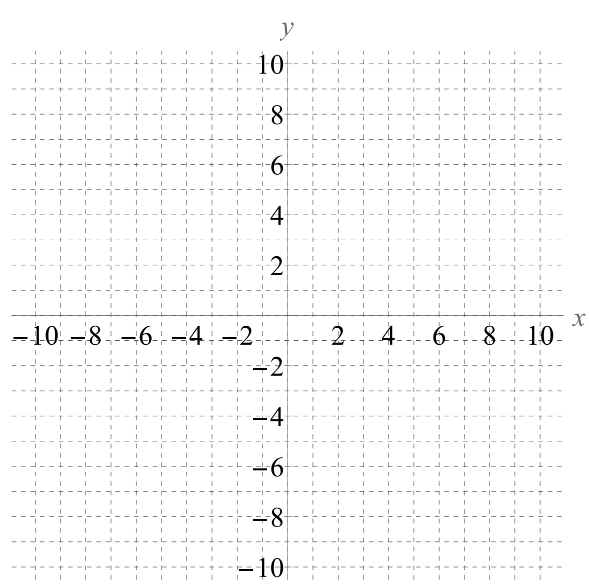
§3.2 EXAMPLE 2

Are the points $(-5, -8)$, $(1, -4)$ and $(10, 1)$ on the same line? Why or why not?

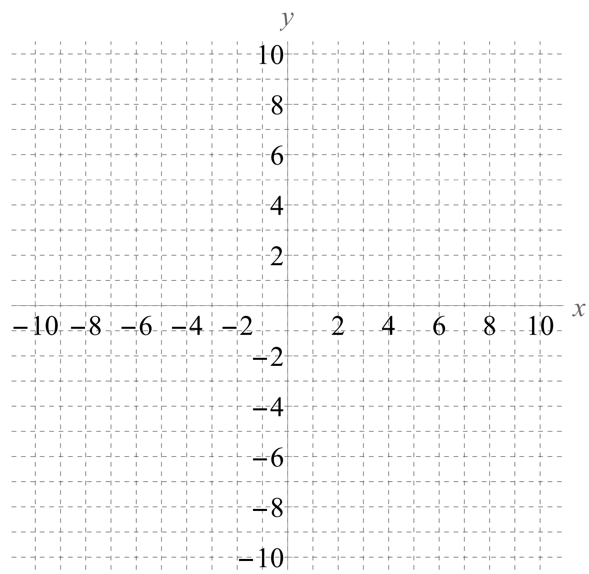
§3.2 EXAMPLE 3

Graph each line:

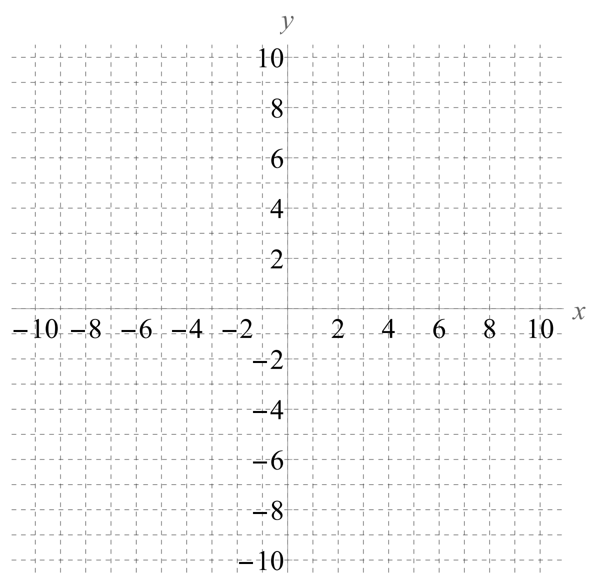
a) $y = 3 - 2(x - 4)$



b) $y = 2 + \frac{3}{4}(x+5)$



c) $y = 4(x-3)$



Other algebraic representations of lines

As we have seen, the line with slope m passing through (x_0, y_0) has equation

There are other ways to rewrite this equation with algebra. For example:

$$y = y_0 + m(x - x_0)$$

Theorem 3.7 (Slope-intercept formula) *The linear function with slope m and y-intercept $(0, b)$ has rule*

$$f(x) = mx + b \qquad \text{a.k.a.} \qquad y = mx + b.$$

WARNING: The slope-intercept formula is highly overrated, because if you started with a point-slope equation, converting it to $y = mx + b$ *obscures the point (x_0, y_0) you used to write the equation* (which you usually don't want to do).

Here is another way to rewrite the equation of a line:

$$\begin{aligned} y &= y_0 + m(x - x_0) \\ y &= y_0 + mx - mx_0 \\ y - mx &= y_0 - mx_0 \\ -mx + y &= y_0 - mx_0 \\ -mBx + By &= B(y_0 - mx_0) \end{aligned}$$

This gives us the equation on the next page:

Definition 3.8 (Standard equation) *The* **standard equation** *of a line is*

$$Ax + By = C$$

where A, B and C are constants.

Advantages of using the standard equation of a line:

- it allows you to write the equation of vertical lines, which are lines but not linear functions (next page);
- it is useful for working with systems of linear equations (coming soon).

A major disadvantage of the standard equation is that y isn't solved for, so it doesn't immediately give you a rule for the linear function you're thinking of. The slope also isn't readily apparent from looking at the standard equation.

§3.2 EXAMPLE 4

Consider the line whose standard equation is $2x - 5y = 17$.

a) Find a rule for this linear function.

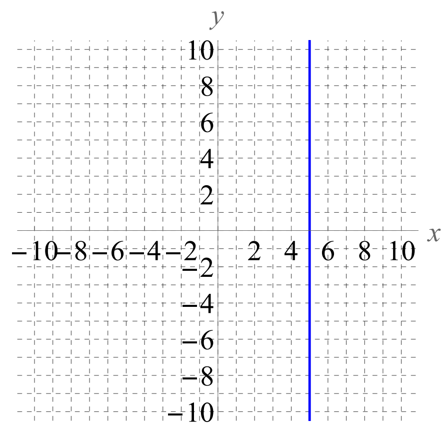
b) Determine the slope of this line.

Solution: From the rule $f(x) = \frac{2}{5}x - \frac{17}{5}$ we found in part (a), we know that the coefficient on the x-term is the slope . Therefore $m = \boxed{\frac{2}{5}}$.

Vertical lines

§3.2 EXAMPLE 5

Here is an example of a vertical line:



All the points on this line have the same ________________________, so an equation that describes this line is ____________ .

Is this line a function $y = f(x)$?

We have seen earlier that vertical lines do not have a defined slope.

You should avoid using the phase "no slope". The reason is that this can be interpreted two ways:

To summarize:

Theorem 3.9 (Vertical lines) *Vertical lines have undefined slope. They have equation $x = c$, where c is a constant. Vertical lines are lines, but not linear functions $y = f(x)$.*

Real-world interpretation of slope; units

Suppose we have some real-world quantities that are related by an equation whose graph is a line. This means that in a linear setting,

the amount the output changes is proportional to the input changes

and the slope of the line is the ____________________________________ .

Since the slope is obtained by dividing the rise (obtained by subtracting outputs) by the run (obtained by subtracting inputs), the units of slope are always

units of slope =

§3.2 EXAMPLE 6

a) Hooke's Law says that the force y needed to keep a spring stretched x units beyond its natural length is directly proportional to x. (Obviously, no force is required to stretch the spring 0 units beyond its natural length.)

If it takes 9 N of force to stretch a particular spring 35 cm beyond its natural length, find the slope of the line relating y and x (with correct units), and interpret the meaning of the slope.

b) At a grocery store, a 90 ounce jug of orange juice costs $\$6.50$, but a 60 ounce jug of orange juice costs $\$5.00$. Assume a linear relationship between the amount of orange juice purchased and the cost of said amount.

i. Find the slope of this linear relationship and interpret the slope.

ii. How much does 190 ounces of orange juice cost?

Various methods of graphing lines

The most efficient method of sketching the graph of a line depends on the equation you are given:

1. **If the line is vertical** (has only x in the equation), **or** if the line is **horizontal** (has only y in the equation), sketch the line by drawing one point on the line and then sketching a vertical or horizontal line as appropriate. "Just do it."

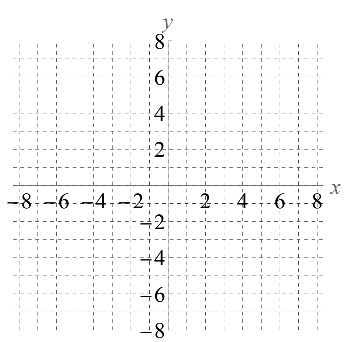
2. If the line isn't vertical or horizontal:

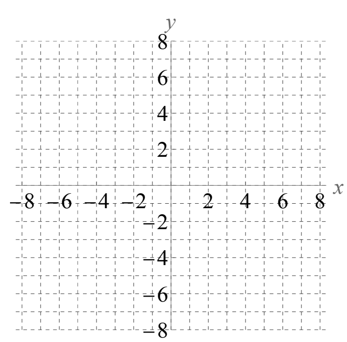
   - **if the line is in slope-intercept form** $f(x) = mx + b$, plot the y-intercept $(0, b)$, use the slope to get a second point, then connect the points;
   - **if the line is in point-slope form** $f(x) = y_0 + m(x - x_0)$, plot the point (x_0, y_0), use the slope to get a second point, then connect the points;
   - **if you are given the standard equation** $Ax+By = C$, find the x-intercept (by setting $y = 0$ and solving for x) and the y-intercept (by setting $x = 0$ and solving for y); connect these points.

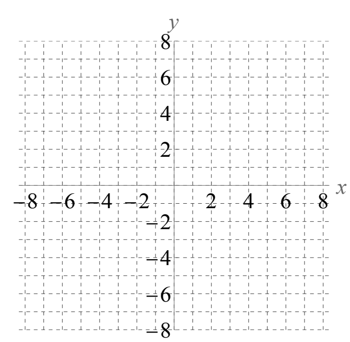
§3.2 EXAMPLE 8

Sketch the graph of each line:

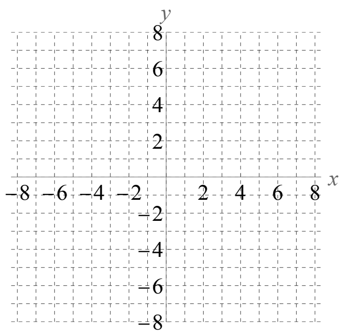
a) $y = -2x + 5$ b) $3x + 4y = 24$ c) $x = -3$

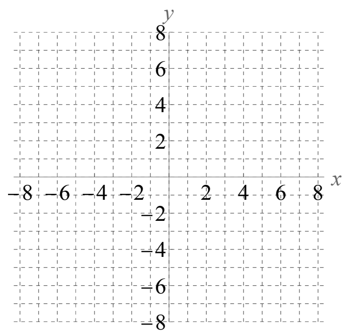


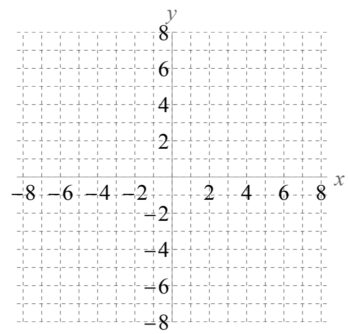




d) $y = -5 + \frac{2}{3}(x - 4)$ e) $y = 0$ f) $x = 3y + 6$

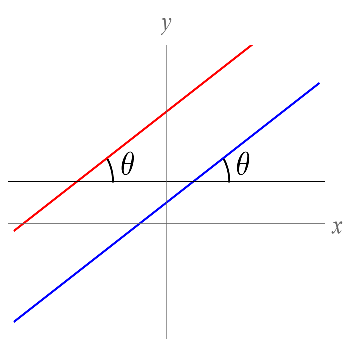






Parallel and perpendicular lines

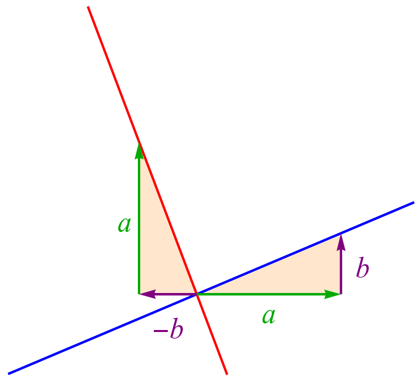
Two parallel lines make the same angle with the horizontal:



Since the slopes of these lines are the tangent of the same angle θ, we see

Theorem 3.10 *Parallel lines have the same slope.*

Two perpendicular lines look like this:



We see from the congruent triangles in this picture that if the first line has slope $m_1 = \frac{b}{a}$, then the second line has slope $m_2 = \frac{a}{-b}$. That means

Put another way,

Theorem 3.11 *Perpendicular lines have slopes that multiply to* -1.
Put another way, perpendicular lines have slopes that are negative reciprocals.

§3.2 EXAMPLE 9

a) Write the equation of the line parallel to $y = 2 + 7(x - 1)$ passing through the point $(5, -3)$.

b) Write the equation of the line perpendicular to $2x - 7y = 11$ passing through the point $(-4, 0)$.

Other useful theoretical facts about linear functions

Inverses

Theorem 3.12 *Any linear function whose slope is nonzero is one-to-one, and the inverse of such a function is linear.*

REASON A line that isn't horizontal will pass the HLT, so it must be one-to-one.

For the inverse, if f is linear then $f(x) = mx + b$. To undo this f, we would first subtract b and then divide by m, so

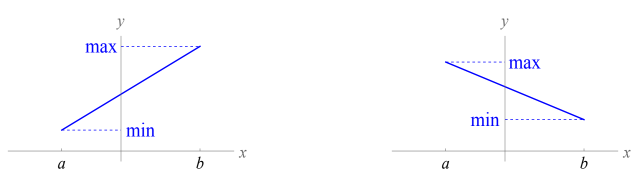
$$f^{-1}(y) = \frac{y-b}{m} = \frac{1}{m}y - \frac{b}{m},$$

which is linear.

Maximum and minimum values

Theorem 3.13 *When the inputs are restricted to a closed interval $[a, b]$, the maximum and minimum values of a linear function must occur at endpoints, meaning when the input is a and/or when the input is b.*

REASON If the slope is positive, the maximum value is at the right-most point (when $x = b$) and the minimum is at the left-most point (when $x = a$). If the slope is negative, the reverse holds.



Closure properties

Theorem 3.14 *Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are linear, and let r be a constant. Then, the following functions are also linear:*

$$f + g \qquad f - g \qquad rf \qquad f \circ g$$

REASON If f and g are linear, then $f(x) = mx + b$ and $g(x) = nx + c$. Then:

$$\begin{aligned}
(f + g)(x) &= (mx + b) + (nx + c) = (m + n)x + (b + c) \\
(f - g)(x) &= (mx + b) - (nx + c) = (m - n)x + (b - c) \\
(rf)(x) &= r(mx + b) = (rm)x + rb \\
(f \circ g)(x) &= f(nx + c) = m(nx + c) + b = (mn)x + (mc + b).
\end{aligned}$$

All these rules are of the form (constant)$x+$ (constant), so they are all linear.

3.3 Solving linear equations

Suppose you have an equation in one variable where both sides of the equation are linear functions.

That means the equation has constant terms and x-terms, but nothing else (no x^2, no e^x, no $\sin x$, no $\sqrt{x}$, no $|x|$, etc.)

Note: most linear equations have 1 solution; there are some "stupid" linear equations with infinitely many solutions ($0x = 0$) or no solution ($0x = 1$).

Such an equation is called a **linear equation** and can be solved by moving all the x-terms to one side and all the constant terms on the other side:

§3.3 EXAMPLE 1

Solve for x:

a) $7x + 9 = 65$

Solution:

$$\begin{aligned} 7x + 9 &= 65 \\ 7x &= 56 \\ x &= \boxed{8} \end{aligned}$$

b) $3(x-1) + 5 = 4(3x-2) + x$

Solution:

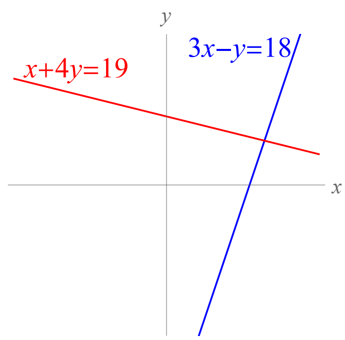
$$\begin{aligned} 3(x-1) + 5 &= 4(3x-2) + x \\ 3x - 3 + 5 &= 12x - 8 + x \\ 3x + 2 &= 13x - 8 \\ 10 &= 10x \\ \boxed{1} &= x \end{aligned}$$

c) $\frac{1}{2}x + \frac{1}{3} = \frac{3}{4}x - \frac{1}{5}$

Systems of linear equations

To find the intersection points of two lines (say the graphs of $y = f(x)$ and $y = g(x)$, based on what we learned in Chapter 2, we would solve the equation

However, sometimes the equation of a line isn't given in $y = f(x)$ form, so we need to know some other techniques for solving for intersection points. For example, suppose want to find the intersection of the lines $3x - y = 18$ and $x + 4y = 19$.



To solve for this point algebraically, we need to find (x, y) which is a solution of the **system of equations**

$$\begin{cases} 3x - y = 18 \\ x + 4y = 19 \end{cases},$$

meaning we need to find (x, y) that works in *both equations*.

Methods of solving a system of equations

METHOD 1: Substitution. Solve for one variable in one equation and substitute into the other equation (usually easier if at least one line is given in slope-intercept form).

$$\begin{cases} 3x - y = 18 \\ x + 4y = 19 \end{cases}$$

METHOD 2: Addition/elimination. Multiply through each equation by a constant, then add the equations to eliminate one variable (usually easier if both equations are in standard form).

$$\begin{cases} 3x - y = 18 \\ x + 4y = 19 \end{cases}$$

§3.3 EXAMPLE 2

Find the point of intersection of the two lines $y = -2x - 1$ and $8x + 5y = -1$.

§3.3 EXAMPLE 3

Find the point of intersection of the two lines $y = 3x - 7$ and $6x - 2y = 5$.

3.4 Introducing transformations

§3.4 EXAMPLE 1

The graph of the linear function in the box is provided for you. On the same axes, sketch the graph of the other function:

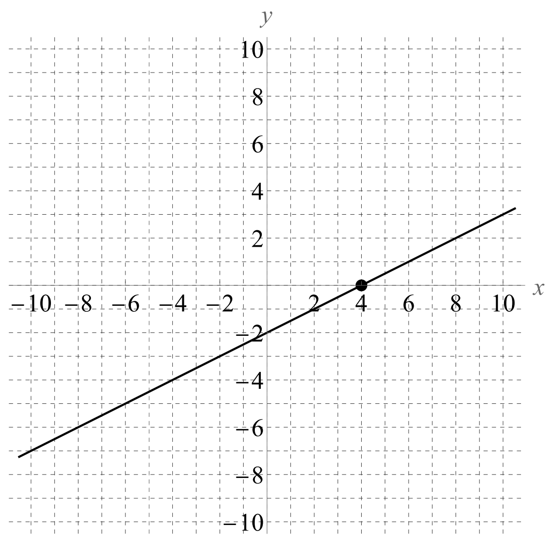
$$\boxed{f(x) = \frac{1}{2}(x-4)}$$

$$f_{-2}(x) = -2 + \frac{1}{2}(x-4)$$

$$f_{-6}(x) = -6 + \frac{1}{2}(x-4)$$

$$f_{3}(x) = 3 + \frac{1}{2}(x-4)$$

$$f_{8}(x) = 8 + \frac{1}{2}(x-4)$$



From these graphs, we see:

- when we add a constant c to the rule of f, the graph ____ .

- when we subtract a constant c from the rule of f, the graph ____ .

This works not just in the example we just did, but for any function:

Theorem 3.15 (Vertical shifts) *Let $c > 0$ be a constant.*

- *The graph of the function $f(x) + c$ is the same as the graph of f, shifted up by c units.*

$$x \xrightarrow{f} \xrightarrow{+c} f(x) + c$$

- *The graph of the function $f(x) - c$ is the same as the graph of f, shifted down by c units.*

$$x \xrightarrow{f} \xrightarrow{-c} f(x) - c$$

§3.4 EXAMPLE 2

The graph of the linear function in the box is provided for you. On the same axes, sketch the graph of each given function:

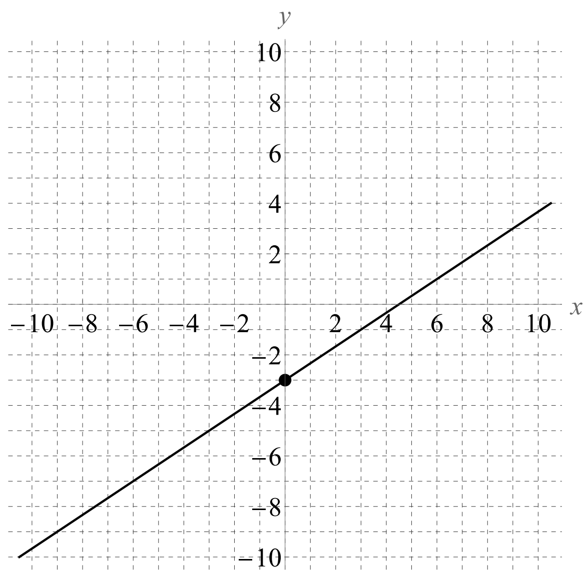
$$\boxed{g(x) = -3 + \frac{2}{3}x}$$

$$g_2(x) = -3 + \frac{2}{3}(x-2)$$

$$g_5(x) = -3 + \frac{2}{3}(x-5)$$

$$g_{-3}(x) = -3 + \frac{2}{3}(x+3)$$

$$g_{-7}(x) = -3 + \frac{2}{3}(x+7)$$



From these graphs, we see:

- when we replace x with $(x-c)$ in the rule of f, the graph ______________________
  ____ .

- when we replace x with $(x+c)$ in the rule of f, the graph ______________________
  ____ .

This works not just in the example we just did, but for any function:

Theorem 3.16 (Horizontal shifts) *Let $c > 0$ be a constant.*

- *The graph of the function $f(x-c)$ is the same as the graph of f, shifted right by c units.*
$$x \xrightarrow{-c} \xrightarrow{f} f(x-c)$$

- *The graph of the function $f(x+c)$ is the same as the graph of f, shifted left by c units.*
$$x \xrightarrow{+c} \xrightarrow{f} f(x+c)$$

§3.4 EXAMPLE 3

The graph of the linear function in the box is provided for you. On the same axes, sketch the graph of each given function:

$$\boxed{h(x) = x - 2}$$

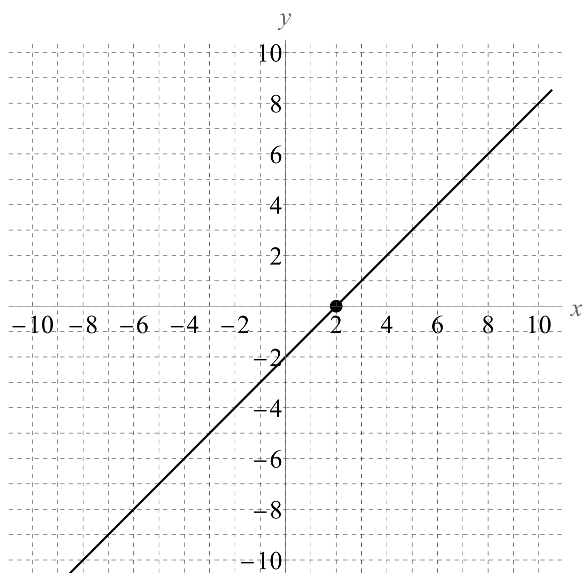
$$h_3(x) = 3(x-2)$$

$$h_5(x) = 5(x-2)$$

$$h_{-2}(x) = -2(x-2)$$

$$h_{-1/2}(x) = -\frac{1}{2}(x-2)$$

$$h_{1/4}(x) = \frac{1}{4}(x-2)$$



From these graphs, we see:

- when we multiply the rule for h by a constant c, the x-intercept of the graph stays the same, and the graph is stretched/compressed vertically by a factor of c.
- if $c < 0$, the graph of h is reflected across the x-axis (flipped vertically).

Theorem 3.17 (Vertical stretch/compression) *Let c be a nonzero constant.*

- *The graph of the function $cf(x)$ is the same as the graph of f, stretched vertically by a factor of c units.*

$$x \xrightarrow{f} \xrightarrow{\times c} cf(x)$$

- *If $|c| > 1$, the graph is stretched; if $|c| < 1$ the graph is compressed.*
- *The x-intercept(s) of $cf(x)$ are the same as the x-intercept(s) of f.*
- *If $c < 0$, the graph of the function $cf(x)$ is the graph of f, stretched vertically by a factor of c and reflected across the x-axis.*

What a transformation is

Definition 3.18 *Let* $f : \mathbb{R} \to \mathbb{R}$. *A* **transformation** *of* f *is any other function whose rule comes from the rule of* f *by including some extra constant(s) via addition, subtraction, multiplication or division.*

EXAMPLES

- Transformations of the function $f(x) = x^2$ include

$$g(x) = x^2 + 2, \quad h(x) = \frac{-1}{4}x^2, \quad k(x) = (x-2)^2, \quad l(x) = (3x)^2 - 1, \quad \text{etc.}$$

- Transformations of the function $F(x) = \sin x$ include

$$G(x) = 2\sin x, \qquad H(x) = \sin(x+3), \qquad K(x) = \frac{2}{3}\sin 2x - 1, \qquad \text{etc.}$$

- Transformations of the function $p(x) = e^x$ include

$$q(x) = \frac{3}{4}e^x, \qquad r(x) = -e^{x-4}, \qquad s(x) = e^{7x} - 3, \qquad u(x) = -8e^{4x}, \quad \text{etc.}$$

The graph of a transformation of f is obtained from the graph of f by doing something relatively "simple". For example, we've seen

TRANSFORMATION OF f	EFFECT ON GRAPH OF f
$f(x) + c$	shifts up c units
$f(x) - c$	shifts down c units
$f(x - c)$	shifts right c units
$f(x + c)$	shifts left c units
$cf(x)$	stretched vertically by factor of c (flipped across the x-axis if $c < 0$)

Note: all these transformations have graphs that have the same general shape as the graph of f (just shifted, stretched or reflected).

Key theme with transformations: when you transform f in a way that takes place after f in an arrow diagram, the graph is affected vertically:

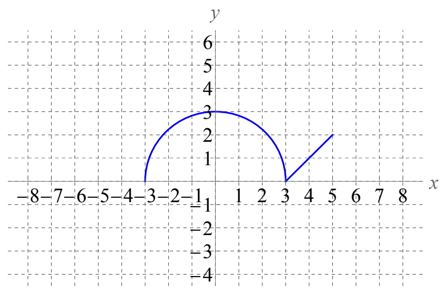
$$x \xrightarrow{f} \xrightarrow{+c} g(x) \qquad\qquad x \xrightarrow{f} \xrightarrow{\times c} g(x)$$

But when you transform f in a way that takes place before f in an arrow diagram, the graph is affected horizontally:

$$x \xrightarrow{+c} \xrightarrow{f} g(x) \qquad\qquad x \xrightarrow{\times c} \xrightarrow{f} g(x)$$

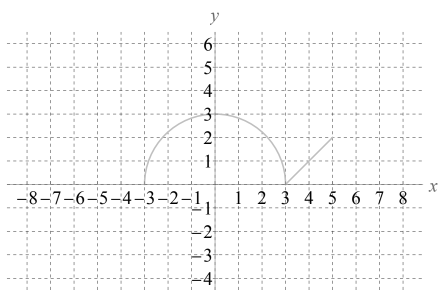
§3.4 EXAMPLE 4

The graph of some unknown function f is given below:

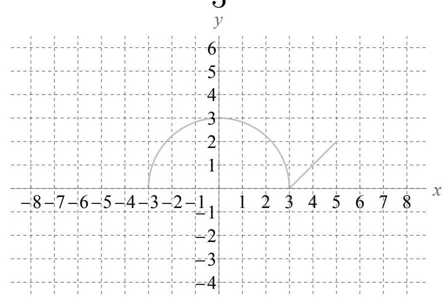


Use this graph to sketch the graph of each function:

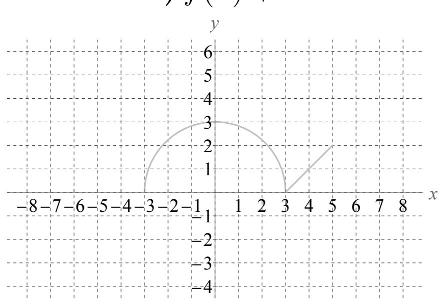
a) $f(x-2)$



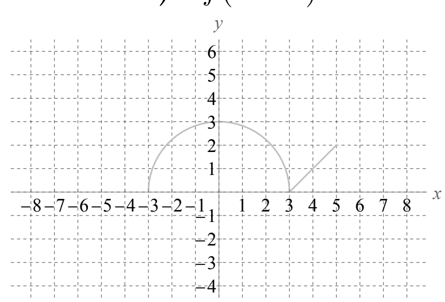
d) $\frac{1}{3}f(x)$



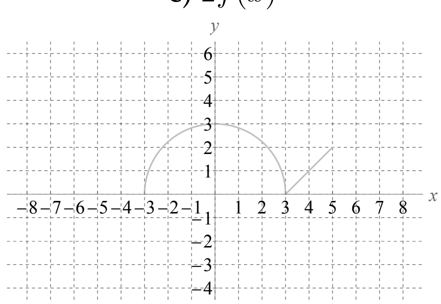
b) $f(x)+1$



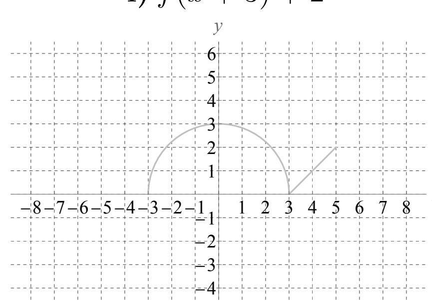
e) $-f(x-1)$



c) $2f(x)$



f) $f(x+3)+2$



3.5 Quadratic functions

CONCEPT

We saw in Sections 3.1 and 3.2 that linear functions are those with a constant rate of change.

Question: What does a function look like if it has a *linear* rate of change?

§3.5 EXAMPLE 1

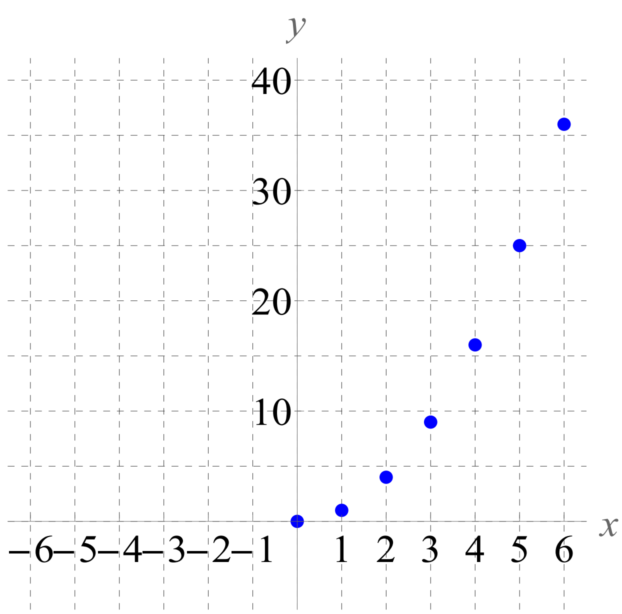
Let f be the function which satisfies

- $f(0) = 0$; and
- the average rate of change of f from x to $x+1$ is $2x+1$.

The second bullet point above means

Using this fact, we can produce a table of values for x, which will lead us to a rule for f and a graph of f:

x	$f(x)$
-1	
0	0
1	
2	
3	
4	
5	
6	
$\vdots$	$\vdots$



In general, if the rate of change of a function f is linear, then f itself will be *quadratic*:

Definition 3.19 *A* **quadratic function** *(or just* **quadratic***) is a function $f : \mathbb{R} \to \mathbb{R}$ whose rule is*

$$f(x) = ax^2 + bx + c$$

for constants a, b and c (where $a \neq 0$).

Theorem 3.20 *Let $f : \mathbb{R} \to \mathbb{R}$. f is quadratic if and only if the rate of change of f is linear.*

(Proving this theorem rigorously requires calculus.)

§3.5 EXAMPLE 2

A beanbag is tossed out of a window so that its height (in meters) at time t, in seconds, is given by $h(t) = -5t^2 + 15t + 40$.

1. Compute the average rate of change of the height of the beanbag between times $t = 0$ and $t = 1$.

2. Compute the average rate of change of the height of the beanbag between times $t = 1$ and $t = 2$.

3. Compute the average rate of change of the height of the beanbag between times $t = 2$ and $t = 3$.

   Solution:

$$\begin{aligned} \frac{h(3) - h(2)}{3 - 2} &= \frac{[-5(3^2) + 15(3) + 40] - [-5(2^2) + 15(2) + 40]}{1} \\ &= [-45 + 45 + 40] - [-20 + 30 + 40] \\ &= 40 - 50 = \boxed{-10 \text{ m/sec}}. \end{aligned}$$

4. Compute the average rate of change of the height of the beanbag between times $t = 3$ and $t = 4$.

   Solution: Mimicking what was done in Questions 1-3, we get

$$\frac{h(4) - h(3)}{4 - 3} = \frac{20 - 40}{1} = \boxed{-20 \text{ m/sec}}.$$

5. Explain why h is **not** a linear function.

6. Verify the average rate of change of the height across a 1 second interval of time *is* a linear function.

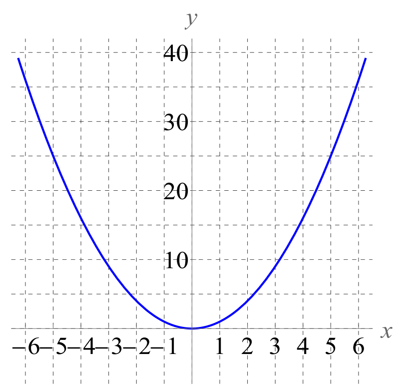
7. Verify the average rate of change of the height across an s second interval of time is also a linear function.

   Solution: Similar to what was done in Question 6, we have

$$\begin{aligned}\frac{f(t+s) - f(t)}{(t+s) - t} &= \frac{[-5(t+s)^2 + 15(t+s) + 40] - [-5t^2 + 15t + 40]}{s} \\ &= \frac{[-5(t^2 + 2st + s^2) + 15t + 15s + 40] + 5t^2 - 15t - 40}{s} \\ &= \frac{-5t^2 - 10st - 5s^2 + 15t + 15s + 40] + 5t^2 - 15t - 40}{s} \\ &= \frac{-10st - 5s^2 + 15s}{s} \\ &= \frac{s(-10t - 5s + 15)}{s} = \boxed{-10t + (-5s + 15)}.\end{aligned}$$

The two forms of a quadratic function

We've seen that the graph of $f(x) = x^2$ is a parabola:



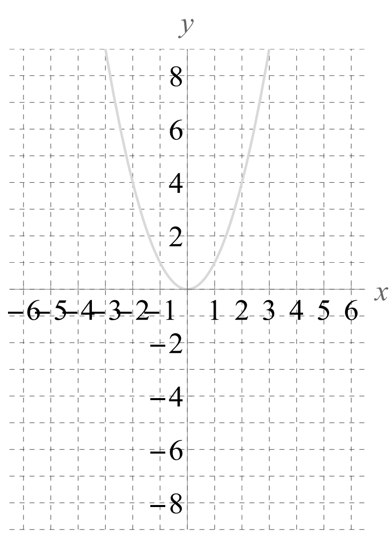
Our next task is to see why the graph of *any* quadratic is a parabola. To do this, we need to find a second version of a rule that describes quadratic function. And to do this, we'll use transformations.

§3.5 EXAMPLE 3

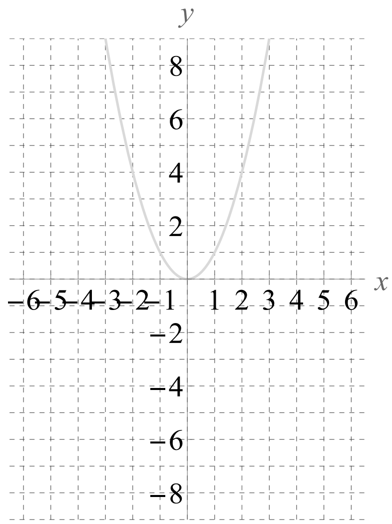
For each given function:

1. Sketch the graph of the function by transforming the graph of $f(x) = x^2$.
2. Write the rule for the function in the form $ax^2 + bx + c$.

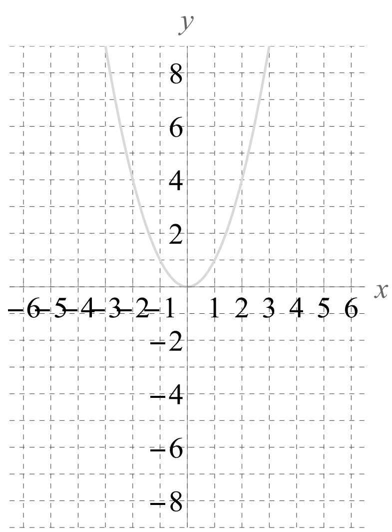
a) $g(x) = (x - 3)^2 + 1$



b) $h(x) = 2(x+3)^2$



c) $k(x) = -\dfrac{1}{2}(x-1)^2 + 5$



Observations about the graph of $f(x) = a(x-h)^2 + k$:

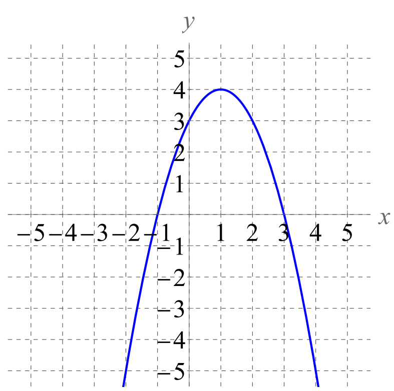
- All these graphs are parabolas;
- the vertical line $x = h$ is an axis of symmetry for the parabola;
- the parabola "turns around" at the point (h, k);
- the domain of f is the set $\mathbb{R}$ of all real numbers;
- if $a > 0$, then:
  - the parabola opens upward,
  - the function f has a minimum value of k, and
  - the range of f is $[k, \infty)$;
- if $a < 0$, then:
  - the parabola opens downward,
  - the function f has a maximum value of k, and
  - the range of f is $(-\infty, k]$.

The vertex of a parabola

Definition 3.21 *The point where a parabola "turns around" is called the* **vertex** *of the parabola. This point is usually denoted (h, k).*

§3.5 EXAMPLE 4

Identify the x-intercepts, the y-intercept and the vertex of this parabola:



x-ints:

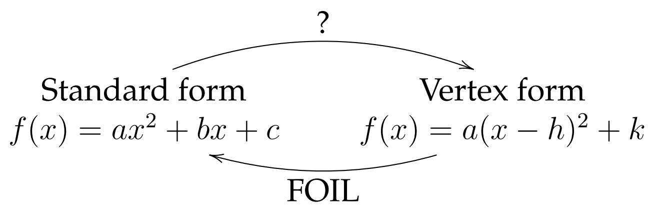
y-int:

vertex:

Completing the square

QUESTION

Can you write *any* quadratic function in the form $a(x-h)^2+k$?



Answer: Yes; to do this you use a technique called **completing the square**.

§3.5 EXAMPLE 4

Write each quadratic function in vertex form:

$$f(x) = 2x^2 + 16x - 10 \qquad\qquad f(x) = ax^2 + bx + c$$

Theorem 3.22 (Completing the square) *Given a quadratic expression ax^2+bx+c, we can rewrite this expression as*

$$ax^2 + bx + c = a\left(x - h\right)^2 + k$$

where $h = -\dfrac{b}{2a}$ and $k = ah^2 + bh + c$.
As a consequence, the graph of every quadratic function is a parabola.

Definition 3.23 *The formula $f(x) = a(x-h)^2 + k$ is called the* **vertex form** *of the quadratic.*

Theorem 3.24 *The vertex of $f(x) = ax^2 + bx + c$ has x-coordinate $h = -\dfrac{b}{2a}$.*

§3.5 EXAMPLE 5

Find the coordinates of the vertex of $f(x) = 2x^2 - 12x - 7$.

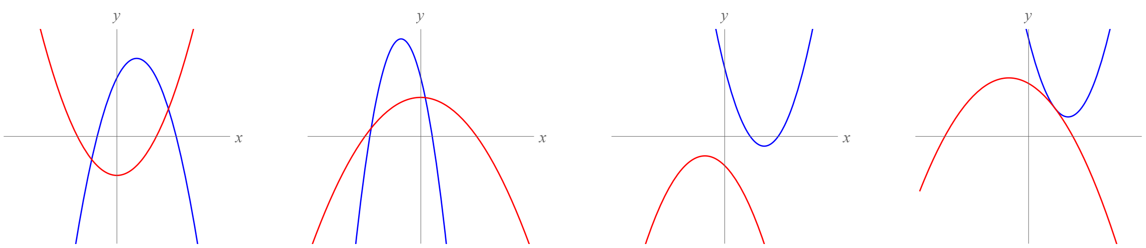
Solving quadratic equations

A quadratic equation is an equation where the LHS and RHS are both quadratic. The most general equation of this type is

which can be thought of as trying to find the (x-coordinates of) intersection points of two parabolas.

Number of solutions

Here are some pictures which show the ways two parabolas can intersect:



Theorem 3.25 *A quadratic equation has* 0, 1 *or* 2 *solutions.*

Solving quadratic equations by factoring

To solve a quadratic equation by factoring, we

1. move all the terms to one side of the equation (i.e. make one side zero), then
2. factor the non-zero side, then
3. set each factor equal to zero and solve for the variable.

This method is based on the following algebraic concept:

If two (or more) terms multiply to make zero,
at least one of the terms must itself be zero.

§3.5 EXAMPLE 6

Solve for x in each equation:

a) $x^2 - 5x - 14 = 0$

b) $x^2 = 18 + 3x$

c) $2x^2 - 28x + 98 = 0$

WARNING: Factoring is only a useful technique to solve equations if one side of the equation is zero:

$$(\square)(\triangle) = 0 \qquad\qquad (\square)(\triangle) = 4$$

Solving quadratic equations with no x term

If a quadratic equation has x^2 terms but no x terms, we can solve it without factoring. To do this,

1. isolate the x^2 term by itself, then
2. take $\pm\sqrt{\ }$ of both sides.

IMPORTANT: Don't forget the $\pm$ here!

This method takes advantage of the following fact:

Theorem 3.26 (Inverse of a quadratic) *If $f(x) = x^2$, then $f^{-1}(x) = \pm\sqrt{x}$.*

As an arrow diagram, this concept is

§3.5 EXAMPLE 6

Solve for x in each equation:

a) $x^2 - 25 = 0$

b) $2x^2 + 7 = 5(x^2 - 2)$

c) $2x^2 + 7 = 0$

More completing the square

We saw earlier how completing the square could be used to find the vertex of a parabola. The method of completing the square is also useful for solving quadratic equations:

§3.5 EXAMPLE 7

Solve for x in each equation by completing the square:

a) $3x^2 + 12x - 7 = 0$

b) $x^2 + 10x - 3 = 0$

Solution: We have

$$h = -\frac{b}{2a} = -\frac{10}{2(1)} = -5$$

and

$$k = (-5)^2 + 10(-5) + 3 = 25 - 50 + 3 = -22,$$

so by completing the square on the left-hand side we can rewrite the equation as

$$\begin{aligned} (x+5)^2 - 22 &= 0 \\ (x+5)^2 &= 22 \\ x + 5 &= \pm\sqrt{22} \\ x &= \boxed{-5 \pm \sqrt{22}}. \end{aligned}$$

The quadratic formula

What do you do with a quadratic equation if you can't factor it and you don't want to complete the square?

Theorem 3.27 (Quadratic Formula)

$$\textit{If } ax^2 + bx + c = 0, \qquad \textit{then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In particular:

- *If* $b^2 - 4ac > 0$*, then the quadratic equation* $ax^2 + bx + c = 0$ *has two solutions.*
- *If* $b^2 - 4ac = 0$*, then the quadratic equation* $ax^2 + bx + c = 0$ *has one solution.*
- *If* $b^2 - 4ac < 0$*, then the quadratic equation* $ax^2 + bx + c = 0$ *has no solution.*

NOTE: The quadratic formula only works on *quadratic* equations, where *one side of the equation is zero*.

WHERE THE QUADRATIC FORMULA COMES FROM

Suppose you complete the square on an arbitrary quadratic equation:

$$\begin{aligned} ax^2 + bx + c &= 0 \\ a(x-h)^2 + k &= 0 \\ a(x-h)^2 &= -k \\ (x-h)^2 &= -\frac{k}{a} \\ x - h &= \pm\sqrt{-\frac{k}{a}} \\ x &= h \pm \sqrt{-\frac{k}{a}} \end{aligned}$$

Now, we know $h = -\dfrac{b}{2a}$ and $k = ah^2 + bh + c$. If you plug these formulas in for h and k and do algebra to simplify this answer, you get the quadratic formula.

§3.5 EXAMPLE 8

Solve for x in each equation:

a) $2x^2 + 3x + 1 = 6x$

b) $3x^2 - 4x = -11$

c) $-x^2 + 5x + 3 = 0$

Graphing parabolas

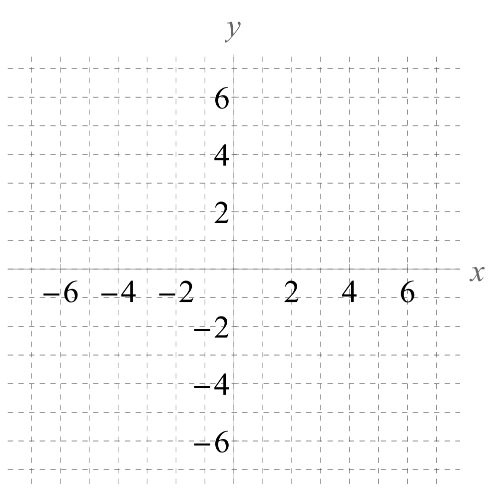
The most efficient method of sketching the graph of a quadratic depends on the equation you are given:

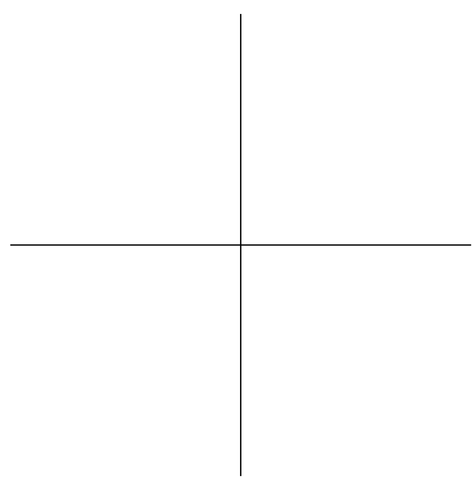
1. **If you are given the standard form** $f(x) = ax^2 + bx + c$, find the x-intercept(s) and the y-intercept and sketch the parabola (it opens up if $a > 0$ and opens down if $a < 0$). The vertex will be halfway between the x-intercepts, by the way. If there are no x-ints, just find the vertex.
2. **If you are given the vertex form** $f(x) = a(x - h)^2 + k$, plot the vertex (h, k) and maybe also the y-intercept, and sketch the parabola (it opens up if $a > 0$ and opens down if $a < 0$). (Alternatively, just shift the parabola $y = x^2$ using transformation methods.)

§3.5 EXAMPLE 9

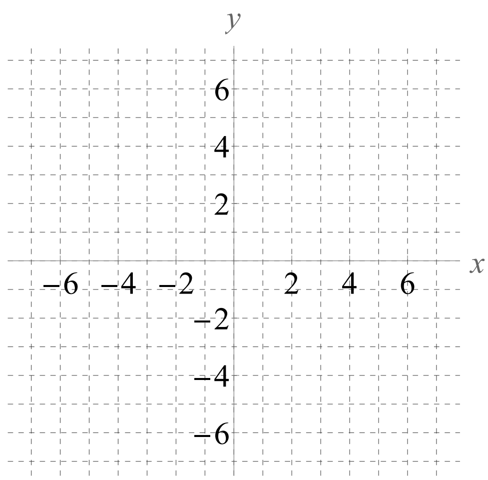
Sketch the graph of each function:

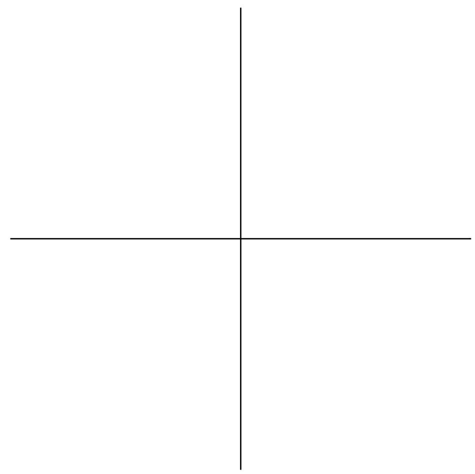
a) $f(x) = 2(x + 3)^2 - 5$



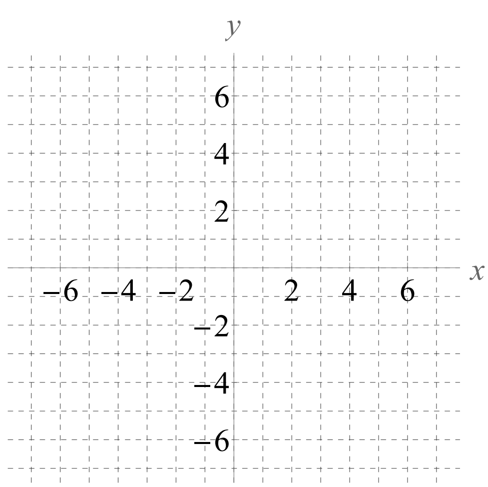


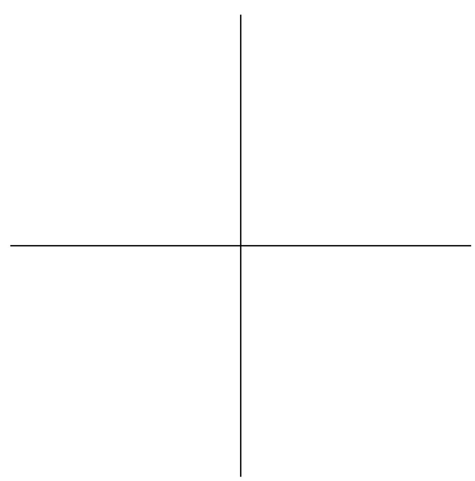
b) $f(x) = -x^2 + 7x - 6$





c) $f(x) = 2x^2 + x - 15$





3.6 Polynomial functions

CONCEPT

IF THE RATE OF CHANGE OF f IS	THEN f IS	AND f HAS THIS RULE
zero	constant	$f(x) = c$
constant	linear	$f(x) = mx + b$
linear	quadratic	$f(x) = ax^2 + bx + c$
quadratic		

If you continue with this chart indefinitely, you get a collection of functions called *polynomials*:

Definition 3.28 *A* **polynomial** *is a function* $f : \mathbb{R} \to \mathbb{R}$ *whose rule can be written as*

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_d x^d$$

for constants $a_0, a_1, ..., a_d$, *where* $a_d \neq 0$.

d *is called the* **degree** *of the polynomial.*

a_d *is called the* **leading coefficient** *of the polynomial.*

Note: Constants (which have degree 0), linear functions (degree 1), and quadratic functions (degree 2) are all polynomials.

Note: Only non-negative powers are allowed in polynomials: $\sqrt{x} = x^{1/2}$, $\frac{1}{x} = x^{-1}$, $\sin x$, etc. are not polynomials.

§3.6 EXAMPLE 1

Determine whether the given function is a polynomial. If it is, identify its degree and leading coefficient.

a) $f(x) = 7x^3 + 2x^6 - 6$

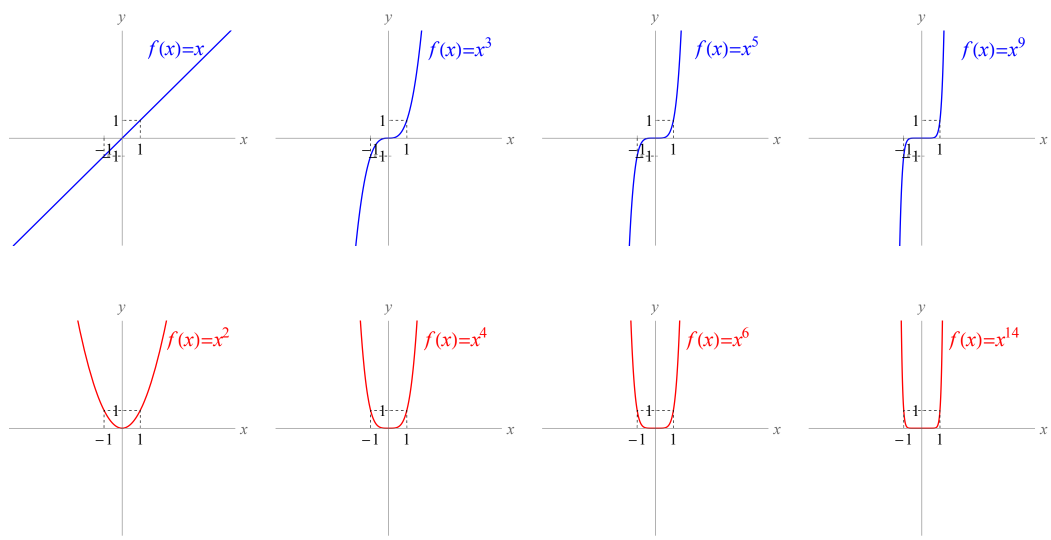
b) $h(x) = 2x^5 + 3x^{3/2} - 4$

c) $r(x) = (4x - 3)(2x^2 + 1)$

d) $f(x) = 4^x - x^4$

Power functions

The simplest polynomials are called **power functions**. These have one term and a leading coefficient of 1, so they have the form $f(x) = x^d$. Here are their graphs:



Theorem 3.29 (Properties of power functions) *Let* $f(x) = x^d$.

Domain: *the domain of every power function is* $\mathbb{R}$.

Range: *if* d *is even, then the range of* f *is* $[0, \infty)$;
if d *is odd, then the range of* f *is* $\mathbb{R}$.

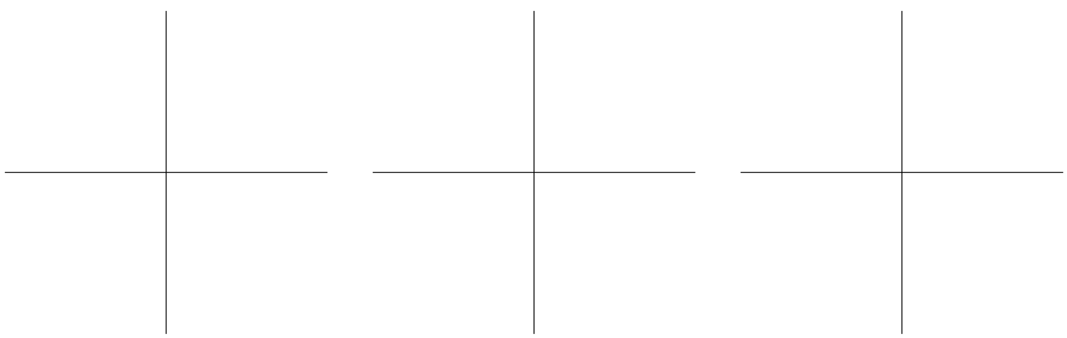
Symmetry: *if* d *is even, then the function* f *is even;*
if d *is odd, then the function* f *is odd.*

Minimum values: *if* d *is even, then* f *has a minimum value of* 0;
if d *is odd then* f *has no maximum or minimum value.*

§3.6 EXAMPLE 2

Sketch a crude graph of each function:

a) $f(x) = -x^3$ b) $g(x) = (x-2)^4$ c) $h(x) = 2x^5 + 3$



Some theory of polynomials

Closure properties

Theorem 3.30 *If* f *and* g *are polynomials, then so are* $f+g$, $f-g$, fg *and* $f \circ g$.

§3.6 EXAMPLE 3

Let $f(x) = x^2 + 4x + 3$ and $g(x) = x^3 + 2$. Compute and simplify $(f \circ g)(x)$.

Continuity and smoothness

Theorem 3.31 *The graph of any polynomial is* **continuous** *(meaning the entire graph can be drawn without lifting your writing instrument from the paper) and* **smooth** *(meaning that the graph has no sharp corners or cusps).*

Turning points

A **turning point** of a polynomial f is an x-coordinate where the graph of f either changes from increasing to decreasing, or decreasing to increasing.

Theorem 3.32 *A polynomial of degree d has at most $d - 1$ turning points.*

Fundamental Theorem of Algebra

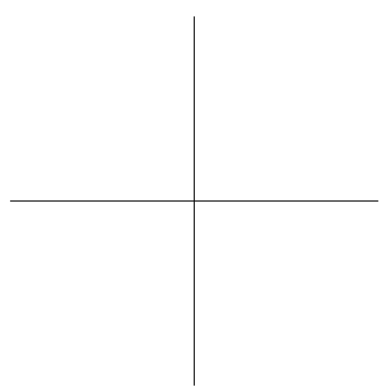
Theorem 3.33 (Fundamental Theorem of Algebra) *If f is a polynomial with degree d, then the equation $f(x) = 0$ has at most d solutions.*

Tail behavior

Earlier, we saw the graphs of $f(x) = ax^d$ (constants multiplied by power functions). As these graphs go to the left and right, the "tails" of these graphs either point upward or downward.

What about the tail behavior of a polynomial? Let's consider an example: let

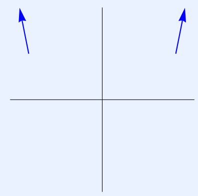
$$f(x) = -4x^3 + 3x^2 + 5x - 1.$$



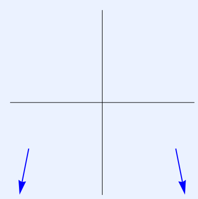
More generally, any polynomial has the same tail behavior as its highest-power term. This means:

Theorem 3.34 (Tail behavior of polynomials) *Suppose f is a polynomial with degree $d \geq 1$ and leading coefficient a_d. Then:*

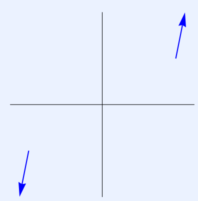
- *if d is even and $a_d > 0$, then both tails of the polynomial point upward, meaning the graph of f looks like this:*



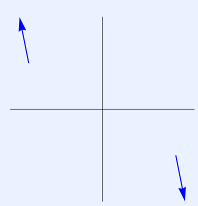
- *if d is even and $a_d < 0$, then both tails of the polynomial point downward, meaning the graph of f looks like this:*



- *if d is odd and $a_d > 0$, then the left tail of the polynomial points downward but the right tail points upward, meaning the graph of f looks like this:*



- *if d is odd and $a_d < 0$, then the left tail of the polynomial points upward but the right tail points downward, meaning the graph of f looks like this:*

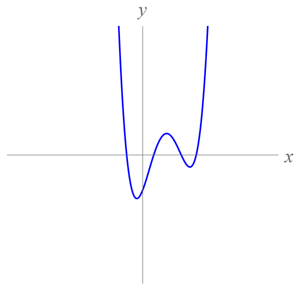


In particular, the tails of a (nonconstant) polynomial function **cannot** point sideways. As you go to the extreme left or right of a polynomial graph, the graph must point upwards or downwards.

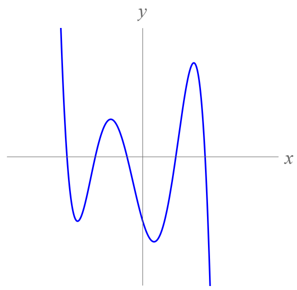
§3.6 EXAMPLE 4

For each given graph, answer the following questions:

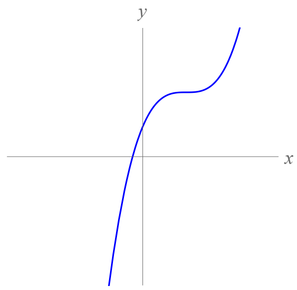
i. Is this the graph of a polynomial?
ii. If it is a polynomial, is its degree even or odd?
iii. If it is a polynomial, is its leading coefficient positive or negative?
iv. If it is a polynomial, what is its smallest possible degree?

a) 

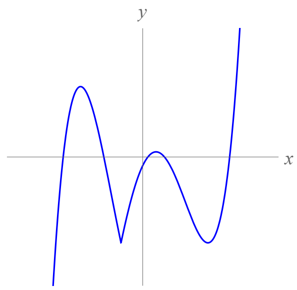
i. Polynomial?
ii. Degree even or odd?
iii. LC positive or negative?
iv. Smallest possible degree?

b) 

i. Polynomial?
ii. Degree even or odd?
iii. LC positive or negative?
iv. Smallest possible degree?

c) 

i. Polynomial?
ii. Degree even or odd?
iii. LC positive or negative?
iv. Smallest possible degree?

d) 

i. Polynomial?
ii. Degree even or odd?
iii. LC positive or negative?
iv. Smallest possible degree?

3.7 Root functions

MOTIVATION

Consider these equations:

$$x^2 = 4 \qquad x^2 = 5 \qquad x^3 = -8 \qquad x^4 = -1$$

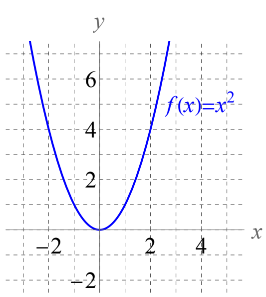
Based on what we learned in Section 2.12, we can interpret these equations graphically: we give a name to each function on the left-hand side, so that these equations become

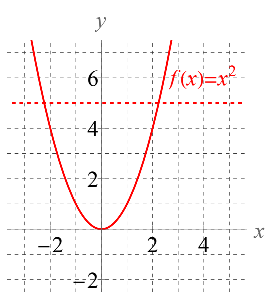
$f(x) = 4$ (where $f(x) = x^2$)

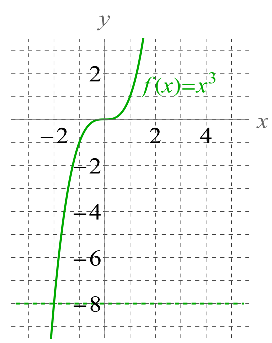
$f(x) = 5$ (where $f(x) = x^2$)

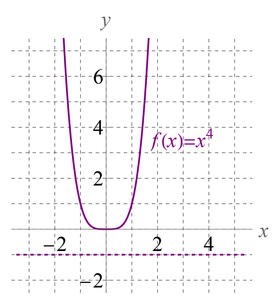
$g(x) = -8$ (where $g(x) = x^3$)

$h(x) = -1$ (where $h(x) = x^4$)









What do these graphs tell us about the solution(s) of these equations?

FOLLOW-UP QUESTION

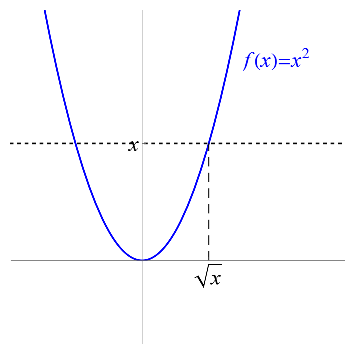
When there is a solution (or solutions), can we write that(those) solution(s) as a whole number? Can we write the solution(s) as a polynomial function of a whole number?

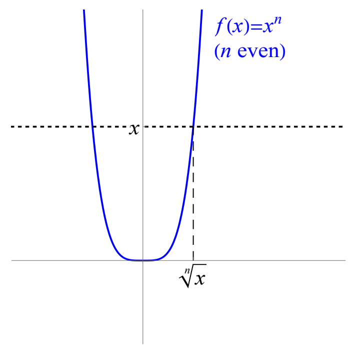
Even roots

Definition 3.35 *Let $x \in \mathbb{R}$. The* **square root** *of x is the non-negative number $\sqrt{x}$ satisfying $(\sqrt{x})^2 = x$ (if such a number exists).*

Definition 3.36 *Let $x \in \mathbb{R}$. For any positive even number n, the n^{th}* **root** *of x is the non-negative number $\sqrt[n]{x}$ satisfying $(\sqrt[n]{x})^n = x$ (if such a number exists).*

Interpretation of even roots





We ignore the negative solution above of to ensure that $\sqrt{\ }$ only has one output. This makes $\sqrt{\ }$ a function $\mathbb{R} \to \mathbb{R}$ (if the input is x, the output is $\sqrt{x}$):

$$x \xrightarrow{\sqrt{\ }} \sqrt{x}$$

If we want all (both) the solutions of $x^2 = constant$, we need the multifunction $\pm\sqrt{\ }$ to capture both the positive and negative solution:

$$x^2 = constant \quad \Rightarrow \quad x = \pm\sqrt{constant}.$$

This leads us to the following interpretation of square root in terms of arrow diagrams:

$$x \underset{\pm\sqrt{\ }}{\overset{\wedge 2}{\rightleftarrows}} y$$

Any even root is interpreted similarly:

$$\begin{array}{c} x^n = y \\ (n \text{ even}) \end{array} \quad \Rightarrow \quad x = \pm\sqrt[n]{y} \qquad\qquad x \underset{\pm\sqrt[n]{\ }}{\overset{\wedge n}{\rightleftarrows}} y$$

§3.7 Example 1

Solve each equation:

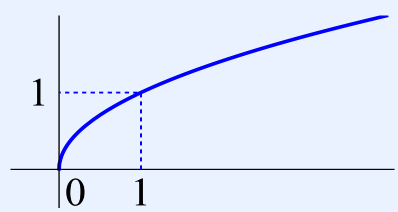
a) $x^2 = 49$

b) $5x^4 = 200$

c) $x^6 = -17$

d) $3x^{12} - 8 = 4$

Theorem 3.37 *Here is a graph of $f(x) = \sqrt{x}$ (the graphs of other even root functions look basically the same as this):*



The key properties shown by this graph are:

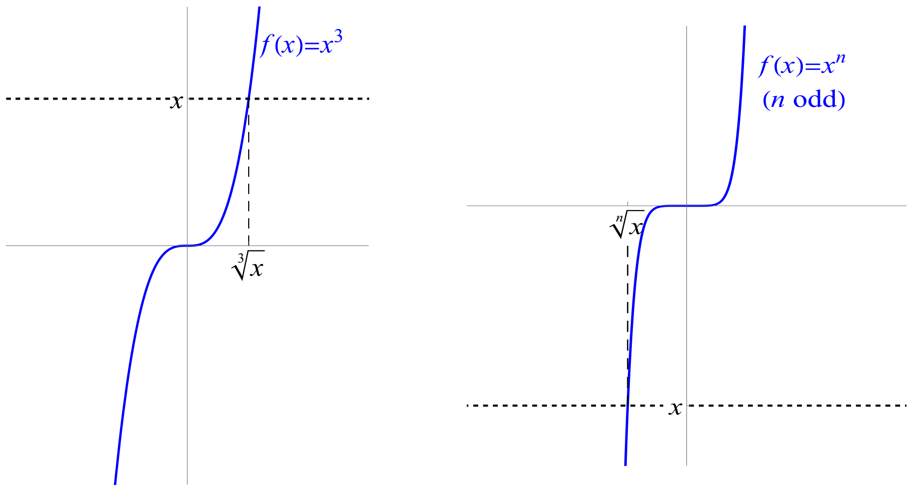
- *The domain is $[0, \infty)$;*
- *the range is $[0, \infty)$;*
- *the graph goes through $(0, 0)$ and $(1, 1)$;*
- *the function increases from left to right;*
- *it is one-to-one; and*
- *it looks like the top half of a "flipped-over" or "sideways" parabola.*

Odd roots

Definition 3.38 *The* **cube root** *of x is the number $\sqrt[3]{x}$ satisfying $(\sqrt[3]{x})^3 = x$.*

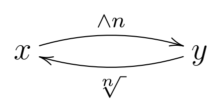
More generally, for any positive odd number n, the n^{th} **root** *of x is the number $\sqrt[n]{x}$ satisfying $(\sqrt[n]{x})^n = x$.*

Interpretation of odd roots



Notice that every real number always has an odd root, since the range of any odd power function is $\mathbb{R}$. This leads us to the following interpretation of odd roots:

$$\begin{array}{c} x^n = y \\ (n \text{ odd}) \end{array} \quad \Rightarrow \quad x = \sqrt[n]{y}$$



§3.7 EXAMPLE 2

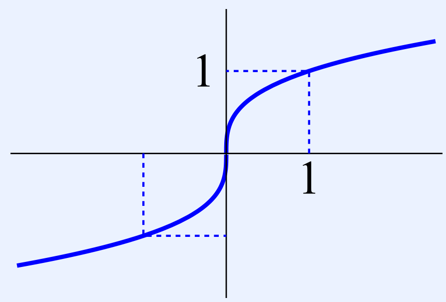
Solve each equation:

a) $x^3 = 64$

b) $x^3 = -17$

c) $3x^9 + 53 = 73$

Theorem 3.39 *Here is a graph of $f(x) = \sqrt[3]{x}$ (the graphs of other odd root functions basically look the same as this):*

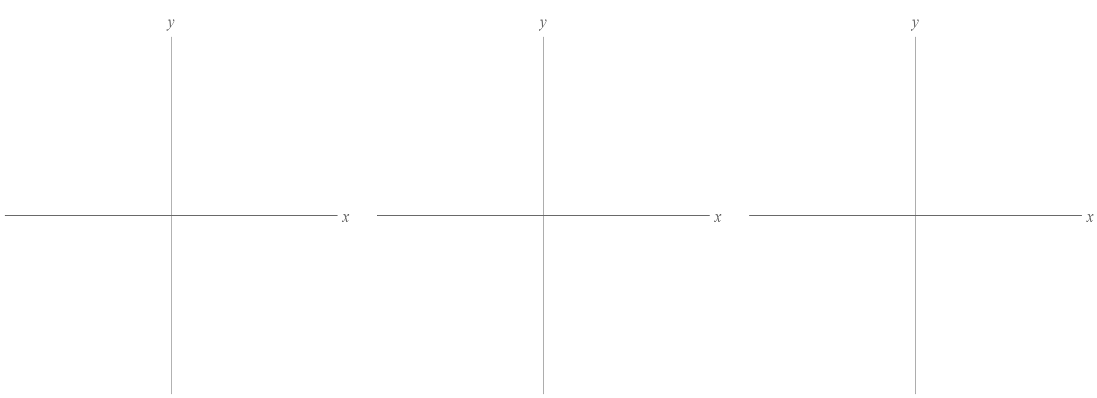


The key properties shown by this graph are:
- *the domain and range are both $\mathbb{R}$;*
- *the function is odd (meaning $\sqrt[n]{-x} = -\sqrt[n]{x}$ if n is odd);*
- *the graph goes through $(-1, -1)$, $(0, 0)$ and $(1, 1)$;*
- *the function increases from left to right;*
- *it is one-to-one; and*
- *it looks like a "flipped-over x^3".*

§3.7 EXAMPLE 3

Sketch a graph of each function:

a) $g(x) = \sqrt[5]{x-4}$ b) $h(x) = \sqrt{x+3} + 2$ c) $k(x) = -\sqrt[3]{x}$



Review of fractional exponents

The symbols $\sqrt{\ }$, $\sqrt[3]{\ }$, etc. are called **radical** signs, and expressions with radical signs in them are called **radical expressions**.

As we saw in Chapter 1 radical expressions can always be rewritten in terms of fractional exponents, allowing us to use exponent rules to manipulate expressions involving radicals. The key rules are:

$$\sqrt{x} = \qquad\qquad \sqrt[n]{x} = \qquad\qquad =$$

§3.7 EXAMPLE 4

Simplify each expression and write it in the form $\square x^{\square}$, where the boxes represent constants:

a) $\sqrt{5x}$

b) $\dfrac{\sqrt{x}}{\sqrt[3]{x}}$

c) $\sqrt{4x^9}$

d) $\left(\sqrt[4]{x}\right)^3$

e) $3\sqrt[3]{-27x^2}\sqrt[6]{x^5}$

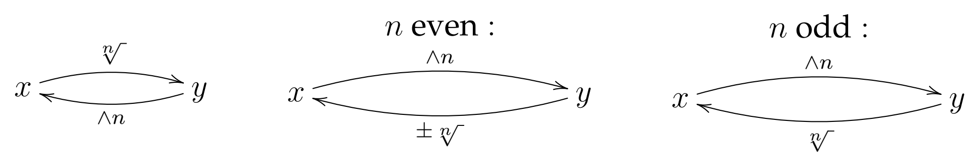
Inverse properties

The arrow diagrams we saw earlier in this section reveal the inverse relationship between power and root functions (since these functions undo each other). More specifically:

Theorem 3.40 (Inverse relationships between power and root functions)

- *The inverse of* $f(x) = \sqrt[n]{x}$ *is the function* $f^{-1}(x) = x^n$.
- *If* n *is even, the inverse of* $f(x) = x^n$ *is the multifunction* $f^{-1}(x) = \pm\sqrt[n]{x}$.
- *If* n *is odd, the inverse of* $f(x) = x^n$ *is the function* $f^{-1}(x) = \sqrt[n]{x}$.

Here are arrow diagrams that best describe these relationships:



In all these diagrams, we think of "undoing" the arrow that goes left-to-right by using the arrow that goes right-to-left.

§3.7 EXAMPLE 5

Compute the inverse of each function:

a) $f(x) = \sqrt{2x+5}$

b) $g(x) = (x-7)^8$

c) $h(x) = \dfrac{3}{4}x^3$

What happens without the $\pm$ on an even root?

§3.7 EXAMPLE 6

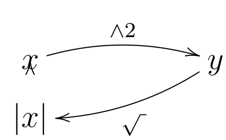
Construct a table of values for the function $f(x) = \sqrt{x^2}$, and use this table to identify f as another more common function:

x	-3	-2	-1	0	1	2	3
$f(x) = \sqrt{x^2}$							

Theorem 3.41 $\sqrt{x^2} = |x|$. *More generally,*

$$\sqrt[n]{x^n} = \begin{cases} x & \text{if } n \text{ is odd} \\ |x| & \text{if } n \text{ is even} \end{cases}.$$

Here's an arrow diagram encapsulating this idea:



3.8 Rational functions

RECALL

Negative exponents are treated as [   ]:

$x^{-3} =$ $\qquad$ $3x + x^{-2} =$

A function made up of integer exponents is called a *rational function*. Any such function can be combined as in the second example above to be put in a standard form:

Definition 3.42 *A* **rational function** $f : \mathbb{R} \to \mathbb{R}$ *is any function whose rule is the quotient of two polynomials.*

In other words, f is rational if it has a rule of the form

$$f(x) = \frac{a_0 + a_1x + a_2x^2 + \ldots + a_mx^m}{b_0 + b_1x + b_2x^2 + \ldots + b_nx^n}.$$

EXAMPLES: $f(x) = \dfrac{x^3 - 3x + 4}{2x^5 - 7x^2 - 3}$ $\qquad g(x) = x^{-3}$ $\qquad h(x) = \dfrac{\pi x^8 + 7x^3 - \sqrt{17}x}{x+4}$

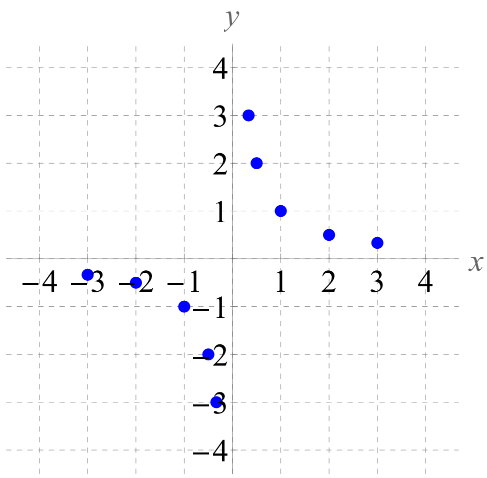
The reciprocal function

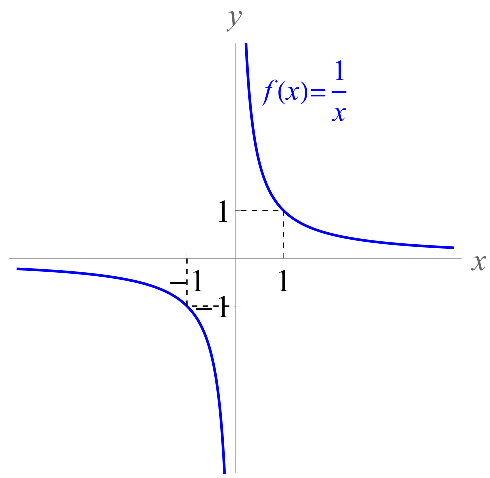
Here is the most important rational function:

Definition 3.43 *The* **reciprocal function** *is the function* $f(x) = \dfrac{1}{x} = x^{-1}$.

Let's graph this function using a table of values:

x	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3
$f(x) = \frac{1}{x}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-3	DNE	3	2	1	$\frac{1}{2}$	$\frac{1}{3}$





The shape of this graph, which has two disconnected pieces, is called a **hyperbola**.

Theorem 3.44 (Properties of the reciprocal function) *Let* $f(x) = \dfrac{1}{x}$. *Then:*

Domain *The domain of* f *is* $\mathbb{R} - \{0\}$.

Range: *The range of* f *is* $\mathbb{R} - \{0\}$.

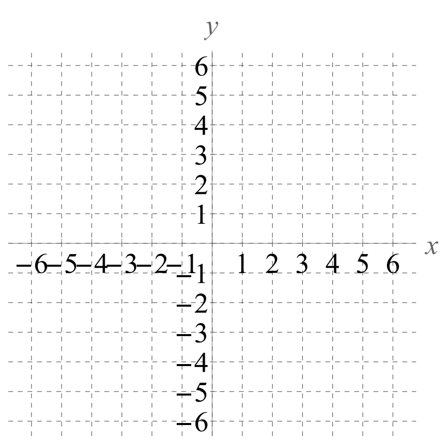
Symmetry: f *is odd.*

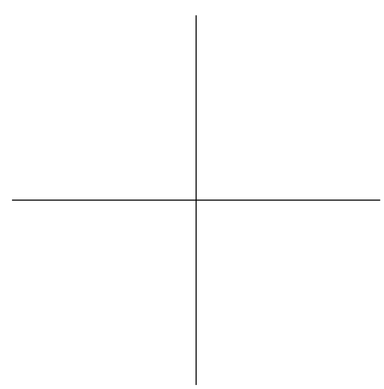
Inverse: f *is one-to-one, and the inverse of* $f(x) = \dfrac{1}{x}$ *is itself:* $f^{-1}(x) = \dfrac{1}{x}$.

§3.8 Example 1

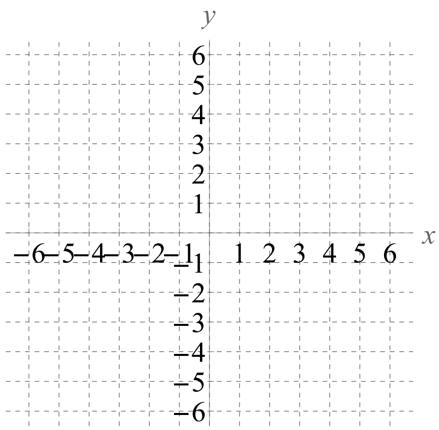
Diagram each function, then sketch its graph:

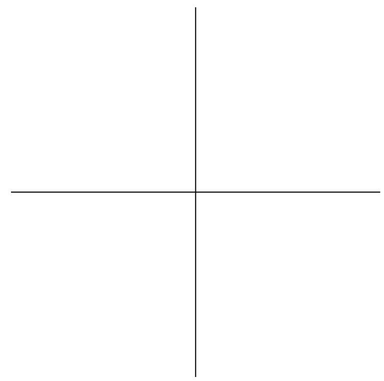
a) $f(x) = \dfrac{1}{x} + 4$



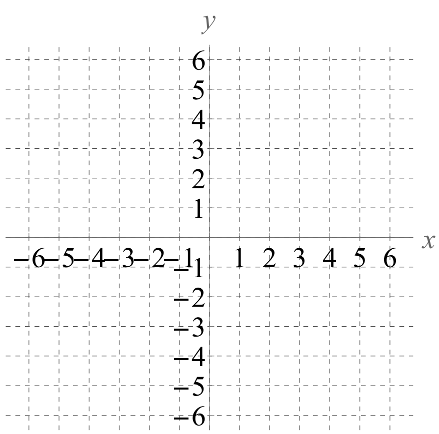


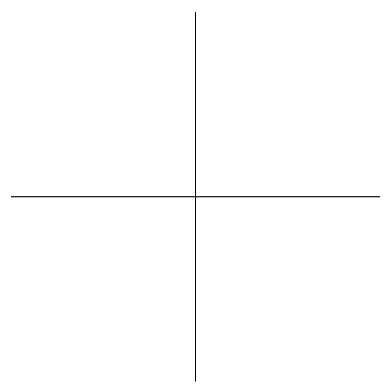
b) $g(x) = \dfrac{1}{x-5}$





c) $k(x) = -\dfrac{1}{x+2} + 3$

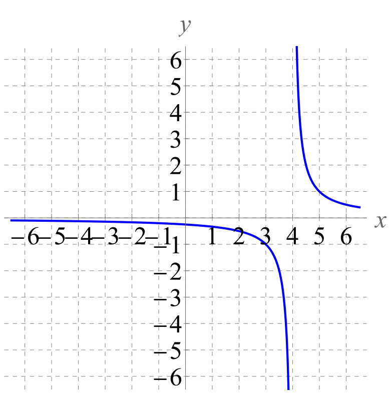




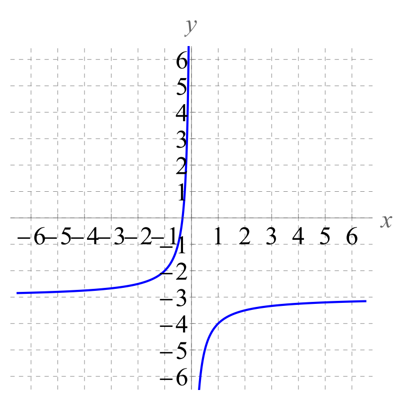
§3.8 EXAMPLE 2

Write a rule for each function whose graph is shown here:

a)

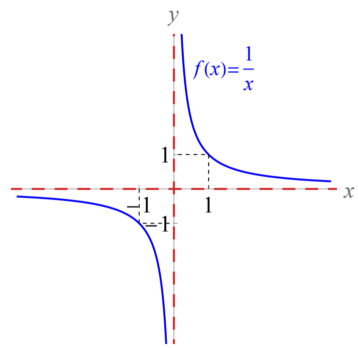


b)



Asymptotes

Let's take another look at the graph of $f(x) = \dfrac{1}{x}$:

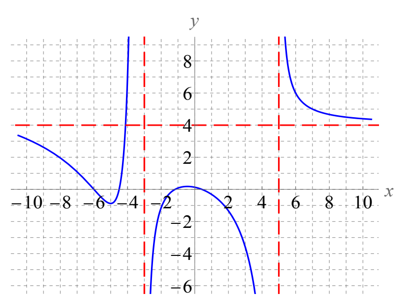


Notice that there are vertical and horizontal lines that the graph of f appears to "merge" into. These lines are called **vertical asymptotes (VA)** and **horizontal asymptotes (HA)** of f. The graph of f *never actually touches the vertical asymptotes*, but gets closer and closer to them.

In MATH 130, we want to be able to write equations of horizontal and vertical asymptotes given a graph:

§3.8 EXAMPLE 3

Write the equations of any horizontal and/or vertical asymptotes of this function:



Computing the asymptotes of a function from its formula is a task you learn how precisely to do in Calculus 1, but here are some general rules for rational functions that, if you remember them, can help you shortcut some calculus computations:

Theorem 3.45 (VA of rational functions) *Let $f : \mathbb{R} \to \mathbb{R}$ be a rational function. The VA of f are lines of the form $x = c$, where c makes the denominator of f zero but makes the numerator of f nonzero.*

A graph of a rational function **never** *touches/crosses any of its VA.*

Theorem 3.46 (HA of rational functions) *Let $f : \mathbb{R} \to \mathbb{R}$ be a rational function, i.e. has form*

$$f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \ldots + a_2 x^2 + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \ldots + b_2 x^2 + b_1 x + b_0}.$$

Then:

1. *If $m < n$ (i.e. largest power in numerator < largest power in denominator), then $y = 0$ is the HA of f.*
2. *If $m = n$ (i.e. largest powers in numerator and denominator are equal), then $y = \frac{a_m}{b_n}$ is the HA of f.*
3. *If $m > n$ (i.e. largest power in numerator > largest power in denominator), then f has no HA.*

In the first and second situations, the graph of f will appear to merge into the HA at the extreme left and extreme right edges of the graph.

It is possible for the graph of a rational function to cross its HA (but not its VA(s)) one or more times.

§3.8 EXAMPLE 4

Compute the horizontal and vertical asymptotes of each function:

a) $f(x) = \dfrac{x-3}{x^2 - 5x + 4}$

b) $g(x) = \dfrac{2x^2 - 18}{x^2 - 5x - 24}$

Closure properties

If you build more complicated functions out of rational functions, you will end up getting a rational function:

Theorem 3.47 *If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both rational functions, and if r is any constant, then the following functions are all rational:*

$$f + g \qquad f - g \qquad fg \qquad \frac{f}{g} \qquad rf \qquad f \circ g.$$

To see why this is, we need to learn how to manipulate rational expressions. These manipulations are based on how we manipulate fractions.

Manipulation of rational expressions

To **simplify** a rational expression, factor the top and bottom and cancel factors.

$$\frac{A}{C} \cdot \frac{B}{A} = \frac{B}{C}.$$

WARNING: In a fraction or rational function, you can only cancel **factors** (things being multiplied), **not terms** (things being added). *You must completely factor both the top and bottom of a rational expression before cancelling anything.*

ALWAYS FACTOR BEFORE YOU CANCEL!

Here are two typical examples of illegal "cancelling":

$$\frac{x^2+3}{x^2} = \qquad\qquad \frac{x^2+3}{x^2} =$$

§3.8 EXAMPLE 4

Simplify each expression:

a) $\dfrac{x^2 - 3x - 10}{x^2 - 12x + 35}$

b) $\dfrac{x^3 + 10x^2 + 24x}{x^4 + x^3 - 12x^2}$

To **multiply** rational expressions, multiply the numerators and multiply the denominators:

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}.$$

It is often easiest to factor A, B, C and D first and cancel, to make the multiplication easier.

§3.8 EXAMPLE 5

a) Compute and simplify $\dfrac{5x+15}{x-2} \cdot \dfrac{x^2-4}{10x-20}$.

b) Suppose $f(x) = \dfrac{x+1}{x-3}$ and $g(x) = \dfrac{x-5}{2x+7}$. Compute and simplify $(fg)(x)$.

To **divide** rational expressions, flip the divisor over and multiply:

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}$$

WARNING: $\dfrac{A/B}{C} = \dfrac{\frac{A}{B}}{C}$ means $\dfrac{A}{B} \div C = \dfrac{A}{B} \cdot \dfrac{1}{C}$;

$\dfrac{A}{B/C} = \dfrac{A}{\frac{B}{C}}$ means $A \div \dfrac{B}{C} = \dfrac{A}{1} \cdot \dfrac{C}{B}$.

So you should **never** write this:

§3.8 EXAMPLE 6

Compute and simplify $\frac{x^2-9}{x+4} \div \frac{x^2+5x+4}{x^2+x-6}$.

To **add** or **subtract** rational expressions, find a common denominator (by creatively multiplying each term by 1) and then add the numerators:

$$\frac{A}{B} \pm \frac{C}{B} = \frac{A \pm C}{B} \qquad \frac{A}{B} + \frac{C}{D} = \frac{AD}{BD} + \frac{CB}{BD} = \frac{AD+CB}{BD}.$$

Factoring helps find the smallest common denominator.

§3.8 EXAMPLE 7

a) Compute and simplify $\frac{x}{x+4} + \frac{3}{x-1}$.

b) Let $f(x) = \dfrac{1}{x^2 + 2x + 1}$ and $g(x) = \dfrac{2}{x^2 - 1}$. Compute and simplify $(f - g)(x)$.

Compound fractions

> **Definition 3.48** *A* **compound fraction** *is a fraction whose numerator and/or denominator themselves contain fractions.*

EXAMPLES: $\dfrac{\frac{3}{5} - \frac{2}{3}}{5 - \frac{1}{9}}$ $\qquad$ $\dfrac{\frac{4}{x+2} + \frac{3x-1}{x+5}}{\frac{x^2+3}{x-5} + \frac{x}{x^2+x-3}}$ $\qquad$ $\dfrac{\frac{3(x+h)}{3(x+h+2)} - \frac{3x}{3x+2}}{h}$

> To **simplify a compound fraction**, creatively multiply through by 1, where the "1" is a fraction containing all the "small denominators" of the compound fraction.

§3.8 EXAMPLE 8

a) Simplify $\dfrac{\frac{3}{x} - 2}{3 + \frac{1}{x+1}}$.

b) Simplify $\dfrac{\frac{2}{x+h}-\frac{2}{x}}{h}$.

c) Let $f(x)=\dfrac{x+1}{x-3}$ and $g(x)=\dfrac{x+2}{x-1}$. Compute and simplify $(g\circ f)(x)$.

d) Let $F(x) = (x+4)^{-1}$ and $H(x) = x+3$. Compute and simplify $(F \circ H \circ F)(x)$.

Solution: Notice first that $F(x) = (x+4)^{-1} = \dfrac{1}{x+4}$. Therefore

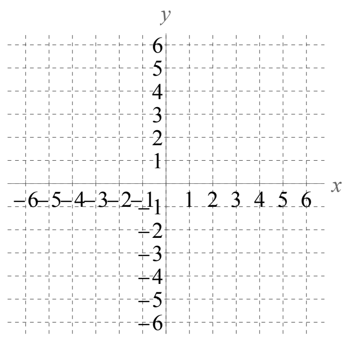
$$\begin{aligned}
(F \circ H \circ F)(x) &= F(H(F(x))) \\
&= F\left(H\left(\frac{1}{x+4}\right)\right) \\
&= F\left(\frac{1}{x+4} + 3\right) \\
&= \frac{1}{\frac{1}{x+4} + 3 + 4} \\
&= \frac{1}{\frac{1}{x+4} + 7} \\
&= \frac{1}{\left(\frac{1}{x+4} + 7\right)} \cdot \frac{(x+4)}{(x+4)} \\
&= \frac{x+4}{1 + 7(x+4)} \\
&= \frac{x+4}{1 + 7x + 28} \\
&= \boxed{\frac{x+4}{7x+29}}.
\end{aligned}$$

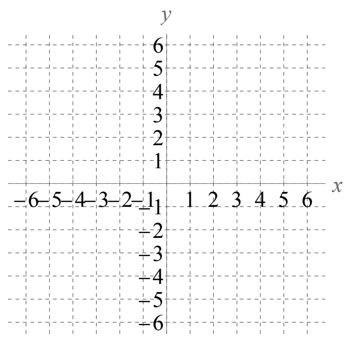
3.9 Absolute value function

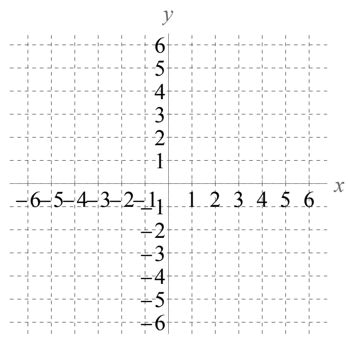
Definition 3.49 *The* **absolute value** *function is the function* $\mathbb{R} \to \mathbb{R}$ *whose rule is*

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

We can sketch the graph of $|x|$ from its piecewise definition:



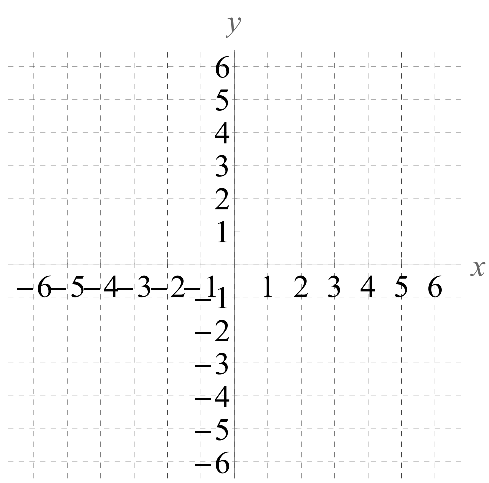


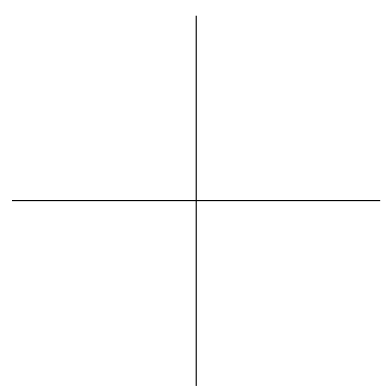


§3.9 EXAMPLE 1

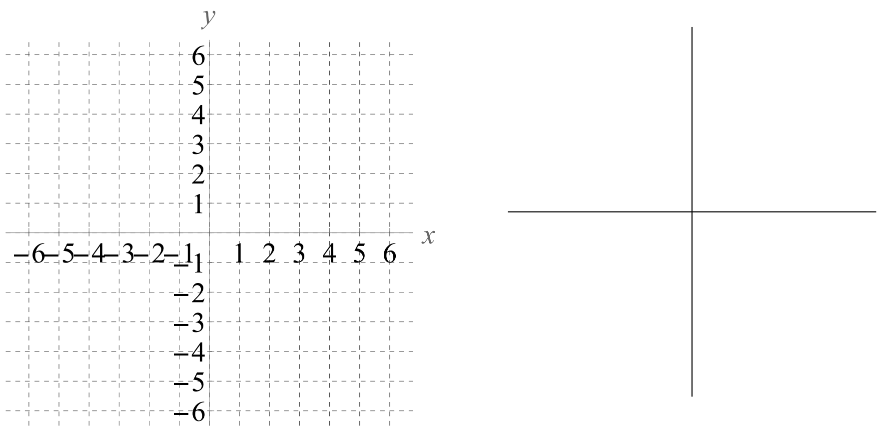
Diagram each function, then sketch its graph:

a) $F(t) = |t - 4|$

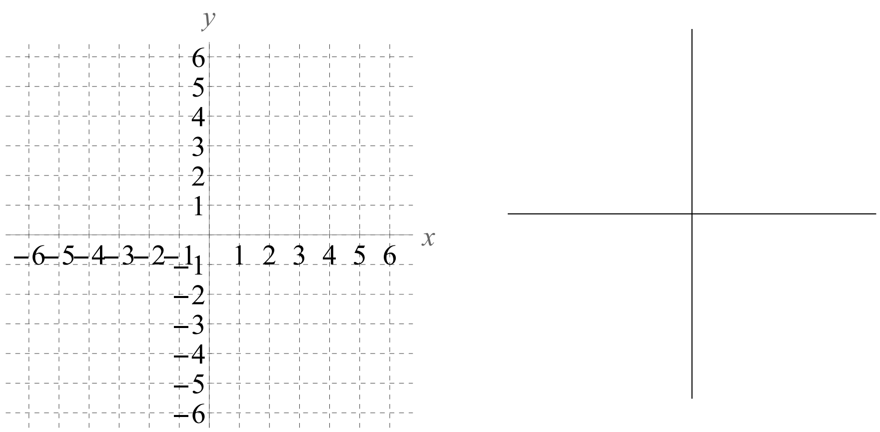




b) $g(x) = |x+3| + 1$



c) $K(x) = -2|x|$



Theorem 3.50 (Properties of absolute value) *Let* $f(x) = |x|$.

Domain *The domain of* f *is* $\mathbb{R}$.

Range: *The range of* f *is* $[0, \infty)$.

Symmetry: f *is even.*

Inverse: *The inverse of* $f(x) = |x|$ *is the multifunction* $f^{-1}(x) = \pm x$.

Multiplicative: $|xy| = |x|\,|y|$.

Relationship with square and square root: $|x| = \sqrt{x^2}$.

Distance computation: $|x| =$ *the distance from* x *to* 0 *on a number line.*

Signum function

Here is another function used to illustrate some calculus concepts:

Definition 3.51 *The* **signum function** *is the function* $f(x) = \dfrac{|x|}{x}$.

OBSERVE

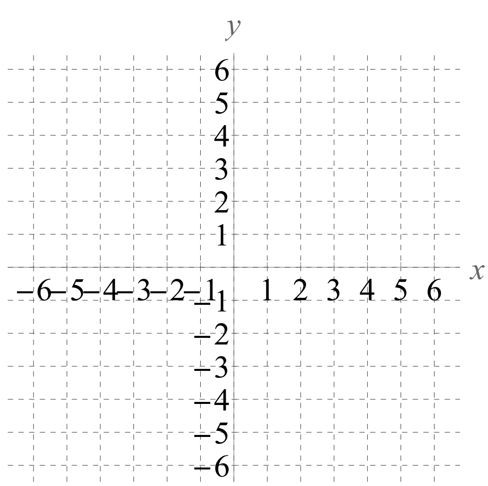
Let f be the signum function.

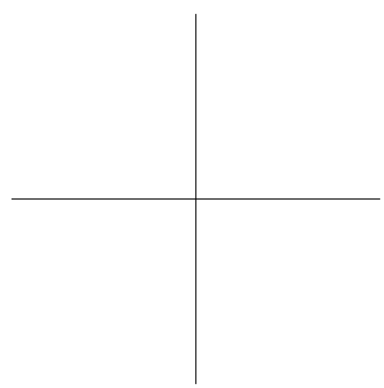
- if $x > 0$, $f(x) = \dfrac{|x|}{x} =$
- if $x < 0$, $f(x) = \dfrac{|x|}{x} =$

so the signum function also has the piecewise definition

$$\frac{|x|}{x} = \begin{cases} & \text{if } x < 0 \\ & \text{if } x > 0 \end{cases}$$

which means its graph looks like this:





3.10 Semicircles

RECALL (FROM CHAPTER 1)

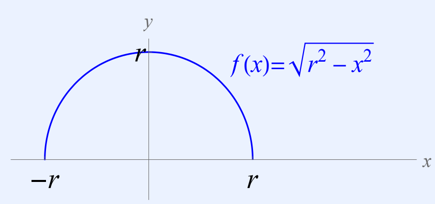
The equation of a circle of radius r centered at $(0, 0)$ is

QUESTION

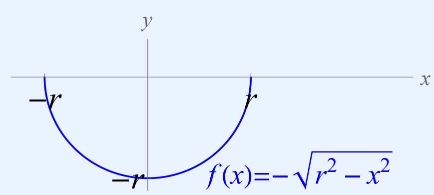
Is a circle the graph of a function $y = f(x)$? Why or why not?

To summarize:

Theorem 3.52 (Semicircles) *The graph of $f(x) = \sqrt{r^2 - x^2}$ is the top half of a circle of radius r centered at the origin:*



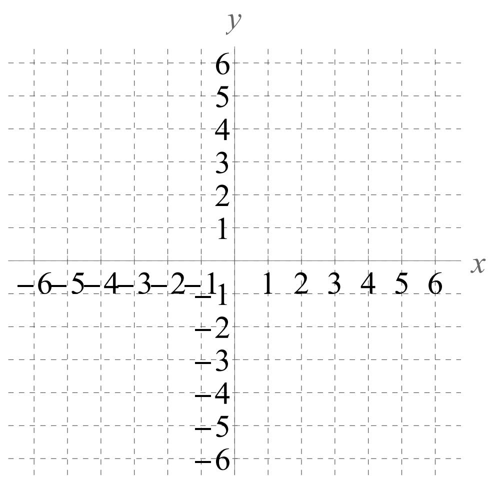
The graph of $f(x) = -\sqrt{r^2 - x^2}$ is the bottom half of the same circle:

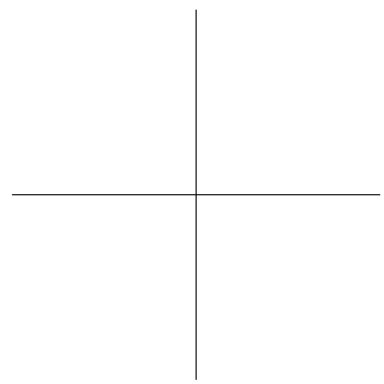


§3.10 EXAMPLE 1

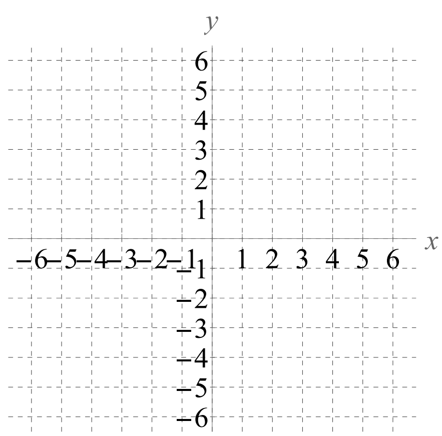
Sketch a graph of each function:

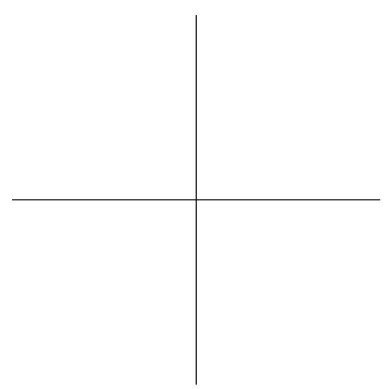
a) $f(x) = \sqrt{25 - x^2}$



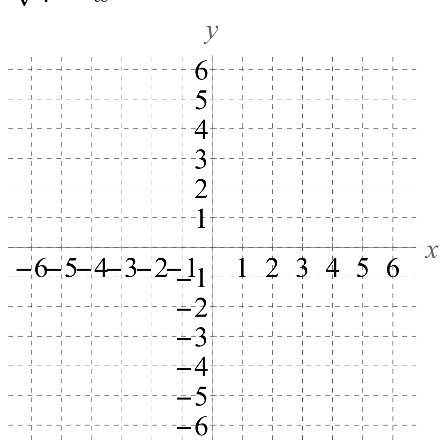


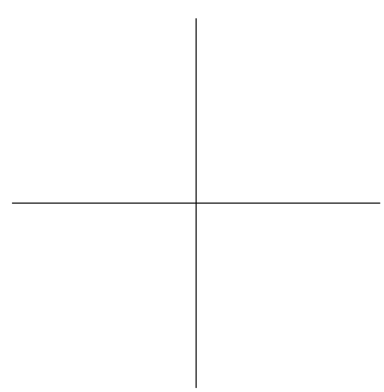
b) $g(x) = -\sqrt{9 - x^2}$



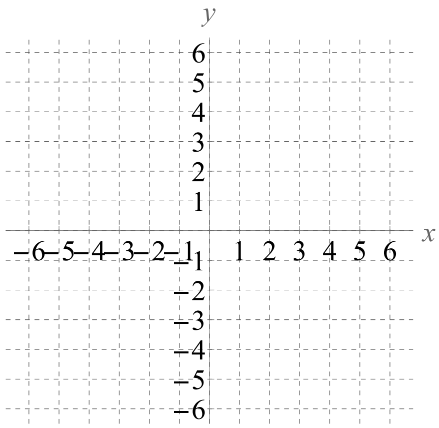


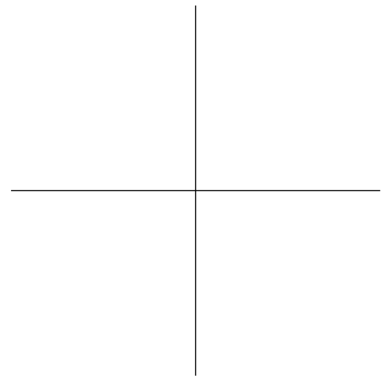
c) $h(x) = \sqrt{7 - x^2}$



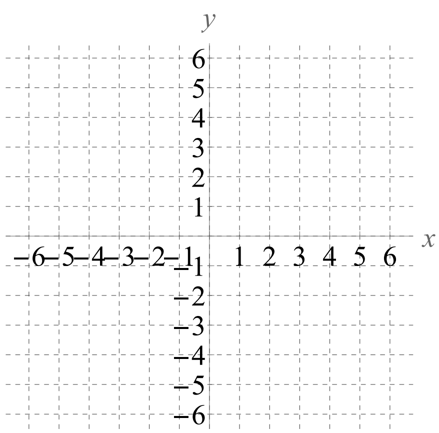


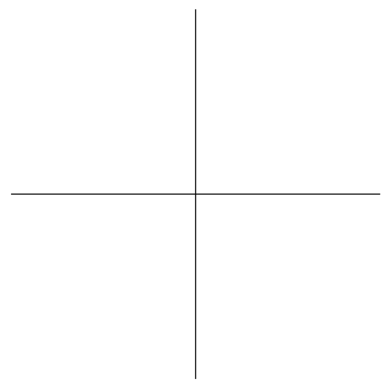
d) $j(x) = \sqrt{16 - (x-2)^2}$





e) $j(x) = \sqrt{4 - x^2} - 3$

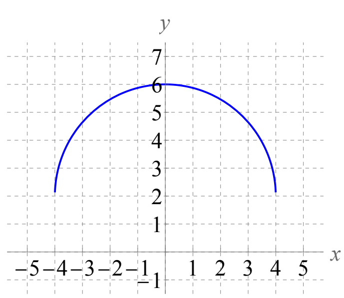




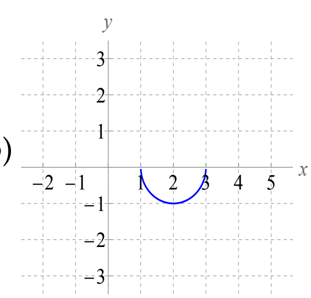
§3.10 EXAMPLE 2

Write a rule for each function whose graph is shown here:

a)



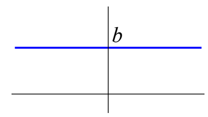
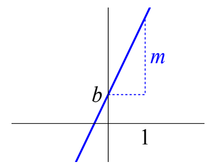
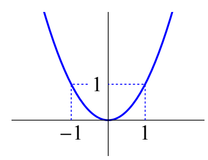
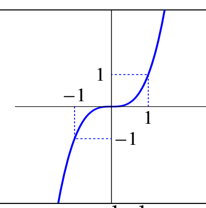
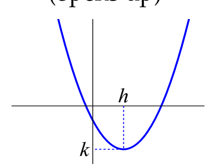
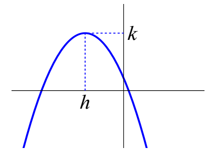
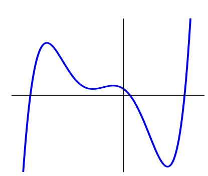
b)



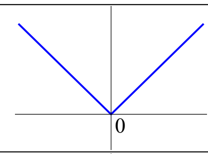
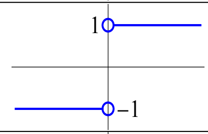
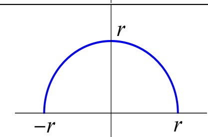
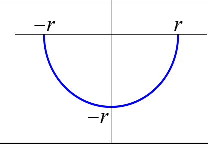
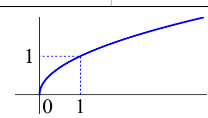
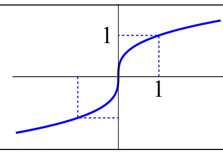
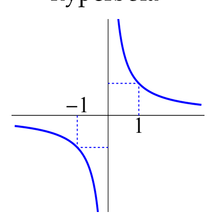
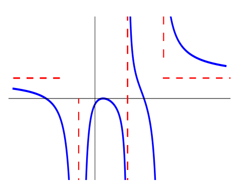
3.11 Summary

A summary of the relevant information for the algebraic functions we studied in Chapter 3 can be found in the charts on the next two pages:

Polynomials

	Rule	Tails	Graph	Domain Range Symmetry Inverse
Constants (zero change)	$f(x) = b$	← →	horizontal line  	**D:** $\mathbb{R}$ **R:** $\{b\}$ **Sym:** even
Linear (output is proportional to input; rate of change is constant)	$f(x) = mx + b$ $(m \neq 0)$	$m > 0$: ↙ ↗ $m < 0$: ↖ ↘	line with slope m and y-int $(0, b)$  	**D:** $\mathbb{R}$ **R:** $\mathbb{R}$ **Sym:** none if $b \neq 0$; odd if $b = 0$ $f^{-1}(x) = \frac{x-b}{m}$
Even powers	$f(x) = x^n$ (n even)	↖ ↗	 	**D:** $\mathbb{R}$ **R:** $[0, \infty)$ **Sym:** even $f^{-1}(x) = \pm\sqrt[n]{x}$
Odd powers	$f(x) = x^n$ (n odd)	↙ ↗	 	**D:** $\mathbb{R}$ **R:** $\mathbb{R}$ **Sym:** odd $f^{-1}(x) = \sqrt[n]{x}$
Quadratic (rate of change is linear)	$f(x) = ax^2 + bx + c$ $= a(x-h)^2 + k$ $(a > 0)$ vertex (h, k) where $h = -\frac{b}{2a}, k = f(h)$	↖ ↗	parabola (opens up)  	**D:** $\mathbb{R}$ **R:** $[k, \infty)$ **Sym:** about $x = h$
	$f(x) = ax^2 + bx + c$ $= a(x-h)^2 + k$ $(a < 0)$ vertex (h, k) where $h = -\frac{b}{2a}, k = f(h)$	↙ ↘	parabola (opens down)  	**D:** $\mathbb{R}$ **R:** $(-\infty, k]$ **Sym:** about $x = h$
Generic	$f(x) =$ $a_0 + a_1x + \ldots + a_nx^n$ $(a_n \neq 0)$ $(n = deg(f))$	n odd, $a_n > 0$: ↙ ↗ n odd, $a_n < 0$: ↖ ↘ n even, $a_n > 0$: ↖ ↗ n even, $a_n < 0$: ↙ ↘		

Non-polynomial algebraic functions

<table>
<tr><th colspan="2"></th><th>Rule</th><th>Graph</th><th>Domain
Range
Symmetry
Asymptote(s)</th><th>Inverse</th></tr>
<tr><td colspan="2">Absolute value</td><td>$f(x) = |x|$</td><td>
</td><td>D: $\mathbb{R}$
R: $[0, \infty)$
Sym: even
no asymptotes</td><td>$f^{-1}(x) = \pm x$</td></tr>
<tr><td colspan="2">Signum</td><td>$f(x) = \frac{|x|}{x}$</td><td>
</td><td>D: $\mathbb{R} - \{0\}$
R: $\{-1, 1\}$
Sym: odd
no asymptotes</td><td></td></tr>
<tr><td rowspan="2">Semicircles</td><td>Top half</td><td>$f(x) = \sqrt{r^2 - x^2}$</td><td>
</td><td>D: $[-r, r]$
R: $[0, r]$
Sym: even
no asymptotes</td><td rowspan="2"></td></tr>
<tr><td>Bottom half</td><td>$f(x) = -\sqrt{r^2 - x^2}$</td><td>
</td><td>D: $[-r, r]$
R: $[-r, 0]$
Sym: even
no asymptotes</td></tr>
<tr><td rowspan="2">Roots</td><td>Even roots</td><td>$f(x) = \sqrt[n]{x} = x^{1/n}$
(n even)</td><td>
</td><td>D: $[0, \infty)$
R: $[0, \infty)$
no symmetry
no asymptotes</td><td rowspan="2">$f^{-1}(x) = x^n$</td></tr>
<tr><td>Odd roots</td><td>$f(x) = \sqrt[n]{x} = x^{1/n}$
(n odd)</td><td>
</td><td>D: $\mathbb{R}$
R: $\mathbb{R}$
Sym: odd
no asymptotes</td></tr>
<tr><td rowspan="2">Rational</td><td>Reciprocal function</td><td>$f(x) = \frac{1}{x}$
$= x^{-1}$</td><td>hyperbola

</td><td>D: $\mathbb{R} - \{0\}$
R: $\mathbb{R} - \{0\}$
Sym: odd
HA: $y = 0$
VA: $x = 0$</td><td>$f^{-1}(x) = \frac{1}{x}$</td></tr>
<tr><td>Generic</td><td>$f(x) = \frac{\text{polynomial}}{\text{polynomial}}$</td><td></td><td>HA:
$deg(\text{top}) < deg(\text{bot})$:
$y = 0$
$deg(\text{top}) = deg(\text{bot})$:
$y = \frac{\text{LC(top)}}{\text{LC(bot)}}$
$deg(\text{top}) > deg(\text{bot})$:
none
VA: x-values where
bottom $= 0$, top $\neq 0$</td><td></td></tr>
</table>

3.12 Chapter 3 Homework

Exercises from Section 3.1

In Exercises 1-6, you are given a table of values for some function $\mathbb{R} \to \mathbb{R}$. Given that table of values:

a) Determine if the table of values represents a linear function.
b) If the table represents a linear function, compute the rate of change of the function.
c) If the table represents a linear function, compute the output of the function when its input is 9.
d) If the table represents a linear function, compute the output of the function when the input is -6.
e) If the table represents a linear function, determine how much the output changes when the input is increased by 8.

1.

x	0	1	2	3	4	5	6
$f(x)$	3	9	15	21	27	33	39

2.

x	1	2	3	4	5
$G(x)$	2	4	8	16	32

3.

x	0	4	8	12	16	24	28
$F(x)$	23	21	19	17	15	11	9

4.

x	-3	-1	1	3	5	7
$g(x)$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{8}{9}$

5.

x	$-\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2
$H(x)$	$\frac{5}{6}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{5}{6}$

6.

x	2	4	6	8	10	16
$h(x)$	0	3	6	9	12	21

7. Determine whether you think it is reasonable for the given function to be linear:

   a) The function P, where $P(x)$ is the price of purchasing x pounds of potatoes at Meijer.

b) The function p, where $p(x)$ is the price of purchasing x boxes of popcorn under the situation where the popcorn is "buy one, get one free"

c) The function P, where $P(x)$ is the price of staying x days at a hotel.

d) The function g, where $g(x)$ is your grade on the next exam if you study x minutes for it.

e) The function d, where $d(t)$ is the number of miles you have travelled after t minutes while driving around an urban city.

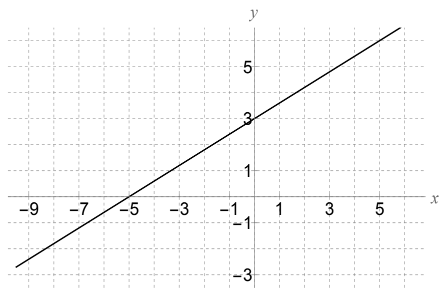
f) The function D, where $x(t)$ is the number of miles you have travelled after t minutes while driving on an expressway with your cruise-control on.

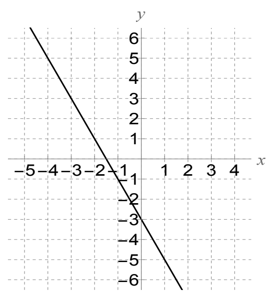
g) The function c, where $c(x)$ is the number of centimeters in x inches.

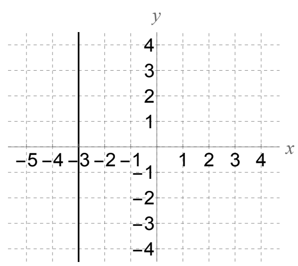
h) The function v, where $v(x)$ is the value of a car x years after it is driven off the lot brand-new.

In Exercises 8-22, compute the slope of the line described in that exercise:

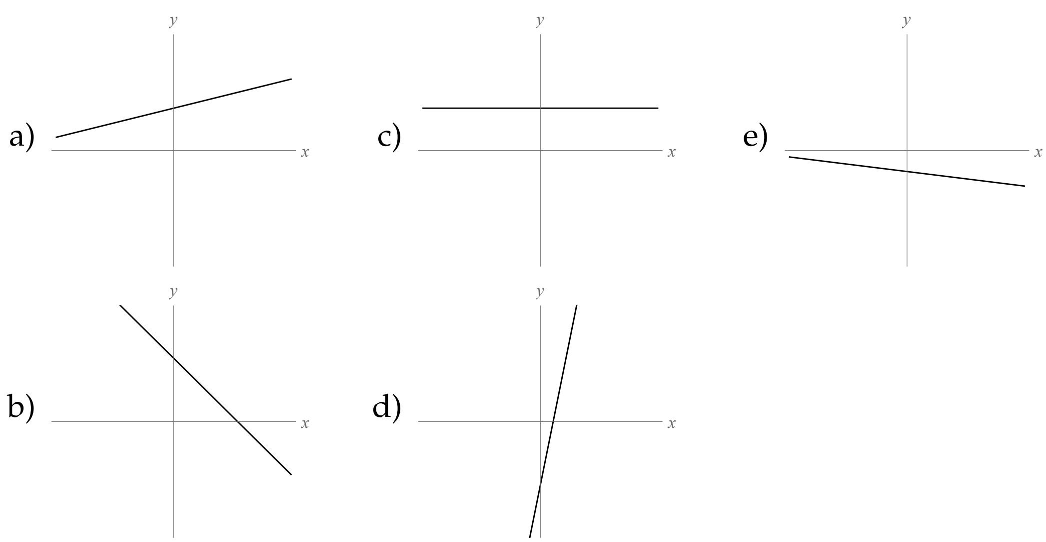
8. the line passes through $(-3, 7)$ and $(4, 11)$
9. the line passes through $(-2, -5)$ and $(-5, -3)$
10. the line passes through $\left(\frac{11}{7}, \frac{3}{8}\right)$ and $(3, -1)$
11. the line passes through (t, t^2) and (r, r^3)
12. the line passes through $(3a + b, 5a - 2c)$ and $(4b - d, 6c + 5d)$
13. for every 8 units you move to the right, the line goes up 3 units
14. for every 4 units you move to the left, the line goes up 5 units
15. for every 2 units you move to the right, the line goes down 14 units
16. the line is horizontal
17. the line makes an angle of $\frac{\pi}{4}$ with the horizontal
18. the line makes an angle of $\frac{5\pi}{6}$ with the horizontal
19. the line makes an angle of 2 radians with the horizontal

20. the line has this graph: 

21. the line has this graph: 

22. the line has this graph: 

23. Estimate the slope of each line graphed here (assuming that the scales on the x-and y-axes are the same):



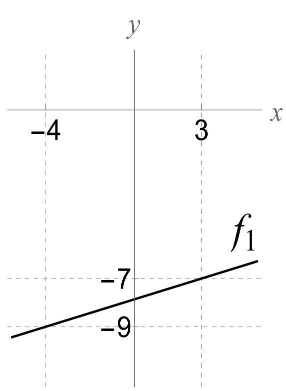
Answers

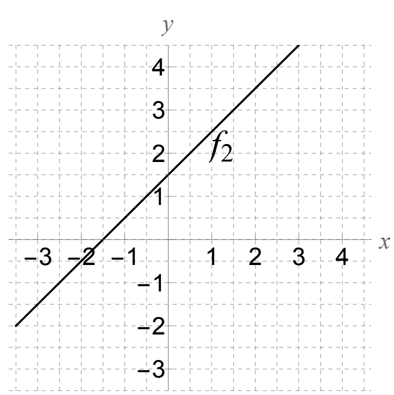
1. a) f is a linear function.
   b) 6
   c) 57
   d) -33
   e) 48
2. G is not a linear function.
3. F is not a linear function (notice that the x-values skip from 16 to 24, but the y-values don't skip accordingly).
4. a) g is a linear function.
   b) $\frac{1}{18}$
   c) 1
   d) $\frac{1}{6}$
   e) $\frac{4}{9}$
5. a) H is a linear function.
   b) $-\frac{2}{3}$
   c) $-\frac{11}{2}$
   d) $\frac{9}{2}$
   e) $-\frac{16}{3}$
6. a) h is a linear function.
   b) $\frac{3}{2}$
   c) $\frac{21}{2}$
   d) -12
   e) 12
7. The functions in (a), (c), (f) and (g) are (probably) linear; the functions in (b), (d), (e) and (h) aren't linear.
8. $\frac{4}{7}$
9. $-\frac{2}{3}$
10. $-\frac{77}{80}$
11. $\frac{r^3 - t^2}{r - t}$
12. $\frac{5a - 8c - 5d}{3a - 3b + d}$
13. $\frac{3}{8}$
14. $-\frac{5}{4}$
15. -7
16. 0
17. 1
18. $-\frac{1}{\sqrt{3}}$
19. $\tan 2$
20. $\frac{3}{5}$
21. -2
22. DNE
23. a) $m \approx \frac{1}{3}$
    b) $m \approx -1$
    c) $m = 0$
    d) $m \approx 3$
    e) $m \approx -\frac{1}{5}$

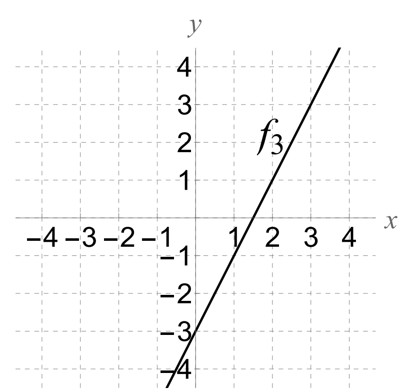
Exercises from Section 3.2

In Exercises 1-14, write the equation of the line with the given properties (note: Exercises 13 and 14 require material covered near the end of Section 3.2):

1. the line passes through $(8, 3)$ and has slope 2
2. the line passes through $\left(-\frac{17}{6}, \frac{25}{8}\right)$ and has slope $-\frac{13}{4}$
3. the line passes through $(\sqrt{5}, 3\sqrt{11})$ and has slope 0
4. the line passes through $(4, -2)$ and has a rate of change of $\frac{1}{4}$
5. the line has x-intercept $(3, 0)$ and y-intercept $(0, -6)$
6. the line passes through $(8, -5)$ and $(-7, 0)$
7. the line passes through $(5, -3)$ and $(5, -1)$
8. the line passes through $(\sqrt{5}, \sqrt{11})$ and $(3\sqrt{5}, \sqrt{19})$
9. the line passes through $\left(-\frac{5}{3}, \frac{3}{8}\right)$ and $\left(\frac{11}{6}, -\frac{5}{4}\right)$
10. the line f_1 graphed below Exercise 11, at left
11. the line f_2 graphed below, in the middle







12. the line f_3 graphed above, at right
13. the line passes through $(-7, -1)$ and is parallel to the line $3x - 4y = 19$
14. the line passes through $(3, -2)$ and is perpendicular to the line $y = 7 - 6(x + 4)$

In Exercises 15-20, we will combine the methods of writing equations of lines with questions related to material about functions we encountered in Chapter 2. Let's see how well you can combine this new material with what we learned in Chapter 2!

15. Suppose f and g are functions with constant rate of change. f passes through $(2,8)$ and $(8,4)$; g passes through $(1,6)$ and $(5,-2)$. Compute and simplify the rule for $6f+2g$.
16. Suppose F is the linear function passing through $(3,-5)$ with slope 4, and G is the linear function passing through $(-1,1)$ with slope $\frac{1}{2}$. Compute and simplify the rule for $F \circ G$.
17. Let l be the linear function with rate of change 4 passing through $(1,3)$. Determine the x-coordinate of the point where the graph of l has height 33.
18. Find the coordinates of the point where the lines F and G intersect, where F and G are as in Exercise 16.
19. Determine the x- and y-intercepts of the function F described in Exercise 16.
20. Compute the net change in the function G described in Exercise 16, from $x=-11$ to $x=5$.

In Exercises 21-22, determine whether or not the given points all lie on the same line:

21. $(-3,0)$, $(9,8)$ and $(12,10)$
22. $\left(\frac{1}{2},-2\right)$, $(4,-9)$ and $\left(\frac{17}{4},-\frac{19}{2}\right)$
23. Write the slope-intercept equation of the line $y=5+2(x-3)$.
24. Write a standard equation of the line $y=\frac{2}{5}\left(x-\frac{1}{4}\right)+\frac{3}{10}$.
25. Write the line $5x-10y=14$ in slope-intercept form.
26. Write the equation $y=3x$ in standard form.

In Exercises 27-37, sketch a graph of each line:

27. $y=\frac{3}{4}x$
28. $L(x)=3x-5$
29. $f(x)=-\frac{1}{4}x+3$
30. $\lambda(x)=2$

31. $y = -x$

32. $y = 2 + 1(x - 5)$

33. $y = -1 - \frac{2}{3}\left(x - \frac{1}{2}\right)$

34. $y = -8 - 2(x - 3)$

35. $4x - 3y = 24$

36. $x = -3$

37. $2x + 5y = 19$

38. Sketch a graph of the function $f(x) = \begin{cases} 2x - 5 & x < 3 \\ 7 - \frac{2}{3}x & x \geq 3 \end{cases}$.

39. A drug is administered intravenously to a patient at a constant rate of $\frac{3}{100}$ cc/min.

   a) How much of the drug will the patient receive in 1 hour?

   b) How long (in minutes) will it take for the patient to receive $\frac{39}{100}$ cc of the drug?

40. A taxi charges a customer $\$12$ to drive a passenger 20 miles and charges $\$13.50$ to drive a passenger 30 miles. Assume that the taxi fare f is a linear function of the distance travelled.

   a) What is the slope of f?

   b) Interpret the meaning of the slope in the context of this problem.

   c) Write a rule for f.

   d) Write the rule for f in slope-intercept form.

   e) What is the y-intercept of f?

   f) Interpret the meaning of the y-intercept in the context of this problem.

   g) Compute the fare that would be charged for a 8 mile ride.

   h) How far would the taxi be willing to drive you, if you have $\$50$?

41. Dr. McClendon decides to walk home from campus, which is 3 miles from his house. 10 minutes after leaving campus, he is $\frac{7}{3}$ mi from his house. Assuming Dr. McClendon walks at a constant rate, let $g(t)$ be the distance Dr. McClendon is from his house, t minutes after leaving campus.

   a) Write a rule for g.

   b) How far is Dr. McClendon from his house after 18 minutes of walking?

   c) How long will it take him to get home?

d) How would you interpret what is being asked for in part (c), in terms of the graph of g?

42. Suppose that a 45 watt incandescent light bulb emits 450 lumens of light (a lumen is a unit of measure of brightness), and suppose that a 75 watt incandescent light bulb emits 1100 lumens of light. Assuming that the relationship between the energy of the light bulb (energy is measured in watts) and its brightness is linear,

    a) Write a rule for the function that computes the brightness of the light bulb as a function of its energy.
    b) How many lumens of light will a 65 watt light bulb emit?
    c) Write a rule for the function that computes the energy of the light bulb as a function of its brightness.
    d) How many watts will a light bulb use, if it emits 800 lumens of light?
    e) What is the relationship between the functions you wrote down in parts (a) and (c)?

    P.S.: in practice the relationship between watts and lumens isn't actually linear. If I used the actual relationship here, the question would be too hard.

43. Determine if each given pair of lines is parallel, perpendicular, or neither:

    a) $f(x) = 3x - 7$ and $g(x) = \frac{1}{3}x + 2$
    b) $16x - 12y = 27$ and $15y - 20x = 11$
    c) $y = 3 + \frac{2}{5}(x - 1)$ and $5x + 2y = -11$
    d) $x = 3$ and $x = -2$
    e) $y = x$ and $y = -x$

44. Assume that the slope of line Λ is -3.

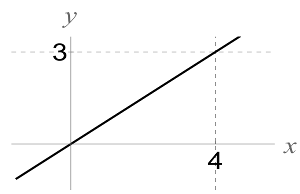
    a) If Φ is a line perpendicular to Λ, what is the slope of Φ?
    b) If Γ is a line parallel to Λ, what is the slope of Γ?

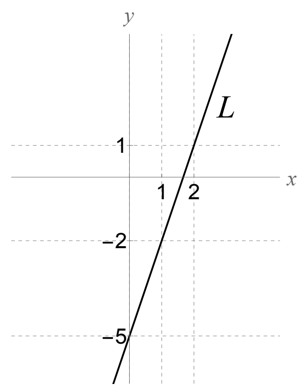
45. Write down a rule for the inverse of each given function:

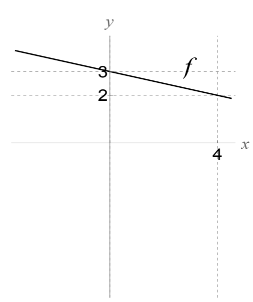
    a) $G(x) = \frac{3}{4}x - \frac{1}{4}$
    b) $H(x) = 7x - 5$
    c) $K(x) = 2 - x$

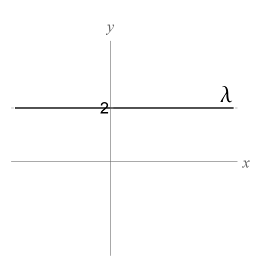
Answers

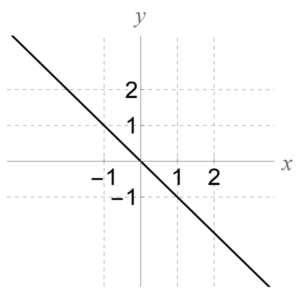
Keep in mind that when writing the equation of a line, answers can vary.

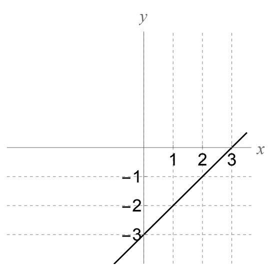
1. $y = 3 + 2(x - 8)$
2. $y = \frac{25}{8} - \frac{13}{4}\left(x + \frac{17}{6}\right)$
3. $y = 3\sqrt{11})$
4. $y = -2 + \frac{1}{4}(x - 4)$
5. $y = 2x - 6$
6. $y = -\frac{1}{3}(x + 7)$
7. $x = 5$
8. $y = \sqrt{11} + \frac{\sqrt{19} - \sqrt{11}}{2\sqrt{5}}\left(x - \sqrt{5}\right)$
9. $y = -\frac{5}{4} - \frac{13}{28}\left(x - \frac{11}{6}\right)$
10. $f_1(x) = -7 + \frac{2}{7}(x - 3)$
11. $f_2(x) = x + \frac{3}{2}$
12. $f_3(x) = 2x - 3$
13. $y = -1 + \frac{3}{4}(x + 7)$
14. $y = -2 + \frac{1}{6}(x - 3)$
15. $(6f + 2g)(x) = -8x + 72.$
16. $F \circ G(x) = 2x - 11.$
17. $\frac{17}{2}$
18. $\left(\frac{37}{7}, \frac{29}{7}\right)$
19. x-int: $\left(\frac{17}{4}, 0\right)$ and y-int: $(0, -17)$
20. 8
21. These are all on the same line (that line is $y = \frac{2}{3}x + 2$).
22. These are all on the same line (that line is $y = -2x - 1$).
23. $y = 2x - 1$
24. $-2x + 5y = 1$
25. $y = \frac{1}{2}x - \frac{7}{5}$
26. $3x - y = 0$
27. 

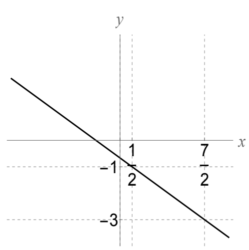
28. 

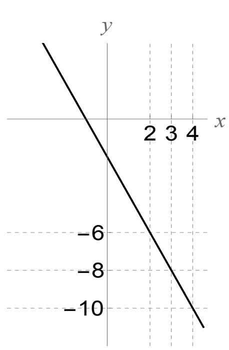
29. 

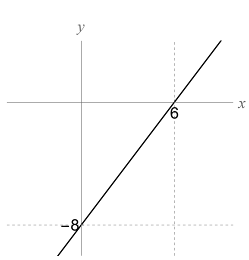
30. 

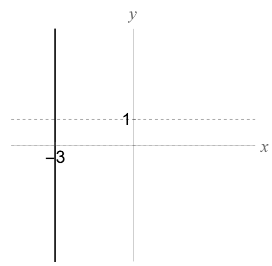
31. 

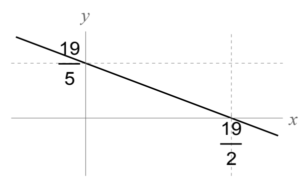
32. 

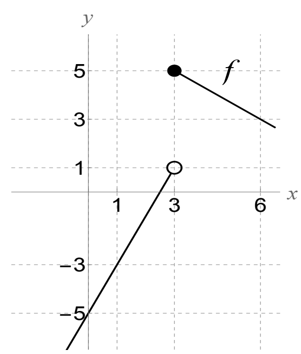
33. 

34. 

35. 

36. 

37. 

38. 

39. a) $\frac{9}{5}$ cc
   b) 13 min

40. a) $\frac{3}{20}$ dollars/mi $= \$0.15/\text{mi}$.
   b) For every additional mile driven, the taxi charges an additional $\$0.15$.
   c) $f(x) = 12 + .15(x - 20)$
   d) $f(x) = .15x + 9$
   e) $(0, 9)$
   f) The taxi driver charges a flat fee of $\$9$ on top of the $\$0.15$ per mile.
   g) $\$10.20$
   h) $\frac{820}{3}$ miles.

41. a) $g(t) = -\frac{1}{15}t + 3$
   b) $\frac{9}{5}$ mi.

c) 45 min

d) The x-intercept of g is $(45, 0)$.

42. a) $B(x) = 450 + \frac{65}{3}(x - 45)$

b) $\frac{2650}{3}$ lumens

c) $E(x) = 45 + \frac{3}{65}(x - 450)$.

d) $\frac{795}{13}$ W

e) B and E are inverses: $E = B^{-1}$.

43. a) neither

b) parallel

c) perpendicular

d) parallel

e) perpendicular

44. a) $\frac{1}{3}$

b) -3

45. a) $G^{-1}(x) = \frac{4x+1}{3}$

b) $H^{-1}(x) = \frac{x+5}{7}$

c) $K^{-1}(x) = 2 - x$

Exercises from Section 3.3

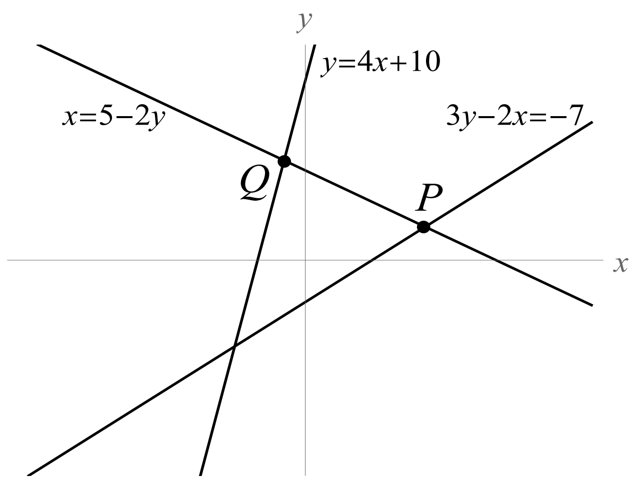
In Exercises 1-6, solve each equation for the indicated variable:

1. $8(4 - 3x) + 2 = 7(2x + 1) - 5$
2. $\frac{2}{7}\theta + \frac{3}{4} = \frac{11}{2}$
3. $\frac{3}{4} - \frac{2}{3}w = \frac{1}{2}\left(\frac{7}{8} + \frac{5}{3}w\right)$
4. $3\sqrt{2} + 2\sqrt{3}\,x = 5\sqrt{11}$
5. $\sqrt[3]{7}\,x = 7$
6. $2 - \frac{3}{10}x = -\frac{5}{2} + \frac{4}{5}x$

In Exercises 7-12, solve each system of equations:

7. $\begin{cases} x &= y + 2 \\ 3y - 2x &= 15 \end{cases}$
8. $\begin{cases} x + 3y &= 12 \\ 2x - y &= 5 \end{cases}$
9. $\begin{cases} 2s + 7t &= -3 \\ s + 4t &= -2 \end{cases}$
10. $\begin{cases} 3x - 7y &= 4 \\ 9x - 21y &= 10 \end{cases}$
11. $\begin{cases} \frac{1}{2}x + 2y &= \frac{13}{8} \\ 3x - \frac{7}{3}y &= -\frac{7}{6} \end{cases}$
12. $\begin{cases} \frac{3}{4}q + \frac{3}{5}r &= -\frac{1}{10} \\ \frac{3}{2}q - \frac{3}{4}r &= \frac{53}{8} \end{cases}$
13. Find the point of intersection of the two lines $3x + 2y = 16$ and $7x + y = 19$.
14. Find the point of intersection of the two lines $3x - y = 23$ and $4x = 48 - 3y$.
15. Find the coordinates of the point P indicated in the picture below.

16. Find the coordinates of the point Q indicated in the picture below.



Answers

1. $x = \frac{16}{19}$
2. $t = \frac{133}{8}$
3. $w = \frac{5}{24}$
4. $x = \frac{5\sqrt{11} - 3\sqrt{2}}{2\sqrt{3}}$
5. $x = 7^{2/3}$
6. $x = \frac{45}{11}$
7. $(21, 19)$
8. $\left(\frac{27}{7}, \frac{19}{7}\right)$
9. $(2, -1)$
10. DNE
11. $\left(\frac{35}{172}, \frac{131}{172}\right)$
12. $\left(\frac{8}{3}, -\frac{7}{2}\right)$
13. $(2, 5)$
14. $(9, 4)$
15. $\left(\frac{29}{7}, \frac{3}{7}\right)$
16. $\left(-\frac{5}{3}, \frac{10}{3}\right)$

Exercises from Section 3.4

1. In each part of this exercise, describe how you would get the graph of the given function from the graph of $\sin x$ (example answers would be: "shift it left 3 units", "vertically stretch by a factor of 2", "reflect across the y-axis", etc.):

   a) $f(x) = \sin x + 4$
   b) $f(x) = 3 + \sin x$
   c) $f(x) = \sin(x - 8)$
   d) $f(x) = 4 \sin x$
   e) $f(x) = \frac{1}{4} \sin x$
   f) $f(x) = \sin\left(x + \frac{\pi}{6}\right)$
   g) $f(x) = 2+\sin(x+5)$
   h) $f(x) = \sin(x-3)-4$
   i) $f(x) = -\sin x$

2. In each part of this exercise, describe how you would get the graph of the given function from the graph of x^4:

a) $g(x) = \left(x + \frac{1}{2}\right)^4 + \frac{2}{5}$

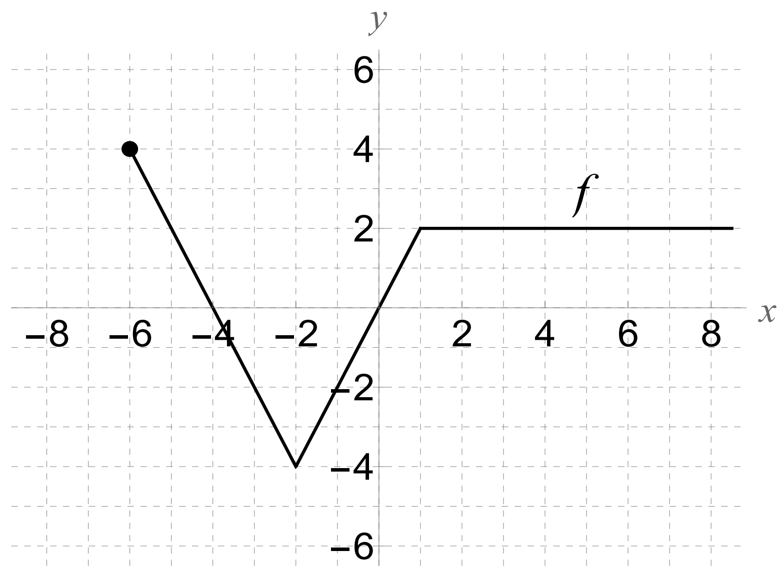
b) $g(x) = x^4 - 3$

c) $g(x) = (x - 8)^4$

d) $g(x) = (x + \sqrt{3})^4 - 5\sqrt{7}$

e) $g(x) = -7x^4$

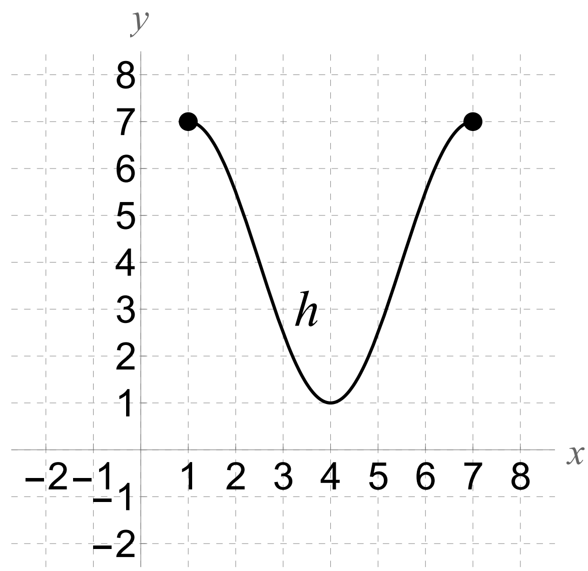
In Exercises 3-10, suppose that f is an unknown function whose graph is given here:



Use this graph to sketch the graph of each function:

3. $f(x-3)$
4. $3f(x)$
5. $f(x)+2$
6. $f(x-4)-1$
7. $\frac{1}{2}f(x)$
8. $f(x+5)$
9. $-f(x)$
10. $-f(x)+2$

In Exercises 11-16, suppose that h is an unknown function whose graph is given here:



Use this graph to sketch the graph of each function:

11. $-h(x)$
12. $h(x+4)$
13. $h(x-3)-2$
14. $3h(x)$
15. $-2h(x-3)$
16. $h(x+1)+5$

Continuing with the function h graphed before Exercise 11, answer the following questions:

17. Give the maximum value and minimum value of each function:

    a) $h(x-5)$
    b) $h(x)+5$
    c) $\frac{1}{4}h(x)$
    d) $-2h(x)$

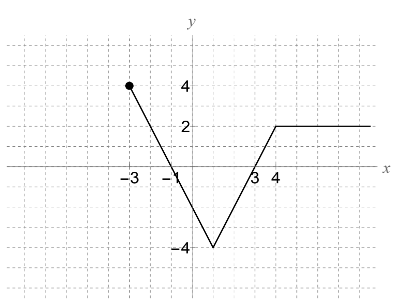
18. For each function, determine how many x-intercepts the function has and how many y-intercepts the function has:

    a) $h(x-3)$
    b) $h(x+3)$
    c) $h(x)-3$
    d) $h(x)+3$
    e) $h(x+3)+2$
    f) $h(x-2)-1$
    g) $h(x+10)$
    h) $h(x+4)-3$
    i) $h(x)-9$
    j) $-h(x)$
    k) $\frac{2}{3}h(x)$
    l) $2h(x)$
    m) $4-h(x)$

19. Give the domain and range of each function:

a) $h(x-4)$
b) $h(x)+7$
c) $h(x+3)+2$
d) $-h(x)$
e) $6h(x)$

Answers

1. a) shift it up 4 units
   b) shift it up 3 units
   c) shift it right 8 units
   d) stretch it vertically by a factor of 4
   e) shrink it vertically by a factor of $\frac{1}{4}$
   f) shift it left $\frac{\pi}{6}$ units
   g) shift it left 5 units and up 2 units
   h) shift if right 3 units and down 4 units
   i) reflect it across the x-axis
2. a) shift it left $\frac{1}{2}$ unit and up $\frac{2}{5}$ units
   b) shift it down 3 units
   c) shift it right 8 units
   d) shift it left $\sqrt{3}$ units and down $5\sqrt{7}$ units
   e) reflect it across the x-axis and stretch it vertically by a factor of 7
3. 

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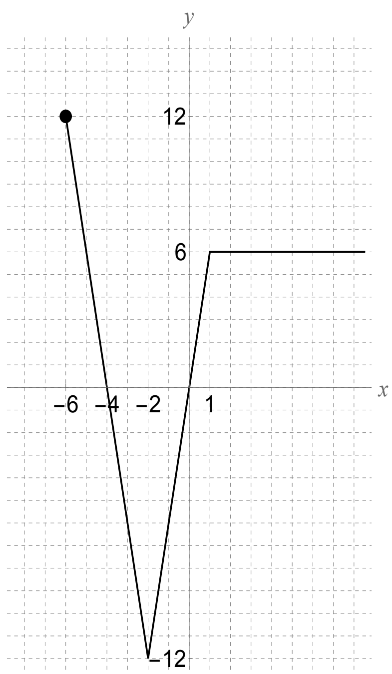

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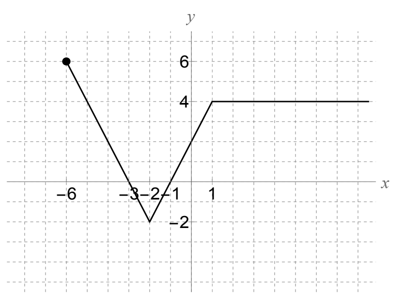

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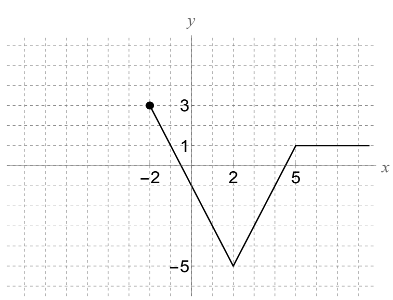

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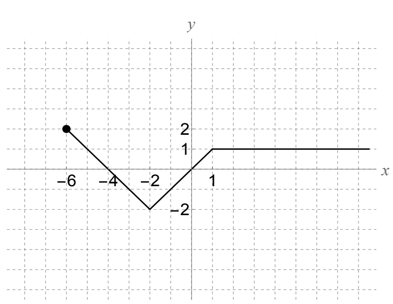

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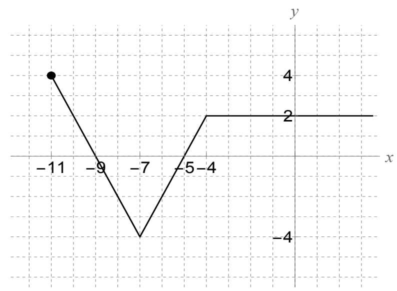

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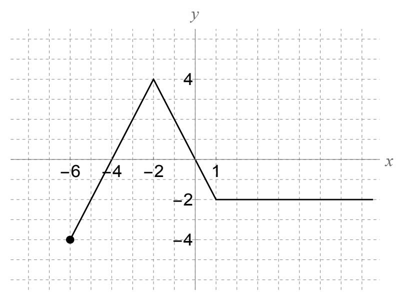

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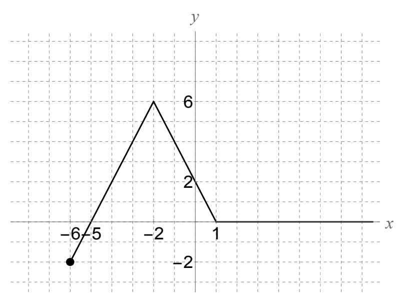
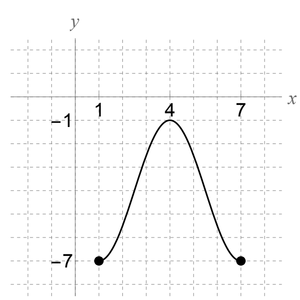


11.



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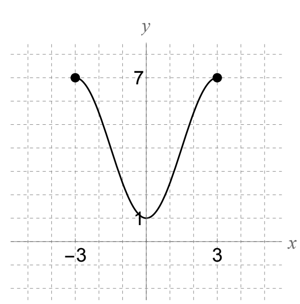
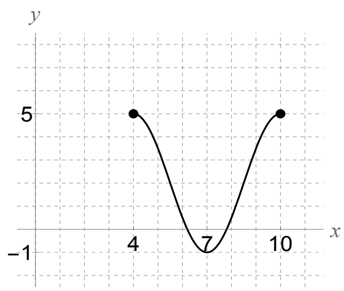
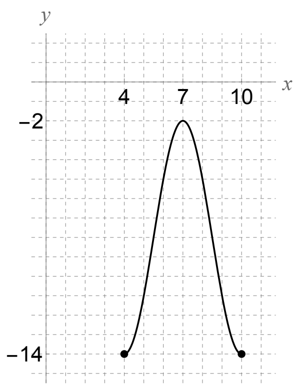
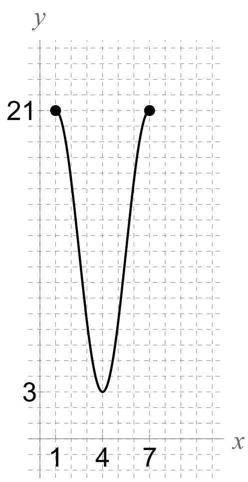
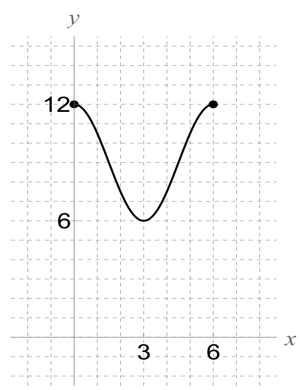


13. 

15. 

14. 

16. 

17. a) maximum value 7; minimum value 1
    b) maximum value 12; minimum value 6
    c) maximum value $\frac{7}{4}$; minimum value $\frac{1}{4}$
    d) maximum value -2; minimum value -14

18. a) no x-ints; no y-ints
    b) no x-ints; 1 y-int
    c) 2 x-ints; no y-int
    d) no x-ints; no y-ints
    e) no x-ints; 1 y-int
    f) 1 x-int; no y-int
    g) no x-ints; no y-ints
    h) 2 x-ints; 1 y-int
    i) no x-ints; no y-ints
    j) no x-ints; no y-ints
    k) no x-ints; no y-ints
    l) no x-ints; no y-ints
    m) 2 x-ints; no y-ints

19. a) domain $[5, 11]$; range $[1, 7]$
    b) domain $[1, 7]$; range $[8, 14]$
    c) domain $[-2, 5]$; range $[3, 9]$
    d) domain $[1, 7]$; range $[-7, -1]$
    e) domain $[1, 7]$; range $[6, 42]$

Exercises from Section 3.5

1. Let $f(x) = 3x^2 - x + 2$.

    a) Compute the average rate of change of f from $x = 3$ to $x = 5$.

    b) Compute the average rate of change of f from $x = -\frac{1}{2}$ to $x = \frac{3}{2}$.

    c) Compute the average rate of change of f from any x to $x + 2$, and verify that this average rate of change is a linear function of x.

2. Suppose that f is a quadratic function. Complete the following table of values:

x	2	3	4	5	6	7
$f(x)$	6	9	15			

In Exercises 3-6, do each of the following:
a) Sketch a graph of the given quadratic function.
b) Give the coordinates of its vertex.
c) Give the range of the function.
d) Write the quadratic function in standard form ("standard form" means $ax^2 + bx + c$).

3. $g(x) = -(x-4)^2$

4. $h(x) = 3\left(x + \frac{1}{3}\right)^2 + 1$

5. $k(x) = x^2 + 5$

6. $l(x) = -2(x - \sqrt{6})^2 + 3$

7. Sketch a graph of

$$F(x) = \begin{cases} -(x-3)^2 + 4 & x < 3 \\ 4 & x \geq 3 \end{cases}$$

8. Let $F(x) = (x-5)^2 + 3$.

    a) Sketch a graph of F.

    b) Based on the graph you drew in part (a), determine the number of solutions of the equation $F(x) = 4$.

    c) Based on the graph you drew in part (a), determine the number of solutions of the equation $F(x) = 1$.

9. Let $G(x) = -(x-8)^2 + 5$.

    a) By sketching a graph of G, determine the number of solutions of the equation $G(x) = 10$.

    b) Determine the number of solutions of the equation $G(x) = 7$.

c) Determine the number of solutions of the equation $G(x) = 5$.

d) Determine the number of solutions of the equation $G(x) = 1$.

e) Determine the number of solutions of the equation $G(x) = -2$.

10. Let $f(x) = (x-3)^2 + 1$ and let $g(x) = -x + 2$.

    a) Sketch graphs of f and g on the same coordinate plane.

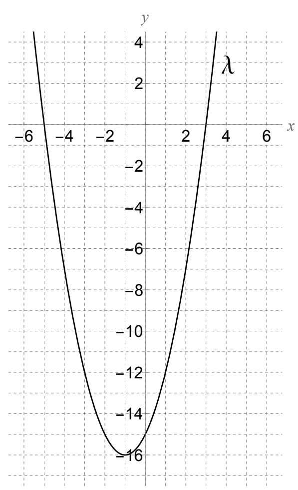
    b) Based on the picture you drew in part (a), determine the number of solutions of the equation $f(x) = g(x)$.

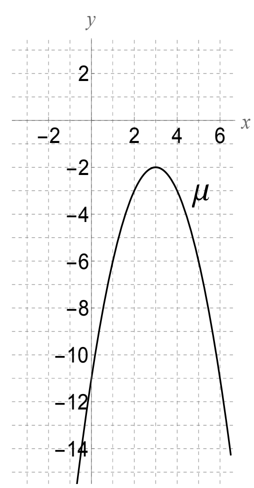
11. Let $\alpha(x) = (x+2)^2 - 1$ and let $\beta(x) = 5 - x^2$.

    a) Sketch graphs of α and β on the same coordinate plane.

    b) Based on the picture you drew in part (a), determine the number of solutions of the equation $\alpha(x) = \beta(x)$.

12. Give the x-intercept(s), y-intercept(s) and vertex of λ, where λ is the quadratic function graphed below at left.





13. Give the x-intercept(s), y-intercept(s) and vertex of μ, where μ is the parabola shown above at right.

In Exercises 14-17, complete the square on each given quadratic function. (This means you are to write the rule for the function as $a(x-h)^2 + k$.)

14. $h(x) = x^2 - 8x + 5$

15. $k(x) = 2x^2 + 12x$

16. $p(x) = -4x^2 + 6x - 7$

17. $q(x) = \frac{2}{3}x^2 - \frac{1}{3}x + \frac{5}{12}$

18. Find (the coordinates of) the vertex of the parabola $y = 4x^2 - 18x + 5$.

19. Find the vertex of the quadratic function $f(x) = -\frac{1}{2}x^2 + \frac{7}{2}x - 2$.

20. If the vertex of the parabola $y = ax^2 - 14x + 3$ has x-coordinate -5, what is a?

Let's review some exponent rules! In Exercises 21-31, rewrite each expression in the form $\square x^{\square}$, where the boxes represent constants:

21. $\frac{4}{x^3}$
22. $\left(\sqrt{3x}\right)^8$
23. $\sqrt[3]{(8x)^2}$
24. $x\sqrt[3]{x}$
25. $(x\sqrt{7x})^4$
26. $5x^3\sqrt{2x}$
27. $\frac{\sqrt{x}}{4x}$
28. $(x^3)^4$
29. $(x^2)^3(2\sqrt{x})^5$
30. $\frac{4(2x^2)^3}{(8x)^2}$
31. $\left(\frac{3}{x^2}\right)^{-3}$

In Exercises 32-45, solve each equation:

32. $x^2 + 8x - 9 = 0$
33. $2x^2 + 5x = -3$
34. $(x - 10)^2 - 64 = 0$
35. $x^2 - 4x = 9$
36. $x^2 - 18x + 81 = 0$
37. $6x^2 + 11x = 30 - 13x$
38. $25x^2 + 100 = 0$
39. $4x^2 - 1 = 0$
40. $3x(x + 2) = 2(x - 10)(x + 4)$
41. $2x^2 + 2x + 5 = x^2 - 2x$
42. $x(x - 3) - 5(x + 1) + 2 = 0$
43. $-12x + 7 = 5 - 2x^2$
44. $x^2 + \frac{3}{8}x + \frac{1}{8} = 0$
45. $3x^2 + 9x - 6 = 0$

In Exercises 46-49, use the quadratic formula to solve for x in terms of the other variables:

46. $x^2 + bx + 5 = 0$
47. $2x^2 - rx + s = 0$
48. $x^2 + 4yx + 2t^2 = 0$
49. $5ax^2 + 3b^2x - 20c = 0$

In Exercises 50-57, sketch a graph of each function:

50. $g(x) = x^2 - 8x$
51. $G(x) = x^2 - 6x - 7$
52. $h(x) = x^2 - 8x + 15$
53. $H(x) = 3x - x^2$
54. $j(x) = 2x^2 - 2x - 24$
55. $J(x) = x^2 - 6x + 11$

56. $k(x) = 24 - 2x - x^2$

57. $K(x) = \frac{1}{2}x^2 + \frac{1}{12}x - 1$

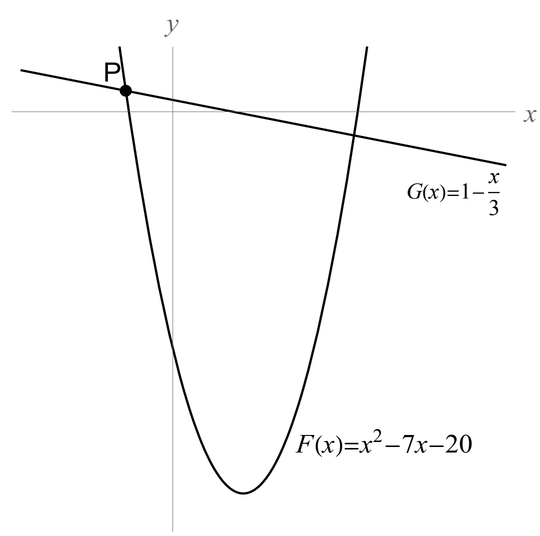
58. Compute the intersection point(s) of the graphs of $f(x) = x^2 - 8x + 3$ and $g(x) = 3x - 7$.

59. Compute the intersection point(s) of the graphs of $f_1(x) = x^2 + \frac{1}{2}x - \frac{1}{4}$ and $f_2(x) = -x^2 - x + 1$.

60. Compute the intersection point(s) of the graphs of $\psi(x) = 3x^2 + 8x - 7$ and $\phi(x) = x^2 + 6x + 5$.

61. Compute the intersection point(s) of the graphs of $g(x) = x(x-2)$ and $\widehat{g}(x) = 3x^2 + 5$.

62. Functions F and G are graphed below:



a) Compute the coordinates of point P.
b) Solve the inequality $F(x) \leq G(x)$.

63. Find the x-coordinates of all points on the parabola $y = x^2 - 8x + 5$ which have y-coordinate 7.

64. Suppose a parabola has x-intercepts $(-5, 0)$ and $(11, 0)$. What is the x-coordinate of its vertex?

65. Suppose a parabola has vertex $(-3, -4)$. If one of its x-intercepts is $(7, 0)$, what is the other x-intercept?

66. Suppose a quadratic function f has its vertex at $(2, 4)$. If $f(5) = 1$, what is a rule of f?

Let's do some more review of exponent rules! In Exercises 67-75, assume A, B and C are the functions

$$A(x) = x^3 + 1 \qquad B(x) = 2\sqrt[3]{x} \qquad C(x) = 3x^3$$

Compute and simplify the rule for each of these functions:

67. $4A - C$
68. $5B^2$
69. $B - 2A$
70. $B \circ A$
71. $A \circ B$
72. $A \circ C$
73. $C \circ 2C$
74. C^4
75. $B \circ C$

Continuing with the functions A, B and C as above, evaluate each expression. If possible, write the answer as $\square x^{\square}$, where the boxes are constants:

76. $B(8x)$
77. $C(5x)$
78. $C(x+2)$
79. $B(x-27)$
80. $A(2x)$
81. $A(x^2)$
82. $B\left(\frac{1}{x}\right)$
83. $C(\sqrt{x})$

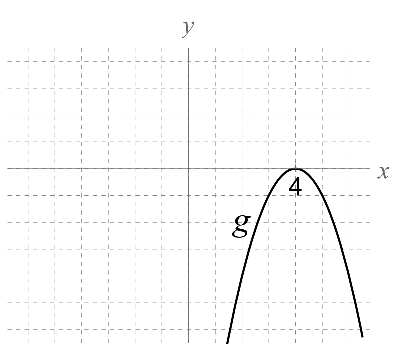
Answers

1. a) 23
   b) 2
   c) $6x + 5$

2.

x	2	3	4	5	6	7
$f(x)$	6	9	15	24	36	51

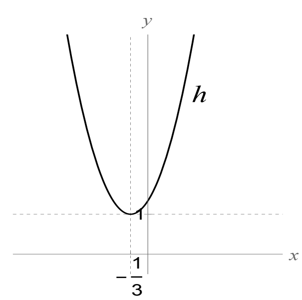
3. $g(x) = -(x-4)^2$

   a) 

   b) $(4, 0)$

   c) $g(x) = -x^2 + 8x - 16$

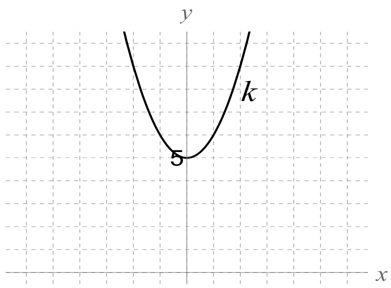
4. $h(x) = 3\left(x + \frac{1}{3}\right)^2 + 1$

   a) 

   b) $\left(-\frac{1}{3}, 1\right)$

   c) $h(x) = 3x^2 + 2x + \frac{4}{3}$

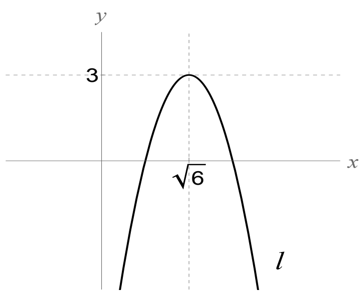
5. $k(x) = x^2 + 5$

a) 

b) $(0, 5)$

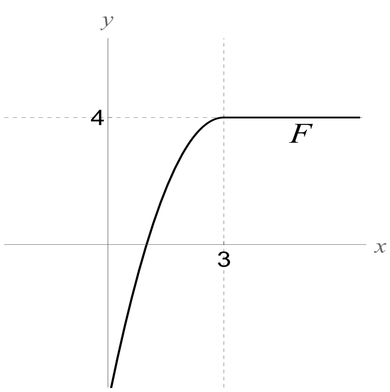
c) $k(x) = x^2 + 5$

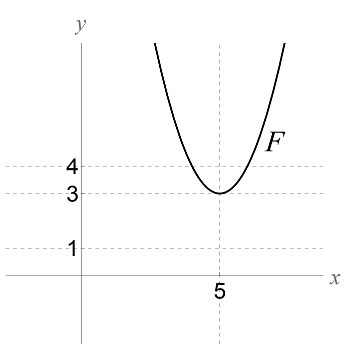
6. $l(x) = -2(x - \sqrt{6})^2 + 3$

a) 

b) $\left(\sqrt{6}, 3\right)$

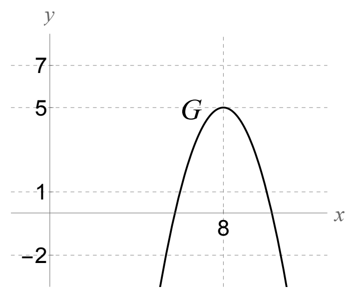
c) $l(x) = -2x^2 + 4\sqrt{6}\,x - 9$

7. 

8. a) 

b) 2

c) 0

9. a) 

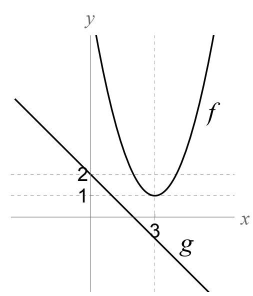
b) 0

c) 0

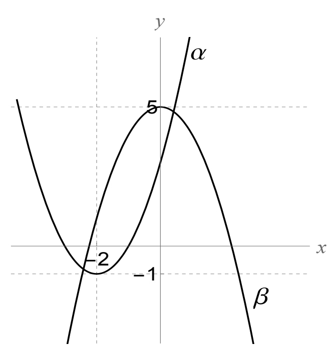
d) 1

e) 2

f) 2

10. a) 

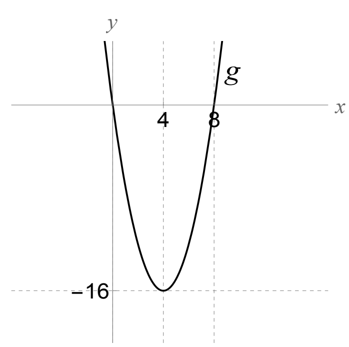
b) 0

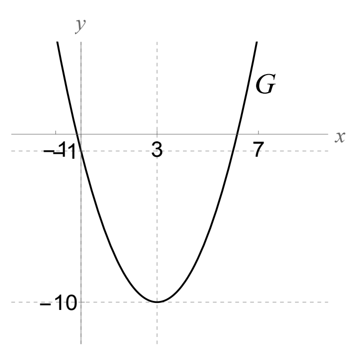
11. a) 

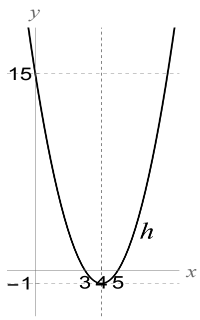
b) 2

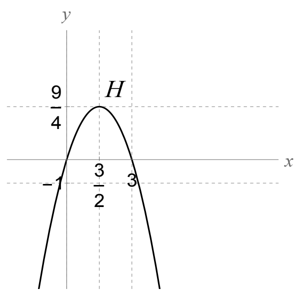
12. x-ints: $(-5, 0)$, $(3, 0)$
y-int: $(0, -15)$
vertex: $(-1, -16)$

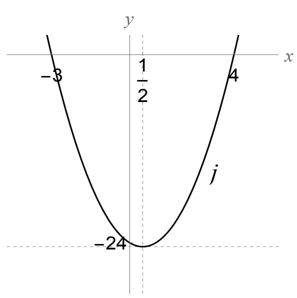
13. x-ints: none;
y-int: $(0, -11)$;
vertex: $(3, -2)$

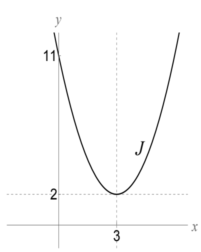
14. $h(x) = (x-4)^2 - 11$
15. $k(x) = 2(x+3)^2 - 18$
16. $p(x) = -4\left(x - \frac{3}{4}\right)^2 - \frac{19}{4}$
17. $q(x) = \dfrac{2}{3}\left(x - \dfrac{1}{4}\right)^2 + \dfrac{3}{8}$
18. $\left(\dfrac{9}{4}, -\dfrac{61}{4}\right)$
19. $\left(\dfrac{7}{2}, \dfrac{33}{8}\right)$
20. $a = \dfrac{7}{5}$
21. $4x^{-3}$
22. $81x^4$
23. $4x^{2/3}$
24. $x^{4/3}$
25. $49x^6$
26. $5\sqrt{2}\,x^{7/2}$
27. $\dfrac{1}{4}x^{-1/2}$
28. x^{12}
29. $32x^{17/2}$
30. $\dfrac{1}{2}x^4$
31. $\dfrac{1}{27}x^6$
32. $\{-9, 1\}$
33. $\left\{-\dfrac{3}{2}, -1\right\}$
34. $\{2, 18\}$
35. $\dfrac{4 \pm \sqrt{52}}{2}$
36. $\{-9, 9\}$
37. $\{-5, 1\}$
38. no solution
39. $\left\{-\dfrac{1}{2}, \dfrac{1}{2}\right\}$
40. $\{-10, -8\}$
41. no solution
42. no solution
43. $\dfrac{6 \pm \sqrt{32}}{2}$
44. no solution
45. $\dfrac{-3 \pm \sqrt{17}}{2}$
46. $x = \dfrac{-b \pm \sqrt{b^2 - 20}}{2}$
47. $x = \dfrac{r \pm \sqrt{r^2 - 8s}}{4}$
48. $x = \dfrac{-4y \pm \sqrt{16y^2 - 8t^2}}{2}$
49. $x = \dfrac{-3b^2 \pm \sqrt{9b^4 + 400ac}}{10a}$
50. 

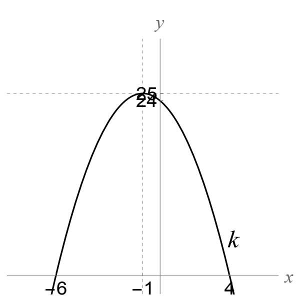
51. 

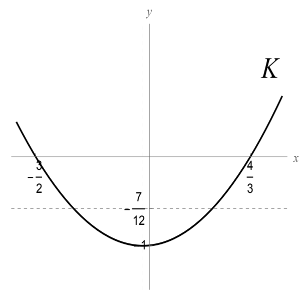
52. 

53. 

54. 

55. 

56. 

57. 

58. $(1, -4)$ and $(10, 23)$

59. $\left(-\dfrac{5}{4}, \dfrac{11}{16}\right)$ and $\left(\frac{1}{2}, \frac{1}{4}\right)$

60. $(-3, -4)$ and $(2, 21)$

61. There are no intersection points.

62. a) $\left(-\dfrac{7}{3}, \dfrac{16}{9}\right)$

b) $\left[-\dfrac{7}{3}, 9\right]$

63. $x = \dfrac{8 \pm \sqrt{72}}{2}$

64. 3

65. $(-13, 0)$

66. $f(x) = -\dfrac{1}{3}(x-2)^2 + 4$

67. $(4A - C)(x) = x^3 + 4$

68. $5B^2(x) = 20x^{2/3}$

69. $(B - 2A)(x) = 2\sqrt[3]{x} - 2x^3 - 2$

70. $B \circ A(x) = 2\sqrt[3]{x^3 + 1}$
(this doesn't simplify further)

71. $A \circ B(x) = 8x + 1$

72. $A \circ C(x) = 27x^9 + 1$

73. $C \circ 2C = 216x^9$

74. $C^4 = 81x^{12}$

75. $B \circ C(x) = 2\sqrt[3]{3}\,x$

76. $B(8x) = 4x^{1/3}$

77. $C(5x) = 375x^3$
78. $C(x+2) = 3(x+2)^3$
79. $B(x-27) = 2\sqrt[3]{x-27}$
80. $A(2x) = 8x^3 + 1$
81. $A(x^2) = x^6 + 1$
82. $B\left(\frac{1}{x}\right) = 2x^{-1/3}$
83. $C(\sqrt{x}) = 3x^{3/2}$

Exercises from Section 3.6

In Exercises 5-15, you are given the rule for a function. Determine whether or not the function is a polynomial; if it is, identify its degree and leading coefficient, and describe its tail behavior.

1. $f(x) = -4x^3 - 3x^2 + x - 5$
2. $f(x) = 8 - 3x^{12} + 17x^5 + 24x^{19} - 37x^8$
3. $f(x) = 3x^4 - 2x + 7x^{-1}$
4. $f(x) = 2^x - 3 \cdot 4^x$
5. $f(x) = \sqrt{8}x^5 - \sqrt{19}x^3 + \sqrt{7}x^2$
6. $f(x) = \frac{19}{5}x^2 - \frac{2}{7}x + \frac{35}{19}$
7. $f(x) = 2x^{3/2} + 5x$
8. $f(x) = (2x-2)(-3x+5)$
9. $f(x) = (\sqrt[3]{x} - 13)(x^2 + 7)$
10. $f(x) = (x-5)(x-3)(x-11)(x+4)$
11. $f(x) = \left(x^{5/2} - 3\right)\left(x^{5/2} + 3\right)$
12. $f(x) = (4x^6 - 3x^3 + 2x^2 - 1)(7x^9 - 5x^5 + 3x^4 - 3)$

In Exercises 13-20, sketch a crude graph of each function:

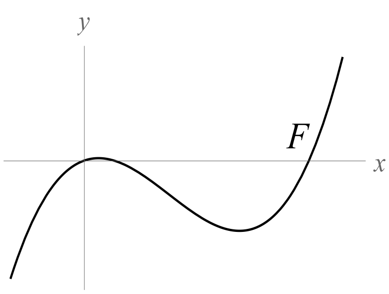
13. $g(x) = x^5 - 3$
14. $g(x) = (x+2)^3 + 4$
15. $g(x) = -x^4$
16. $g(x) = (x-3)^8$
17. $g(x) = x^3 + 1$
18. $g(x) = 6 - x^5$
19. $g(x) = \frac{1}{4}x^6$
20. $g(x) = -\frac{1}{2}(x-5)^7$

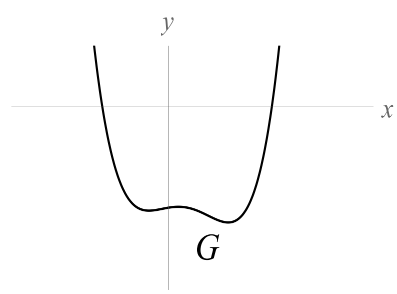
In Exercises 21-28, let f, g, h and k be the following functions:

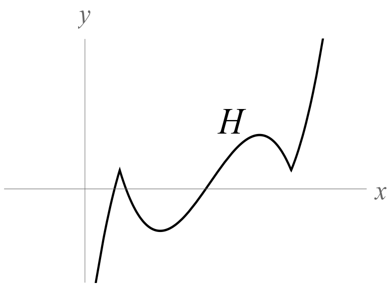
$$f(x) = (x-2)^4 + 5 \qquad g(x) = x^3 + 2 \qquad h(x) = x^2 - 3x + 1 \qquad k(x) = x^2 - 5$$

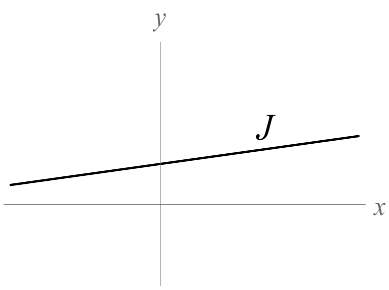
21. What is the domain and range of f?
22. What is the domain and range of g?
23. Compute and simplify the rule for $h - g$.
24. Compute the rule for $2hk$, and simplify by distributing and combining like terms.
25. Compute and simplify the rule for $f \circ g$.
26. Compute and simplify the rule for $h \circ g$.
27. Compute the net change of f from $x = 2$ to $x = 5$.
28. Compute the average rate of change of g from $x = -\frac{1}{2}$ to $x = \frac{3}{2}$.

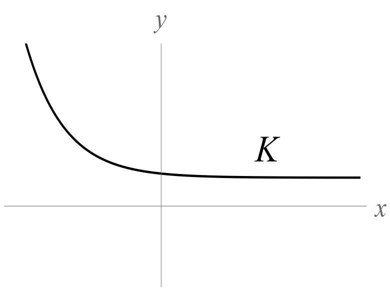
In Exercises 29-38, you are given the graph of some function. For each function:
(a) Determine if the function graphed is a polynomial. (If it isn't, ignore parts (b) and (c).)
(b) If the function is a polynomial, determine whether its leading coefficient is positive or negative.
(c) If the function is a polynomial, write down what you know about its degree. (Do you know what the degree is? If not, do you know the minimum possible degree of the function? Do you know if the degree is even or odd? Etc.)

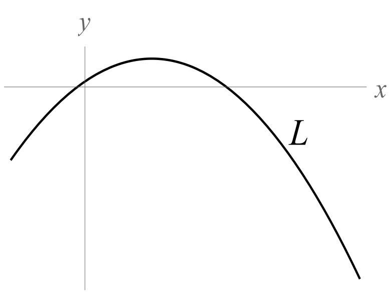
29. 

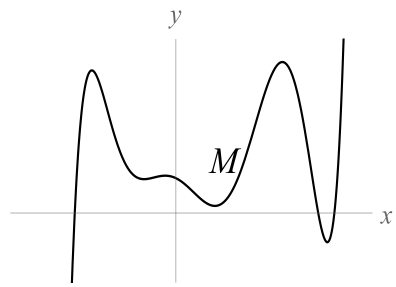
30. 

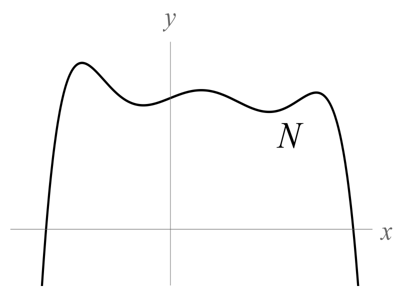
31. 

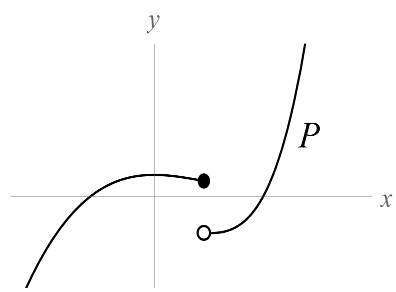
32. 

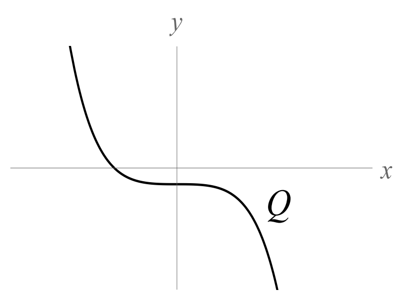
33. 

34. 

35. 

36. 

37. 

38. 

Answers

1. polynomial; degree 3; LC -4; tails $\nwarrow$ $\searrow$
2. polynomial; degree 19; LC 24; tails $\swarrow$ $\nearrow$
3. not a polynomial
4. not a polynomial
5. polynomial; degree 5; LC $\sqrt{8}$; tails $\swarrow$ $\nearrow$
6. polynomial; degree 2; LC $\frac{19}{5}$; tails $\nwarrow$ $\nearrow$

7. not a polynomial

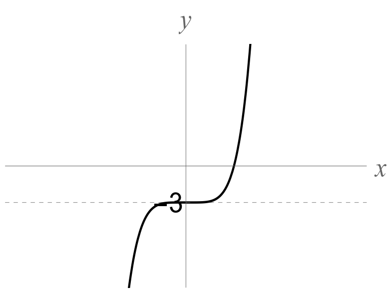
8. polynomial; degree 2; LC −6;
   tails ↙ ↘

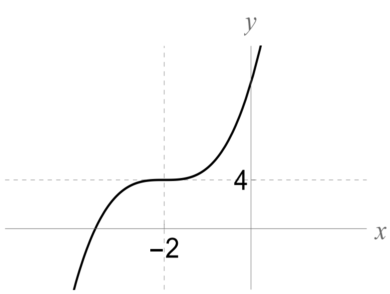
9. not a polynomial

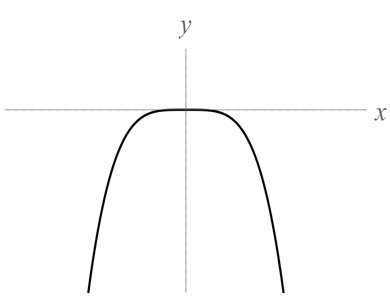
10. polynomial; degree 4; LC 1;
    tails ↖ ↗

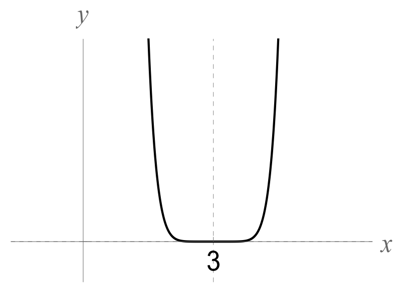
11. polynomial; degree 5; LC 1;
    tails ↙ ↗

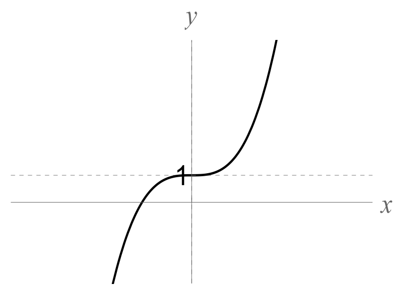
12. polynomial; degree 15; LC 28;
    tails ↙ ↗

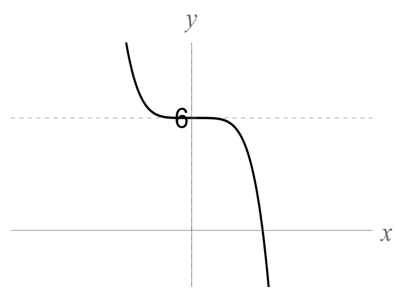
13. 

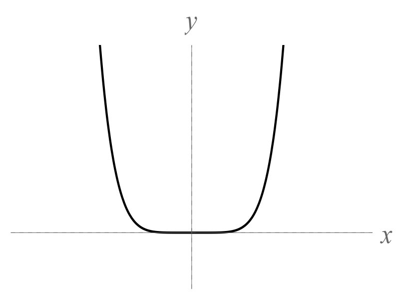
14. 

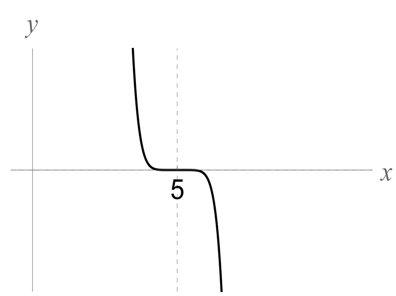
15. 

16. 

17. 

18. 

19. 

20. 

21. Domain is $\mathbb{R}$ (all real numbers);
    range is $[5, \infty)$

22. Domain is $\mathbb{R}$; range is $\mathbb{R}$
23. $(h-g)(x) = -x^3 + x^2 - 3x - 1$
24. $2hk(x) = 2x^4 - 6x^3 - 8x^2 + 30x - 10$
25. $f \circ g(x) = x^{12} + 5$
26. $h \circ g(x) = x^6 + x^3 - 1$
27. 81
28. $\frac{7}{4}$
29. polynomial; LC positive; degree odd and at least 3
30. polynomial; LC positive; degree even and at least 4
31. not a polynomial
32. polynomial; LC positive; degree 1
33. not a polynomial
34. polynomial; LC negative; degree 2
35. polynomial; LC positive; degree odd and at least 7
36. polynomial; LC negative; degree even and at least 6
37. not a polynomial
38. polynomial; LC negative; degree odd and at least 3

Exercises from Section 3.7

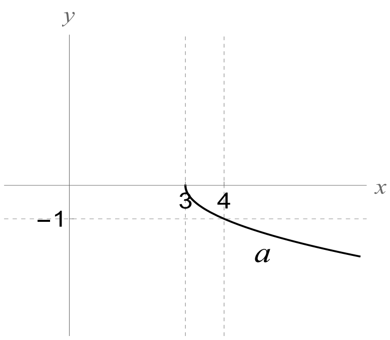
In Exercises 1-8, solve each equation:

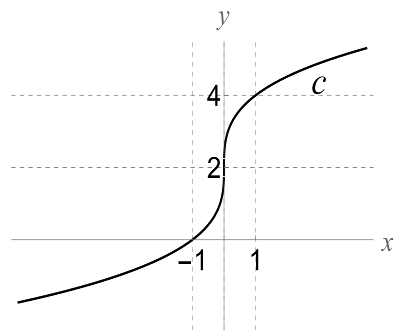
1. $3x^5 = 27$
2. $x^6 + 3 = 2$
3. $4x^8 - 19 = 33$
4. $\frac{2}{3}x^3 + \frac{5}{6} = \frac{8}{3}$
5. $\frac{1}{2}x^4 - 2 = \frac{9}{4}$
6. $x^6 - 8 = -5$
7. $7 - x^5 = 3$
8. $2x^7 = -22$

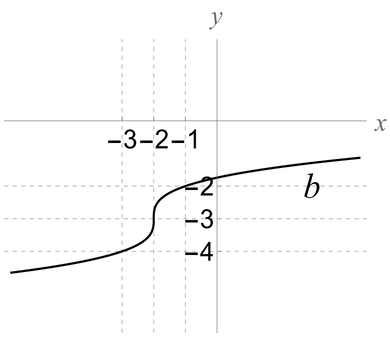
In Exercises 9-14, sketch a crude graph of each function:

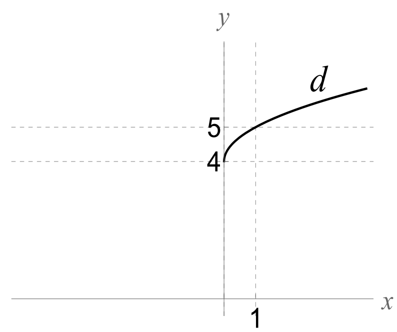
9. $h(x) = 2\sqrt{x-4}$
10. $k(x) = \sqrt[3]{x+5} - 4$
11. $l(x) = x^{1/4} - 5$
12. $m(x) = 3 - \sqrt[3]{x}$
13. $n(x) = -\sqrt{x}$
14. $p(x) = \frac{1}{3}\sqrt[5]{x+1} - 2$

In Exercises 15-18, write a rule for the function whose graph is given. You may assume that each graph is a transformation of either $f(x) = \sqrt{x}$ or $f(x) = \sqrt[3]{x}$.

15. 

17. 

16. 

18. 

In Exercises 19-30, simplify each expression, writing it in the form $\square x^{\square}$ if possible, where the boxes represent constants:

19. $\sqrt{\dfrac{x}{3}}$

20. $\dfrac{x}{4\sqrt{x}}$

21. $5x\sqrt[3]{x}$

22. $x^6\sqrt{x}$

23. $\dfrac{3x}{\sqrt[3]{2x}}$

24. $\sqrt{9(\sqrt{x})^3}$

25. $x^{3/5}\sqrt{x}$

26. $\dfrac{\sqrt{28x^3}}{\sqrt{7x}}$

27. $\left(\sqrt[3]{-8x^4}\right)^2$

28. $\sqrt{16x^2}$

29. $\left(\sqrt[5]{x^2}\right)^3(6\sqrt[3]{x})^2$

30. $5\sqrt{6x}\sqrt{3x}\sqrt{2x^3}$

In Exercises 31-34, compute the inverse of each function:

31. $F(x) = \sqrt{x-5}$

32. $G(x) = x^7 + 11$

33. $H(x) = (x-5)^4$

34. $K(x) = \dfrac{\sqrt[3]{x}}{4}$

Answers

1. $x = \sqrt[5]{9}$

2. no solution

3. $x = \pm\sqrt[8]{13}$

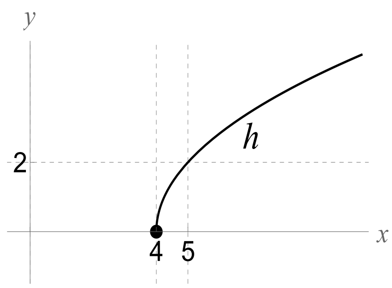
4. $x = \sqrt[3]{\dfrac{11}{4}}$

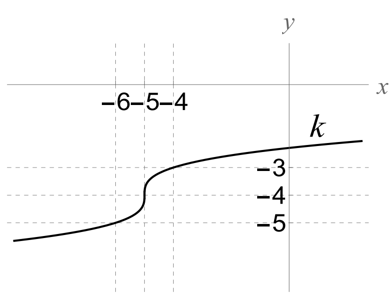
5. $x = \pm\sqrt[4]{\dfrac{17}{2}}$

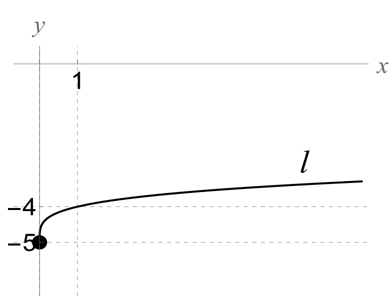
6. $x = \pm\sqrt[6]{3}$

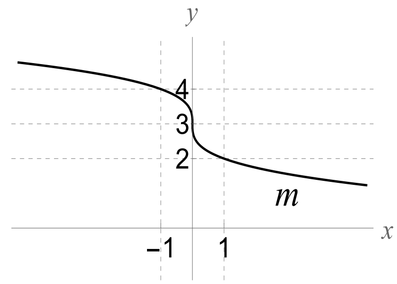
7. $x = \sqrt[5]{4}$

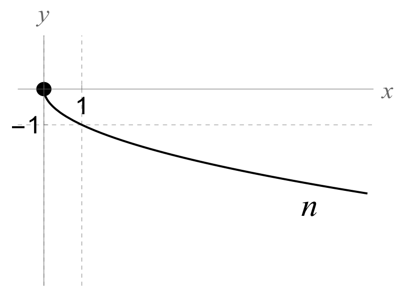
8. $x = \sqrt[7]{-11}$

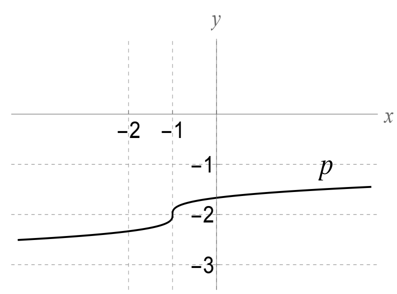
9. 

10. 

11. 

12. 

13. 

14. 

15. $a(x) = -\sqrt{x-3}$

16. $b(x) = \sqrt[3]{x+2} - 3$

17. $c(x) = 2\sqrt[3]{x} + 2$

18. $d(x) = \sqrt{x} + 4$

19. $\dfrac{1}{\sqrt{3}}x^{1/2}$

20. $\dfrac{1}{4}x^{1/2}$

21. $5x^{4/3}$

22. $x^{13/2}$

23. $\dfrac{3}{\sqrt[3]{2}}x^{2/3}$

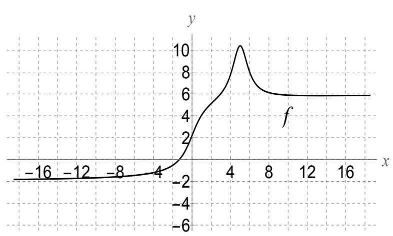
24. $3x^{3/4}$
25. $x^{11/10}$
26. $2|x|$
27. $4x^{8/3}$
28. $4|x|$
29. $36x^{28/15}$
30. $30x^{5/2}$
31. $F^{-1}(x) = x^2 + 5$
32. $G^{-1}(x) = \sqrt[7]{x - 11}$
33. $H^{-1}(x) = \sqrt[4]{x} + 5$
34. $K^{-1}(x) = 64x^3$

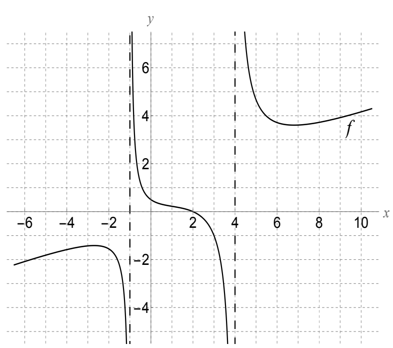
Exercises from Section 3.8

In Exercises 1-4, sketch a graph of each function:

1. $f(x) = \dfrac{1}{x+2}$
2. $f(x) = -2x^{-1}$
3. $f(x) = \dfrac{1}{x} - 5$
4. $f(x) = \dfrac{1}{x-2} + 3$

In Exercises 5-12, identify all horizontal and/or vertical asymptotes of the indicated function:

5. 

6. 

7. $f(x) = \dfrac{3x^2}{x^2 - 8x + 15}$
8. $f(x) = \dfrac{x}{2x^2 - 8}$
9. $f(x) = \dfrac{x^2 - 5x + 6}{x^2 + 2x - 8}$
10. $f(x) = 3(x + 14)^{-2}$
11. $f(x) = \dfrac{2}{x+3} - 4$
12. $f(x) = \dfrac{3x^2}{x+5}$

In Exercises 13-28, simplify each expression (writing it in the form $\frac{\square}{\square}$):

13. $\dfrac{x^2 - 4x - 21}{x^2 - 11x + 28}$
14. $\dfrac{x^2 - 7x - 44}{4x^2 + 8x - 32}$

15. $\dfrac{x-5}{x^2+4x-12} \cdot \dfrac{x^2+x-6}{x^2-9x+20}$

16. $\dfrac{2x-4}{x+3} \cdot \dfrac{x+6}{3x+9}$

17. $\dfrac{x}{x^2+x-12} \cdot (x-3)x^{-2}$

18. $\dfrac{x^2-x-2}{x-3} \div \dfrac{x-2}{x-4}$

19. $\dfrac{3x^2-6x-189}{x^2-3x-40} \div \dfrac{6x+42}{x^2+9x+20}$

20. $\dfrac{2}{x+6} + \dfrac{3}{x+4}$

21. $\dfrac{x+1}{x-5} - \dfrac{x}{x+5}$

22. $\dfrac{3}{x^2-5x+4} + \dfrac{5}{x^2-4x}$

23. $2x^{-1} + 3(x-1)^{-1} + 4(x+3)^{-1}$

24. $4x + 3x^{-3}$

25. $x^2 - 2x^{-1}$

26. $\dfrac{\dfrac{1}{x+4} - 1}{\dfrac{1}{x+2} + 2}$

27. $\dfrac{\dfrac{5}{x+h} - \dfrac{5}{x}}{h}$

28. $\dfrac{\dfrac{1}{x+2} + \dfrac{3}{x-1}}{\dfrac{4x}{x+2} + \dfrac{5}{x-1}}$

In Exercises 29-36, simplify each expression, writing it in the form $\square x^{\square}$ if possible, where the boxes represent constants:

29. $\dfrac{1}{x^4}$

30. $\dfrac{4}{x^{-5}}$

31. $17x^{-4}\sqrt{4x}$

32. $\dfrac{3}{2x^5}$

33. $\dfrac{7}{x\sqrt[3]{x^2}}$

34. $\dfrac{2}{(5x^2)^3}$

35. $\dfrac{\sqrt[3]{8x}}{x^{-1}}$

36. $\left(x\sqrt[4]{x^{-5}}\right)^3$

In Exercises 37-47, let F, G and H be these functions:

$$F(x) = \frac{x-3}{x+4} \qquad G(x) = 2(x+1)^{-1} \qquad H(x) = \frac{x^2+5x+4}{x-2}$$

In Exercises 37-44, write the rule for each given function, simplifying it so that it has the form $\dfrac{\square}{\square}$:

37. $F + 3G$

38. $G - F$

39. FH

40. G^2

41. $F \circ G$

42. $F \circ F$

43. $\dfrac{H}{G}$

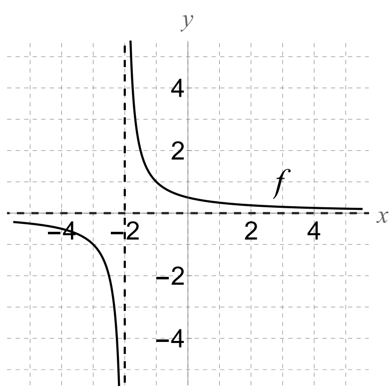
44. $F + G \circ F$

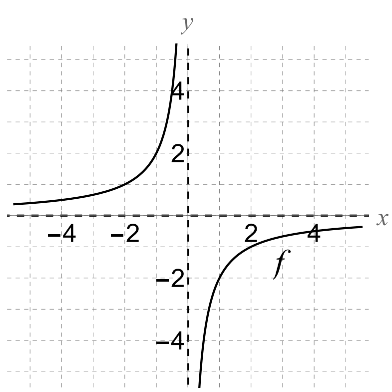
45. Identify all horizontal and/or vertical asymptotes of $F + G$.

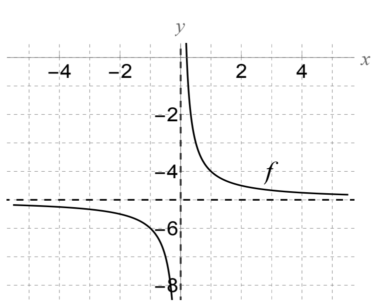
46. Identify all vertical and/or horizontal asymptotes of $F \circ G$.

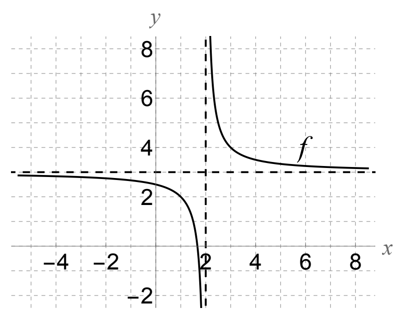
47. Identify all vertical and/or horizontal asymptotes of $G \circ G$.

Answers

1. 

2. 

3. 

4. 

5. HA: $y = -2$, $y = 6$;
   VA: none

6. HA: none;
   VA: $x = -1$, $x = 4$

7. $y = 3$;
   VA: $x = 3$, $x = 5$

8. HA: $y = 0$;
   VA: $x = -2$, $x = 2$

9. HA: $y = 1$;
   VA: $x = -4$

10. HA: $y = 0$;
    VA: $x = -14$

11. HA: $y = -4$;
    VA: $x = -3$

12. HA: none;
    VA: $x = -5$

13. $\dfrac{x+3}{x-4}$

14. $\dfrac{x-11}{4(x-2)}$

15. $\dfrac{x+3}{(x+6)(x-4)}$

16. $\dfrac{2(x-2)(x+6)}{3(x+3)^2}$

17. $\dfrac{1}{x(x+4)}$

18. $\dfrac{(x+1)(x-4)}{x-3}$

19. $\dfrac{(x+4)(x-9)}{2(x-8)}$

20. $\dfrac{5x+26}{(x+4)(x+6)}$

21. $\dfrac{11x+5}{(x-5)(x+5)}$

22. $\dfrac{8x-5}{x(x-1)(x-4)}$

23. $\dfrac{9x^2+9x-6}{x(x-1)(x+3)}$

24. $\dfrac{4x^4+3}{x^3}$

25. $\dfrac{x^3-2}{x}$

26. $\dfrac{-x^2-5x-6}{2x^2+13x+20}$

27. $\dfrac{-5}{x(x+h)}$

28. $\dfrac{4x+5}{4x^2+x+10}$

29. x^{-4}

30. $4x^5$

31. $34x^{-7/2}$

32. $\dfrac{3}{2}x^{-5}$

33. $7x^{-5/3}$

34. $\dfrac{2}{125}x^{-6}$

35. $2x^{4/3}$

36. $x^{-3/4}$

37. $(F+3G)(x) = \dfrac{x^2+4x+21}{(x+4)(x+1)}$

38. $(G-F)(x) = \dfrac{-x^2+4x+11}{(x+4)(x+1)}$

39. $(FH)(x) = \dfrac{(x+1)(x-3)}{(x-2)}$

40. $G^2(x) = \dfrac{4}{(x+1)^2}$

41. $F \circ G(x) = \dfrac{-3x-1}{2(2x+3)}$

42. $F \circ F(x) = \dfrac{-2x-15}{5x+13}$

43. $\dfrac{H}{G}(x) = \dfrac{(x+4)(x+1)^2}{2(x-2)}$

44. $(F + G \circ F)(x) = \dfrac{4x^2+11x+29}{2x^2+9x+4}$

45. HA: $y = 1$;
    VA: $x = -1$, $x = -4$

46. HA: $y = -\dfrac{3}{4}$;
    VA: $x = -\dfrac{3}{2}$

47. HA: $y = 2$;
    VA: $x = -3$

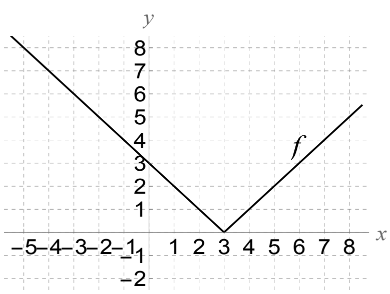
Exercises from Sections 3.9 and 3.10

In Exercises 1-14, sketch a graph of each function:

1. $f(x) = \sqrt{49 - x^2}$
2. $F(x) = |x - 3|$
3. $g(x) = -|x| - 5$
4. $G(x) = -\sqrt{16 - (x-2)^2}$
5. $\alpha(x) = \dfrac{|x|}{x}$
6. $\beta(x) = \dfrac{1}{4}|x + 2| - 1$
7. $\Gamma(x) = \sqrt{21 - x^2}$
8. $\Psi(x) = -\sqrt{25 - x^2}$
9. $\Phi(x) = \dfrac{|x+3|}{x+3}$
10. $\Theta(x) = \sqrt{36 - x^2} + 3$
11. $\Delta(x) = \sqrt{x^2}$
12. $\Omega(x) = \dfrac{|x|}{x} + 5$
13. $\Phi(x) = 2\sqrt{4 - x^2}$
14. $\Xi(x) = 2|x + 5| + 4$

In Exercises 15-22, write a rule for the function whose graph is given.

15.



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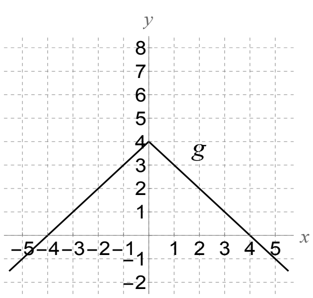

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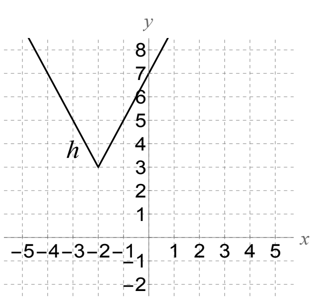

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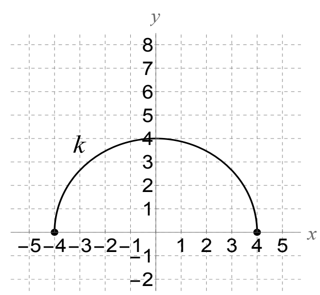

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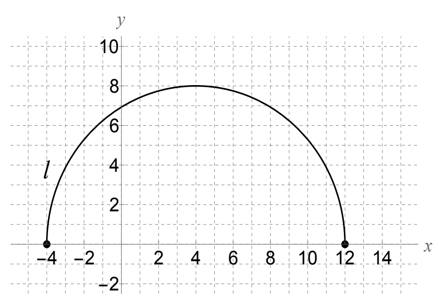

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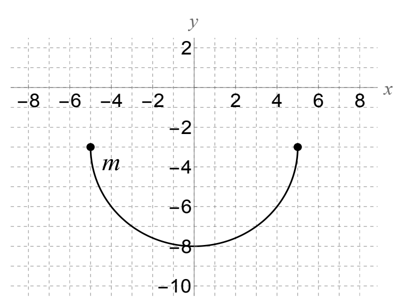
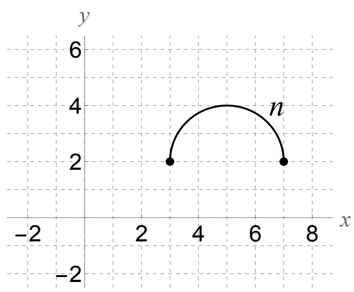
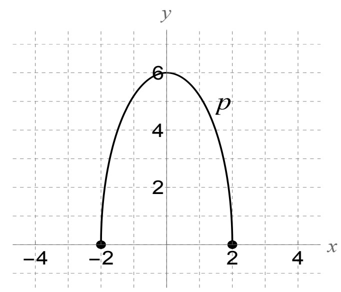


21. 

22. 

In Exercises 23-27, you are given a function f. For each function:

(a) Identify the domain of f.
(b) Identify the range of f.
(c) Does f have a maximum value? If so, what is it?
(d) If f has a maximum value, at what value(s) of x is $f(x)$ maximized?
(e) Does f have a minimum value? If so, what is it?
(f) If f has a minimum value, at what value(s) of x is $f(x)$ minimized?

23. $f(x) = 2|x - 4| + 3$

24. $f(x) = 5 - |x + 2|$

25. $f(x) = \sqrt{100 - x^2}$

26. $f(x) = 5 - \sqrt{9 - x^2}$

27. $f(x) = 3\sqrt{4 - (x + 4)^2} - 7$

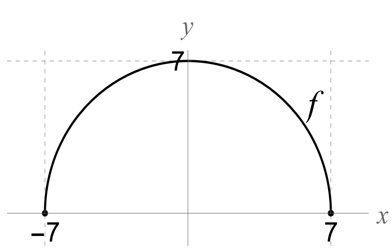
28. Write a rule for the inverse of $g(x) = |x - 5|$.

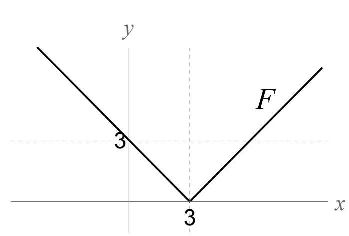
29. Write a rule for the inverse of $h(x) = |x| + 7$.

30. Write a rule for the inverse of $k(x) = -3|x|$.

31. Compute the average rate of change of $l(x) = |x - 3| + 2$ from $x = -4$ to $x = 8$.

Answers

1. 

2. 

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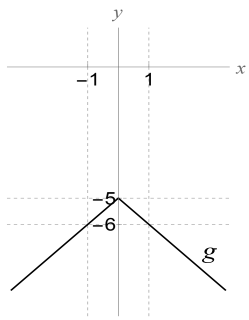

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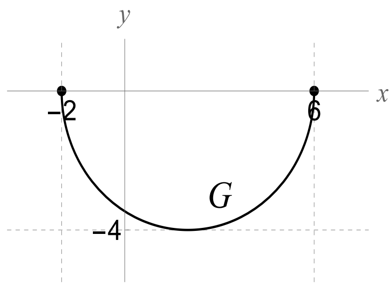

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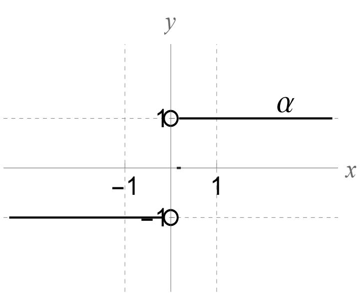

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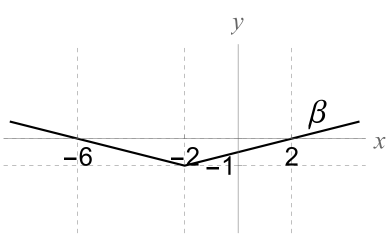

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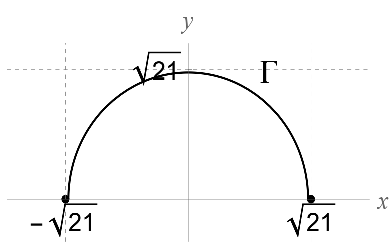

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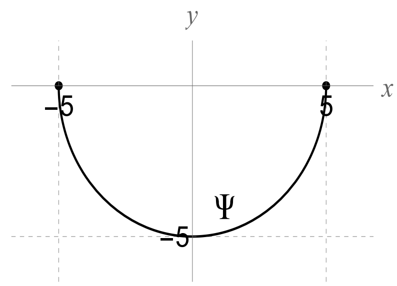

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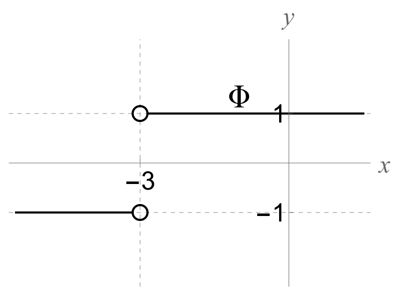

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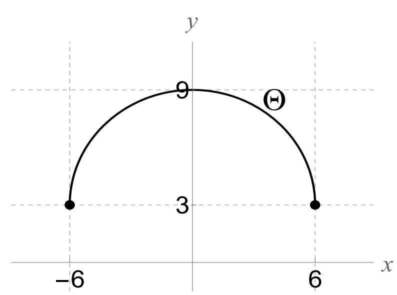

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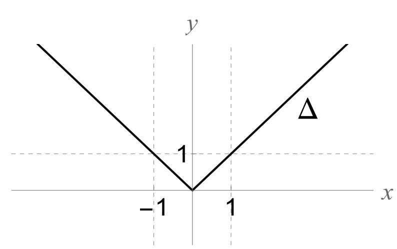

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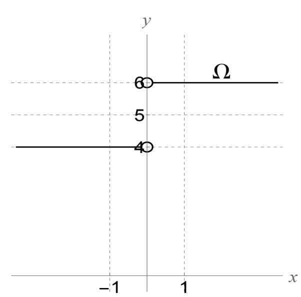
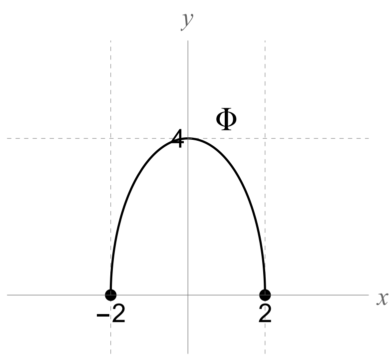
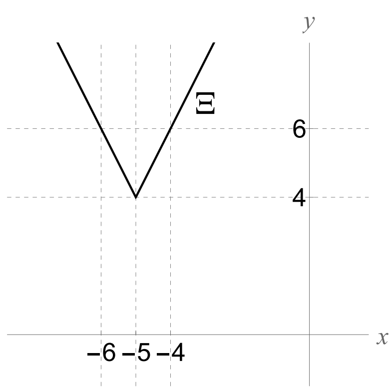


13. 

14. 

15. $f(x) = |x - 3|$

16. $g(x) = -|x| + 4$

17. $h(x) = 2|x + 2| + 3$

18. $k(x) = \sqrt{16 - x^2}$

19. $l(x) = \sqrt{64 - (x - 4)^2}$

20. $m(x) = -\sqrt{25 - x^2} - 2$

21. $n(x) = \sqrt{4 - (x - 5)^2} + 2$

22. $p(x) = 3\sqrt{4 - x^2}$

23. a) $\mathbb{R}$
    b) $[3, \infty)$
    c) no maximum value
    d) N/A
    e) 3
    f) $x = 4$

24. a) $\mathbb{R}$
    b) $(-\infty, 5]$
    c) 5
    d) $x = -2$
    e) no minimum value
    f) N/A

25. a) $[-10, 10]$
    b) $[0, 10]$
    c) 10
    d) $x = 0$
    e) 0
    f) $x = -10$, $x = 10$

26. a) $[-3, 3]$
    b) $[2, 5]$
    c) 5
    d) $x = 3$, $x = -3$
    e) 2
    f) $x = 0$

27. a) $[-6, -2]$
    b) $[-7, -1]$
    c) -1
    d) $x = -4$
    e) -7
    f) $x = -6$, $x = -2$

28. $g^{-1}(x) = \pm x + 5$

29. $h^{-1}(x) = \pm(x - 7)$

30. $k^{-1}(x) = \pm\left(\dfrac{x}{-3}\right)$

31. $-\dfrac{1}{6}$

Chapter 4

A library of transcendental functions

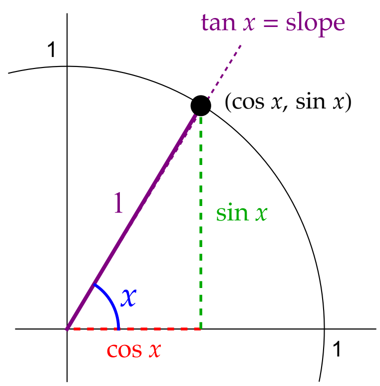
In Chapter 3, we studied functions that are *algebraic* (made up of the arithmetic operations $+$, $-$, $\times$, $\div$, powers, $\sqrt{\ }$, $\sqrt[n]{\ }$ and $|\cdot|$.

In this chapter, we turn to useful functions that cannot be made of these algebraic operations. Such functions are called *transcendental* because they "transcend" algebra; they include trigonometric and inverse trigonometric functions, exponential functions and logarithms.

4.1 Graphs of trigonometric functions

RECALL

We use the trig functions $\cos$, $\sec$ and $\tan$ to convert rotational measurements (angles) to lengths (coordinates). They are defined using the unit circle:



The three other trig functions $\sec$, $\csc$ and $\cot$ are the reciprocals of these functions:

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x} \qquad \cot x = \frac{1}{\tan x}$$

In an ideal world, we would be able to write down a formula for the trig functions using mathematical language we've already seen (powers, roots, absolute value, and/or other algebraic stuff). But unfortunately, *no formula for any of the trigonometric functions can be made from any finite-length string of algebraic formulas*. That's why we have to give these functions the names "$\sin$", "$\cos$", etc.

As an aside, I note here that if you allow an *infinite-length* string of algebraic formulas, you can create formulas for the trig functions using powers:

Theorem 4.1 *For any whole number n, let*

$$n! = n(n-1)(n-2)(n-3)\cdots 3 \cdot 2 \cdot 1.$$

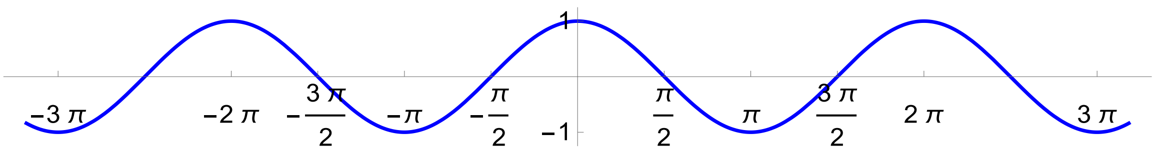
Then, for any real number x,

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots \pm \frac{x^{2n}}{(2n)!} \pm \ldots$$
$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots \pm \frac{x^{2n+1}}{(2n+1)!} \pm \ldots$$

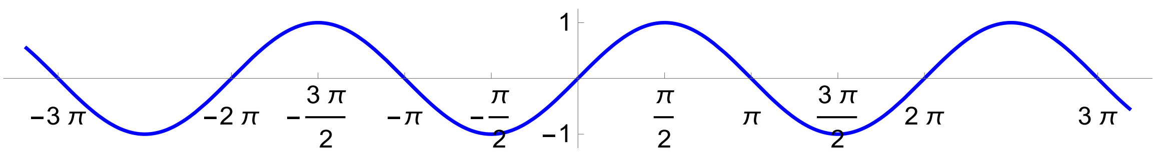
You don't need to know these formulas in MATH 130; you learn about them in Calculus 2.

Graphs of cosine and sine

Here is the graph of $f(x) = \cos x$:



Here is the graph of $f(x) = \sin x$:



IMPORTANT OBSERVATIONS ABOUT THESE GRAPHS

General shape: The graphs of $\cos$ and $\sin$ are smooth (no sharp corners) and continuous (no breaks or jumps), and they look like waves.

Domain and range: The domain of both $\cos$ and $\sin$ is $\mathbb{R}$ (so $\cos x$ and $\sin x$ always exist);

the range of both $\cos$ and $\sin$ is $[-1, 1]$.

Periodicity: Every 2π units (horizontally), the graphs repeat themselves (i.e the functions have a period of 2π).

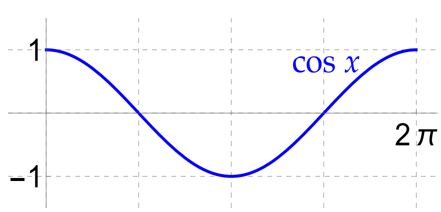
y**-intercept:** The graph of $f(x) = \sin x$ goes through $(0, 0)$ and begins by going up and to the right from $(0, 0)$;

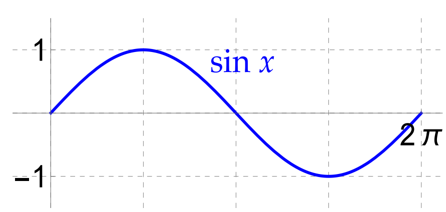
the graph of $f(x) = \cos x$ goes through $(0, 1)$ and begins by going down on both sides of $(0, 1)$;

Symmetry: $\cos$ is even (i.e. the y-axis is an axis of symmetry for its graph); this is because $\cos(-x) = \cos x$;

$\sin$ is odd (i.e. its graph has rotational symmetry about the origin); this is because $\sin(-x) = -\sin x$.

WHAT YOU NEED TO DRAW WHEN ASKED TO SKETCH THESE





Periodicity

Definition 4.2 *Let T be a real number. A function $f : \mathbb{R} \to \mathbb{R}$ is called* **periodic** *with period T if $f(x+T) = f(x)$ for every x; the number T is called the* **period** *of f.*

This means that the graph of f repeats itself every T units horizontally.

PROTOTYPE EXAMPLE

cos and sin are both periodic with period 2π.

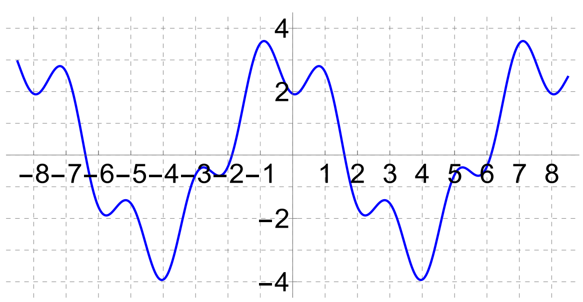
Graphical reason:

Algebraic reason:

Application:

§4.1 EXAMPLE 1

The graph of an unknown function f is given below.



If f is periodic,

a) What is the period of f?

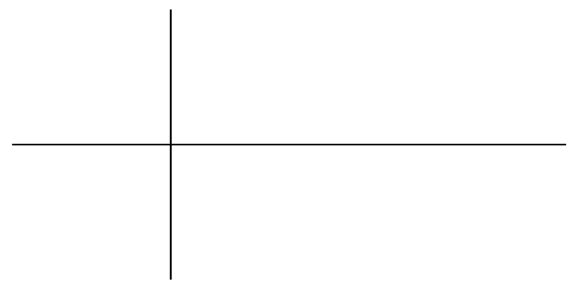
b) Estimate $f(20)$.

c) Estimate $f(-18)$.

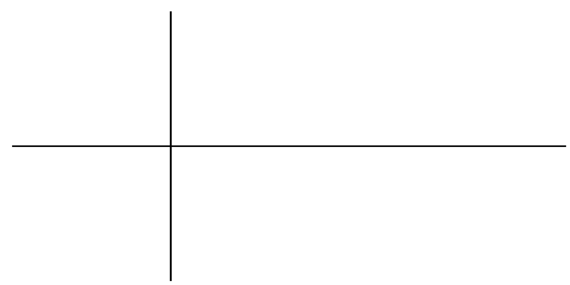
§4.1 EXAMPLE 2

Sketch the graph of each function:

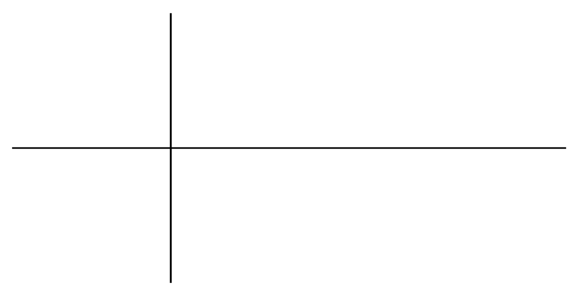
a) $f(x) = 3\sin x$



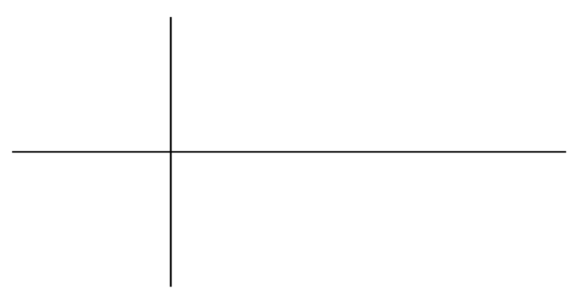
b) $g(t) = -\cos t + 5$



c) $h(x) = \cos(x + \pi)$



d) $r(x) = \sin(x - 9) - 2$



A new transformation: horizontal stretch

QUESTIONS

What does the graph of $F(x) = \sin 2x$ look like?

What does the graph of $G(x) = \cos \frac{1}{2}x$ look like?

Why weren't functions like F and G included in our preceding example?

§4.1 EXAMPLE 3

Diagram each of these functions:

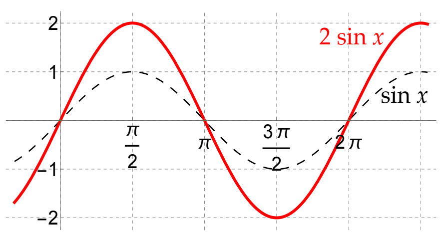
$$f(x) = 2\sin x \qquad g(x) = \frac{1}{2}\cos x \qquad F(x) = \sin 2x \qquad G(x) = \cos\frac{1}{2}x$$

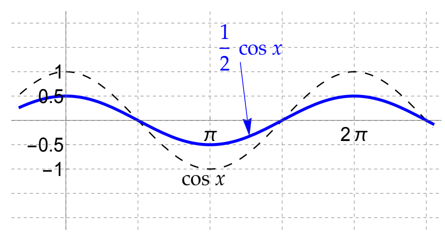
Solution:

$$x \longrightarrow \quad \longrightarrow f(x) \qquad\qquad x \longrightarrow \quad \longrightarrow F(x)$$

$$x \longrightarrow \quad \longrightarrow g(x) \qquad\qquad x \longrightarrow \quad \longrightarrow G(x)$$

Notice that for f and g, the multiplication comes [   ] the trig function. This means the graphs of these functions are obtained from the basic trig graph by a [   ]. In fact, here are the graphs of f and g:





But for F and G, the multiplication comes [   ] the trig function. This has a different effect on the graph of the basic trig function.

SOME MOTIVATION

Notice that $|3x|$ is a transformation of $|x|$ in the same way $\sin 2x$ is a transformation of $\sin x$, since the multiplication comes before the elementary function:

$$x \xrightarrow{\times 3} \xrightarrow{|\cdot|} |3x| \qquad\qquad x \xrightarrow{\times 2} \xrightarrow{\sin} \sin 2x$$

But, for the absolute value transformation, we can rewrite it:

Notice also that $h(x) = \left(\frac{1}{3}x\right)^2$ is literally this:

$$x \xrightarrow{\div 3} \xrightarrow{\wedge 2} \left(\frac{1}{3}x\right)^2 = h(x)$$

but by algebra, we can rewrite h as

which diagrams like this:

$$x \xrightarrow{\wedge 2} \xrightarrow{\div 9} \frac{1}{9}x^2 = h(x)$$

PUNCHLINE

For an absolute value, quadratic or other algebraic function, the transformation of multiplication/division before the function can be effectively removed (or turned into a vertical stretch and/or other shifts) by doing some algebra.

But for a trig function, this doesn't work. There's no way to rewrite $\sin 2x$ or $\cos \frac{1}{2}x$ that is useful for drawing graphs. Thus, for trig graphs we need to learn one more transformation.

To do this, recall that $F(x) = \sin 2x$ diagrams as

$$x \xrightarrow{\times 2} \xrightarrow{\sin} \sin 2x = F(x)$$

Since this multiplication takes place **before** the trig function, we expect that this 2 will affect the graph __________. Since this is **multiplication** by 2, we expect that the graph will be affected by a __________, not a __________ (that would come from addition or subtraction).

Last question

On the previous page, we saw that $F(x) = \sin 2x$ is a horizontal compression/stretch of $\sin x$ by a factor of 2. Is this compression, or is it stretching?

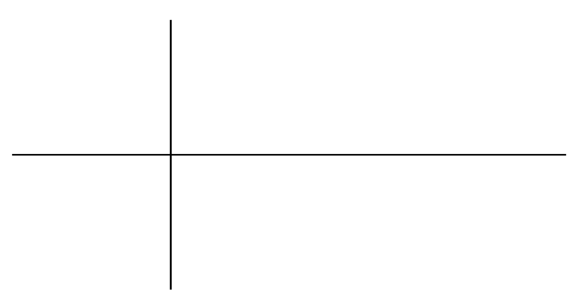
More generally:

> **Theorem 4.3** *The graph of $f(x) = \cos Bx$ is obtained from the graph of $\cos x$ by compressing (if $B > 1$) or stretching (if $0 < B < 1$) it horizontally, so that its period is $T = \dfrac{2\pi}{B}$.*
>
> *Similarly, the graph of $f(x) = \sin Bx$ is obtained from the graph of $\sin x$ by compressing (if $B > 1$) or stretching (if $0 < B < 1$) it horizontally, so that its period is $T = \dfrac{2\pi}{B}$.*
>
> *More generally, if f is any function, then the graph of $g(x) = f(Bx)$ is obtained from the graph of f by compressing/stretching it horizontally by a factor of B (compression if $B > 1$, stretching if $0 < B < 1$).*
>
> *If B is negative, that reflects the graph of f horizontally across the y-axis. (More on this later.)*

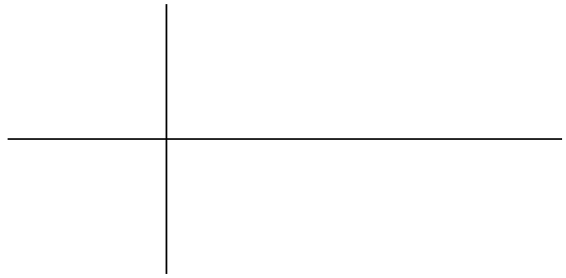
§4.1 Example 4

Compute the period of each function, and then sketch its graph:

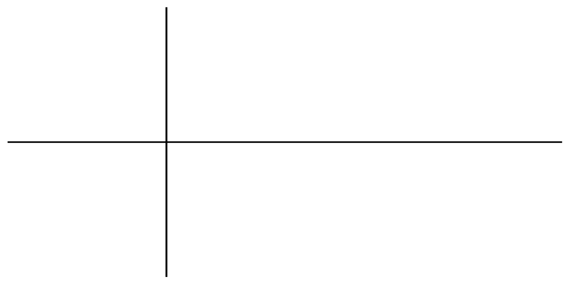
a) $f(x) = \sin 3x$



b) $g(x) = \cos \dfrac{x}{8}$

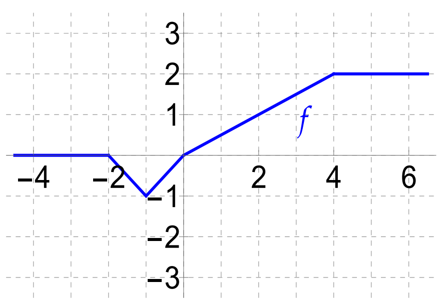


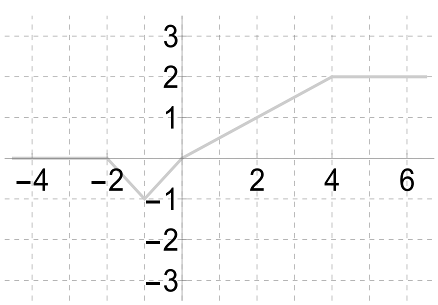
c) $h(x) = \sin 6\pi x$

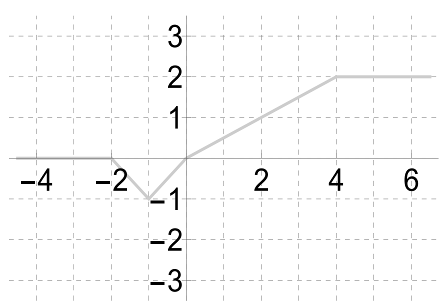


§4.1 EXAMPLE 5

Suppose the graph of some unknown function f is given below. Sketch the graphs of $g(x) = f(4x)$ and $h(x) = f\left(\dfrac{x}{2}\right)$.







A summary of transformations on graphs

A generic transformation of function f looks like

$$g(x) = Af(B(x - C)) + D$$

and diagrams like this:

$$x \xrightarrow{-C} \xrightarrow{\times B} \xrightarrow{f} \xrightarrow{\times A} \xrightarrow{+D} g(x)$$

THEMES

1. The stuff that comes **before** f in the order of operations (the B **and** C) affects the graph **horizontally** because they affect the value that gets plugged into f (the input or the "x");
2. The stuff that comes **after** f in the order of operations (the A **and** D) affects the graph **vertically** because they affect the value that is obtained from f (the output or the "y").
3. Any multiplication/division (the B and A) is a stretch or compression (and a $-$ sign on these numbers is a reflection.
4. Any addition/subtraction (the C and D) is a shift (a.k.a. translation).

In the chart below, let's indicate whether the quantity in the left-hand column can be changed by the transformation indicated in the other four columns:

	Changed by vertical stretch (by the A)?	Changed by horiz. stretch (by the B)?	Changed by horiz. shift (by the C)?	Changed by vertical shift (by the D)?
x-int(s)				
y-int(s)				
HA				
VA				
period				
evenness				
oddness				

Note: in the chart above, a "yes" doesn't mean those things will be changed; it just means they might be changed by the corresponding transformation.

Sinusoidal graphs

Definition 4.4 *A* **sinusoidal** *graph is any graph of a function that is a transformation of either* $\sin x$ *or* $\cos x$*. In other words, it is the graph of a function of the form*

$$f(x) = A\cos B(x - C) + D \quad \text{or} \quad f(x) = A\sin B(x - C) + D.$$

Procedure to draw a sinusoidal graph

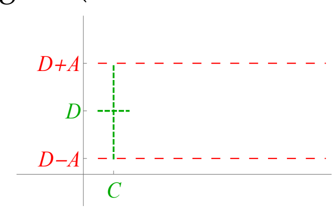
1. Write the function in the standard form

$$y = A\sin(B(x - C)) + D \quad \text{or} \quad y = A\cos(B(x - C)) + D$$

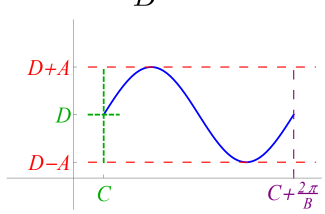
   Once in this form:

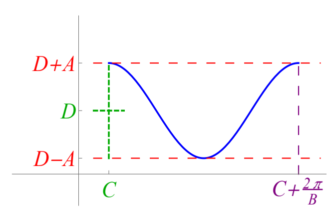
   - A is the vertical stretch (a.k.a. the amplitude),
   - B is the horizontal stretch,
   - C is the horizontal shift, and
   - D is the vertical shift.

2. Draw dashed "crosshairs" at the point (C, D). (This accounts for the horizontal and vertical shift.)
3. From the vertical crosshair, go up and down A units and mark the y-coordinates of those heights. (This accounts for the vertical stretch.)



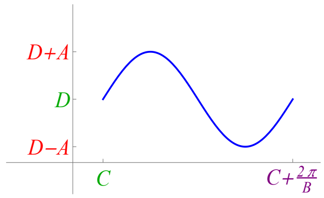
4. Starting in the appropriate place, draw one period of the graph.
5. Then, mark where the period finished. This should be at $x = C + \dfrac{2\pi}{B}$, since the period is $\dfrac{2\pi}{B}$).

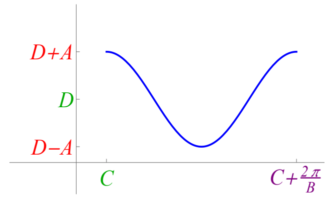




6. If your picture is too messy, erase any crosshairs and the dashed lines you drew (but not the numbers you marked on the x- and y-axes).

If you carry out these steps, you'll get a graph like one of these:



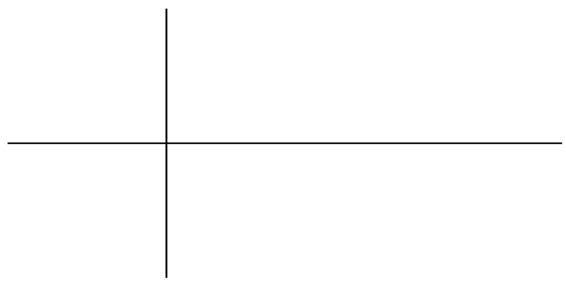


The rest of the graph just repeats itself periodically, but you generally don't have to draw that.

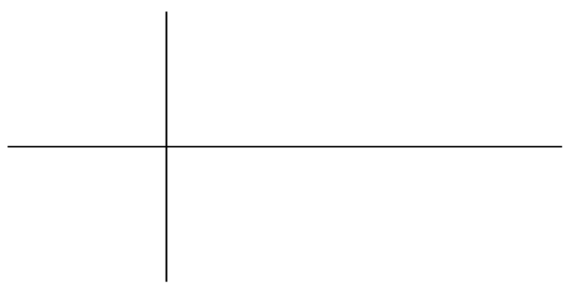
§4.1 Example 6

Sketch the graph of each function:

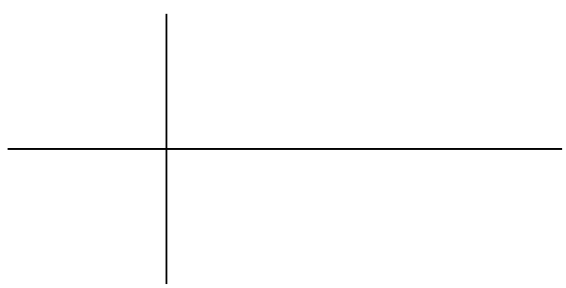
a) $f(x) = 3\cos 2(x-7)$



b) $g(t) = -\dfrac{1}{4}\sin\dfrac{t}{6} + \dfrac{1}{2}$



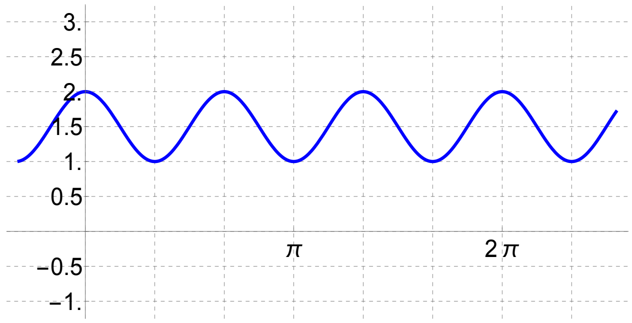
c) $h(x) = -4\cos x + 3$



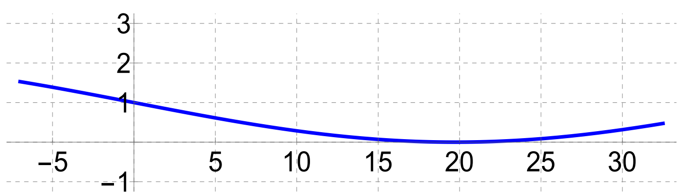
§4.1 EXAMPLE 7

Write a rule for each function whose graph is shown here (you may assume the graph is sinusoidal):

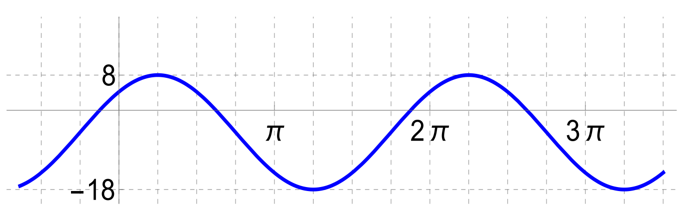
a)



b)

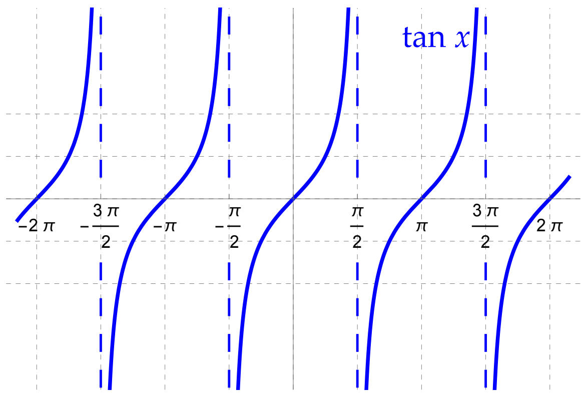


c)



The graph of tangent

Here is the graph of $f(x) = \tan x$:



IMPORTANT OBSERVATIONS ABOUT THE GRAPH OF TANGENT

General shape: The graph of $\tan$ is not continuous–it consists of infinitely many pieces, each of which kind of looks like the graph of $y = x^3$ (going up from left to right).

Domain and range: The domain of $\tan$ is all real numbers except $\frac{(\text{odd } \#)\pi}{2}$ (meaning $\tan x$ DNE if $x = \frac{(\text{odd } \#)\pi}{2}$); the range of $\tan$ is $\mathbb{R}$.

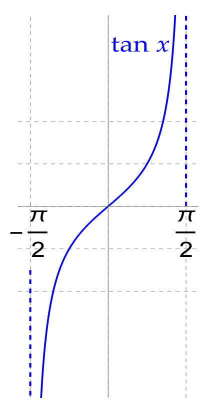
Periodicity: The period of $\tan$ is π (not 2π).

y-intercept: The graph of $f(x) = \tan x$ goes through $(0, 0)$.

Symmetry: $\tan$ is odd (i.e. its graph has rotational symmetry about the origin); this is because $\tan(-x) = -\tan x$.

Asymptotes: $\tan$ has VA at every $x = \frac{(\text{odd } \#)\pi}{2}$ ($\tan$ has no HA).

WHAT YOU NEED TO DRAW WHEN ASKED TO SKETCH THIS GRAPH



A transformed tangent graph has the formula

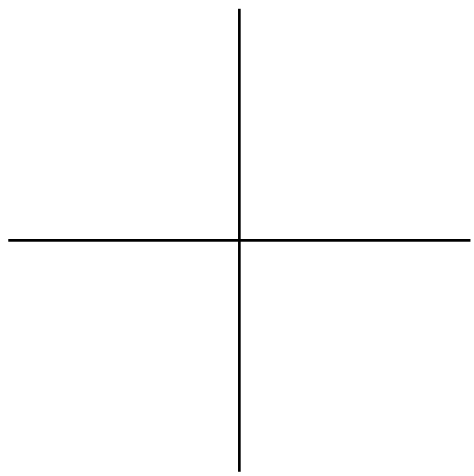
$$f(x) = A \tan B(x - C) + D,$$

and the transformations work the same way as they do for other functions. The biggest difference is that the period of a transformed tangent function is _____ (not $\frac{2\pi}{B}$), because the period of the basic tangent graph is _____ (not 2π).

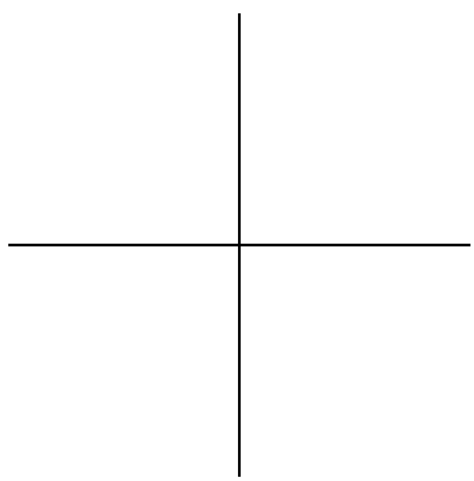
§4.1 EXAMPLE 8

Sketch the graph of each function:

a) $f(x) = \tan 4x$



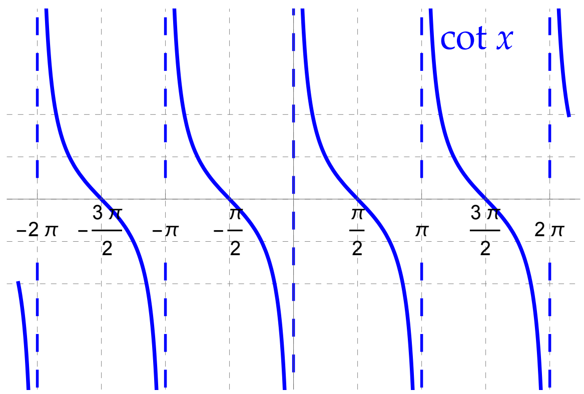
b) $g(\theta) = -2 \tan \theta + 5$



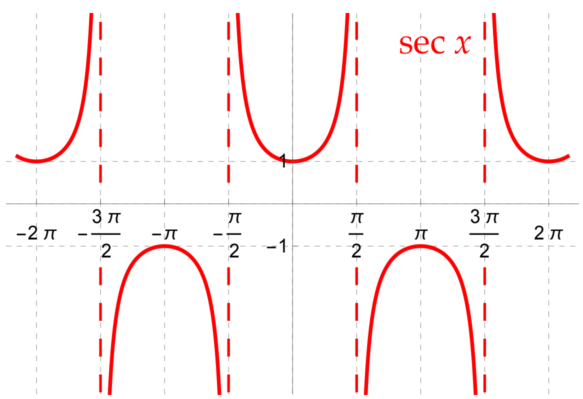
Graphs of secant, cosecant and cotangent (presented without comment)

I won't hold you responsible for knowing these, but you might see them again one day:

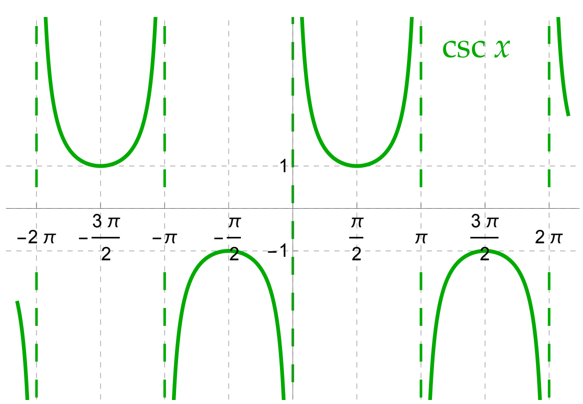
$f(x) = \cot x$



$f(x) = \sec x$



$f(x) = \csc x$



4.2 Trig identities

Here is a list of trig identities that you "should" know:

Theorem 4.5 (Reciprocal Identities)

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\sin x = \frac{1}{\csc x} \qquad \cos x = \frac{1}{\sec x} \qquad \tan x = \frac{1}{\cot x}$$

Theorem 4.6 (Quotient Identities)

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Theorem 4.7 (Pythagorean Identities)

$$\sin^2 x + \cos^2 x = 1 \qquad \sec^2 x = 1 + \tan^2 x \qquad \csc^2 x = 1 + \cot^2 x$$

Theorem 4.8 (Odd-even Identities)

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

$$\csc(-x) = -\csc x \qquad \sec(-x) = \sec x \qquad \cot(-x) = -\cot x$$

In other words, cosine and secant are even functions (graphs are symmetric across the y-axis) and the other four trig functions are odd (graphs have rotational symmetry about the origin).

Theorem 4.9 (Cofunction identities)

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \sin\left(\frac{\pi}{2} - x\right) = \cos x$$

Why do we need to know these identities?

They allow you to simplify or rewrite more complicated trig expressions that may arise in a problem. For example, let's say you do some math and come up with an answer like

$$\tan x + \frac{\cos x}{1 + \sin x}.$$

I can work out that this simplifies to $\sec x$, and soon you will be able to do this too!

Alternate forms of the Pythagorean identities

Consider the statement "$3 + 5 = 8$". There are several ways to restate this fact, all of which are equivalent:

There are also some incorrect ways to restate this:

Each of the Pythagorean identities can be rewritten in similar ways:

$$\sin^2 x + \cos^2 x = 1 \qquad \sec^2 x = 1 + \tan^2 x \qquad \csc^2 x = 1 + \cot^2 x$$

§4.2 EXAMPLE 1

If the following expressions can be simplified, simplify them. If they cannot be simplified, do nothing.

$\tan^2 x + 1$	$1 + \cos^2 x$	$\csc^2 x - 1$
$\tan^2 x - 1$	$1 + \tan^2 x$	$\csc^2 x - \sec^2 x$
$1 - \sec^2 \theta$	$1 - \csc^2 x$	$1 - \cos^2 x$
$\sin^2 \theta + \cos^2 \theta$	$\csc^2 w + \sin^2 w$	$\sec^2 x - 1$
$\sec^2 x - \tan^2 x$	$\cot^2 \theta + 1$	$\cot^2 \theta - 1$
$1 + \csc^2 x$	$1 - \sin^2 t$	$1 + \cot^2 x$

Simplifying trig expressions

Procedure to simplify a trig expression

1. Use odd-even identities to remove any $-$ signs from inside any trig functions.
2. If possible, simplify (part of) the expression using a Pythagorean identity (similar to how things worked on the previous page).
3. Write whatever is left in terms of sines and cosines, using the quotient and reciprocal identities.
4. Simplify using algebra.

§4.2 EXAMPLE 2

Simplify each expression as much as possible, and write your answer so that no quotients appear in the final answer.

a) $\tan x \cos x$

b) $\csc x \cos x \tan x$

c) $\dfrac{\tan(-t)}{\sec t}$

d) $\dfrac{1-\sin^2(-x)}{1+\cot^2(-x)}$

e) $\sin^2 x(1+\cot^2 x)$

f) $\tan x + \dfrac{\cos x}{1+\sin x}$

Other trig identities

There are a bunch of other trig identities that you do not need to know; I list a bunch of them below. The main takeaway from these identities is:

> Trig expressions like
>
> $$\sin(x+y) \qquad \tan(x-y) \qquad \cos 2x \qquad \tan\frac{x}{2}$$
>
> that have arithmetic inside the trig function **can be rewritten, but not in obvious ways**.

You have to look up formulas if you have to rewrite these types of expressions; here, I give some (not all) of the formulas you might find if you did some research. I will not hold you responsible for knowing these in MATH 130.

Theorem 4.10 (Power-reducing Identities)

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

The power-reducing identities are especially useful in Calculus 2 and 3.

Theorem 4.11 (Sum and Difference Identities)

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta & \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta & \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} & \tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}
\end{aligned}$$

Theorem 4.12 (Double- and half-angle Identities)

$$\begin{aligned}
\sin 2x &= 2\sin x\cos x \\
\cos 2x &\begin{cases} = 2\cos^2 x - 1 \\ = 1 - 2\sin^2 x \\ = \cos^2 x - \sin^2 x \end{cases} \\
\tan 2x &= \frac{2\tan x}{1-\tan^2 x}
\end{aligned}
\qquad
\begin{aligned}
\sin\frac{x}{2} &= \pm\sqrt{\frac{1-\cos x}{2}} \\
\cos\frac{x}{2} &= \pm\sqrt{\frac{1+\cos x}{2}} \\
\tan\frac{x}{2} &= \frac{1-\cos x}{\sin x}
\end{aligned}$$

4.3 Arctangent

Motivation: solving trig equations

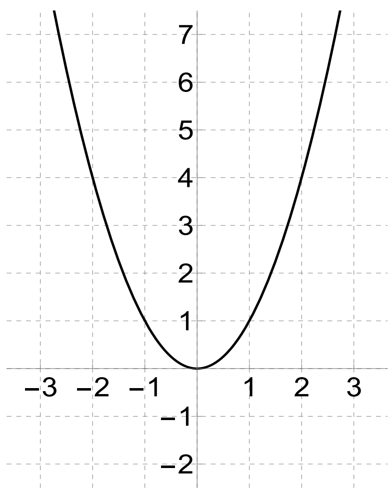
A NON-TRIGONOMETRIC EXAMPLE

Consider these three equations:

$$x^2 = 4 \qquad x^2 = 7 \qquad x^2 = -2$$

Back in Section 2.12, we learned how to interpret these equations graphically. Let $f(x) = x^2$; then these three equations become

$$f(x) = 4 \qquad f(x) = 7 \qquad f(x) = -2$$



Notice:

- Sometimes, there is no solution (this corresponds to b being not in the range of $f(x) = x^2$).

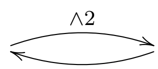
- Sometimes, we can write the answer(s) to the equations without introducing new notation.

- In other situations, we can't write the answer(s) without inventing a new symbol or word to describe at least one of the answers.

After having invented the new symbol $\sqrt{\ }$, we see that the solutions to $x^2 = b$ can be written as $x = \pm\sqrt{b}$. In the language of functions, this means

$$\text{if } f(x) = x^2, \text{ then}$$

which as an arrow diagram can be thought of as



Keep this example in mind as a prototype of what we do in this section...

we're going to do the same thing as above, but with trig equations like

$$\sin x = \frac{1}{2} \qquad \tan x = 3 \qquad \cos x = -\frac{7}{5}$$

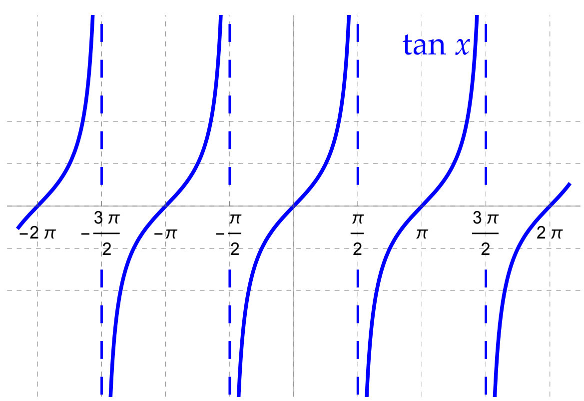
so that instead of thinking of the function $f(x) = x^2$, we'll think of $f(x) = \cos x$, $f(x) = \sin x$ or $f(x) = \tan x$.

§4.3 Example 1

Consider the equation $\tan x = \dfrac{1}{3}$.

a) If this equation is thought of as $f(x) = b$, what is f and what is b?

b) Interpret this equation in the context of the graph of the f you wrote down in part (a) (by drawing a picture). Make it clear what the x you are solving for has to do with the picture.

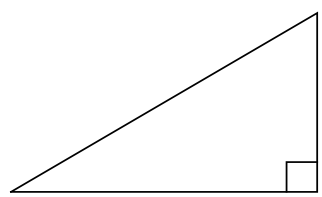


§4.3 EXAMPLE 1, CONTINUED

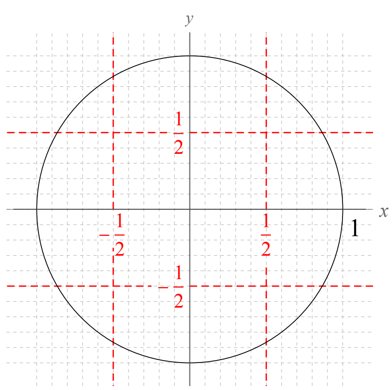
Recall that we are considering the equation $\tan x = \frac{1}{3}$.

c) Based on the picture you drew in part (b), how many solutions does this equation have? How can you write all the solutions in terms of one of the solutions?

d) Use SOHCAHTOA to interpret this equation in the context of a right triangle (by labelling the picture below). Make it clear what the x you are solving for has to do with the right triangle.



e) Use the unit circle definition of $\tan$ to interpret this equation in the context of the unit circle (by labelling the picture below). Make it clear what the x you are solving for has to do with the picture.



f) What conclusions can we draw from what we did in parts (a)-(e) of this question?

In the same way that we have to invent the notation $\sqrt{}$ to describe the positive solution of $x^2 = 7$, we have to invent notation to describe one of the infinitely many solutions x of the equation $\tan x = \frac{1}{3}$.

First, we have to pick out a solution we want to describe. In general, for any y, $\tan x = y$ always has a solution in either Quadrant I or Quadrant IV, i.e. one solution between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, so let's describe that one particular solution of $\tan x = y$ by calling it $x = \boxed{\arctan y}$.

In general, you can't describe $\arctan y$ *using previously known math langauge like* $+, -, \times, \div$, *powers, roots,* $\sin$, $\cos$ *and/or other trig functions.*

This gives us a new function $\arctan : \mathbb{R} \to \mathbb{R}$ whose main purpose is to provide notation for the x in the first or fourth quadrant that solves $\tan x = y$.

Later, we'll repeat this reasoning with sine and cosine equations to get two other functions called $\arcsin$ and $\arccos$. Collectively, $\arctan$, $\arcsin$ and $\arccos$ are called the **arc-trigonometric functions** or the **inverse trigonometric functions**.

Arctangent

Definition 4.13 *Let x be any number. The* **arctangent** *of x, denoted $\arctan x$, is the number y satisfying $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ whose tangent is x, i.e.*

$$y = \arctan x \quad \textit{means} \quad \begin{cases} \tan y = x \\ \quad \textit{and} \\ -\frac{\pi}{2} \le y \le \frac{\pi}{2} \end{cases}.$$

Since there is only one such y for each x, this defines a function $\arctan : \mathbb{R} \to \mathbb{R}$.

WARNING: The function $\arctan$ is also denoted $\tan^{-1}$, especially on calculators. This makes sense, because $\arctan$ is (sort of) an inverse function of $\tan$, and we denote the inverse of function f by f^{-1} in general.

However, I **intensely** dislike the notation $\tan^{-1}$. The reason is that it is easy to get this notation confused. As we learned in Section 2.7,

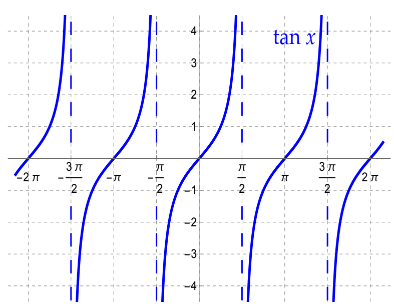
$$\tan^3 x \text{ means } (\tan x)^3$$
$$\tan^{1/2} x \text{ means } (\tan x)^{1/2} = \sqrt{\tan x}$$
$$\tan^{-8} x \text{ means } (\tan x)^{-8} = \frac{1}{(\tan x)^8} = (\cot x)^8 = \cot^8 x$$

but

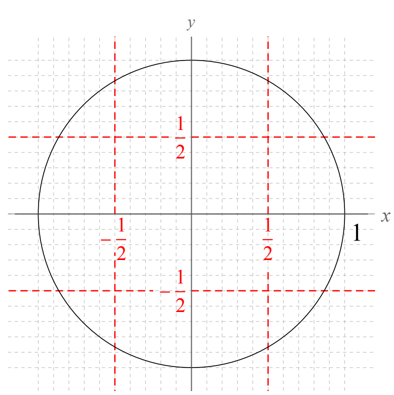
$$\tan^{-1} x \text{ does NOT mean } (\tan x)^{-1} = \frac{1}{\tan x} = \cot x.$$

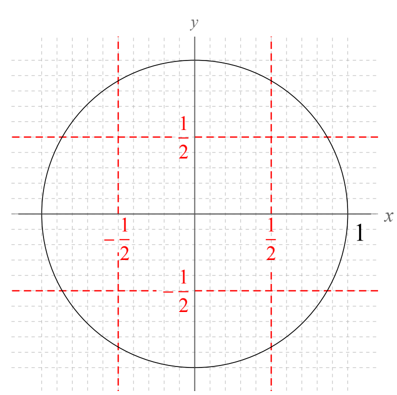
Don't confuse $\boxed{\arctan x = \tan^{-1} x}$ **with** $\boxed{\cot x = \dfrac{1}{\tan x}}$ **!**

Interpretation of $\arctan$ **in terms of graphs**

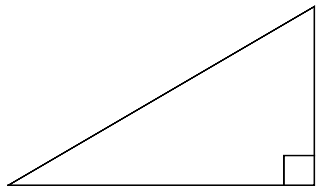


Interpretation of $\arctan$ **on the unit circle**



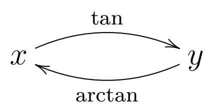


Interpretation of $\arctan$ **in a right triangle (only valid for** $x > 0$**)**



Temporary interpretation of $\arctan$ **with arrow diagrams**

(this diagram will be upgraded later, after Theorem 4.16)



§4.3 EXAMPLE 2

Compute each quantity, if it can be simplified easily (if not, do nothing).

a) $\arctan 1$

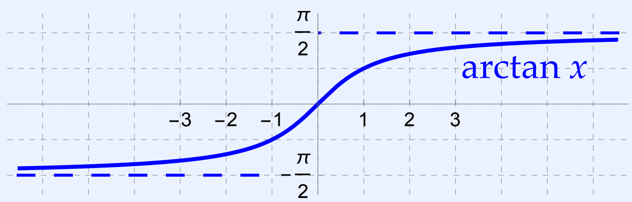
b) $\arctan \frac{1}{2}$

c) $\arctan -\sqrt{3}$

Theorem 4.14 (Properties of arctangent)
Domain: *The domain of* $\arctan$ *is* $\mathbb{R}$. *This means that* $\arctan x$ *always exists.*
Range: *The range of* $\arctan$ *is* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. *This means that for any* x, $\arctan x$ *is always between* $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$.
Symmetry: $\arctan$ *is odd. This means that for any* $x \in \mathbb{R}$,

$$\arctan(-x) = -\arctan x.$$

Asymptotes: $\arctan$ *has two HAs:* $y = \frac{-\pi}{2}$ *(to the left) and* $y = \frac{\pi}{2}$ *(to the right).* $\arctan$ *has no VA.*
Graph: *here is a graph of* $f(x) = \arctan x$*:*

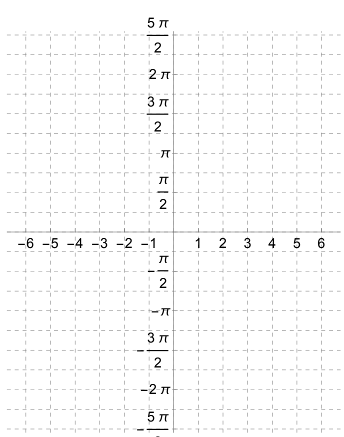


The key properties of this graph are:
- *the graph looks kind of like the graph of* $y = \sqrt[3]{x}$*;*
- *the graph goes through* $(0, 0)$*;*
- *the graph goes up from left to right;*
- *the graph has the symmetry of an odd function; and*
- *the graph has the HAs described above.*

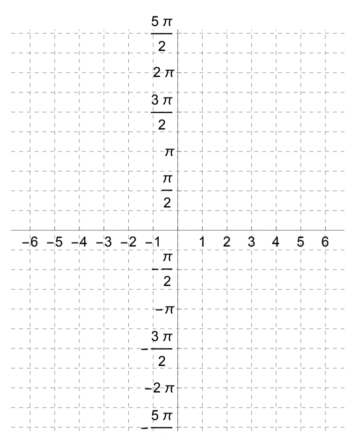
§4.3 Example 3

Sketch the graph of each function:

a) $f(x) = -4 \arctan x$



b) $F(t) = \arctan(t + 3) + \frac{\pi}{2}$



Evaluating special arctan values

To understand the values of arctan we can actually simplify, let's use the tangents of special angles:

TANGENT VALUE WE ALREADY KNOW	$\tan 0 = 0$	$\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$	$\tan \frac{\pi}{4} = 1$	$\tan \frac{\pi}{3} = \sqrt{3}$
ARROW DIAGRAM	$0 \xrightarrow{\tan} 0$, $0 \xleftarrow{\arctan} 0$	$\frac{\pi}{6} \xrightarrow{\tan} \frac{1}{\sqrt{3}}$, $\frac{\pi}{6} \xleftarrow{\arctan} \frac{1}{\sqrt{3}}$	$\frac{\pi}{4} \xrightarrow{\tan} 1$, $\frac{\pi}{4} \xleftarrow{\arctan} 1$	$\frac{\pi}{3} \xrightarrow{\tan} \sqrt{3}$, $\frac{\pi}{3} \xleftarrow{\arctan} \sqrt{3}$
ARCTAN VALUE WE NOW KNOW	arctan $\quad =$	arctan $\quad =$	arctan $\quad =$	arctan $\quad =$

Together with $\arctan(-x) = -\arctan x$, these four values of arctan are sufficient to evaluate any arctangent you will encounter (that can actually be evaluated; if it can't, you just leave the answer with an arctan in it).

Theorem 4.15 (Values of arctan to know)

x	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
$\arctan x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$

§4.3 EXAMPLE 4

Compute each quantity:

a) $\arctan 0 + 1$

b) $12 \arctan -1$

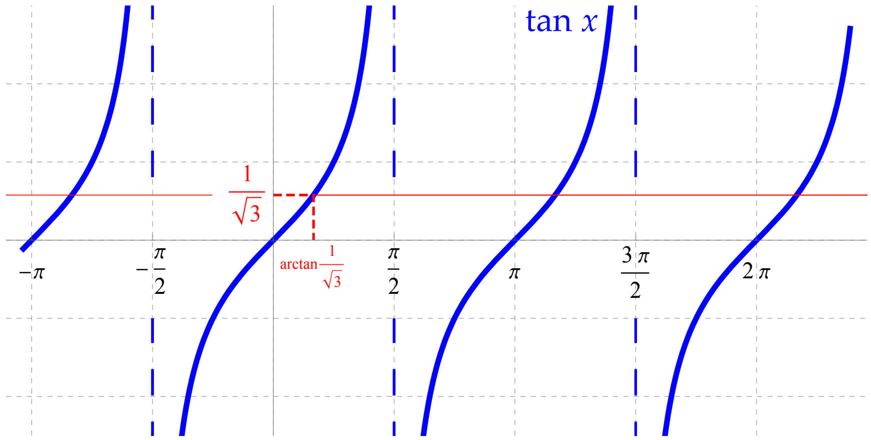
c) $\arctan^2(5 - 4)$

d) $\arctan(-\sqrt{3})$

Back to solving equations

MOTIVATING EXAMPLE

Solve the equation $\tan x = \dfrac{1}{\sqrt{3}}$.



One solution of the equation $\tan x = \dfrac{1}{\sqrt{3}}$ *is...*

All solutions of the equation $\tan x = \dfrac{1}{\sqrt{3}}$ *are...*

This example generalizes: to find **all** the solutions of $\tan x = b$, you need to take [blank] to get one solution, then add [blank] to that one solution to get all of them.

Theorem 4.16 (Inverse relationships between tan and arctan)

- *The inverse of* $f(x) = \arctan x$ *is the function* $f^{-1}(x) = \tan x$.

  In other words, there is one solution of the equation $\arctan x = y$, *namely* $x = \tan y$.

- *The inverse of* $f(x) = \tan x$ *is the multifunction* $f^{-1}(x) = \pi n + \arctan x$. *In other words, the equation* $\tan x = y$ *has infinitely many solutions, which are*

$$\arctan y,\ \arctan y \pm \pi,\ \arctan y \pm 2\pi,\ \arctan y \pm 3\pi, \ldots$$

  and which are collectively written as

$$\arctan y + \pi n \quad \text{or} \quad \pi n + \arctan y.$$

  The n *here represents an arbitrary integer like* $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$.

The arrow diagrams that best describe these relationships are:

In both of these diagrams, we think of "undoing" the arrow that goes left-to-right by using the arrow that goes right-to-left.

§4.3 EXAMPLE 5

Solve each equation:

a) $\tan x = -\sqrt{3}$

b) $\tan x = \dfrac{2}{\sqrt{5}}$

4.4 Arcsine and arccosine

The function $\arcsin$ is created as an inverse (of sorts) to $\sin$, in the same way $\arctan$ was created as an inverse (of sorts) to $\tan$.

Definition and interpretation of arcsine

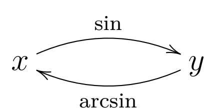
Definition 4.17 *Let x be a number in the interval $[-1,1]$. The* **arcsine** *of x, denoted $\arcsin x$, is the number y satisfying $-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$ whose sine is x, i.e.*

$$y = \arcsin x \quad \textit{means} \quad \begin{cases} \sin y = x \\ \quad \textit{and} \\ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \end{cases}.$$

Since there is only one such y for each x, this defines a function $\arcsin : [-1,1] \to \mathbb{R}$.

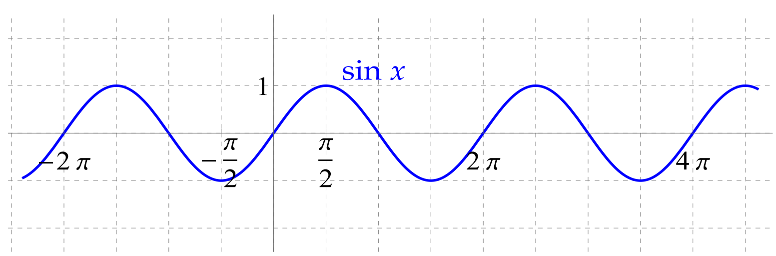
Basic idea of $\arcsin$

$x = \arcsin y$ is a solution of the equation $\sin x = y$, in the same way $x = \arctan y$ is a solution of the equation $\tan x = y$.

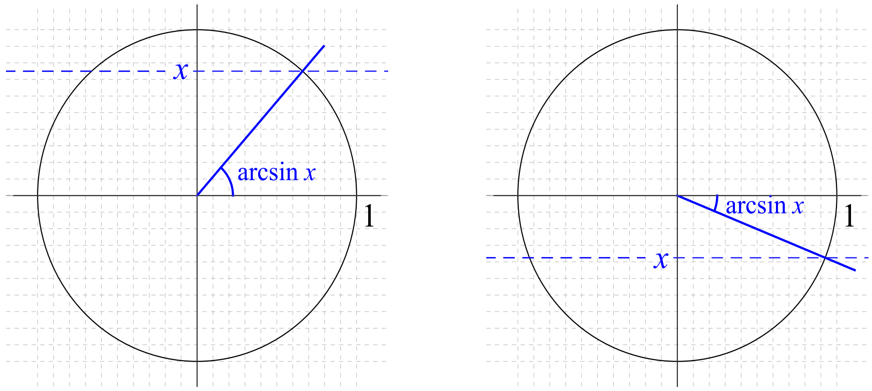


$\longleftarrow$ (this will be improved later)

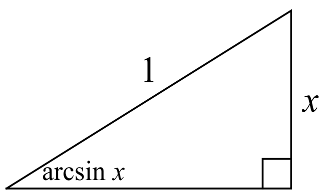
Interpretation of $\arcsin$ in terms of graphs



Interpretation of $\arcsin$ **on the unit circle**



Interpretation of $\arcsin$ **in a right triangle (only valid for** $x > 0$**)**



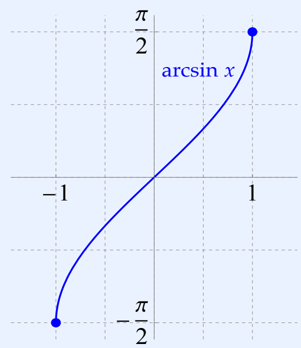
Properties of arcsine

Theorem 4.18 (Properties of arcsine)
Domain: *The domain of* $\arcsin$ *is* $[-1, 1]$.
This means that if $x > 1$ *or* $x < -1$, *then* $\arcsin x$ *DNE.*
Range: *The range of* $\arcsin$ *is* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
This means that for any x, $\arcsin x$ *is always between* $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$.
Symmetry: $\arcsin$ *is an odd function. This means that for any* $x \in \mathbb{R}$,

$$\arcsin(-x) = -\arcsin x.$$

Asymptotes: $\arcsin$ *has no asymptotes.*

Theorem 4.19 (Graph of arcsine) *Here is a graph of* $f(x) = \arcsin x$*:*



The key properties of this graph are:

- *the graph looks like a piece of* $y = \tan x$ *or* $y = x^3$*;*
- *the graph goes through* $(0, 0)$*;*
- *the graph goes up from left to right, from the point* $\left(-1, -\frac{\pi}{2}\right)$ *to the point* $\left(1, \frac{\pi}{2}\right)$*;*
- *the graph does not continue indefinitely to the left and right; and*
- *the graph has the symmetry of an odd function.*

Evaluating special $\arcsin$ values

To understand the values of $\arcsin$ we can actually simplify, let's use the sines of special angles that we already know to figure out some arcsine values:

$\sin 0 =$	$\sin \frac{\pi}{6} =$	$\sin \frac{\pi}{4} =$	$\sin \frac{\pi}{3} =$	$\sin \frac{\pi}{2} =$
0 $\xrightarrow{\sin}$ $\xleftarrow{\arcsin}$	$\frac{\pi}{6}$ $\xrightarrow{\sin}$ $\xleftarrow{\arcsin}$	$\frac{\pi}{4}$ $\xrightarrow{\sin}$ $\xleftarrow{\arcsin}$	$\frac{\pi}{3}$ $\xrightarrow{\sin}$ $\xleftarrow{\arcsin}$	$\frac{\pi}{2}$ $\xrightarrow{\sin}$ $\xleftarrow{\arcsin}$
$\arcsin \quad =$	$\arcsin \quad =$	$\arcsin \quad =$	$\arcsin \quad =$	$\arcsin \quad =$

Together with $\arcsin(-x) = -\arcsin x$, these values are sufficient to evaluate any arcsine that can actually be evaluated.

Theorem 4.20 (Values of $\arcsin$ to know)

x	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\arcsin x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

§4.4 EXAMPLE 1

Compute each quantity:

a) $\arcsin 0$

b) $\arcsin(-1)^2$

c) $2 - 4\arcsin\left(-\frac{\sqrt{2}}{2}\right)$

d) $\arcsin\frac{3}{2}$

e) $\arcsin\frac{2}{5}$

f) $\arcsin\frac{1}{2}\arctan 1$

Theorem 4.21 (Inverse relationships between sin and arcsin)

- *The inverse of* $f(x) = \arcsin x$ *is the function* $f^{-1}(x) = \sin x$.

  In other words, there is one solution of the equation $\arcsin x = y$*, namely* $x = \sin y$.

- *The inverse of* $f(x) = \sin x$ *is the multifunction* $f^{-1}(x) = 2\pi n \arcsin x, 2\pi n + \pi - \arcsin x$. *In other words, the equation* $\sin x = y$ *has infinitely many solutions, which are*

$$\arcsin y,\ \arcsin y \pm 2\pi,\ \arcsin y \pm 4\pi,\ \arcsin y \pm 6\pi, ...$$

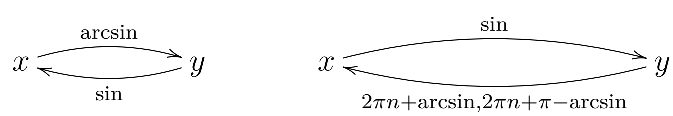
$$\pi - \arcsin y,\ 3\pi - \arcsin y,\ 5\pi - \arcsin y,\ 7\pi - \arcsin y, ...$$

$$-\pi - \arcsin y,\ -3\pi - \arcsin y,\ -5\pi - \arcsin y,\ -7\pi - \arcsin y, ...$$

and these solutions are collectively written as

$$\arcsin y + 2\pi n, \pi - \arcsin y + 2\pi n$$
or
$$2\pi n + \arcsin y, 2\pi n + \pi - \arcsin y.$$

The arrow diagrams that best describe these relationships are:



In both of these diagrams, we think of "undoing" the arrow that goes left-to-right by using the arrow that goes right-to-left.

§4.4 Example 2

Solve each equation:

a) $\sin x = -\dfrac{\sqrt{3}}{2}$

b) $\sin x = 1$

Arccosine

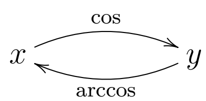
Definition 4.22 *Let x be a number in the interval $[-1, 1]$. The* **arccosine** *of x, denoted* $\arccos x$*, is the number y satisfying $0 \le y \le \pi$ whose cosine is x, i.e.*

$$y = \arccos x \quad \textit{means} \quad \begin{cases} \cos y = x \\ \quad \textit{and} \\ 0 \le y \le \pi \end{cases}.$$

Since there is only one such y for each x, this defines a function $\arccos : [-1, 1] \to \mathbb{R}$.

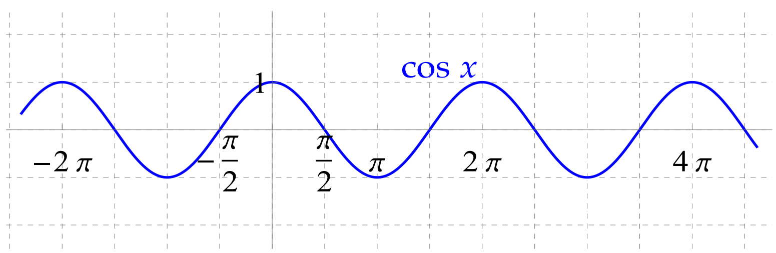
Basic idea of $\arccos$

$x = \arccos y$ is a solution of the equation $\cos x = y$:

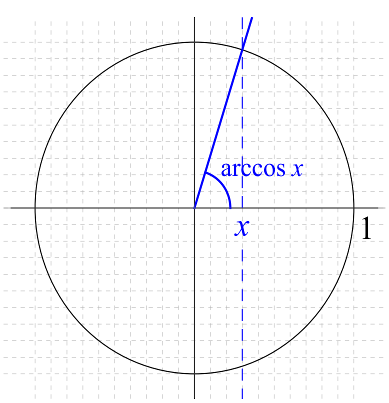


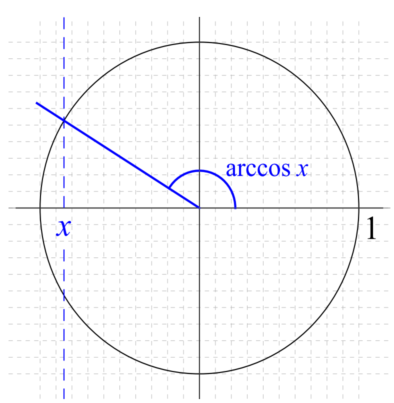
$\longleftarrow$ (this will be improved later)

Interpretation of $\arccos$ in terms of graphs

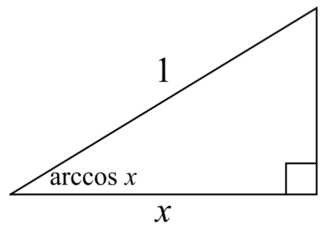


Interpretation of $\arccos$ on the unit circle



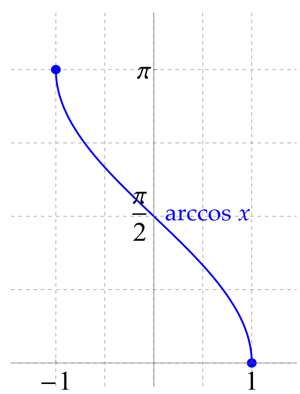


Interpretation of $\arccos$ **in a right triangle (only valid for** $x > 0$**)**



> **Theorem 4.23 (Properties of arccosine)**
> **Domain:** *The domain of* $\arccos$ *is* $[-1, 1]$.
> *This means that if* $x > 1$ *or* $x < -1$, *then* $\arccos x$ *DNE.*
> **Range:** *The range of* $\arccos$ *is* $[0, \pi]$.
> *This means that for any* x, $\arccos x$ *is always between* 0 *and* π.
> **Symmetry:** $\arccos$ *has no useful symmetry.*
> **Asymptotes:** $\arccos$ *has no asymptotes.*

Here's the graph of $\arccos$, but I don't care if you remember what this graph looks like:



Evaluating special $\arccos$ values

Similar to how we learned values of $\arctan$ and $\arcsin$, we can figure out values of $\arccos$ using the arrow diagram

$$x \underset{\arccos}{\overset{\cos}{\rightleftarrows}} y$$

If we do this, we get the chart on the next page (which is longer than the charts for $\arctan$ and $\arcsin$ because $\arccos$ isn't odd, so we can't use symmetry to get the negative values):

Theorem 4.24 (Values of $\arccos$ to know)

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\arccos x$	π	$\frac{5\pi}{6}$	$\frac{3\pi}{4}$	$\frac{2\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	0

§4.4 EXAMPLE 3

Compute each quantity:

a) $\arccos 2$

b) $\arccos \frac{4-5}{2}$

c) $\arccos \arctan 0$

d) $\arccos -1 + \arcsin -1$

Theorem 4.25 (Inverse relationships between $\cos$ and $\arccos$)

- *The inverse of* $f(x) = \arccos x$ *is the function* $f^{-1}(x) = \cos x$.

  In other words, there is one solution of the equation $\arccos x = y$, *namely* $x = \cos y$.

- *The inverse of* $f(x) = \cos x$ *is the multifunction* $f^{-1}(x) = 2\pi n \pm \arccos x$. *In other words, the equation* $\cos x = y$ *has infinitely many solutions, which are*

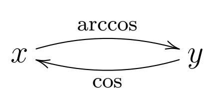
$$\arccos y,\ \arccos y \pm 2\pi,\ \arccos y \pm 4\pi,\ \arccos y \pm 6\pi, ...$$

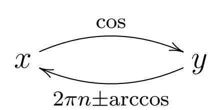
$$-\arccos y,\ -\arccos y \pm 2\pi,\ -\arccos y \pm 4\pi,\ -\arccos y \pm 6\pi, ...$$

  and these solutions are collectively written as

$$\arccos y + 2\pi n, -\arcsin y + 2\pi n \text{ or } 2\pi n \pm \arccos y.$$

The arrow diagrams that best describe these relationships are:





As usual, we think of "undoing" the arrow that goes left-to-right by using the arrow that goes right-to-left.

§4.4 EXAMPLE 4

Solve each equation:

a) $\cos x = \dfrac{\sqrt{3}}{2}$

b) $\arccos x = \dfrac{\pi}{4}$

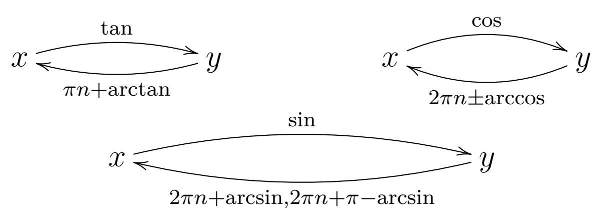
Solving simple trig equations

How to solve a simple trigonometric equation

1. Isolate the trig function on one side of the equation.
2. If the trig function is cot, sec or csc, take reciprocals of both sides. This makes the equation look like one of these three forms:

$$\sin x = y \qquad \cos x = y \qquad \tan x = y$$

3. Check if the equation has a solution (any tangent equation is solvable; sine and cosine equations have solutions if and only if $-1 \leq y \leq 1$).
4. If the equation has a solution, solve it with an arctrig function, using one of these arrow diagrams we learned earlier:



§4.4 EXAMPLE 5

Solve each equation:

a) $6\cos x - 2 = 4$

b) $7 \sin x + 1 = 2$

Solution: Isolate the $\sin x$ term by subtracting 1 from both sides, then dividing through by 7:

$$\begin{aligned} 7 \sin x + 1 &= 2 \\ 7 \sin x &= 1 \\ \sin x &= \frac{1}{7} \end{aligned}$$

That means $x = \boxed{2\pi n + \arcsin \frac{1}{7}, 2\pi n + \pi - \arcsin \frac{1}{7}}$ (and this can't be simplified further).

c) $\sin x = \sqrt{3} - \sin x$

d) $4 \csc x = 1$

e) $\sec x + 5 = 7$

f) $3 \cot x = \sqrt{3}$

Solution: Isolate the $\cot x$ term by dividing both sides by 3 to get

$$\cot x = \frac{\sqrt{3}}{3}.$$

Now, take reciprocals of both sides (flipping over the right-hand side) to get

$$\tan x = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

That means $x = \pi n + \arctan \sqrt{3} = \boxed{\pi n + \frac{\pi}{3}}$.

g) $\frac{1}{4} + \frac{1}{2} \sin x = 1$

Solution: Isolate the $\sin x$ term by subtracting $\frac{1}{4}$ from both sides, then multiplying through by 2:

$$\begin{aligned} \frac{1}{4} + \frac{1}{2} \sin x &= 1 \\ \frac{1}{2} \sin x &= 1 - \frac{3}{4} = \frac{3}{4} \\ \sin x &= 2\left(\frac{3}{4}\right) \\ \sin x &= \frac{3}{2} \end{aligned}$$

Since $\frac{3}{2} > 1$, $\arcsin \frac{3}{2}$ DNE, so there is $\boxed{\text{no solution}}$.

§4.4 EXAMPLE 6

a) Find all x-coordinates of points where the graphs of $y = \tan x$ and $y = -1$ intersect.

b) Compute the x-intercepts of $f(x) = 2\sin x + 1$.

§4.4 EXAMPLE 7

Solve each equation:

a) $3 \arcsin x - \pi = 0$

b) $2 \arccos x = 0$

c) $7 \arctan x - 5 = 3$

Solution: Isolate the $\arctan x$ term by adding 5 to both sides, then dividing through by 7:

$$\begin{aligned} 7 \arctan x - 5 &= 3 \\ 7 \arctan x &= 8 \\ \arctan x &= \frac{8}{7} \end{aligned}$$

We undo $\arctan$ with $\tan$, so that means $x = \boxed{\tan \frac{8}{7}}$.

4.5 Exponential functions

MOTIVATING EXAMPLE: COMPOUND INTEREST

Investment accounts (like 401(k)s or savings accounts) and debt (like mortgages and credit card debt) change as time passes according to an interest rate. For example, if you have a credit card that you owe $\$1000$ on (after making this month's payment), and the card charges you 2% interest monthly, then after one month that $\$1000$ you owe will turn into

(plus whatever you charge over the next month).

More generally, if an account (or an amount owed) is accumulating interest at rate i per unit of time and currently has value p, then its value one unit of time from now is [ ] In other words, the value of the account is [ ] by [ ] for every one unit of time that passes.

FOLLOW-UP QUESTION

What will the value of the account be two units of time from now? Three units of time from now? x units of time from now?

In the context of our credit card example, after 6 months the $\$1000$ will turn into

$$\approx \$1126.16$$

and after 3 years, the $\$1000$ will turn into

$$\approx \$2039.89.$$

This example leads us to a new class of functions called *exponential functions* that model not only interest-bearing accounts, but lots of other other stuff (population dynamics, physics of nuclear reactions, etc.)

Definition 4.26 *An* **exponential function** *is a function* $f : \mathbb{R} \to \mathbb{R}$ *whose rule is*

$$f(x) = a \cdot b^x$$

for a nonzero constant a *and a positive constant* $b \neq 1$. b *is called the* **base** *of the exponential function.*

If $b > 1$, f *is called an* **exponential growth** *function or model; if* $0 < b < 1$, *then* f *is called an* **exponential decay** *function or model.*

Note: we don't allow the base b to be 1. For if it was, $f(x) = a \cdot b^x = a \cdot 1^x = a$ which is a constant function (not classified as exponential).

The same thing would happen if $b = 0$.

We don't allow the base b to be negative either, because you can't take a negative number to any power: for instance, $(-2)^{1/2} = \sqrt{-2}$ DNE.

When we use an exponential function to model an interest-bearing account,

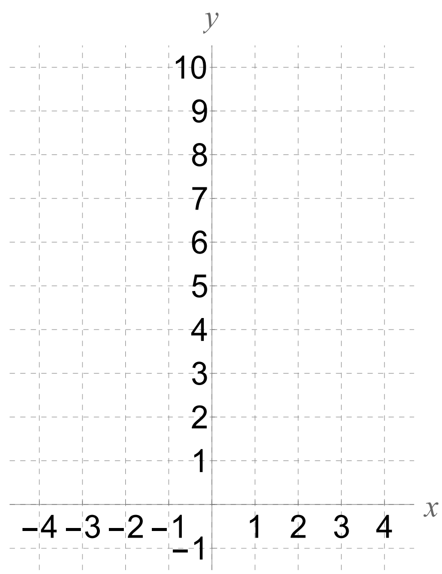
$$\begin{aligned} x &= \text{ the elapsed time} \\ f(x) &= \text{ size of account at time } x \\ b &= 1 + i = 1 + \text{ interest rate per unit of time} \\ a &= p = \text{ principal} = \text{initial account size.} \end{aligned}$$

Graphs of exponential functions

§4.5 EXAMPLE 1

Complete the following table of values for the function $f(x) = 2^x$, and use that table to sketch the graph of f.

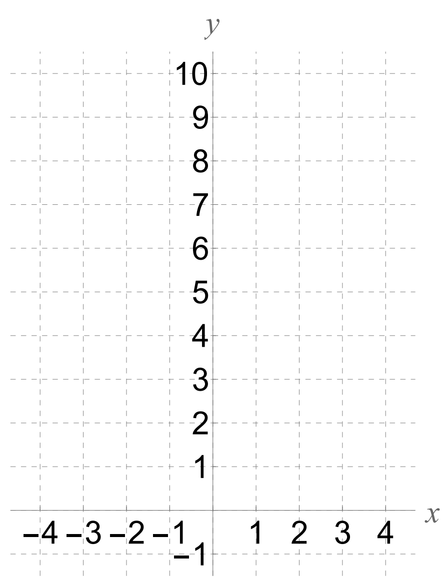
x	$f(x) = 2^x$
-3	
-2	
-1	
0	
1	
2	
3	



§4.5 EXAMPLE 2

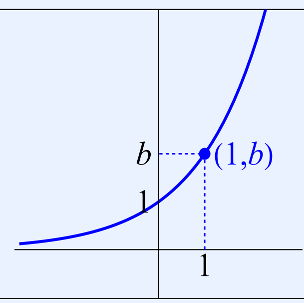
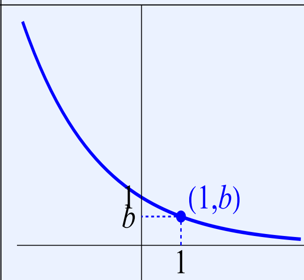
Complete the following table of values for the function $g(x) = \left(\frac{2}{3}\right)^x$, and use that table to sketch the graph of g.

x	$f(x) = \left(\frac{2}{3}\right)^x$
-3	
-2	
-1	
0	
1	
2	
3	



We can use Examples 1 and 2 as prototypes for what the graph of any exponential function looks like:

Theorem 4.27 (The graph of an exponential function) *Let $b > 0$. The graph of $f(x) = b^x$ looks like one of two types, depending on whether $b > 1$ or $0 < b < 1$:*

	$b > 1$	$0 < b < 1$
GRAPH		
POINTS ON GRAPH	$(0,1)$ *and* $(1,b)$ *(also* $\left(-1, \frac{1}{b}\right)$ *)*	
VA	*none*	
HA	$y = 0$	
GRAPH APPROACHES HA TO THE ...	*... left*	*... right*
FROM LEFT TO RIGHT, THE GRAPH ...	*... increases*	*... decreases*
THIS SITUATION IS CALLED ...	*exponential* ***growth***	*exponential* ***decay***

Theorem 4.28 (Properties of exponential functions) *Let $b > 0$.*

Domain: *The domain of $f(x) = b^x$ is $\mathbb{R}$.*
This means that you can always compute b^x, for any b and x.

Range: *The range of $f(x) = b^x$ is $(0, \infty)$.*
This means that $b^x > 0$ for any x.

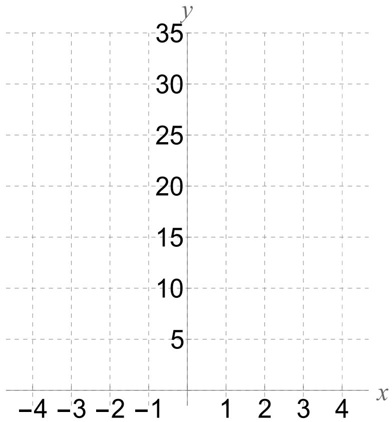
Symmetry: *Exponential functions have no useful symmetry.*

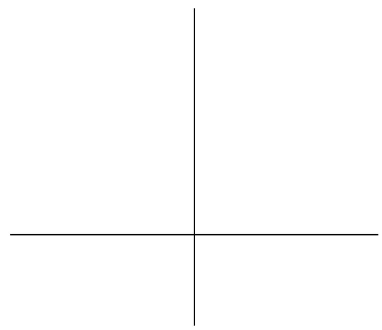
Asymptotes: *$f(x) = \cdot b^x$ has HA $y = 0$, but no VA.*

Rate of change of an exponential function

§4.5 EXAMPLE 3

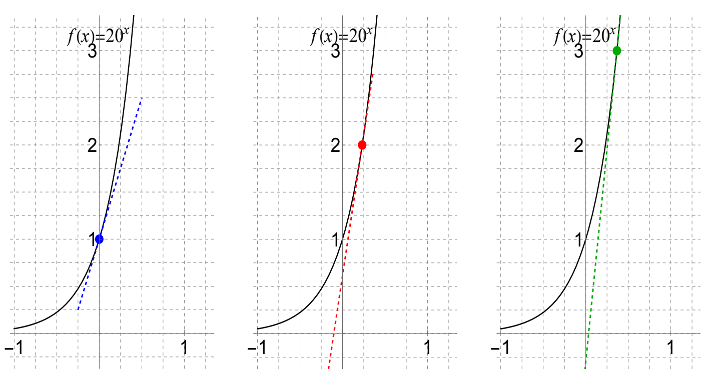
Sketch a crude graph of the function $f(x) = 5^x$.





§4.5 EXAMPLE 4

Here are computer-generated graphs of $f(x) = 20^x$, where x goes from -1 to 1. Determine the instantaneous rate of change of f at each value of (x, y) in the indicated chart, and then try to estimate the instantaneous rate of change of f at the point $(2, 400)$.



x	y	inst. rate of change at (x, y)
0	1	
.2313...	2	
.3667...	3	
x	y	
2	400	

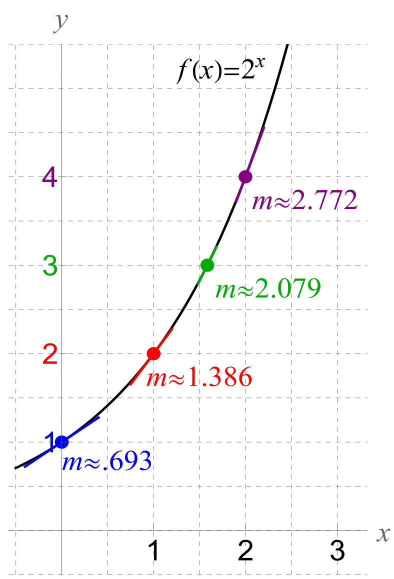
This example demonstrates the following important fact:

> **Theorem 4.29** *Exponential functions are exactly those whose instantaneous rate of change is proportional to their value/output.*

P.S. What kind of functions have a rate of change that is proportional to their input?

The natural exponential function

We just learned that the instantaneous rate of change of an exponential function $f(x) = b^x$ is proportional to the output y. In Example 4, we studied how this worked for $f(x) = 20^x$; here's the same stuff for $f(x) = 2^x$:



y	inst. rate of change at (x, y)
1	.693147
2	1.386294
3	2.07944
4	2.772588
⋮	⋮
y	$.693147\,y$

If we chose a different base, we'd get a different proportionality constant:

base	exponential function	instantaneous rate of change at (x, y) (must be proportional to y)
1.5	$f(x) = (1.5)^x$	$.405465\,y$
2	$f(x) = 2^x$	$.693147\,y$
2.5	$f(x) = (2.5)^x$	$.916291\,y$
3	$f(x) = 3^x$	$1.09861\,y$
3.5	$f(x) = (3.5)^x$	$1.25276\,y$

Conclusion: Somewhere between 2.5 and 3, there is a special number e so that the instantaneous rate of change of e^x at the point (x, y) is $1y = y$.

It would be great if this number e could be written easily in terms of whole numbers using $+$, $-$, $\times$, $\div$, roots, trig functions, etc., but it can't. So we just call this number e:

Definition 4.30 Euler's number, *denoted e, is the constant*

$$e = \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$$

e is an irrational number which is about 2.71828182845....

You should remember that $e \approx 2.7$ in the same way you remember $\pi = 3.14$, but in math classes, you leave e (and π) in your answers to any question unless told otherwise.

Definition 4.31 *The* **natural exponential** *function is the function $f(x) = e^x$. We also write this function as $f(x) = \exp(x)$ or $f(x) = \exp x$, and represent it with the arrow $\xrightarrow{\exp}$.*

The reason for the "exp" notation is that in many applications, we'll need to take the exponential not of x, but of some complicated formula. So instead of writing, for example,

$$e^{\frac{2x^2(1+\sin 2x)}{4\sqrt[3]{x}+\arctan x^4}},$$

we'd write this as

$$\exp\left(\frac{2x^2(1+\sin 2x)}{4\sqrt[3]{x}+\arctan x^4}\right) \quad \text{or} \quad \exp\frac{2x^2(1+\sin 2x)}{4\sqrt[3]{x}+\arctan x^4}$$

to keep from having to write stuff that is really small.

The point of e: e is a number so that $f(x) = e^x$ has an instantaneous rate of change of y at any point (x, y) on its graph (i.e. the rate of change equals the output).

This property makes e^x **much** easier to work with than 2^x or b^x when doing calculus.

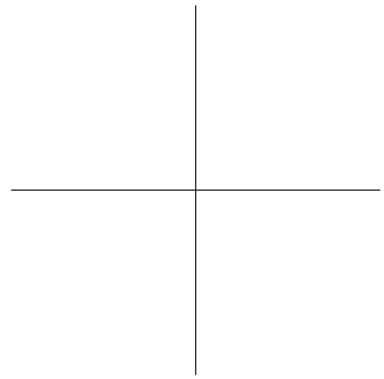
So if you hear a mathematician talk about "the exponential function" or "an exponential function" without mentioning its base, they mean that the base is e. In fact, we will later learn that **every** exponential function can be rewritten so that it has base e.

Transformed exponential graphs

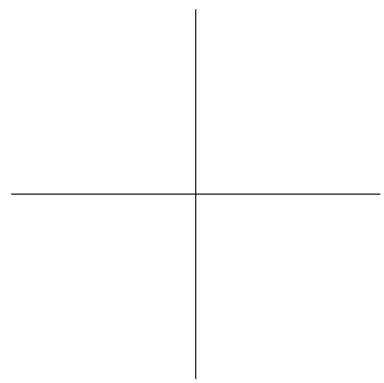
§4.5 EXAMPLE 5

For each function, diagram it and sketch its graph:

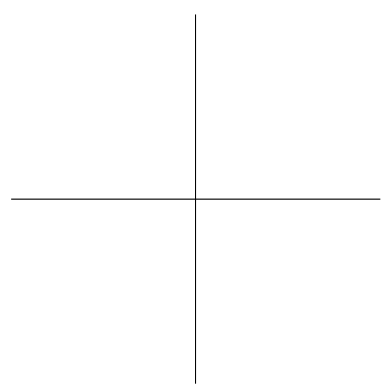
a) $f(x) = e^x$



b) $h(x) = \frac{1}{2}e^{x+3}$



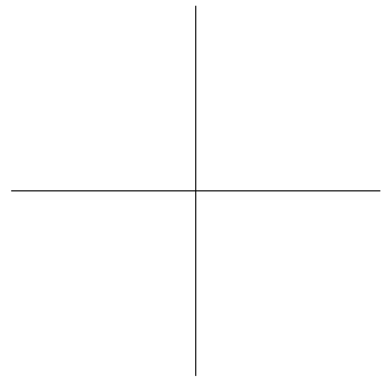
c) $l(x) = 5\exp\frac{x}{4}$



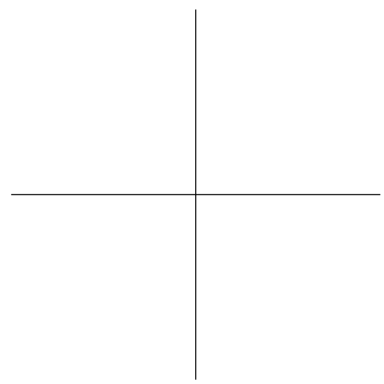
§4.5 EXAMPLE 6

For each given function, identify its horizontal and/or vertical asymptotes and sketch its graph:

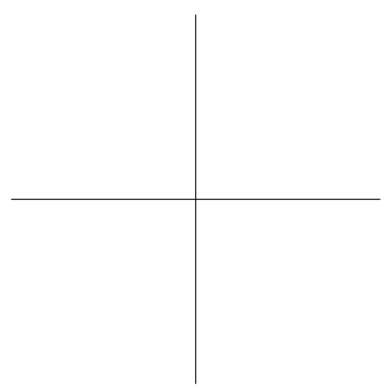
a) $f(x) = 4 \cdot 3^x$



b) $g(x) = -2^x$

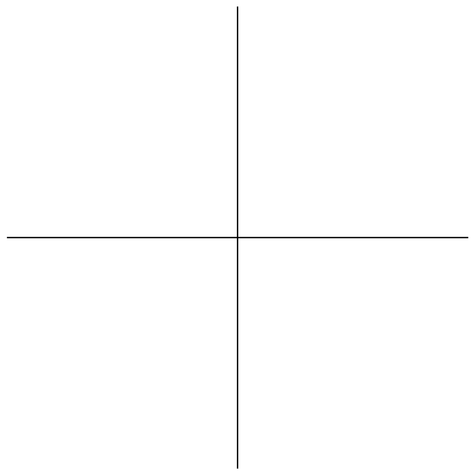


c) $k(x) = \left(\frac{2}{3}\right)^x$



§4.5 EXAMPLE 7

Consider the functions $f(x) = 2^x$ and $g(x) = f(-x) = 2^{-x}$. Sketch crude graphs of these functions, and then describe the relationship between the two graphs.

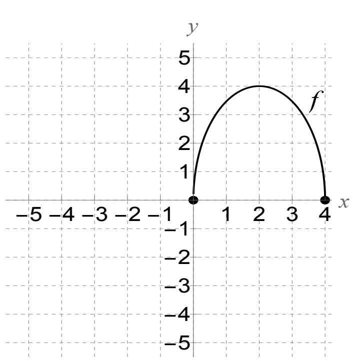


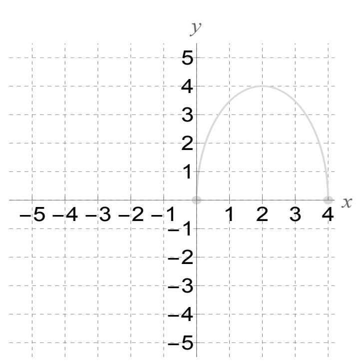
Follow-up: the $-$ sign used to create g from f comes [ ] the exponential, so it affects the graph [ ].

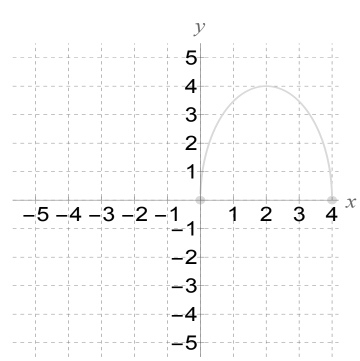
> **Theorem 4.32 (Horizontal reflections)** *The graph of $f(-x)$ is obtained from the graph of f by reflecting it horizontally across the y-axis.*

§4.5 EXAMPLE 8

The graph of some unknown function f is given below. Sketch the graphs of $g(x) = f(-x)$ and $h(x) = -f(x)$ on the provided axes.







§4.5 EXAMPLE 9

Write a function of the form $f(x) = a \cdot b^x$ that goes through each given pair of points:

a) $(0, 1)$ and $(1, 7)$

b) $(0, 3)$ and $(1, 6)$

c) $(3, 11)$ and $(7, 39)$

Exponential functions vs. polynomials

§4.5 EXAMPLE 10

In each part of this problem, you are given a table of values for a function. Determine whether the function is *linear*, *exponential*, or *neither*:

a)

x	2	3	4	5
$f(x)$	4	12	36	108

b)

x	2	3	4	5
$G(x)$	18	23	28	33

c)

x	2	3	4	5
$p(x)$	10	20	40	70

d)

x	2	3	4	5
$k(x)$	27	25	23	21

Solution: For each increase of the input by 1 unit, the output decreases by 2 units. Therefore k is $\boxed{\text{linear}}$.

§4.5 EXAMPLE 11

In each part of this problem, you are given two functions. Circle the function that has bigger values (i.e. outputs) when x is very large:

a) $f(x) = x^4 \qquad g(x) = x^3$

b) $f(x) = 5x^4 \qquad g(x) = 9x^4$

c) $f(x) = 8x^4 \qquad g(x) = 2x^5$

d) $f(x) = 100000x^4 \qquad g(x) = \dfrac{1}{1000000}x^{4.000001}$

e) $f(x) = 2^x \qquad g(x) = 3^x$

f) $f(x) = 100 \cdot 2^x \qquad g(x) = 2.25^x$

g) $f(x) = e^x \qquad g(x) = x^{100}$

h) $f(x) = (1.000000001)^x \qquad g(x) = 1000000000x^{100000000}$

Exponentiation rules as arrow diagrams

Remark: in this subsection, we'll focus on natural exponentials (base e), but the ideas work for exponential functions with any base.

RECALL

Earlier in the course we learned a bunch of exponent rules, including:

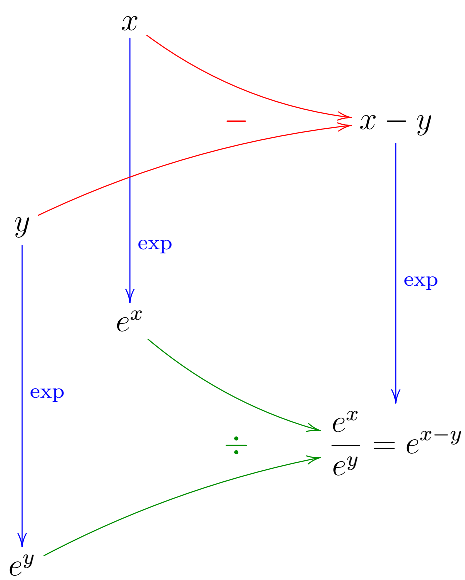
Here, we'll focus on two of these rules and discuss what they mean in terms of arrow diagrams. This won't directly lead to any homework right now, but mastering this provides a foundation for learning logarithm rules (which a lot of students have trouble with), and there will be homework on that.

Let's start with the rule $e^{x+y} = e^x e^y$. To understand this rule better, let's diagram both sides of this rule:

LHS: RHS:

Since these two sides are always equal, we can combine them into a single arrow diagram like this:

Similarly, if we start with the rule $e^{x-y} = \dfrac{e^x}{e^y}$, we get this picture:



All together, we have this important concept:

Key conceptual idea of exponentiation:

Exponentiation converts $\begin{matrix}\text{addition}\\ \text{subtraction}\end{matrix}$ to $\begin{matrix}\text{multiplication}\\ \text{division}\end{matrix}$.

More simply: $\begin{matrix}+\\ -\end{matrix} \xrightarrow{\exp} \begin{matrix}\times\\ \div\end{matrix}$

Keep this in mind as we move to the next couple of sections (and beyond).

Enrichment: a theoretical issue with exponential functions

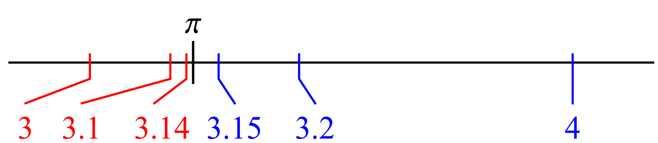
Defining a function like $f(x) = 3^x$ seems harmless enough. In many cases, we already know exponent and radical rules that tell us what 3^x means:

- if x is a whole number, like say $x = 5$, then $3^x = 3^5 =$
- if $x = 0$, then $3^x = 3^0 =$
- if x is a negative whole number, like say $x = -3$, then $3^x = 3^{-3} =$
- if x is a rational number, like say $x = \dfrac{8}{5}$, then $3^x = 3^{8/5} =$
- if x is a decimal, like say $x = 2.72$, then we can write $x =$ so $3^x = 3^{2.72} =$

But there's a problem: what if x is irrational (not a whole or rational number or a terminating decimal), what happens? For instance, what is meant by this expression:

$$3^{\pi} = ?$$

Idea:

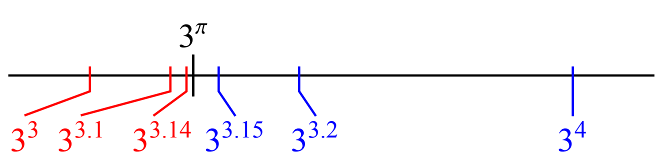


$$3.1 < \pi < 3.2 \quad \Rightarrow \quad 30.1353 \approx 3^{3.1} < 3^{\pi} < 3^{3.2} \approx 33.6347$$

$$3.14 < \pi < 3.15 \quad \Rightarrow \quad 30.4891 \approx 3^{3.14} < 3^{\pi} < 3^{3.15} \approx 31.837$$

$$3.141 < \pi < 3.142 \quad \Rightarrow \quad 31.5237 \approx 3^{3.141} < 3^{\pi} < 3^{3.142} \approx 31.5584$$

$$3.1415 < \pi < 3.1416 \quad \Rightarrow \quad 31.5411 \approx 3^{3.1415} < 3^{\pi} < 3^{3.1416} \approx 31.5445$$



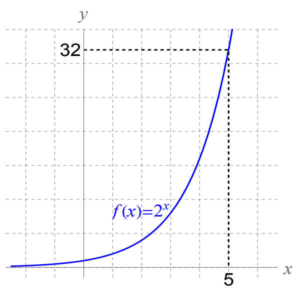
To make this precise, you need some sophisticated mathematics taught in MATH 430 (Advanced Calculus). For now, we just leave terms like "3^{π}" alone, and use a calculator to estimate them if we need a decimal approximation.

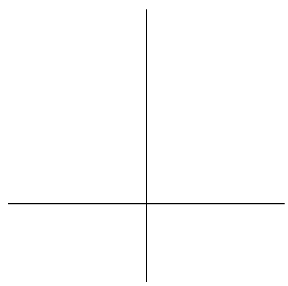
4.6 Logarithmic functions

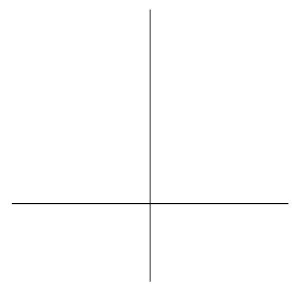
MOTIVATION: SOLVING AN EXPONENTIAL EQUATION

Solve for x in each equation:

$2^x = 32$ $\qquad$ $2^x = 23$ $\qquad$ $e^x = 6$







<u>Observe</u>: each of these equations has one solution. But except in freak circumstances (like $2^x = 32$), we can't write down exactly what this solution is.

This setup is similar to what we encountered when trying to solve equations like $x^2 = 7$ or $\sin x = .37$ or $\tan x = \frac{7}{4}$. We couldn't solve those equations with previously defined math, so we <u>invented a new function</u> to describe the solution(s) of these equations:

EQUATION	NEW FUNCTION INVENTED	SOLUTION OF EQUATION
$x^2 = 7$		
$\sin x = .37$		
$\tan x = \frac{7}{4}$		
$e^x = 6$	?	?

To solve $e^x = 6$, we invent a new function called *natural logarithm*.

Definition and interpretation of natural logarithm

Definition 4.33 *Let $x > 0$. The* **natural logarithm** *of x, denoted $\ln x$, is the number y satisfying $e^y = x$, i.e.*

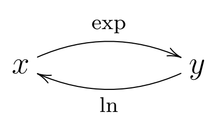
$$y = \ln x \text{ means } e^y = x.$$

Since there is only one such y for each x, this defines a function $\ln : (0, \infty) \to \mathbb{R}$.

NOTE: if $x \leq 0$, then $\ln x$ DNE.

Basic idea of $\ln$

$x = \ln y$ is the (only) solution of the equation $e^x = y$, in the same way $x = \sqrt{y}$ is a solution of $y^2 = x$, $x = \arctan y$ is a solution of $\tan x = y$ and $x = \arcsin y$ is a solution of $\sin x = y$:



§4.6 EXAMPLE 1

Compute each quantity, if possible (if the quantity can't be reasonably simplified, say so):

a) $\ln 0$

b) $\ln 1$

c) $\ln 5$

d) $\ln e^8$

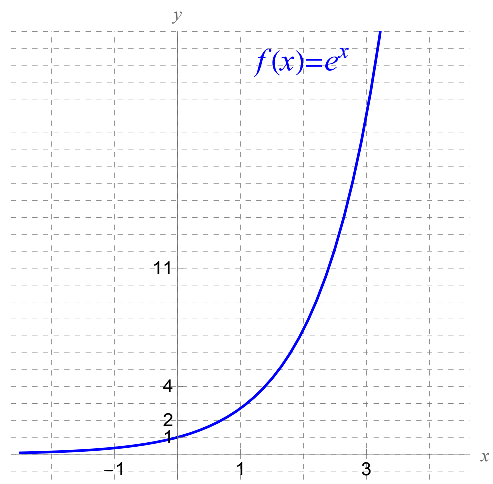
e) $\ln \sqrt{e}$

The arrow diagram given above suggests the inverse relationship between $\exp$ and $\ln$, which is made precise with these cancellation laws:

Theorem 4.34 (Cancellation laws for natural logarithm)

$$\ln e^x = x \qquad\qquad e^{\ln x} = x \text{ if } x > 0$$

Interpretation of $\ln$ in terms of graphs



Unnatural logarithms

Definition 4.35 *Let $b > 0$ and let $x > 0$. The* **logarithm base** *b of x, denoted $\log_b x$, is the number y satisfying $b^y = x$, i.e.*

$$y = \log_b x \text{ means } b^y = x.$$

For each b, we therefore get a function $\log_b : (0, \infty) \to \mathbb{R}$. If b isn't stated, then we assume assume $b = 10$ by default. Thus

$$y = \log x \text{ means } 10^y = x;$$

$\log x$ is called the **common logarithm** *of x; this is a function $\log : (0, \infty) \to \mathbb{R}$.*

$\log_e x$ is the same thing as $\ln x$, so e is the base of the natural logarithm.

Logarithms with bases other than e are *unnatural* because calculus with them is harder. In fact, we'll see later how to rewrite unnatural logs in terms of natural logs, and in general, **the first thing you usually want to do with an unnatural log is rewrite it in terms of natural logarithms**.

Also: when a math person says "log", they mean $\underline{\ln}$ unless they say otherwise. (In fact, in advanced classes, when math people write "log" they actually mean $\ln$ (not $\log$ base 10), despite what I've told you.)

Basic idea of $\log_b$

$$x \underset{\log_b}{\overset{b^x}{\rightleftarrows}} y$$

§4.6 EXAMPLE 2

Compute each quantity, if possible (if the quantity can't be reasonably simplified, say so):

a) $\log_7 1$

b) $\log_5(-3)$

c) $\log_3 81$

d) $\log_6 48$

e) $\log_5 25$

f) $\log 1000$

g) $\log_2 \dfrac{1}{8}$

h) $\log_{1/2} \dfrac{1}{4}$

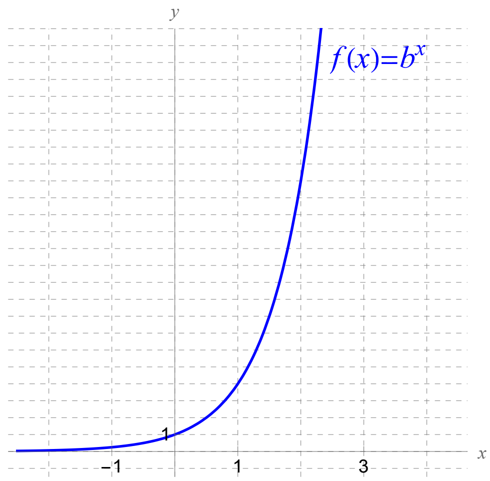
i) $\log_{17} 17^9$

Theorem 4.36 (Cancellation laws for logs with unnatural base)

$$\log_b b^x = x \qquad b^{\log_b x} = x \text{ if } x > 0$$

$$\log 10^x = x \qquad 10^{\log x} = x \text{ if } x > 0$$

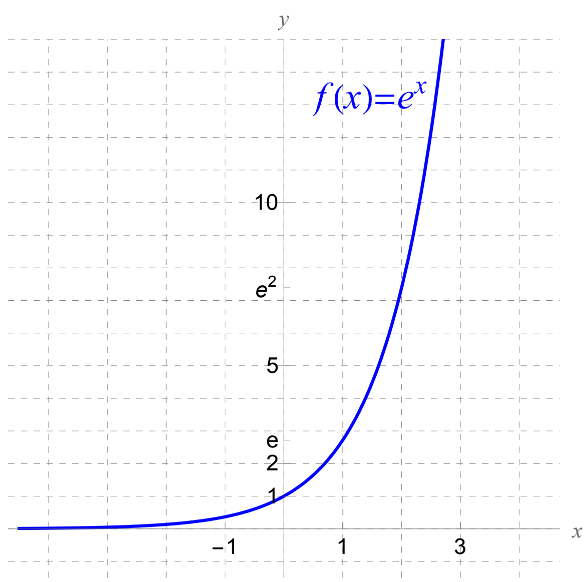
Interpretation of $\log_b$ in terms of graphs



Graphs of shifted logarithm functions

Elementary log graphs

Let's start by figuring out what the graph of $\ln x$ looks like, using a table of values that we'll build by looking at a graph of e^x:

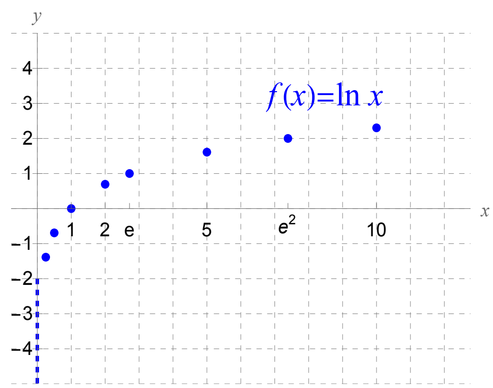


x	$\ln x$
$\frac{1}{4}$	
$\frac{1}{2}$	
1	
2	
$e \approx 2.72$	
5	
$e^2 \approx 7.39$	
10	

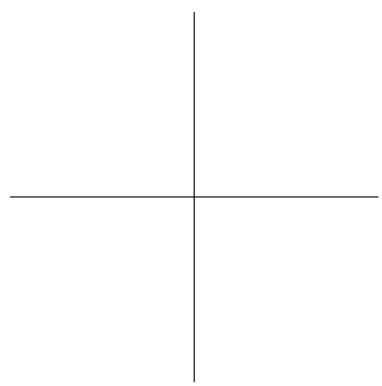
On the previous page, we had (roughly) this table of values for $\ln$:

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	$e \approx 2.72$	5	$e^2 \approx 7.39$	10
$\ln x$	-1.5	$-.7$	0	$.7$	1	1.6	2	2.3

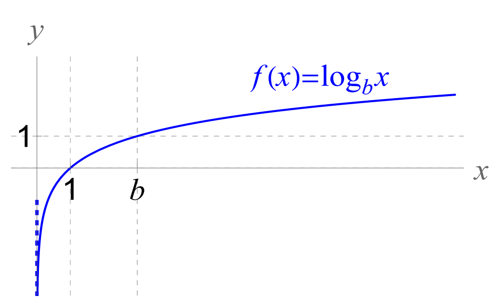
Let's plot these points:



Here is the crude picture of the graph of $\ln x$ you need to be able to draw:



The graph of $\log_b x$ is similar, but goes through $(b, 1)$ instead of $(e, 1)$:



Theorem 4.37 (Properties of $\ln$)

Domain: *The domain of* $\ln$ *is* $(0, \infty)$.
This means that if $x \leq 0$, $\ln x$ *DNE.*

Range: *The range of* $\ln$ *is* $\mathbb{R}$.

Symmetry: $\ln$ *has no useful symmetry.*

Increasing: *The graph of* $\ln$ *goes up from left to right.*

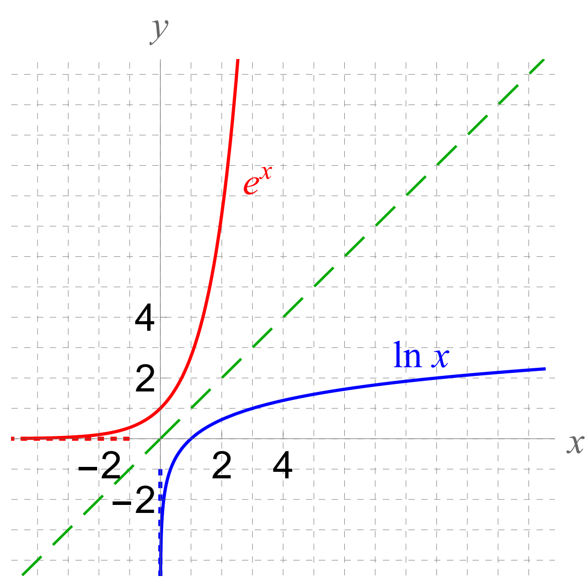
Asymptotes: $\ln$ *has one VA:* $x = 0$, *but no HA.*
This means that to the extreme right, the graph of $\ln$ *points slightly north of east, not due east.*

One-to-one: $\ln$ *is one-to-one, and its inverse function is* $\exp$:

$$x \underset{\exp}{\overset{\ln}{\rightleftarrows}} y \qquad\qquad x \underset{\ln}{\overset{\exp}{\rightleftarrows}} y$$

All these facts also apply to $\log_b$ *and* $\log$, *except that the inverse of* $\log_b x$ *is* b^x *and the inverse of* $\log x$ *is* 10^x:

Relationship between graphs of $\exp$ and $\ln$

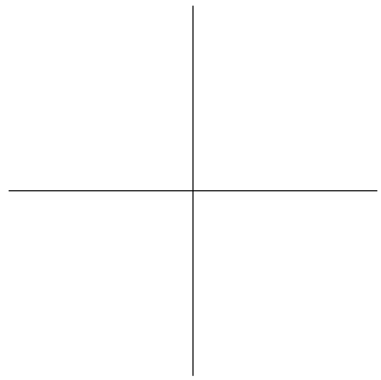


Transformed log graphs

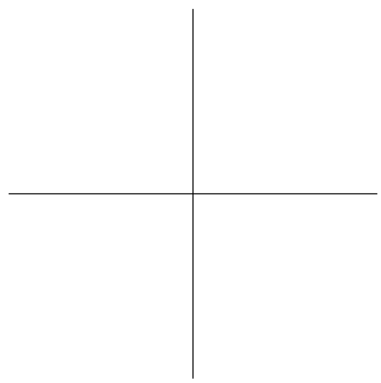
§4.6 EXAMPLE 3

Sketch a crude graph of each function:

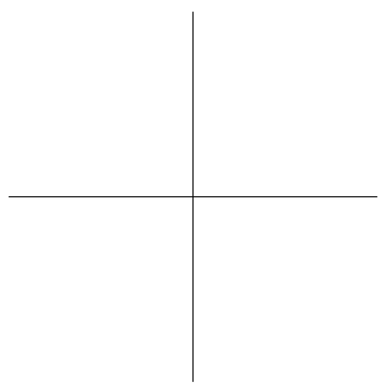
a) $f(x) = \log_4(x-5)$



b) $G(t) = -\ln t$

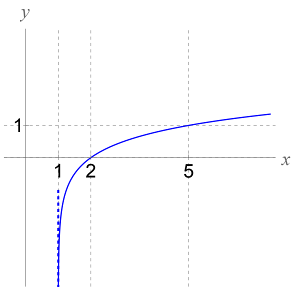


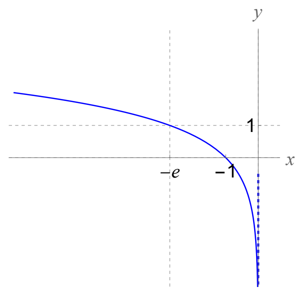
c) $h(x) = 3\ln x$

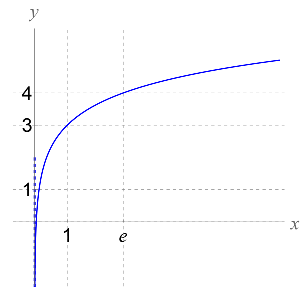


§4.6 EXAMPLE 4

Write a rule for each function whose graph is given (you may assume the function has not been stretched):

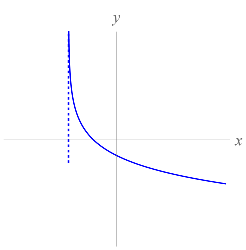
a) 

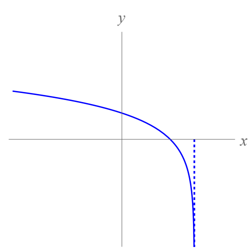
b) 

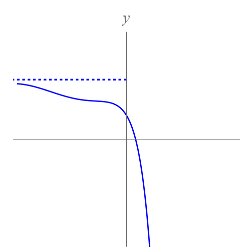
c) 

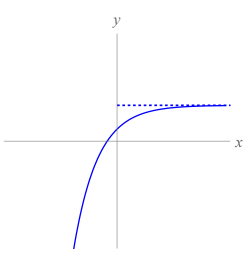
§4.6 EXAMPLE 5

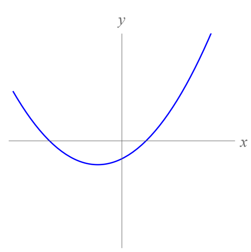
Determine whether each graph is that of a transformed exponential function, or transformed logarithmic function, or neither:

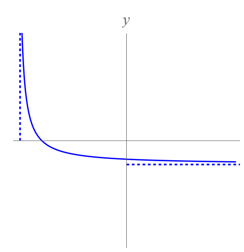
a) 

c) 

e) 

b) 

d) 

f) 

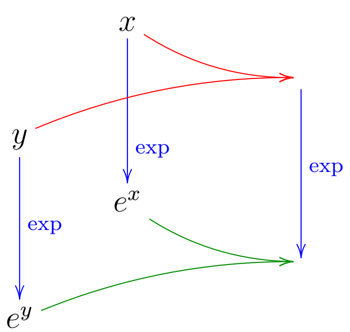
4.7 More on logarithms

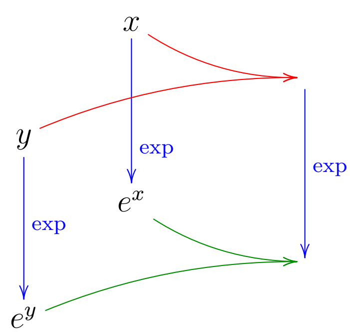
Log rules

In this section, we'll talk about some rules that you can use to rewrite expressions with logarithms in different ways. These rules have a reputation as being confusing, but they really aren't if you just remember two things:

1. You have to remember the key conceptual idea of exponentiation (two sections ago):

   This conceptual idea manifests itself as two exponent rules:

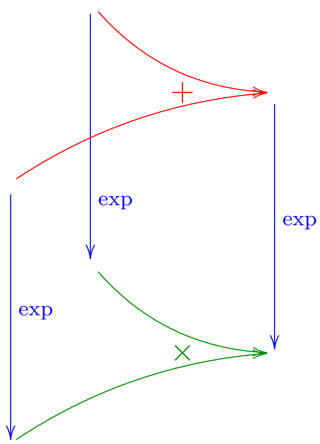


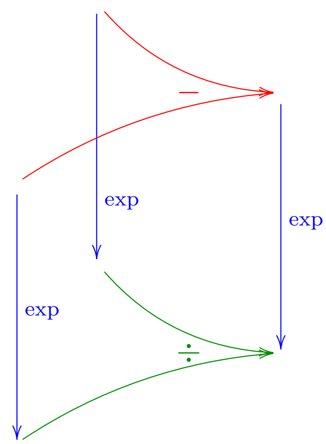


2. You have to remember the relationship between $\exp$ and $\ln$:

That means that since exponentiation converts addition to multiplication (see picture below at left), **logarithms must convert** ________ **to** ________.

Similarly, since exponentiation converts subtraction to division (see picture below at right), **logarithms must convert** ________ **to** ________:





Now let's turn these pictures into formulas:

Theorem 4.38 (Log rules)

Log of a product is the sum of the logs: $\ln(xy) = \ln x + \ln y$.

Log of a quotient is the difference of the logs: $\ln\left(\dfrac{x}{y}\right) = \ln x - \ln y$.

Powers can be pulled out in front of a log: $\ln x^a = a \ln x$.

(The same rules work for $\log_b$ *and* $\log$*.)*

The only rule that needs explanation is the last one. But that rule works because

$$\ln x^a = \ln(x \cdot x \cdot x \cdots x) = \ln x + \ln x + \ldots + \ln x = a \ln x.$$

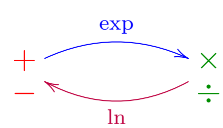
WARNING: there is no log rule for a log of a sum or difference:

$$\ln(x+y) \quad \text{and} \quad \ln(x-y)$$

can **NOT** be rewritten in a useful way. (Recall that this is typical for $+/-$ signs inside parentheses.)

These rules can be internalized with the following diagram:

Key conceptual idea of exponents and logs:



Exponentiation converts addition / subtraction to multiplication / division;

logarithm works in the opposite direction.

§4.7 EXAMPLE 1

Expand each expression as much as possible, simplifying where appropriate:

a) $\ln 25x^4$

b) $\ln\left(\dfrac{x^5}{6t^2}\right)$

c) $\ln\dfrac{xy\sqrt{z}}{w^3}$

d) $\ln(x+1)$

e) $\log_3\dfrac{a(x^2+1)}{9}$

f) $\ln\dfrac{(xy^2)^4}{z}$

Solution:

$$\begin{aligned}\ln\frac{(xy^2)^4}{z} &= \ln(xy^2)^4 - \ln z = 4\ln(xy^2) - \ln z\\ &= 4[\ln x + \ln y^2] - \ln z\\ &= 4\ln x + 4\ln y^2 - \ln z\\ &= 4\ln x + 4(2\ln y) - \ln z\\ &= \boxed{4\ln x + 8\ln y - \ln z}.\end{aligned}$$

§4.7 EXAMPLE 2

Suppose $a = \ln x$, $b = \ln y$ and $c = \ln z$. Rewrite each expression in terms of a, b and c:

a) $\ln x^2 e^5 \sqrt{y}$

b) $\ln \left(\frac{z}{x}\right)^6$

Solution: $\ln \left(\frac{z}{x}\right)^6 = 6 \ln \frac{z}{x} = 6(\ln z - \ln x) = 6 \ln z - 6 \ln x = \boxed{6c - 6a}$.

c) x

§4.7 EXAMPLE 3

Write as a single logarithm, simplifying where appropriate:

a) $\log 500 + \log 2$

b) $7 \ln w$

c) $8 \ln x + \ln 17$

d) $\log_2 60 - \log_2 30$

e) $\ln b - 5\ln d + 2\ln(3a)$

Solution: $\ln b - 5\ln d + 2\ln(3a) = \ln b - \ln d^5 + \ln(3a)^2 = \boxed{\ln \frac{b(3a)^2}{d^5}}$.

f) $\ln(x-1) + \ln(x+1)$

g) $3\ln 2 + 2\ln x - \frac{2}{3}\ln(x+4)$

Change of base

Earlier, I mentioned that when you have an exponential or logarithmic formula with a base other than e, you can rewrite it so that the base is e. Here's how:

Theorem 4.39 (Change of base)

$$A^B = e^{B\ln A} \qquad \log_b x = \frac{\ln x}{\ln b}$$

PROOF For the first statement,

$$e^{B\ln A} =$$

For the second statement, write $y = \log_b x$. That means $\boxed{}$. At the same time,

$$x = b^y = e^{y\ln b}$$

so $y\ln b = \ln x$. Divide both sides of the blue equation by $\ln b$ to get $y = \frac{\ln x}{\ln b}$ as wanted. □

Tip: The change of base formulas are **highly underrated** bits of math. They're super-useful in calculus and more advanced courses–when you have an exponential or log expression with a base other than e, often the first thing you want to do is change the base to write it in terms of natural exponentials/logs.

§4.7 EXAMPLE 4

Write each expression as a single exponential term with base e:

a) 4^x

b) 17^{19}

c) $\sqrt[7]{853q}$

d) $5^{3x}7^x$

Part (d) of Example 4 illustrates how we can use the change of base formula to write expressions containing exponential functions with different bases as a single exponent. This can be very useful in certain computations.

§4.7 EXAMPLE 5

Write each expression in terms of natural exponentials and/or natural logarithms:

a) $\log_5 37$

b) $\log 58$

c) $\log_{1/6} 14$

d) x^{x-1}

e) 7^{4x}

f) $(3 \ln x)^{\cos x}$

The method of Example 5 gives us a way of using a calculator to get a decimal approximation of logs with strange bases. Most calculators have a $\boxed{\text{LN}}$ button but not a $\boxed{\text{LOG}_b}$ button. So if we needed a decimal approximation to $\log_5 37$, we would proceed as in part (a) above:

$$\log_5 37 = \frac{\ln 37}{\ln 5} \approx \frac{3.61092}{1.60944} = \boxed{2.24359}.$$

§4.7 EXAMPLE 6

Simplify each expression:

a) $\dfrac{\ln 8}{\ln 2}$

b) $\dfrac{\ln 54 - \ln 6}{\ln 3}$

c) $\dfrac{-\ln 9}{3 \ln 3}$

d) $e^{5 \ln 2}$

e) $10e^{2 \ln 3} - 8$

f) $e^{1/2 \ln x}$

g) $e^{x \ln(1/2)}$

§4.7 EXAMPLE 7

Diagram each given function. **Catch:** the only exponential and/or logarithmic functions allowed in your diagrams are natural exponentials ($\exp$) and natural logarithms ($\ln$):

a) $f(x) = 5^x$

b) $g(x) = \log x$

c) $F(x) = 3e^{2x-5}$

Solution: $x \xrightarrow{\times 2} \xrightarrow{-5} \xrightarrow{\exp} \xrightarrow{\times 3} F(x)$

d) $G(x) = \arctan e^{\cos^2 x}$

Solution: $x \xrightarrow{\cos} \xrightarrow{\wedge 2} \xrightarrow{\exp} \xrightarrow{\arctan} G(x)$

e) $h(x) = \ln\left(\frac{\ln^2 7x}{4}\right)$

f) $k(x) = e^{\ln(x+1)} \ln(e^x + 1)$

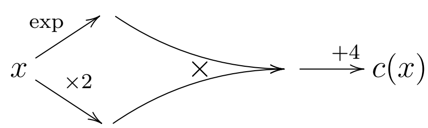
§4.7 Example 8

Reverse-diagram each function, and if possible, simplify its rule:

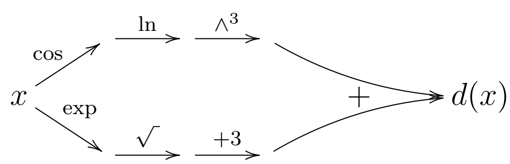
a) $x \xrightarrow{\times \ln 3} \xrightarrow{\exp} \xrightarrow{\div 5} a(x)$

b) $x \xrightarrow{\cos} \xrightarrow{\ln} \xrightarrow{\div \ln 6} b(x)$

c)

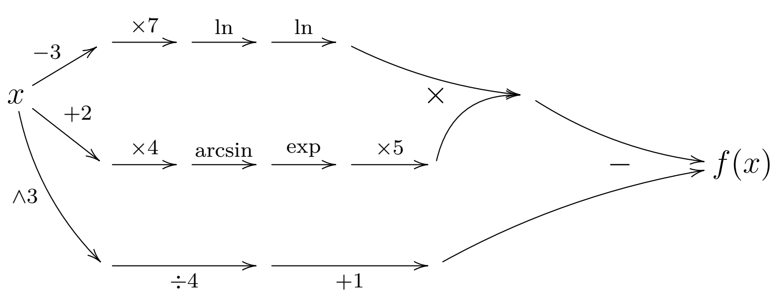


d)



e) $x \xrightarrow{\ln} \xrightarrow{1/\cdot} \xrightarrow{\ln} e(x)$

f)



Exponential and logarithmic equations

Although it's easy to forget, the purpose of inventing $\ln$ was to provide notation for solving equations like

$$e^x = 6.$$

(The solution of this is called $x = \ln 6$.) This led us to the arrow diagram

$$x \underset{\ln}{\overset{\exp}{\rightleftarrows}} y$$

which we are now going to use to solve basic exponential and logarithmic equations.

§4.7 EXAMPLE 9

Solve each equation:

a) $e^x = 15$

b) $\ln x = -4$

c) $e^x = \sqrt{8}$

d) $e^x = 0$

Theorem 4.40 (Basic exponential and log equations) *Let $y \in \mathbb{R}$.*

- *If $y > 0$, the equation $e^x = y$ has solution $x = \ln y$ (otherwise it has no solution).*
- *The equation $\ln x = y$ has solution $x = e^y$.*

In a more complicated equation, you isolate the exponential term (perhaps by combining like terms), and then apply the method of the previous examples.

§4.7 EXAMPLE 9

Solve each equation:

a) $3e^x + 8 = 19$

b) $\frac{1}{2}\ln x - \frac{2}{3} = \frac{1}{7}\ln x + \frac{1}{2}$

c) $4(3 - e^x) = 8 + 7e^x$

4.8 Summary

In this chapter we learned about transcendental functions. These include:

Important trig functions: $\sin, \cos, \tan$

Less important trig functions: $\csc, \sec, \cot$

Important inverse trig functions: $\arctan, \arcsin$

Less important inverse trig function: $\arccos$

Important exponential function: $\exp$

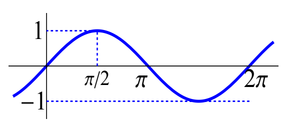
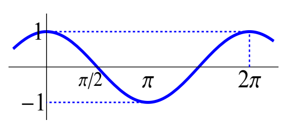
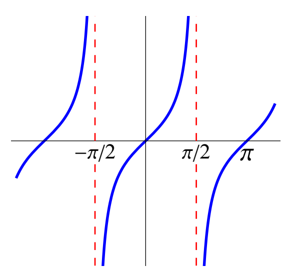
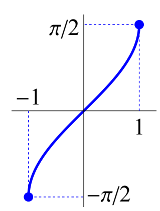
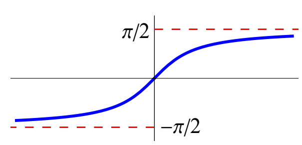
Less important exponential function: b^x

Important logarithm: $\ln$

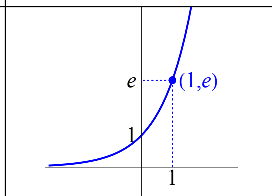
Less important logarithms: $\log_b, \log$

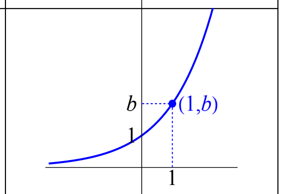
Much of the information you should know about these functions can be found in the charts on the next two pages.

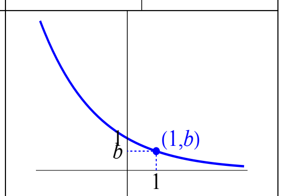
Trigonometric functions

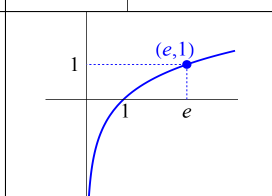
RULE	CONCEPT	GRAPH	PERIOD DOMAIN RANGE SYMMETRY ASYMPTOTES INVERSE
$f(x) = \sin x$	gives the y-coordinate of point on unit circle at angle x		**Period:** 2π **D:** $\mathbb{R}$ **R:** $[-1, 1]$ **Sym:** odd no asymptotes $f^{-1}(x) = \begin{cases} \arcsin x + 2\pi n, \\ 2\pi n + \pi - \arcsin x \end{cases}$
$f(x) = \cos x$	gives the x-coordinate of point on unit circle at angle x		**Period:** 2π **D:** $\mathbb{R}$ **R:** $[-1, 1]$ **Sym:** even no asymptotes $f^{-1}(x) = 2\pi n \pm \arccos x$
$f(x) = \tan x$	gives slope of line that has angle x to the horizontal		**Period:** π **D:** $\mathbb{R} - \left\{ \frac{(\text{odd})\pi}{2} \right\}$ **R:** $\mathbb{R}$ **Sym:** odd **HA:** none **VA:** $x = \frac{(\text{odd})\pi}{2}$ $f^{-1}(x) = \pi n + \arctan x$
$f(x) = \arcsin x$	produces an angle that has height x on the unit circle		**Period:** none **D:** $[-1, 1]$ **R:** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **Sym:** odd no asymptotes $f^{-1}(x) = \sin x$
$f(x) = \arctan x$	produces an angle that has slope x		**Period:** none **D:** $\mathbb{R}$ **R:** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **Sym:** odd **HA:** $y = \pm\frac{\pi}{2}$ **VA:** none $f^{-1}(x) = \tan x$

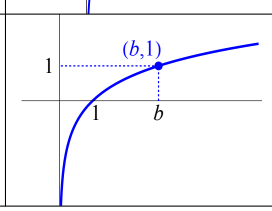
Exponential and logarithmic functions

<table>
<tr><th></th><th></th><th>RULE</th><th>GRAPH</th><th>DOMAIN
RANGE
SYMMETRY
ASYMPTOTE</th><th>INVERSE</th></tr>
<tr><td rowspan="3">EXPONENTIAL
(rate of change proportional to output)</td><td>NATURAL</td><td>$f(x) = e^x$</td><td>


</td><td rowspan="3">D: $\mathbb{R}$
R: $(0, \infty)$
no symmetry
HA: $y = 0$
VA: none</td><td>$f^{-1}(x) = \ln x$</td></tr>
<tr><td rowspan="2">GENERIC</td><td>$f(x) = b^x$
$= e^{x \ln b}$
$(b > 1)$</td><td>


</td><td rowspan="2">$f^{-1}(x) = \log_b x$</td></tr>
<tr><td>$f(x) = b^x$
$= e^{x \ln b}$
$(0 < b < 1)$</td><td>


</td></tr>
<tr><td rowspan="2">LOGARITHMIC</td><td>NATURAL</td><td>$f(x) = \ln x$</td><td>


</td><td rowspan="2">D: $(0, \infty)$
R: $\mathbb{R}$
no symmetry
HA: none
VA: $x = 0$</td><td>$f^{-1}(x) = e^x$</td></tr>
<tr><td>GENERIC</td><td>$f(x) = \log_b x$
$= \dfrac{\ln x}{\ln b}$</td><td>


</td><td>$f^{-1}(x) = b^x$</td></tr>
</table>

4.9 Chapter 4 Homework

Exercises from Section 4.1

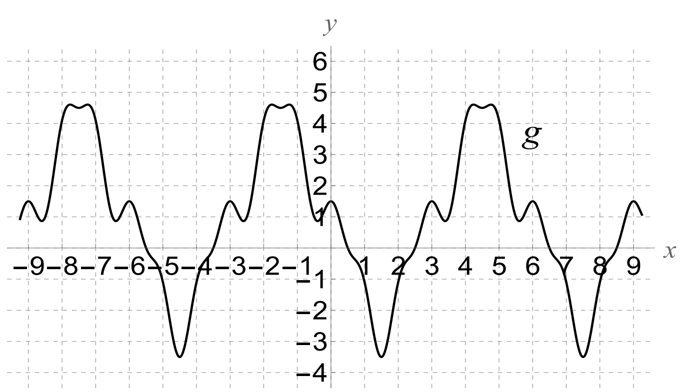
Let's review some trig functions of special angles. In Exercises 1-16, compute the indicated quantity:

1. $\sin \frac{2\pi}{3}$
2. $\tan \frac{3\pi}{4}$
3. $\cos 7\pi$
4. $\tan \frac{\pi}{3}$
5. $\sin \frac{5\pi}{6}$
6. $\cos -\frac{\pi}{4}$
7. $\sin -\frac{\pi}{2}$
8. $\tan 0$
9. $\cos \frac{\pi}{6}$
10. $\cos -\frac{2\pi}{3}$
11. $\sin \frac{7\pi}{4}$
12. $\sin \frac{13\pi}{6}$
13. $\sin 0$
14. $\cos -\frac{7\pi}{3}$
15. $\tan -\frac{11\pi}{6}$
16. $\cos \frac{\pi}{2}$

Recall that $n!$ refers to the number $n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$; for example, $4! = 4(3)(2)(1) = 24$. In Exercises 17-22, compute each quantity:

17. $5!$
18. $5!3!$
19. $(5\cdot 3)!$
20. $5\cdot 3!$
21. $\frac{101!}{100!}$
22. $\frac{8!}{5!3!}$

In Exercises 23-31, suppose that g is a periodic function graphed below (and assume that the portion of the graph of g that you see includes at least one complete period):



Use this graph to answer the following questions:

23. What is the period of g?
24. What is $g(29)$?

25. What is $g(12001)$?

26. What is $g(-47)$?

27. Between $x = 74$ and $x = 75$, is g increasing or decreasing?

28. How many x-intercepts does g have between $x = 57$ and $x = 60$?

29. How many x-intercepts does g have between $x = 17000$ and $x = 17024$?

30. The largest x-intercept shown on this graph is at about $(8.5, 0)$. What are the coordinates of the next three x-intercepts (after $(8.5, 0)$ that would occur off the right edge of this graph?

31. Give six positive values of x so that $g(x) = 4$.

In Exercises 32-45, sketch a graph of each function:

32. $f(x) = 4\sin x$
33. $g(x) = \cos\left(x - \frac{\pi}{2}\right)$
34. $h(x) = -\frac{1}{3}\cos x + \frac{2}{3}$
35. $j(x) = \cos x - 3$
36. $k(x) = \sin(x - \pi) + 4$
37. $l(x) = -3\sin x + 2$
38. $m(x) = \frac{3}{2}\cos\left(x + \frac{1}{4}\right)$
39. $n(x) = 1 - \sin x$
40. $p(x) = \cos 8x$
41. $q(x) = \sin\frac{x}{2}$
42. $r(x) = \cos 4\pi x$
43. $s(x) = 7 + \cos\frac{2x}{3}$
44. $t(x) = -\sin\frac{5\pi x}{8}$
45. $u(x) = 2\sin\frac{5x}{2}$

Let's do some more review of trig functions of special angles (and order of operations). In Exercises 46-62, compute the indicated quantity:

46. $\sin\frac{10\pi}{3}$
47. $\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}$
48. $\cos\left(\frac{7\pi}{4} - \frac{3\pi}{2}\right)$
49. $\tan^4\frac{\pi}{3}$
50. $\cos 0 \sin\frac{5\pi}{3}\tan\frac{\pi}{3}$
51. $4\sin\frac{\pi}{3} - 2\cos\frac{\pi}{3}$
52. $3\sec^2\pi$
53. $\csc\frac{\pi}{6}$
54. $\sin 5 \cdot \frac{\pi}{6}$

55. $\sin^3 \frac{\pi}{4} \cos^3 \frac{\pi}{6}$

56. $4 \tan \frac{5\pi}{6} + \frac{\pi}{6}$

57. $2 \cot \frac{\pi}{4}$

58. $\cos \left(\pi \sin \frac{\pi}{2} \right)$

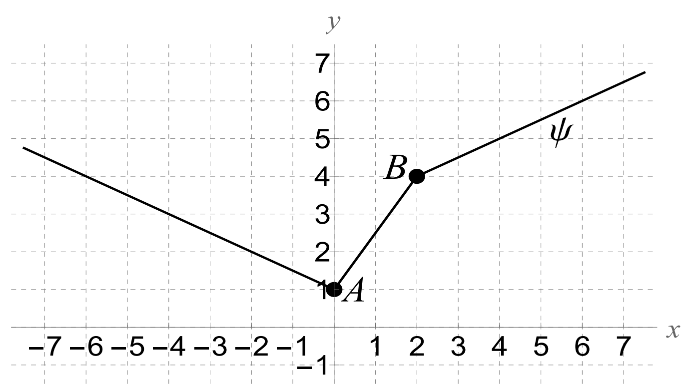
59. $\frac{2}{3} \sec \pi$

60. $2 \sin^2 \left(-\frac{\pi}{2} \right) - \cos^2 \left(\frac{\pi}{6} \right)$

61. $14 \sin \frac{5\pi}{6} - \sqrt{3} \sec \frac{\pi}{6}$

62. $\tan 3\pi$

Suppose that ψ is the function whose graph is given here:



Use this graph to sketch the graph of each function given in Exercises 63-70. Keep careful track of where the points A and B shown on the graph above are moved to.

63. $\psi(2x)$

64. $\psi\left(\frac{x}{3}\right)$

65. $\psi(4x) - 2$

66. $3\psi\left(\frac{x}{2}\right)$

67. $\psi(2(x-3))$

68. $\psi(2x+4)$

69. $-\psi\left(\frac{x}{4}\right)$

70. $\psi\left(\frac{2x}{3}\right)$

In Exercises 71-82, sketch a graph of each function:

71. $a(x) = 3 \cos 4x - 1$

72. $b(x) = \sin \frac{x}{2} + 1$

73. $c(x) = -\cos \pi x$

74. $d(x) = \frac{1}{4} \cos \left(x - \frac{\pi}{4} \right) + \frac{1}{2}$

75. $e(x) = -2 \sin 4\pi x + 3$

76. $f(x) = \sqrt{5} \cos \frac{2}{3} x$

77. $g(x) = \tan \frac{x}{8}$

78. $h(x) = \sqrt{13} - \cos 8x$

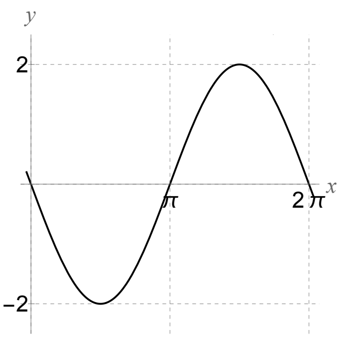
79. $j(x) = -\tan 2x$

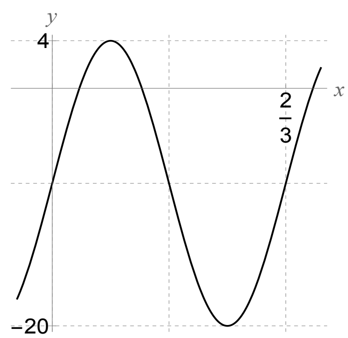
80. $k(x) = \tan\left(x - \frac{\pi}{4}\right)$

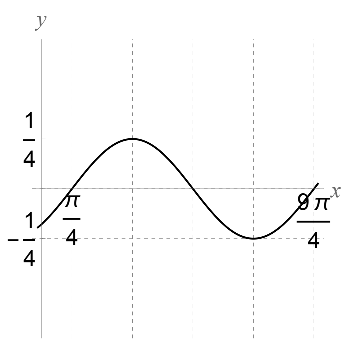
81. $l(x) = 3\tan\frac{\pi x}{6}$

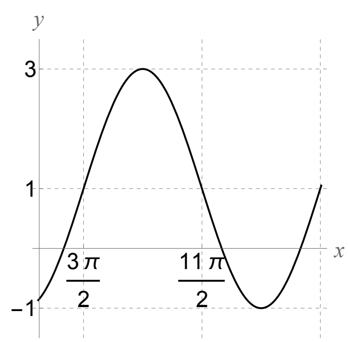
82. $m(x) = \sin 2\left(x + \frac{\pi}{8}\right)$

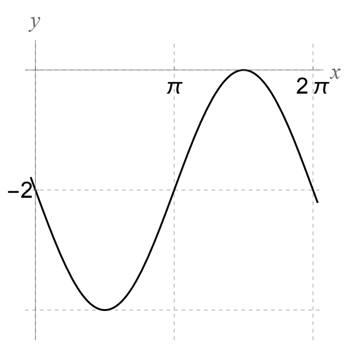
In Exercises 83-88, write a rule for the sinusoidal function whose graph is given:

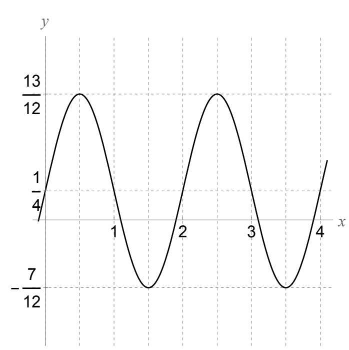
83.


86.


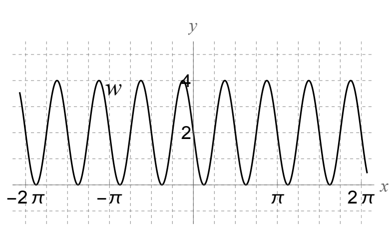
84.


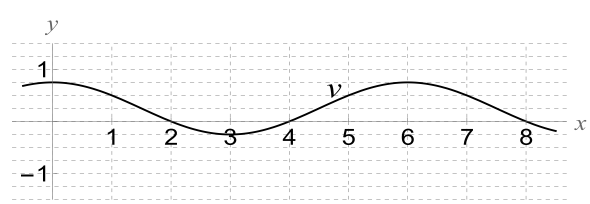
87.


85.


88.


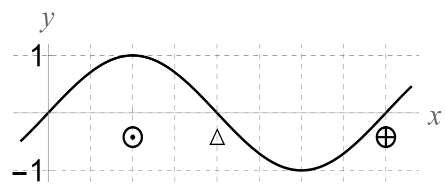
89. An unknown sinusoidal function w is graphed below at left. What is $w\left(\frac{25\pi}{12}\right)$?





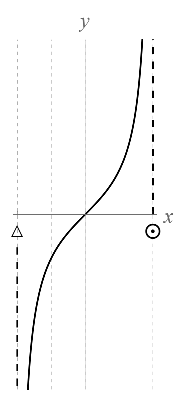
90. An unknown sinusoidal function v is graphed above at right. What is $v\left(\frac{105}{2}\right)$?

91. Suppose that this is the graph of a sinusoidal function:



a) If this is the graph of $\sin 4x$, what is the x-coordinate indicated by the $\oplus$?
b) If this is the graph of $\sin \frac{x}{3}$, what is the x-coordinate indicated by the $\oplus$?
c) If this is the graph of $\sin \frac{x}{3}$, what is the x-coordinate indicated by the $\triangle$?
d) If this is the graph of $\sin \frac{x}{3}$, what is the x-coordinate indicated by the $\odot$?
e) If this is the graph of $\sin \frac{3\pi}{2}x$, what is the x-coordinate indicated by the $\triangle$?
f) If the $\odot$ is x-coordinate 7π, what is the rule of the function being graphed?
g) If the $\triangle$ is x-coordinate 2, what is the rule of the function being graphed?

92. Suppose this is the graph of a function of the form $\tan bx$, where b is a constant:



a) If this is the graph of $\tan 8x$, what x-coordinate is the $\odot$?
b) If this is the graph of $\tan \frac{x}{3}$, what x-coordinate is the $\triangle$?
c) If the $\odot$ is x-coordinate 5, what is the rule of the function being graphed?
d) If the $\triangle$ is x-coordinate -12π, what is the rule of the function being graphed?
e) If the $\triangle$ is x-coordinate $-\frac{\pi}{7}$, what is the rule of the function being graphed?

For each function given in Exercises 93-100, give the maximum value of the function, the minimum value of the function, and the period of the function.

93. $F(x) = 3\cos 2x$

94. $G(x) = -\sin x - 5$

95. $H(x) = \frac{1}{4}\cos(x-2)+1$

96. $J(x) = -2\sin 3\left(x+\frac{\pi}{4}\right) - \frac{1}{2}$

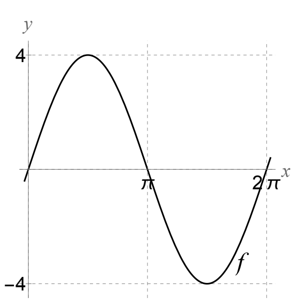
97. $K(x) = 17\cos\frac{x}{4} + 11$

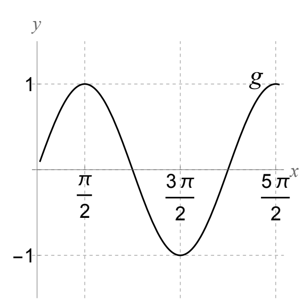
98. $L(x) = 3\tan\frac{x}{3}$

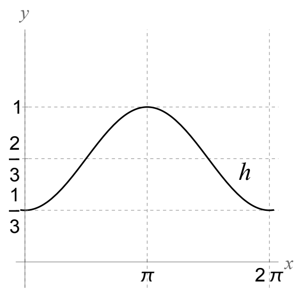
99. $M(x) = -\tan 4\pi(x-2) - 3$

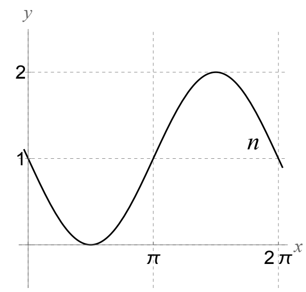
100. $N(x) = \frac{9}{4} - \cos 10\pi x$

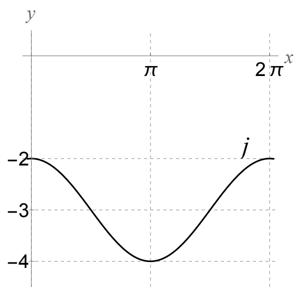
Answers

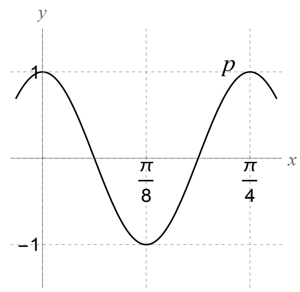
1. $\frac{\sqrt{3}}{2}$
2. -1
3. -1
4. $\sqrt{3}$
5. $\frac{1}{2}$
6. $\frac{\sqrt{2}}{2}$
7. -1
8. 0
9. $\frac{\sqrt{3}}{2}$
10. $-\frac{1}{2}$
11. $-\frac{\sqrt{2}}{2}$
12. $\frac{1}{2}$
13. 0
14. $\frac{1}{2}$
15. $\frac{1}{\sqrt{3}}$
16. 0
17. 120
18. 720
19. 1307674368000
20. 30
21. 101
22. 56
23. 6
24. 4
25. -1
26. -1
27. increasing
28. none
29. 8
30. $(12.5, 0)$, $(14.5, 0)$, $(18.5, 0)$
31. 4, 5, 10, 11, 16, 17
32. 

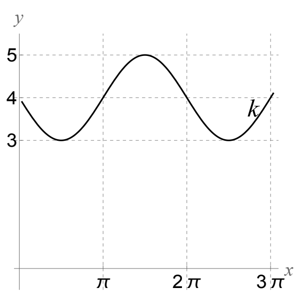
33. 

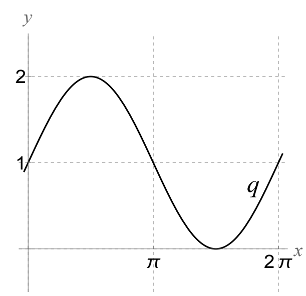
34. 

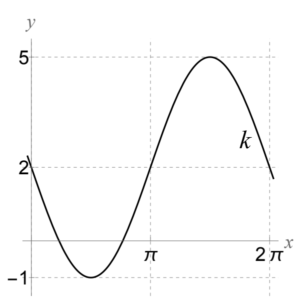
39. 

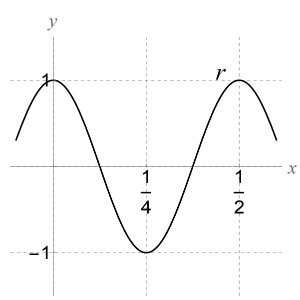
35. 

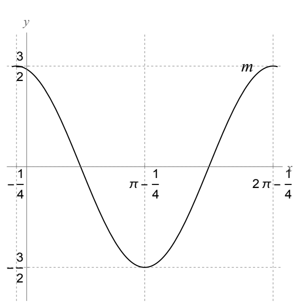
40. 

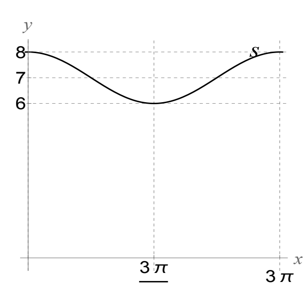
36. 

41. 

37. 

42. 

38. 

43. 

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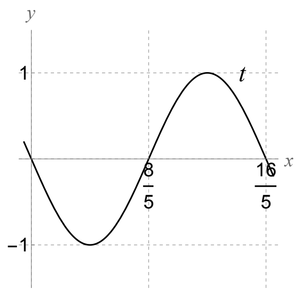

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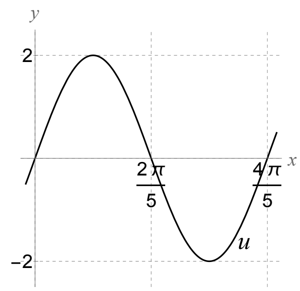


46. $-\dfrac{\sqrt{3}}{2}$

47. 0

48. $\dfrac{\sqrt{2}}{2}$

49. 9

50. $-\dfrac{3}{2}$

51. $-1+2\sqrt{3}$

52. 3

53. 2

54. $\dfrac{1}{2}$

55. $\dfrac{3\sqrt{6}}{32}$

56. $-\dfrac{4}{\sqrt{3}}+\dfrac{\pi}{6}$

57. 2

58. -1

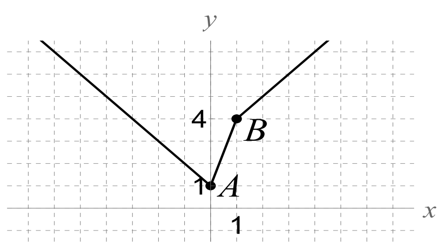
59. $-\dfrac{2}{3}$

60. $\dfrac{5}{4}$

61. 5

62. 0

63.



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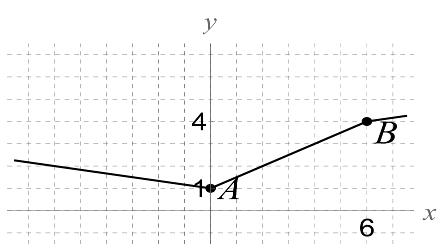

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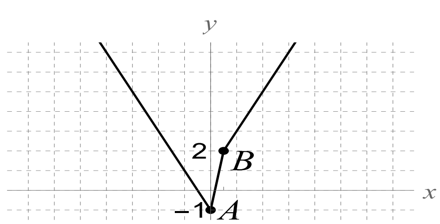

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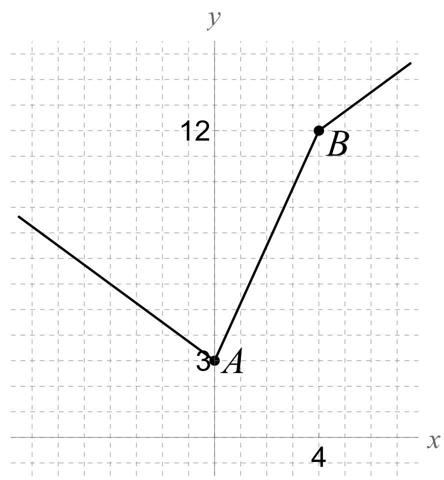

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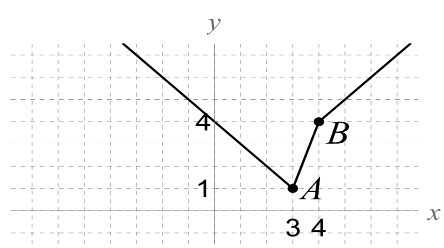

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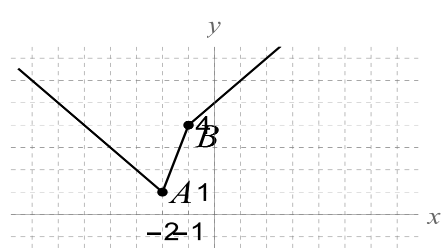

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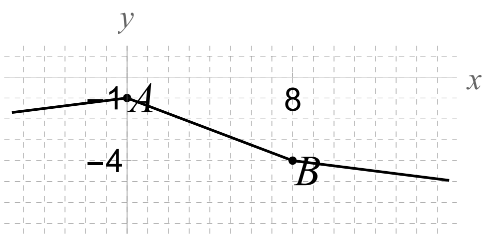

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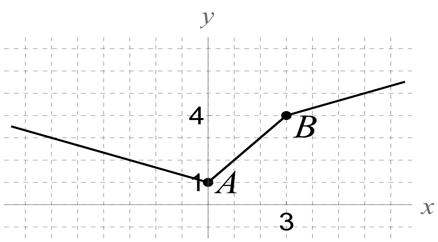

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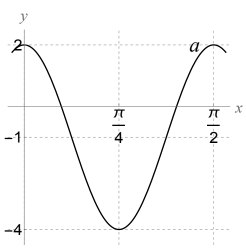

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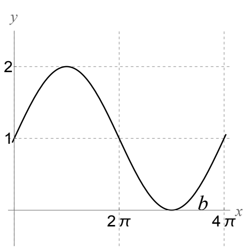

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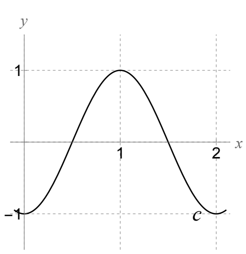

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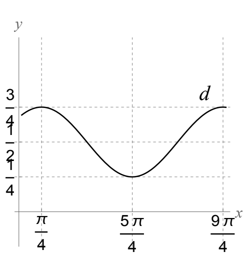

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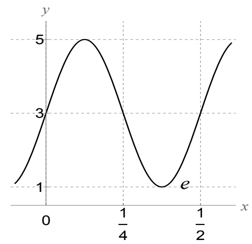

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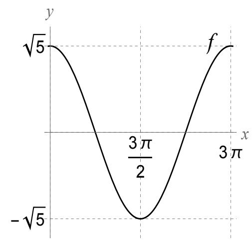

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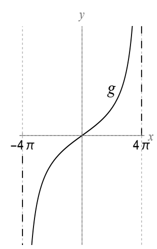

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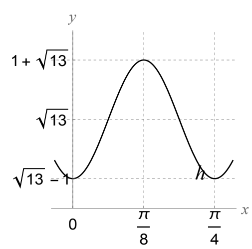

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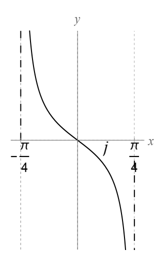

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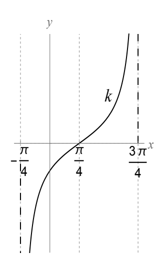
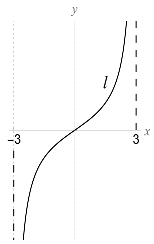
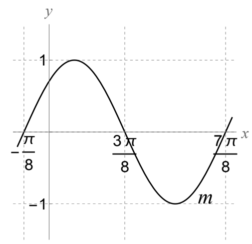


81. 

82. 

NOTE: In Exercises 83-88, multiple answers are possible:

83. $f(x) = -2\sin x$
84. $f(x) = \frac{1}{4}\sin\left(x - \frac{\pi}{4}\right)$
85. $f(x) = -2\sin x - 2$
86. $f(x) = 12\sin 3\pi x - 8$
87. $f(x) = -2\sin\frac{1}{4}\left(x - \frac{3\pi}{2}\right) + 1$
88. $f(x) = \frac{5}{6}\sin\pi x + \frac{1}{4}$
89. $2 - \sqrt{3}$
90. $\frac{1}{3}$
91. a) $\frac{\pi}{2}$
    b) 6π
    c) 3π
    d) $\frac{3\pi}{2}$
    e) $\frac{2}{3}$
    f) $f(x) = \sin\frac{x}{14}$
    g) $f(x) = \sin\frac{\pi x}{2}$
92. a) $\frac{\pi}{16}$
    b) $-\frac{3\pi}{2}$
    c) $f(x) = \tan\frac{\pi x}{10}$
    d) $f(x) = \tan\frac{x}{24}$
    e) $f(x) = \tan\frac{7x}{2}$
93. min value -3; max value 3; period π
94. min value -6; max value -4; period 2π
95. min value $\frac{3}{4}$; max value $\frac{5}{4}$; period 2π
96. min value $-\frac{5}{2}$; max value $\frac{3}{2}$; period $\frac{2\pi}{3}$
97. min value 6; max value 28; period 8π
98. min value DNE; max value DNE; period 3π
99. min value DNE; max value DNE; period $\frac{1}{4}$
100. min value $\frac{5}{4}$; max value $\frac{13}{4}$; period $\frac{1}{5}$

Exercises from Section 4.2

In each of Exercises 1-20, determine if each given expression can be simplified into a single term using a Pythagorean identity. If so, simplify it; otherwise, do nothing.

1. $\sin^2 x - \cos^2 x$
2. $\tan^2 x + 1$
3. $\cot^2 x + 1$
4. $1 - \sec^2 w$
5. $1 - \tan^2 \frac{x}{2}$
6. $4\csc^2 t - 4$
7. $\cos^2 x + \cot^2 x$
8. $\sec^2 x - \csc^2 x$
9. $\sin^2 3x - 1$
10. $2\cot^2 x - 2\csc^2 x$
11. $\cos^2 x + \cot^2 x$
12. $\sin^2 3x + \cos^2 3x$
13. $\sec^2 x - \csc^2 x$
14. $1 - \cos^2 x$
15. $\sin^2 x + \cos^2 2x$
16. $1 + \sin^2 x$
17. $3 + 3\cot^2 5z$
18. $\cot^2 2x - 1$
19. $3\sec^2 \frac{x}{4} - 3$
20. $\sec^2 x - \tan^2 x$

In Exercises 21-28, reverse the procedure of Exercises 1-20. In other words, given the expression in these exercises, rewrite them as the sum or difference of two terms using a Pythagorean identity. For example, if you are given "$\sec^2 x$", the answer I'm looking for here is "$\tan^2 x + 1$".

21. $\sin^2 x$
22. $\cos^2 2x$
23. $\tan^2 x$
24. $\cot^2 \frac{x}{3}$
25. $\cos^2 2x$
26. $5\csc^2 x$
27. $\sin^2 \frac{x}{4}$
28. $\frac{1}{4}\sec^2 3x$
29. Rewrite the expression $4\tan^2 y$ in terms of only the trig function $\sec$.
30. Rewrite the expression $3\csc^2 2x$ in terms of only the trig function $\cot$.
31. Rewrite the expression $7\sin^2 x + 5\cos^2 x$ in terms of only the trig function $\cos$.
32. Rewrite the expression $2\cos^2 x$ in terms of only the trig function $\sin$.
33. Rewrite the expression $2\sec^2 z + 3$ in terms of only the trig function $\tan$.
34. Rewrite the expression $2\csc^2 4x - 4\cot^2 4x$ in terms of only the trig function $\csc$.
35. Rewrite the expression $2\sin^2 \frac{2x}{3} + 3\cos^2 \frac{2x}{3} + 4$ in terms of only the trig function $\sin$.
36. Rewrite the expression $2\sin^2 \frac{2x}{3} + 3\cos^2 \frac{2x}{3} + 4$ in terms of only the trig function $\cos\theta$.

In each of Exercises 37-55, simplify each expression as much as possible, and write your answer so that no quotients appear in the final answer, and no $-$ signs are inside trig functions:

37. $\sec x \cot x$
38. $\csc(-x) \cos x$
39. $\csc x \tan x$
40. $\dfrac{\sin x}{\csc x}$
41. $\dfrac{\sec(-x)}{\csc x}$
42. $\dfrac{\tan(-x)}{\cot(-x)}$
43. $\sin x \tan x - \sec x$
44. $\sin x(\csc x - \sin x)$
45. $\dfrac{\csc(-x)}{\cot x}$
46. $\dfrac{1+\tan^2(-x)}{1+\cot^2 x}$
47. $\dfrac{1-\csc^2 x}{\sec^2 x - 1}$
48. $\dfrac{1+\tan(-x)}{\tan(-x)}$
49. $\csc x \tan x$
50. $\sin x(\sec x + \csc x)$
51. $\sec(-x) \cot x \sin x$
52. $\dfrac{\tan(-x)}{\sec(-x)}$
53. $\sin x + \cot x \cos x$
54. $\dfrac{\sin x - \cos x}{\sin x \cos x} - \sec x$
55. $\dfrac{1-\cos^2 x}{1+\tan^2 x}$

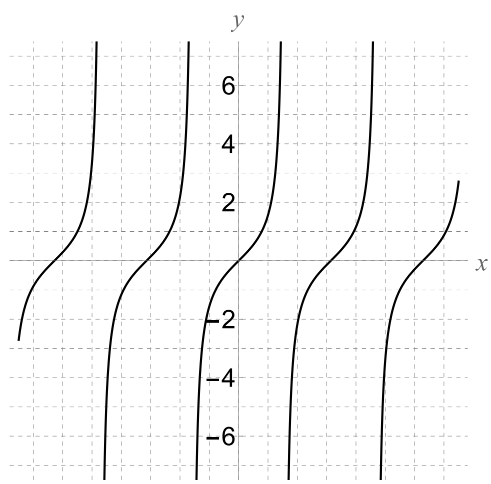
Answers

1. can't be simplified
2. $\sec^2 x$
3. $\csc^2 x$
4. $-\tan^2 w$
5. can't be simplified
6. $4\cot^2 t$
7. can't be simplified
8. can't be simplified
9. $-\cos^2 3x$
10. -2
11. can't be simplified
12. 1
13. can't be simplified
14. $\sin^2 x$
15. can't be simplified
16. can't be simplified
17. $3\csc^2 5z$
18. $-\csc^2 2x$
19. $3\tan^2 \dfrac{x}{4}$
20. 1
21. $1-\cos^2 x$
22. $1-\sin^2 2x$
23. $\sec^2 x - 1$
24. $\sec^2 \dfrac{x}{3} - 1$
25. $1-\sin^2 2x$
26. $5+5\cot^2 x$
27. $1-\cos^2 \dfrac{x}{4}$
28. $\dfrac{1}{4}+\dfrac{1}{4}\tan^2 3x$
29. $4\sec^2 y - 4$
30. $3\cot^2 2x + 3$
31. $7-2\cos^2 x$
32. $2-2\sin^2 x$
33. $2\tan^2 z + 5$
34. $6\csc^2 4x - 4$
35. $7-\sin^2 \dfrac{2x}{3} + 3\cos^2 \dfrac{2x}{3}$
36. $6+\cos^2 \dfrac{2x}{3}$
37. $\csc x$
38. $-\cot x$
39. $\sec x$
40. $\sin^2 x$
41. $\tan x$
42. $\tan^2 x$

43. $-\cos x$
44. $\cos^2 x$
45. $-\sec x$
46. $\tan^2 x$
47. $-\cot^4 x$
48. $1 - \cot x$
49. $\sec x$
50. $1 + \tan x$
51. 1
52. $-\sin x$
53. $\csc x$
54. $-\csc x$
55. $\sin^2 x \cos^2 x$

Exercises from Section 4.3

1. Here is a graph of $\tan x$:



Explain, by drawing a picture on this graph, how you would find each of these values from this graph:

a) $\arctan 2$
b) $\arctan -3$
c) $\arctan 6$

2. Explain, by drawing a picture, how to interpret each of these quantities in the context of the unit circle:

a) $\arctan \frac{1}{4}$
b) $\arctan 2$
c) $\arctan -\frac{5}{3}$

3. Explain, by drawing a picture, how to interpret each of these quantities in the context of a right triangle:

a) $\arctan \frac{2}{7}$
b) $\arctan 4$

In each of Exercises 4-7, sketch a graph of each function:

4. $f(x) = 6 \arctan x$
5. $g(x) = \arctan(x - 5)$
6. $h(x) = 2 \arctan(x + 3) - 3\pi$
7. $k(x) = -\arctan x - \dfrac{\pi}{2}$

In Exercises 8-18, simplify each expression (if possible):

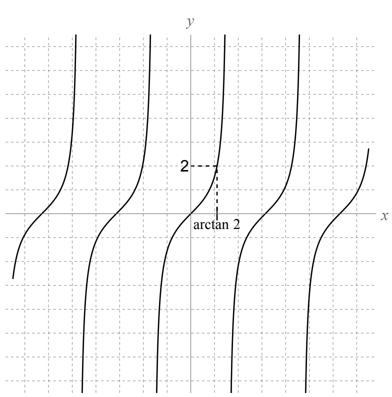
8. $\arctan \sqrt{3}$
9. $\arctan \dfrac{1}{\sqrt{2}}$
10. $\arctan \dfrac{1}{\sqrt{3}}$
11. $3 \arctan 1$
12. $\arctan(-7)$
13. $-4 \arctan^2 \sqrt{3}$
14. $2 \arctan 0 + 5$
15. $\dfrac{1}{5} \arctan \dfrac{-1}{\sqrt{3}}$
16. $\arctan \dfrac{\pi}{4}$
17. $\dfrac{1}{3} \arctan -1$
18. $3 \arctan \sqrt{3} + 8 \arctan 1$

In Exercises 19-24, solve each equation:

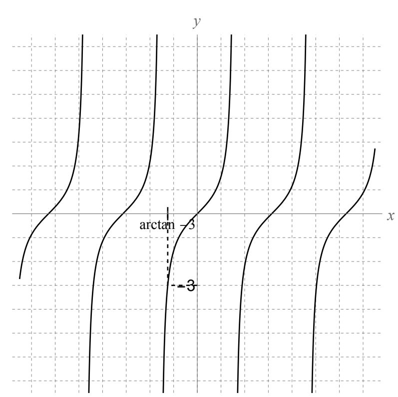
19. $\tan x = -1$
20. $\tan x = \sqrt{11}$
21. $\tan x = \dfrac{-1}{\sqrt{3}}$
22. $\tan x = \sqrt{3}$
23. $\tan x = 5$
24. $\tan x = 0$

Answers

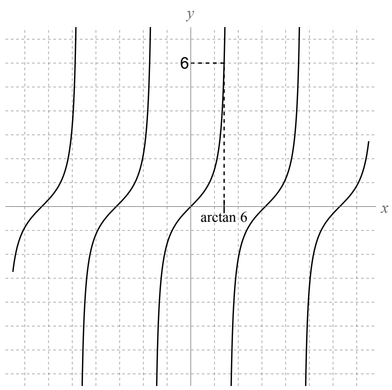
1. a)



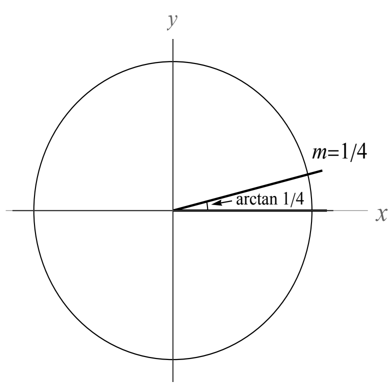
b)



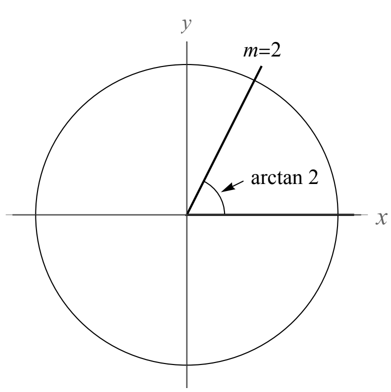
c)



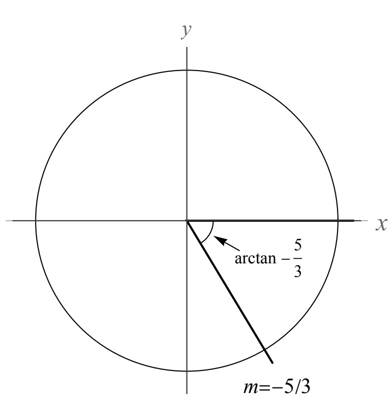
2. a)



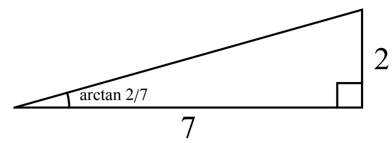
b)



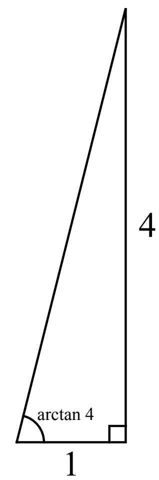
c)



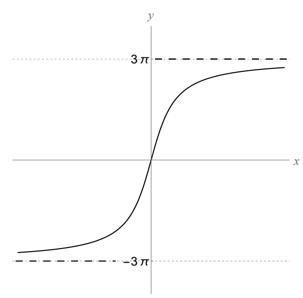
3. a)



b)



4.



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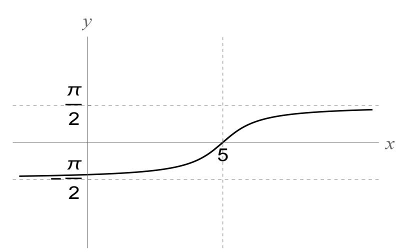

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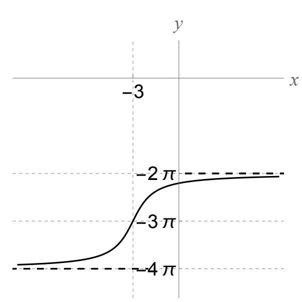
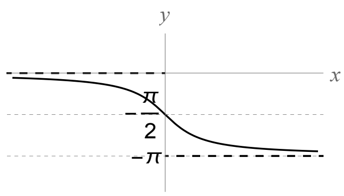


7. 

8. $\frac{\pi}{3}$

9. $\arctan \frac{1}{\sqrt{2}}$

10. $\frac{\pi}{6}$

11. $\frac{3\pi}{4}$

12. $-\arctan 7$

13. $\frac{-4\pi^2}{9}$

14. 5

15. $-\frac{\pi}{30}$

16. $\arctan \frac{\pi}{4}$

17. $-\frac{\pi}{12}$

18. 3π

19. $x = \pi n - \frac{\pi}{4}$

20. $x = \pi n + \arctan \sqrt{11}$

21. $x = \pi n - \frac{\pi}{6}$

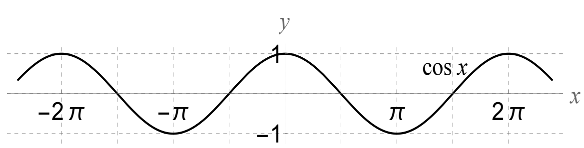
22. $x = \pi n + \frac{\pi}{3}$

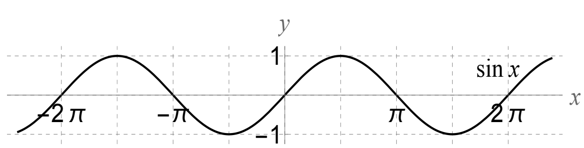
23. $x = \pi n + \arctan 5$

24. $x = \pi n$

Exercises from Section 4.4

1. Here are graphs of $\cos x$ and $\sin x$:





Explain, by drawing a picture on one of these graphs, how you would find each of these values from this graph:

a) $\arccos -\frac{3}{4}$

b) $\arcsin -.65$

c) $\arcsin \frac{1}{3}$

d) $\arccos .85$

2. Explain, by drawing a picture, how to interpret each of these quantities in the context of the unit circle:

   a) $\arcsin \frac{3}{4}$
   b) $\arccos -\frac{1}{5}$
   c) $\arccos \frac{2}{7}$
   d) $\arctan \frac{1}{2}$
   e) $\arcsin 1$
   f) $\arccos 0$

3. Explain, by drawing a picture, how to interpret each of these quantities in the context of a right triangle:

   a) $\arccos \frac{3}{8}$
   b) $\arcsin \frac{11}{15}$
   c) $\arccos \frac{2}{5}$
   d) $\arctan \frac{1}{3}$
   e) $\arcsin \frac{3}{4}$

In each of Exercises 4-6, sketch a graph of each function:

4. $F(x) = -2 \arcsin x$
5. $G(x) = \arcsin(x - 3)$
6. $H(x) = \arcsin x + \frac{3\pi}{2}$

In Exercises 7-27, simplify each expression (if possible):

7. $\arcsin \frac{\sqrt{2}}{2}$
8. $\arccos 0$
9. $\arcsin^4 1$
10. $\frac{1}{\pi} \arcsin \frac{\sqrt{3}}{2}$
11. $-\arcsin^2 -\frac{1}{2}$
12. $\frac{3}{2} \arccos \frac{1}{2}$
13. $\arcsin \frac{2}{3}$
14. $\arcsin 0 \arcsin \frac{1}{5}$
15. $\arccos \frac{7}{8}$
16. $\arcsin \left(\frac{1}{4} + \frac{3}{4}\right)$
17. $4 \arcsin \frac{3}{2}$
18. $\arcsin 1 - \arcsin \frac{1}{2}$
19. $\arccos \frac{1}{2} + \frac{1}{4}$
20. $\arcsin \frac{\pi}{6}$
21. $\arccos -\frac{\sqrt{3}}{2}$
22. $\arccos \left(\cos \frac{3\pi}{4}\right)$
23. $\arcsin \left(\cos \frac{\pi}{6}\right)$
24. $\arcsin \arccos 1$
25. $\arcsin 1 \arctan 1$
26. $2 \arcsin -\frac{1}{2} + 5 \arccos 0$
27. $4 \arcsin \frac{\sqrt{2}}{2} - 8 \arccos \left(-\frac{\sqrt{2}}{2}\right)$

In Exercises 28-46, solve each equation:

28. $4\tan x + 3 = 5$

29. $2\sin x + 1 = 0$

30. $\sin x = -\dfrac{\sqrt{2}}{2}$

31. $\cos x + 2 = 3$

32. $2\sin x = \sqrt{3}$

33. $3\tan x - 5 = 1$

34. $\cos x = 0$

35. $\dfrac{2}{3} + 3\sin x = 2$

36. $\cot x + \sqrt{3} = 0$

37. $3\sin x - 1 = 0$

38. $\sqrt{3}\csc x - 2 = 0$

39. $4\cot x + 7 = -2\cot x$

40. $2\tan x - 2\sqrt{3} = 0$

41. $4\cos x - \sqrt{3} = 2\cos x$

42. $3\cot x = -3$

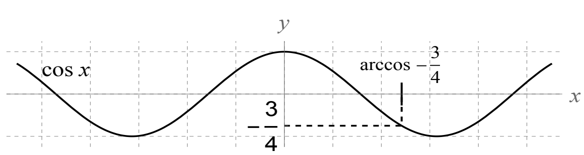
43. $5\cos x + 2\sqrt{3} = \cos x$

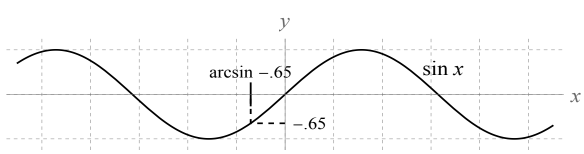
44. $2 + \sin x = 2$

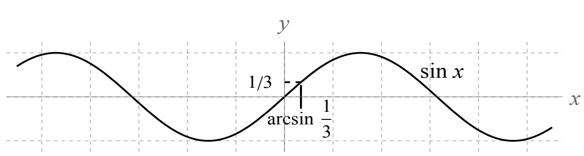
45. $4\sec x - 5 = \sec x + 1$

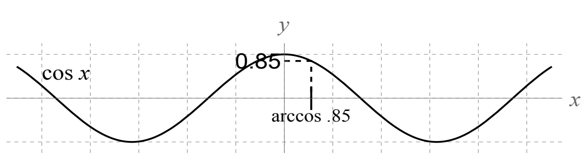
46. $5\tan x - 4 = 2\tan x + 1$

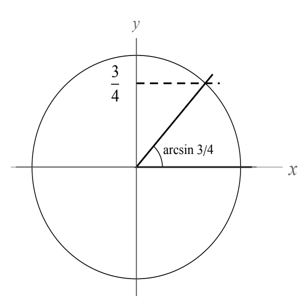
Answers

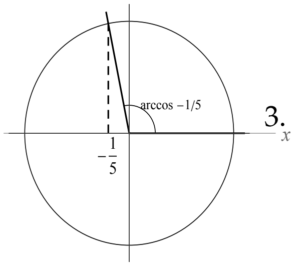
1. a) 

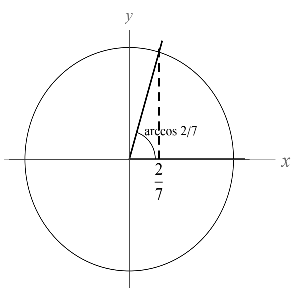
b) 

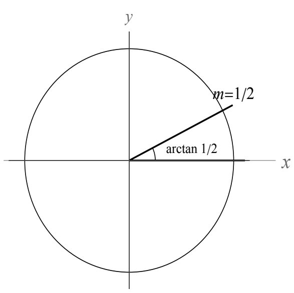
c) 

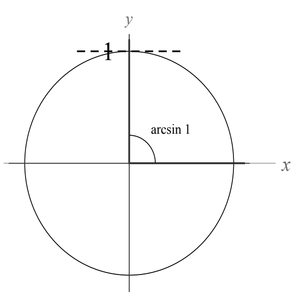
d) 

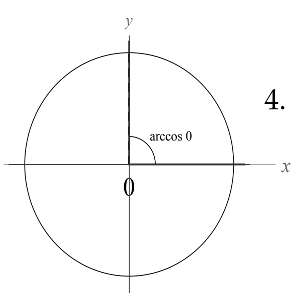
2. a) 

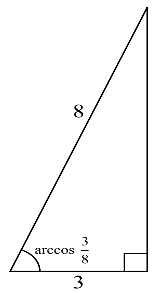
b) 

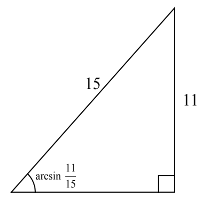
c) 

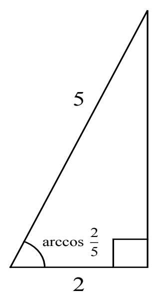
d) 

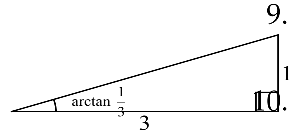
e) 

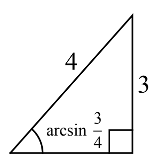
f) 

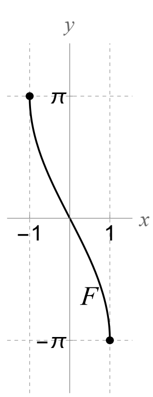
3. a) 

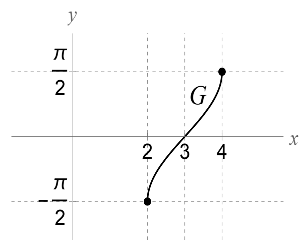
b) 

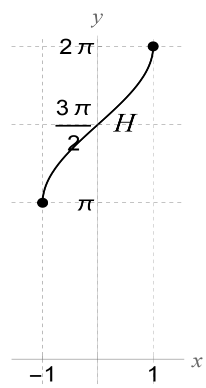
c) 

d) 

e) 

4. 

5. 

6. 

7. $\frac{\pi}{4}$

8. $\frac{\pi}{2}$

9. $\frac{\pi^4}{16}$

10. $\frac{1}{3}$

11. $-\frac{\pi^2}{36}$

12. $\frac{\pi}{2}$

13. this can't be simplified

14. 0

15. this can't be simplified

16. $\frac{\pi}{2}$

17. DNE

18. $\frac{\pi}{3}$

19. $\frac{\pi}{3} + \frac{1}{4}$

20. this can't be simplified
21. $\frac{5\pi}{6}$
22. $\frac{3\pi}{4}$
23. $\frac{\pi}{3}$
24. 0
25. $\frac{\pi^2}{8}$
26. $\frac{13\pi}{6}$
27. -5π
28. $x = \pi n + \arctan \frac{1}{2}$
29. $x = 2\pi n - \frac{\pi}{6}$, $x = 2\pi n + \pi + \frac{\pi}{6}$
30. $x = 2\pi n - \frac{\pi}{4}$, $x = 2\pi n + \frac{5\pi}{4}$
31. $x = 2\pi n$
32. $x = 2\pi n + \frac{\pi}{3}$, $x = 2\pi n + \frac{2\pi}{3}$
33. $x = \pi n + \arctan \frac{5}{3}$
34. $x = 2\pi n \pm \frac{\pi}{2}$
35. $x = 2\pi n + \arcsin \frac{4}{9}$, $x = 2\pi n + \pi - \arcsin \frac{4}{9}$
36. $x = \pi n - \frac{\pi}{6}$
37. $x = 2\pi n + \arcsin \frac{1}{3}$, $x = 2\pi n + \pi - \arcsin \frac{1}{3}$
38. $x = 2\pi n + \frac{\pi}{3}$, $x = 2\pi n + \frac{2\pi}{3}$
39. $x = \pi n - \arctan \frac{6}{7}$
40. $x = \pi n + \frac{\pi}{3}$
41. $x = 2\pi n \pm \frac{\pi}{6}$
42. $x = \pi n - \frac{\pi}{4}$
43. $x = 2\pi n \pm \frac{5\pi}{6}$
44. $x = 2\pi n, x = \pi + 2\pi n$
45. $x = 2\pi n \pm \frac{\pi}{3}$
46. $x = \pi n + \arctan \frac{5}{3}$

Exercises from Section 4.5

1. An investor makes a single deposit of $\$1200$ into an account that earns 5% interest, compounded annually. Let $A(t)$ be the amount of money in the account after t years.

    a) Compute $A(1)$ by hand.
    b) Use a calculator to compute $A(2)$, $A(3)$, $A(4)$ and $A(10)$.
    c) Write a rule for A.
    d) What is the base of the function A?
    e) Does the graph of A increase or decrease as t increases?

2. A person owes $\$500$ on a credit card, and that if the person makes minimum payments every month, the amount the person owes increases by 1%. Assume that the person makes minimum payments every month, and that no further charges are made. Let $f(x)$ be the amount owed after x months.

    a) Compute $f(1)$ by hand.
    b) Use a calculator to compute $f(2)$, $f(3)$ and $f(6)$.

c) Use a calculator to compute the amount the person will owe after six years.

d) Write a rule for f.

e) How does the debt change when any one month passes (i.e. the amount of time increases by 1 month)?

f) How does the debt change when any three months pass?

g) What is the base of the function f?

h) Would you consider the function f to be an example of exponential growth, or exponential decay? Why?

i) If you were to graph f, what would be the graphical significance of the $\$500$ that was the original debt?

3. Complete the following table of values for the function $f(x) = 3^x$, and use that table to sketch the graph of f:

x	-4	-3	-2	-1	0	1	2	3
$f(x)$								

4. Complete the following table of values for the function $g(x) = \left(\frac{1}{4}\right)^x$, and use that table to sketch the graph of g:

x	-3	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	3
$g(x)$											

In Exercises 5-15, you are given a function. For each function:

a) if the base of the exponential function is e, diagram the function;

b) identify any horizontal and/or vertical asymptotes of the function; and

c) sketch a graph of the function.
Note: when graphing an exponential function, your graph should indicate any asymptote(s), as well as the coordinates of where the point $(0, 1)$ is transformed.

5. $\alpha(x) = \exp(x-2)$

6. $\beta(x) = -e^x$

7. $\delta(x) = 3\exp x - 5$

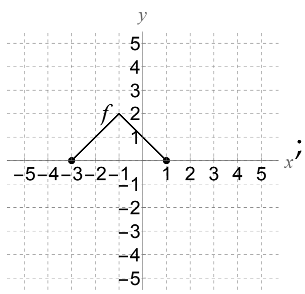
8. $\gamma(x) = 2 - e^x$

9. $\epsilon(x) = \exp 4x$

10. $\zeta(x) = -5^x + 2$

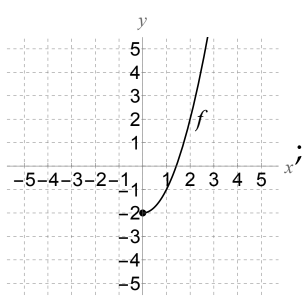
11. $\eta(x) = \left(\frac{1}{7}\right)^{x-4}$
12. $\theta(x) = 7 \cdot 3^x$
13. $\iota(x) = 2^x 4^x$
14. $\kappa(x) = 6 - 2e^{4x}$
15. $\lambda(x) = -2\left(\frac{5}{6}\right)^x$

In each of Exercises 16-23, you are given the graph of a function f, and a rule for a function g. Use the graph of f to sketch a graph of g.

16. 

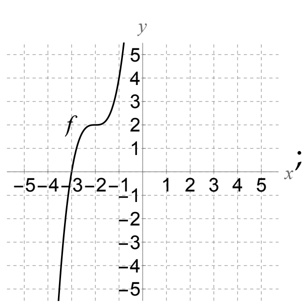
;

$g(x) = f(-x) - 3.$

17. 

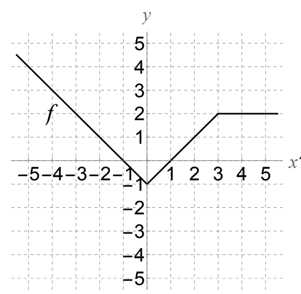
;

$g(x) = -f(x+1).$

18. 

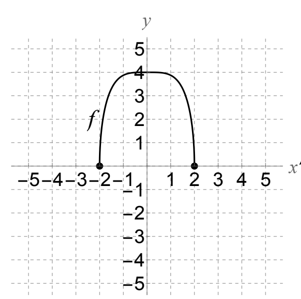
;

$g(x) = -f(-x).$

19. 

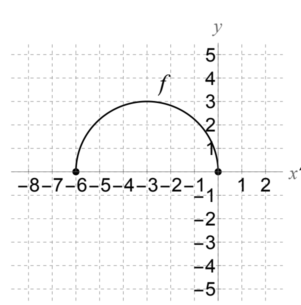
;

$g(x) = f(-(x-2)).$

20. 

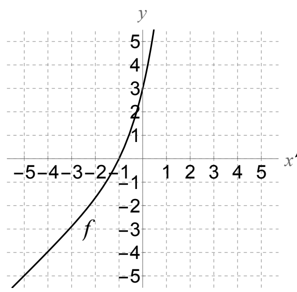
;

$g(x) = f(-x+5).$

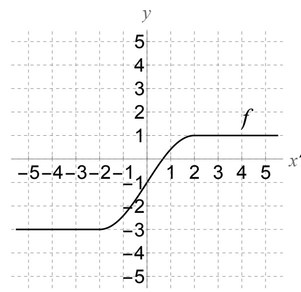
21. 

;

$g(x) = f(-3x).$

22. 

; $g(x) = 2 - f(x)$.

23. 

; $g(x) = 4f(-x)$.

In Exercises 24-30, sketch the graph of each function (as usual with an exponential function, your graph should show any asymptote(s) and label where the point $(0,1)$ is shifted to).

24. $D(x) = e^{-x/3}$
25. $E(x) = e^{-(x-4)}$
26. $F(x) = -6e^{-4x}$
27. $G(x) = 2e^{-x} - 4$
28. $H(x) = -5\exp(x+2)$
29. $I(x) = -\exp(-x) + \dfrac{3}{2}$
30. $J(x) = \dfrac{1}{4}\exp(3-x)$

In Exercises 31-36, write a function of the form $f(x) = a \cdot b^x$ that goes through each given pair of points:

31. $(0,1)$ and $\left(1, \dfrac{5}{2}\right)$
32. $(0,-2)$ and $(1,-6)$
33. $(0,8)$ and $(1,2)$
34. $(2,5)$ and $(7,35)$
35. $(4,39)$ and $(11,3)$
36. $(0,12)$ and $(-1,2)$

37. $\$100$ is deposited into an account that accrues interest at some unknown rate, compounded monthly. Suppose that after 5 months, there is $\$120$ in the account.

    a) Let $A(x)$ be the amount of money in the account after x months. Write a rule for A.

       Hint: This is a story problem version of what you just did in Exercises 31-36.

    b) What is the exact amount ("exact" meaning no decimals allowed) of money that will be in the account after 19 months?

    c) Use a calculator to determine the amount of money that will be in the account after 3 years.

38. Suppose that the amount of a uranium-238 in a nuclear reactor is given by an exponential decay model. After 15 years, there are 70 grams of U-238 in the reactor, and after 45 years, there are 20 grams of U-238 in the reactor.

   a) Let $f(t)$ be the amount of U-238 in the reactor after t years. Write a rule for f.
   b) What is the exact amount of U-238 that was initially in the reactor?
   c) What is the exact amount of U-238 that will be in the reactor after 65 years?

In each of Exercises 39-43, you are given a table of values for a function. Determine if that table of values comes from a linear function, an exponential function, or neither:

39.

x	3	4	5	6
$r(x)$	8	5	2	-1

40.

x	-2	-1	0	1
$s(x)$	4	7	11	16

41.

x	10	20	30	40
$t(x)$	15	45	135	405

42.

x	7	8	9	10
$t(x)$	640	320	160	80

43.

x	0	1	2	3
$t(x)$	5	7	9	11

In each of Exercises 44-52, you are given two functions. Determine which function has larger values when x is very large.

44. $a(x) = 5x^3$;
    $b(x) = \sqrt{20x^6}$.

45. $f(x) = 17x^6$;
    $g(x) = 2\sqrt{x^{13}}$.

46. $F(x) = x^2$;
    $G(x) = 2^x$.

47. $\Psi(x) = 3^x$;
    $\Phi(x) = \left(\frac{1}{3}\right)^x$.

48. $u(x) = 100 \cdot 4^x$;
    $v(x) = 3 \cdot 5^x$.

49. $h(x) = 5^x$;
    $k(x) = 5^{x-2}$.

50. $m(x) = 2^x$;
    $n(x) = 2^{-x}$.

51. $p(x) = \left(\frac{2}{7}\right)^x$;
    $q(x) = \left(\frac{2}{7}\right)^{-x}$.

52. $f(x) = 10^{100}x^{1000}$;
    $g(x) = \frac{1}{10^{100}}2^x$.

Answers

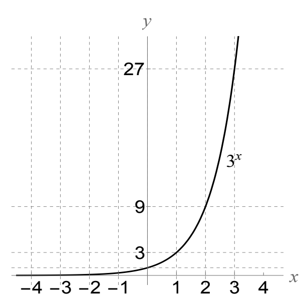
1. a) $A(1) = 1200(1.05) = \$1260$.
   b) $A(2) = \$1323$; $A(3) = \$1389.15$; $A(4) = \$1458.61$; $A(10) = \$1954.67$.
   c) $A(t) = 1200(1.05)^t$.
   d) 1.05
   e) increases

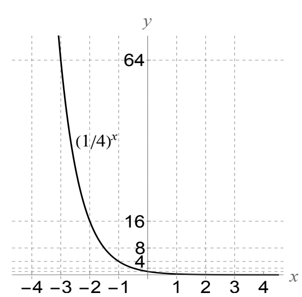
2. a) $f(1) = 500(1.01) = \$505$.
   b) $f(2) = \$510.05$; $f(3) = \$515.15$; $f(6) = \$530.76$.
   c) $f(72) = \$1023.55$.
   d) $f(x) = 500(1.01)^x$.
   e) The debt is multiplied by 1.01.
   f) The debt is multiplied by $(1.01)^3$.
   g) 1.01
   h) f has exponential growth, since the base is larger than 1.
   i) $(0, 500)$ is the y-intercept of f.

3.

x	-4	-3	-2	-1	0	1	2	3
$f(x)$	$\frac{1}{81}$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27

; graph below at left.



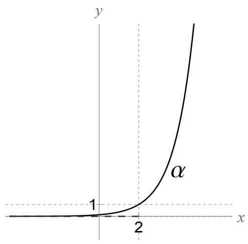


4. Graph is above at right; the table is

x	-3	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	3
$g(x)$	64	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{64}$

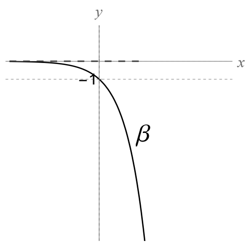
5. a) $x \xrightarrow{-2} \xrightarrow{\exp} \alpha(x)$

   b) HA $y = 0$; no VA

c) 

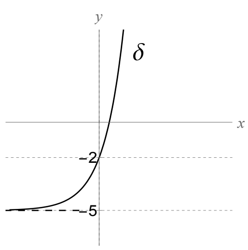
6. a) $x \xrightarrow{\text{exp}} \xrightarrow{\times -1} \beta(x)$

b) HA $y = 0$; no VA

c) 

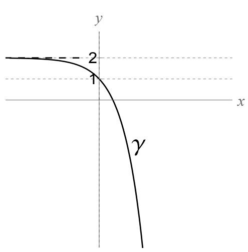
7. a) $x \xrightarrow{\text{exp}} \xrightarrow{\times 3} \xrightarrow{-5} \delta(x)$

b) HA $y = -5$; no VA

c) 

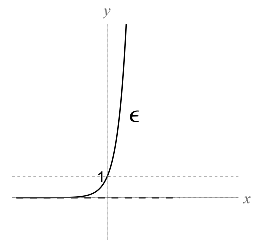
8. a) $x \xrightarrow{\text{exp}} \xrightarrow{\times -1} \xrightarrow{+2} \gamma(x)$

b) HA $y = 2$; no VA

c) 

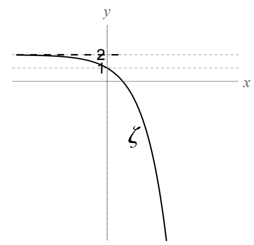
9. a) $x \xrightarrow{\times 4} \xrightarrow{\text{exp}} \epsilon(x)$

b) HA $y = 0$; no VA

c) 

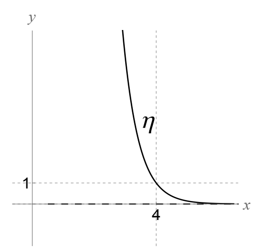
10. a) N/A since base is 5

b) HA $y = 2$; no VA

c) 

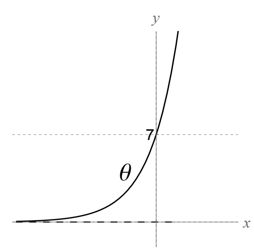
11. a) N/A since base is $\frac{1}{7}$

b) HA $y = 0$; no VA

c) 

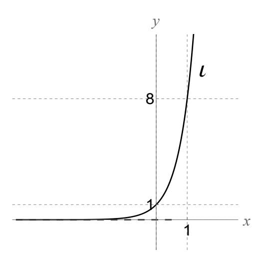
12. a) N/A since base is 3

b) HA $y = 0$; no VA

c) 

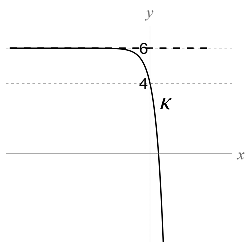
13. a) N/A since base is 8

b) HA $y = 0$; no VA

c) 

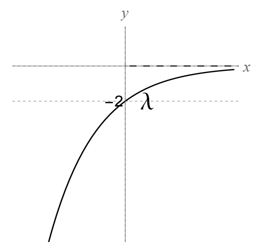
14. a) $x \xrightarrow{\times 4} \xrightarrow{\exp} \xrightarrow{\times -2} \xrightarrow{+6} \gamma(x)$

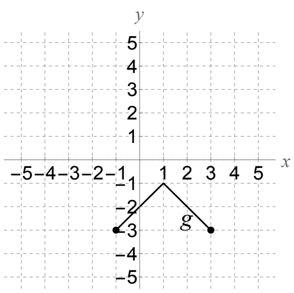
b) HA $y = 6$; no VA

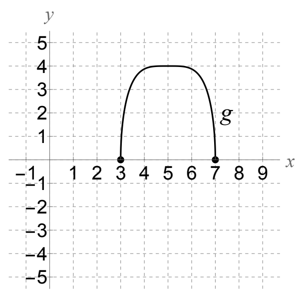
c) 

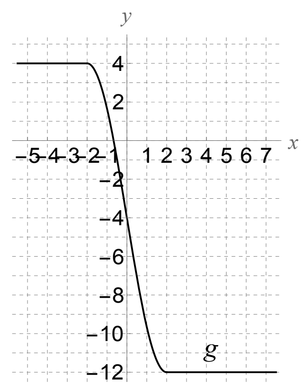
15. a) N/A since base is $\frac{5}{6}$

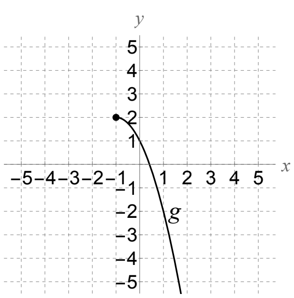
b) HA $y = 0$; no VA

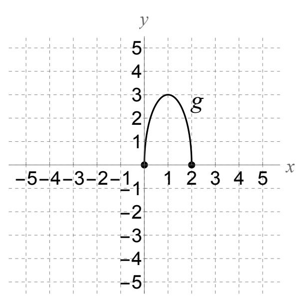
c) 

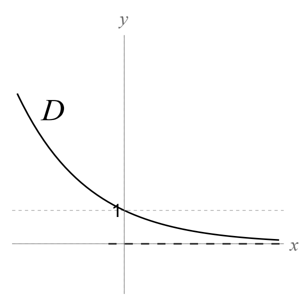
16. 

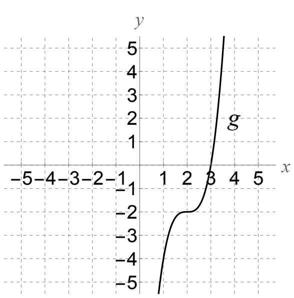
20. 

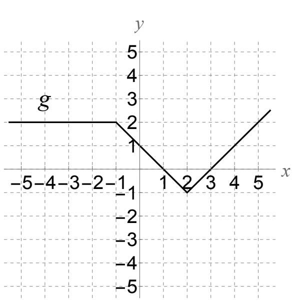
23. 

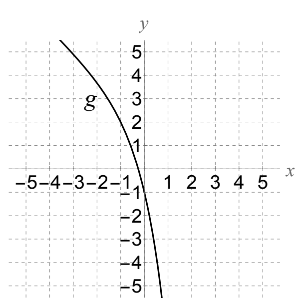
17. 

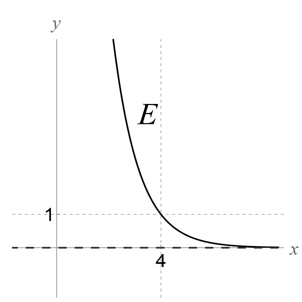
21. 

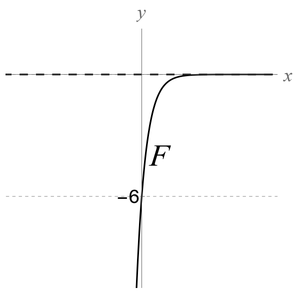
24. 

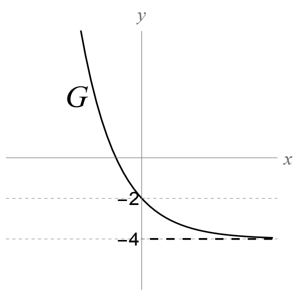
18. 

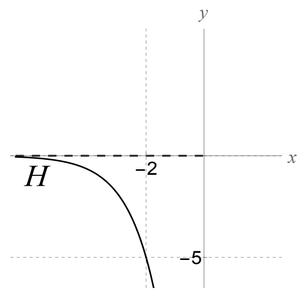
19. 

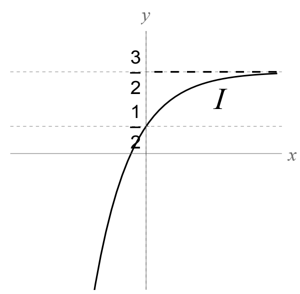
22. 

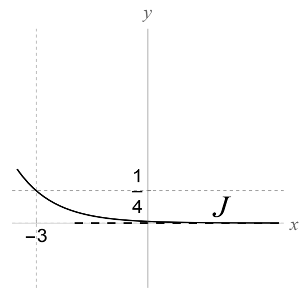
25. 

26. 

27. 

28. 

29. 

30. 

31. $f(x) = \left(\frac{5}{2}\right)^x$

32. $f(x) = -2 \cdot 4^x$

33. $f(x) = 8 \cdot \left(\frac{1}{4}\right)^x$

34. $f(x) = \frac{5}{7^{2/5}} \left(\sqrt[5]{7}\right)^x$

35. $f(x) = (39 \cdot 13^{4/7}) \left(\sqrt[7]{\frac{1}{13}}\right)^x$

36. $f(x) = 12 \cdot 6^x$

37.
   a) $A(x) = 100 \left(\sqrt[5]{\frac{6}{5}}\right)^x$
   b) $A(19) = 100 \left(\sqrt[5]{\frac{6}{5}}\right)^{19}$ dollars.
   c) $A(36) = 100 \left(\sqrt[5]{\frac{6}{5}}\right)^{36} \approx \$371.63$.

38.
   a) $f(t) = \left(70\sqrt{\frac{7}{2}}\right) \left(\sqrt[30]{\frac{2}{7}}\right)^t$.
   b) $f(0) = 70\sqrt{\frac{7}{2}}$ g.
   c) $f(65) = \left(70\sqrt{\frac{7}{2}}\right) \left(\sqrt[30]{\frac{2}{7}}\right)^{65}$ g.

39. linear

40. neither

41. exponential

42. exponential

43. linear

44. a is larger.

45. g is larger.

46. G is larger.

47. Ψ is larger.

48. v is larger.

49. h is larger.

50. m is larger.

51. q is larger.

52. g is larger.

Exercises from Section 4.6

In Exercises 1-46, compute each quantity, simplifying where appropriate. If a quantity does not exist, say so; if it cannot be reasonably simplified, leave it alone.

1. $\ln e^5$
2. $\log_2 32$
3. $2\ln e^7$
4. $\log 100000$
5. $\ln e^9$
6. $\log_{19} 1$
7. $\log_4 -64$
8. $\log_9 \dfrac{1}{3}$
9. $\ln \sqrt[3]{e^2} + 2\ln \sqrt[4]{e^5}$
10. $\log_3 0$
11. $2^{\log_2 -4}$
12. $\ln 7 + \ln e$
13. $\ln 0$
14. $\ln^2 e^5$
15. $6\log_{64} 8$
16. $\log_{57} 57$
17. $7^{\log_7 \frac{1}{4}}$
18. $\ln e^{-2/3}$
19. $\log_4 64$
20. $e^{\ln 5}$
21. $3 - \log_7 49 + \log_2 4$
22. $\ln -e^2$
23. $\log \dfrac{1}{1000}$
24. $\log_{1/4} \dfrac{1}{8}$
25. $\ln \dfrac{1}{4}$
26. $\ln e^{-4}$
27. $4\log_2 8$
28. $2\ln e^7 - \ln 4$
29. $\log_5 \dfrac{1}{125}$
30. $\ln -6$
31. $\log .01$
32. $4\log_2^3 16$
33. $3\log_6 0$
34. $3\ln e^3 + 2\log_6 36$
35. $e^{\ln \frac{1}{4}}$
36. $\log_9 9^5$
37. $\ln \sqrt[11]{e}$
38. $\ln 3$
39. $10\ln^2 \sqrt{e} + 5\log^2 100$
40. $\log_3 3 + 6$
41. $\log_3(3+6)$
42. $\exp \ln 17$
43. $\log_5 5^6$
44. $\ln \sqrt[5]{e}\log_2 16$
45. $\log_3 \sqrt{3}$
46. $4^{\log_4 7}$

In each of Exercises 47-56, you are given a function. For that function:
(a) Sketch its graph. Keep in mind that when graphing a logarithmic function, be sure to indicate any asymptote(s) and indicate where the point $(1, 0)$ is transformed.
(b) Give its domain and range.
(c) Write the equation(s) of any horizontal and/or vertical asymptote(s) of the function, if any.

47. $f(x) = \ln(x-3)$
48. $g(x) = \log_3(-x)$
49. $h(x) = -\ln x$
50. $i(x) = 5\log_2(x+4)$

51. $j(x) = -\log_5(-x)$

52. $k(x) = 4\ln(-(x+1))$

53. $l(x) = \log_{10}(-x+5)$

54. $m(x) = \ln \dfrac{x}{6}$

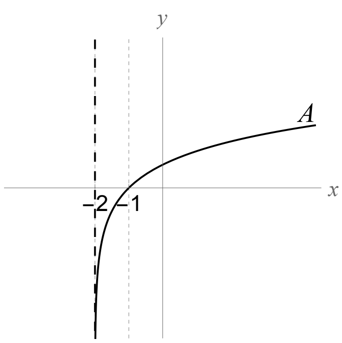
55. $n(x) = \log_3 4x$

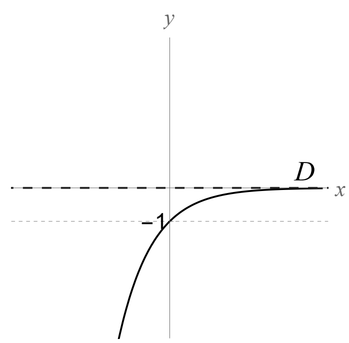
56. $o(x) = \dfrac{2}{3}\log_4(x-2)$

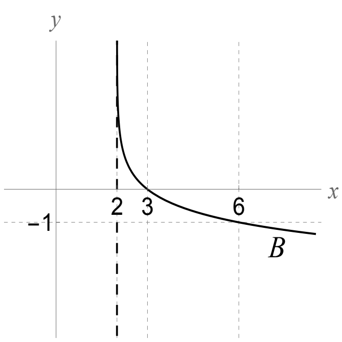
In each of Exercises 57-62, you are given a graph of some unknown function.

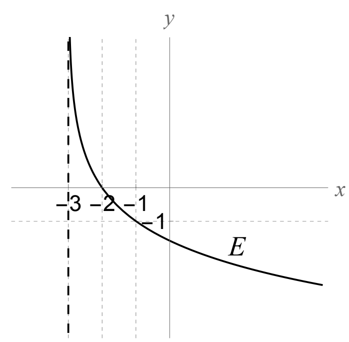
(a) Determine whether the function is exponential, logarithmic, or neither.

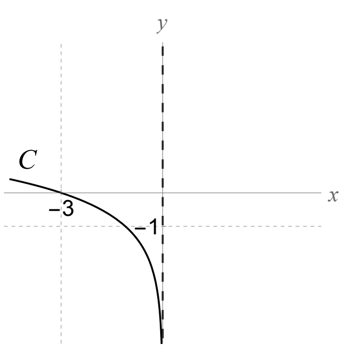
(b) If the function is either exponential or logarithmic, write a rule for the function. Unless the graph provides you with specific information to the contrary, assume the function is <u>natural</u> exponential or <u>natural</u> logarithmic.

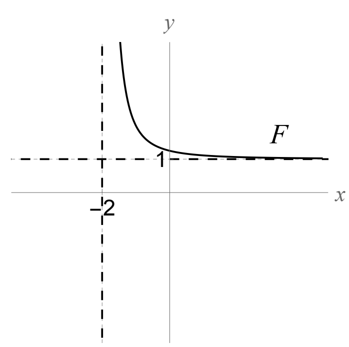
57. 

60. 

58. 

61. 

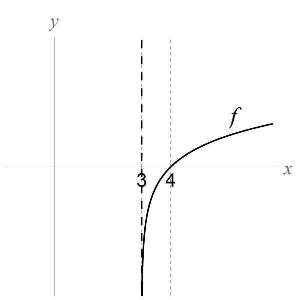
59. 

62. 

Answers

1. 5
2. 5
3. 14
4. 6
5. 9
6. 0
7. DNE
8. $-\frac{1}{2}$
9. $\frac{19}{6}$
10. DNE
11. DNE
12. $\ln 7 + 1$
13. DNE
14. 25
15. 3
16. 1
17. $\frac{1}{4}$
18. $-\frac{2}{3}$
19. 3
20. 5
21. 3
22. DNE
23. -3
24. $\frac{3}{2}$
25. $\ln \frac{1}{4}$
26. -4
27. 12
28. $14 - \ln 4$
29. -3
30. DNE
31. -2
32. 256
33. DNE
34. 13
35. $\frac{1}{4}$
36. 5
37. $\frac{1}{11}$
38. $\ln 3$
39. $\frac{45}{2}$
40. 7
41. 2
42. 17
43. 6
44. $\frac{4}{5}$
45. $\frac{1}{2}$
46. 7

47. a)

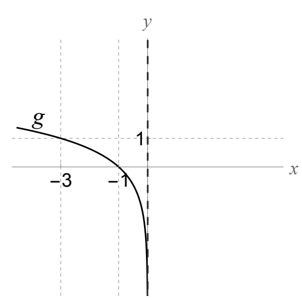


b) domain $(3, \infty)$;

range $\mathbb{R}$ (all real numbers)

c) no HA; VA $x = 3$

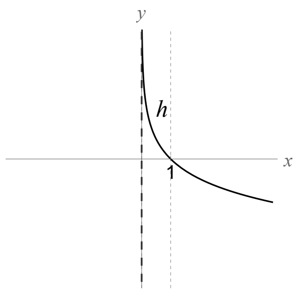
48. a)



b) domain $(-\infty, 0)$;

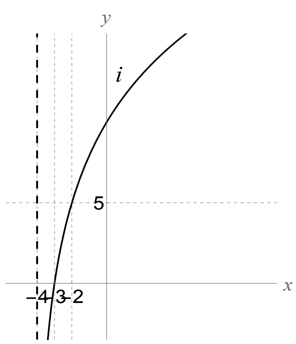
range $\mathbb{R}$ (all real numbers)

c) no HA; VA $x = 0$

49. a) 

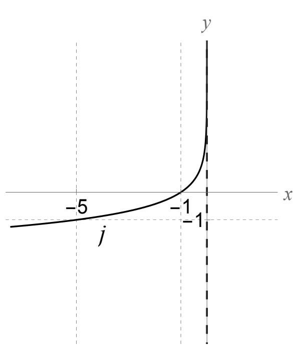
b) domain $(0, \infty)$;
range $\mathbb{R}$ (all real numbers)

c) no HA; VA $x = 0$

50. a) 

b) domain $(-4, \infty)$;
range $\mathbb{R}$ (all real numbers)

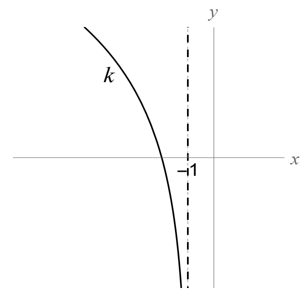
c) no HA; VA $x = -4$

51. a) 

b) domain $(-\infty, 0)$;
range $\mathbb{R}$ (all real numbers)

c) no HA; VA $x = 0$

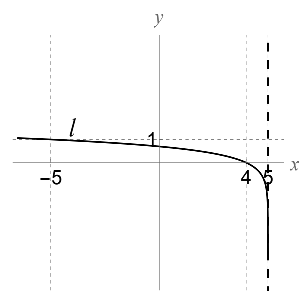
52. $k(x) = 4\ln(-(x+1))$

a) 

b) domain $(-\infty, -1)$;
range $\mathbb{R}$ (all real numbers)

c) no HA; VA $x = -1$

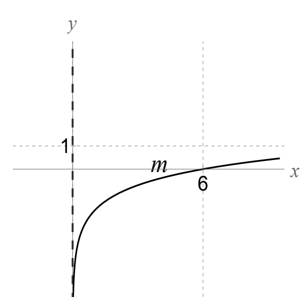
53. $l(x) = \log_{10}(-x+5)$

a) 

b) domain $(-\infty, 5)$;
range $\mathbb{R}$ (all real numbers)

c) no HA; VA $x = 5$

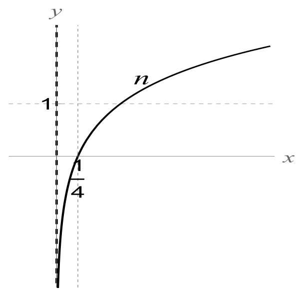
54. $m(x) = \ln \dfrac{x}{6}$

a) 

b) domain $(0, \infty)$;
range $\mathbb{R}$ (all real numbers)

c) no HA; VA $x = 0$

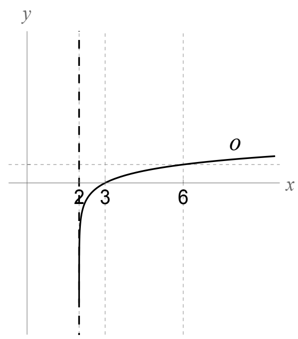
55. $n(x) = \log_3 4x$

a) 

b) domain $(0, \infty)$;
range $\mathbb{R}$ (all real numbers)

c) no HA; VA $x = 0$

56. $o(x) = \frac{2}{3}\log_4(x-2)$

a) 

b) domain $(2, \infty)$;
range $\mathbb{R}$ (all real numbers)

c) no HA; VA $x = 2$

57. a) logarithmic
b) $B(x) = -\log_4(x-2)$

58. a) logarithmic
b) $C(x) = \ln(-x/3)$

59. a) exponential
b) $D(x) = -e^{-x}$

60. a) logarithmic
b) $E(x) = -\log_2(x+3)$

61. a) neither
b) N/A

62. a) logarithmic
b) $G(x) = \ln(x-3)$

Exercises from Section 4.7

In each of Exercises 1-22, expand the given expression as much as possible, simplifying where appropriate:

1. $\ln \frac{x}{y}$
2. $\ln x^5$
3. $\ln \sqrt[3]{x}$
4. $\ln 5^7 e^3$
5. $\log_3 \frac{9y^4}{\sqrt{y}}$
6. $\ln x^3 y$
7. $\log_9 \frac{q^4}{3}$
8. $\ln 5^r$
9. $\ln 8e(x-1)$
10. $\log_2 (a^2 b)^3 c^5$
11. $\ln \frac{x^3}{y}$
12. $\ln(4x)$
13. $\ln 4x$
14. $\log_4 \left(\frac{2x}{y^3}\right)^5$
15. $\log_5 \frac{x^3}{125}$
16. $\ln(w^4+2)^3$
17. $\ln(x^2-1)$
18. $\ln x^2 - 1$
19. $\ln \sqrt{x\sqrt{y}}$
20. $\log_4 16\sqrt{x}$
21. $\log \frac{10000}{t+4}$
22. $\log_2 4\sqrt[3]{xy}$

In Exercises 23-37, suppose that $p = \ln x$, $q = \ln y$ and $r = \ln z$. Rewrite each expression in terms of p, q and/or r:

23. $\ln xy$
24. $\ln \frac{y}{x^2 z}$
25. $\ln \sqrt[3]{xy^2}$
26. $2 \ln 4x^5$
27. $\ln ez$
28. y
29. $\ln y^3 z^2 + \ln 3x^4 y^2$
30. $3z$
31. $\ln e^7 x^4 + 3$
32. $\ln \frac{5\sqrt{y}}{z^3}$
33. $\ln 5\sqrt{\frac{z}{y}} - 4$
34. $\ln \left(\frac{x}{4y}\right)^9$
35. $\ln \frac{z^7}{e^8}$
36. x^4
37. $\ln(x^2 + 1)$

In Exercises 38-53, rewrite the given expression as a single logarithm (meaning your answer should be "$\ln \square$" or "$\log_b \square$" or "$\log \square$"). If possible, simplify the answer:

38. $\ln x + \ln 2$
39. $3 \ln t + 2 \ln u$
40. $5 \ln 4$
41. $2 \ln 7 + 1$
42. $2 \log_4 8$
43. $4 \log_3 z + 2(\log_3 w - 5 \log_3 y)$
44. $2 \log(x^2 + 5)$
45. $\log_6 12 + \log_6 3$
46. $2 \log_5(x + 2)$
47. $\log(x - 2) + \log(x + 3)$
48. $5 \ln 2 - 3 \ln 4 + \ln 5$
49. $2 \log_5 \frac{1}{4} + \log_5 3$
50. $2 \ln(x + 3) - 2 \ln 5$
51. $\log 25 + \log 4$
52. $\log_2 36 - 2 \log_2 3$
53. $5 \ln 3 + 2$

In Exercises 54-71, rewrite each given expression in terms of natural exponentials and/or natural logarithms. If possible, write the expression as a single exponential term with base e:

54. 3^x
55. 13^{-5}
56. $\log_6 17$
57. $\sqrt{x + 5}$
58. $\log 29$
59. $\sqrt[5]{y^4}$
60. $\log_4 z$
61. $\log_r x$
62. 8^{xy}
63. $(x + 3)^{x-4}$
64. $\log 15$
65. $x^{\sin x}$
66. $3^{4x} a^{x-1}$
67. $x^{3 \ln x}$
68. $7 \cdot 2^{5x} 9^{3x}$

69. $\log_2 y^4 - 3\log_3 x$

70. 35

71. $a^{\sqrt[3]{b}}$

In Exercises 72-86, simplify each expression:

72. $e^{3\ln 4}$

73. $7e^{-2\ln 3}$

74. $\dfrac{\ln 9}{\ln 81}$

75. $e^{\ln 3} + 4$

76. $\exp\left[(x-3)\ln(x^2+4)\right]$

77. $\dfrac{\ln 10 + \ln 2 + \ln 5}{\ln 30 - \ln 3}$

78. $e^{5\ln 2x} + 1$

79. $3e^{x\ln 4x} + 7e^{x\ln 5}$

80. $\dfrac{-2\ln 8}{\ln 4}$

81. $e^{a\ln b}e^{c\ln d}$

82. $\ln e^{2\ln 5}$

83. $\log_4\sqrt{2} + \log_4\sqrt{8}$

84. $\exp(4\ln 3 - 2\ln 6)$

85. $\dfrac{\ln 25 + \ln 9}{\ln 5 + \ln 3}$

86. $\dfrac{\ln a^n - \ln b^n}{\ln a - \ln b}$

In each of Exercises 87-94, you are given a function. Diagram the function, keeping in mind that the only exponential and/or logarithmic functions allowed in diagrams are natural exponentials (exp) and natural logarithm (ln):

87. $f(x) = 3\cdot 5^x$

88. $g(x) = \log_7(x-5)$

89. $h(x) = -2^{4x+1}$

90. $i(x) = e^{3\sin^2 x^2}$

91. $j(x) = x^{3x}$

92. $k(x) = \sin 4^x 2^{\cos x}$

93. $l(x) = \ln\log x$

94. $m(x) = 2^{x\cos x} + x^{\ln x} - 5$

In Exercises 95-107, reverse-diagram each function, simplifying its rule if possible:

95. $x \xrightarrow{\ln} \xrightarrow{\times 8} \xrightarrow{\exp} F(x)$

96. $x \xrightarrow{\times 3} \xrightarrow{\ln} \xrightarrow{\times 4} \xrightarrow{\exp} G(x)$

97. $x \xrightarrow{\times \ln 4} \xrightarrow{\exp} \xrightarrow{\times 3} H(x)$

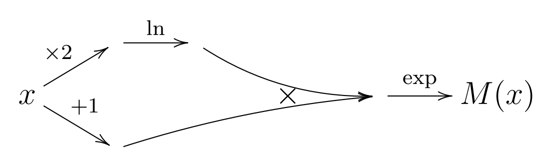
98. $x \xrightarrow{\ln} \xrightarrow{\div 3} \xrightarrow{+7} I(x)$

99. $x \xrightarrow{\ln} \xrightarrow{\div \ln 7} J(x)$

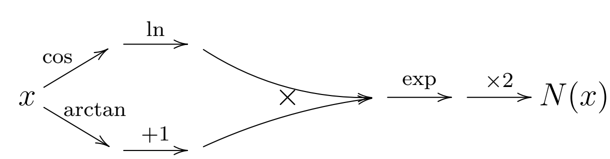
100. $x \xrightarrow{+2} \xrightarrow{\ln} \xrightarrow{\div \ln 10} K(x)$

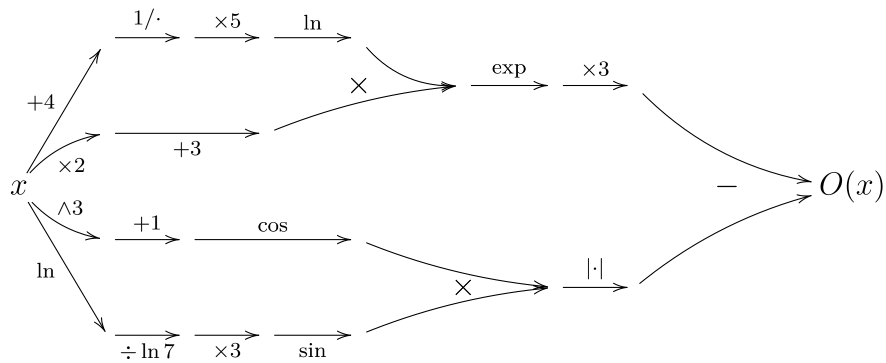
101. $x \xrightarrow{\times -5} \xrightarrow{+1} \xrightarrow{\ln} \xrightarrow{\div \ln 4} L(x)$

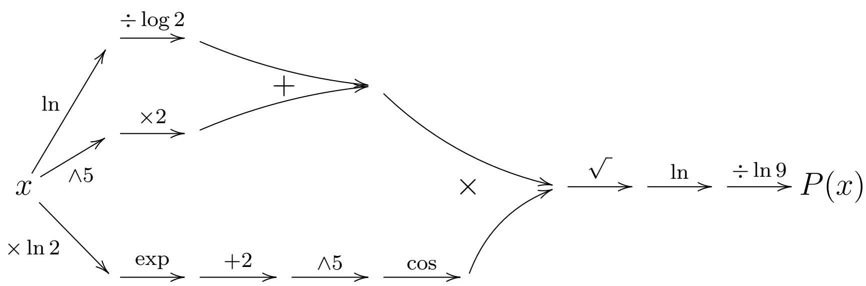
102.

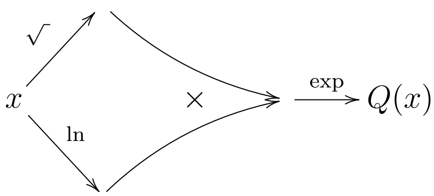


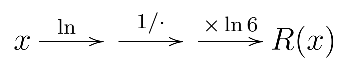
103.



104. 

105. 

106. 

107. 

In Exercises 108-120, solve each equation:

108. $\ln x = 4$

109. $\ln x = -\dfrac{2}{5}$

110. $\dfrac{2}{7} \ln x = 4$

111. $e^x = 17$

112. $5 - 8 \ln x = 2$

113. $\dfrac{2}{5} e^x = \dfrac{11}{7}$

114. $e^x = -2$

115. $5e^x + 17 = 52$

116. $6 + 5 \ln x = 3$

117. $3(\ln x - 5) = 4(2 \ln x + 5)$

118. $\dfrac{1}{2}(e^x + 3) = \dfrac{2}{3} e^x$

119. $\dfrac{2}{3}\left(e^x + \dfrac{2}{5}\right) = \dfrac{1}{4}\left(1 - e^x\right)$

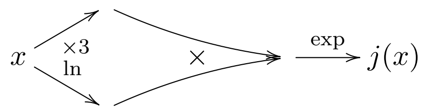
120. $2 - 9e^x = -34$

Answers

1. $\ln x - \ln y$
2. $5 \ln x$
3. $\frac{1}{3} \ln x$
4. $7 \ln 5 + 3$
5. $2 + \frac{7}{2} \log_3 y$
6. $3 \ln x + \ln y$
7. $4 \log_9 q - \frac{1}{2}$
8. $r \ln 5$
9. $\ln 8 + 1 + \ln(x-1)$
10. $6 \log_2 a + 3 \log_2 b + 5 \log_2 c$
11. $3 \ln x - \ln y$
12. $\ln 4 + \ln x$
13. $\ln 4 + \ln x$
14. $5 \log_4 2 + 5 \log_4 x - 15 \log_4 y$
15. $3 \log_5 x$
16. $3 \ln(w^4 + 2)$
17. $\ln(x-1) + \ln(x+1)$
18. $2 \ln x - 1$
19. $\frac{1}{2} \ln x + \frac{1}{4} \ln y$
20. $2 + \frac{1}{2} \log_4 x$
21. $4 - \log(t+4)$
22. $2 + \frac{1}{3} \log_2 x + \frac{1}{3} \log_2 y$
23. $p + q$
24. $q - 2p - r$
25. $\frac{1}{3}p + \frac{2}{3}q$
26. $2 \ln 4 + 10p$
27. $1 + r$
28. e^q
29. $5q + 2r + \ln 3 + 4p$
30. $3e^r$
31. $10 + 4p$
32. $\ln 5 + \frac{1}{2}q - 3r$
33. $\ln 5 + \frac{1}{2}r - \frac{1}{2}q - 4$
34. $9p - 9 \ln 4 - 9q$
35. $7r - 8$
36. e^{4p}
37. $\ln(e^{2p} + 1)$
38. $\ln 2x$
39. $\ln t^3 u^2$
40. $\ln 4^5$
41. $\ln 49 + 1$
42. 3
43. $\log_3 \frac{z^4 w^2}{y^{10}}$
44. $\log(x^2 + 5)^2$
45. 2
46. $\log_5 (x+2)^2$
47. $\log(x^2 + x - 6)$
48. $\ln \frac{5}{2}$
49. $\log_5 \frac{3}{16}$

50. $\ln \frac{(x+3)^2}{25}$
51. 2
52. 2
53. $\ln 729 + 2$
54. $e^{x \ln 3}$
55. $e^{-5 \ln 13}$
56. $\frac{\ln 17}{\ln 6}$
57. $\exp\left(\frac{1}{2}\ln(x+5)\right)$
58. $\frac{\ln 29}{\ln 10}$
59. $\exp\left(\frac{4}{5}\ln y\right)$
60. $\frac{\ln z}{\ln 4}$
61. $\frac{\ln x}{\ln r}$
62. $e^{xy \ln 8}$
63. $e^{(x-4)\ln(x+3)}$
64. $\frac{\ln 15}{\ln 10}$
65. $e^{\sin x \ln x}$
66. $e^{4x \ln 3 + (x-1)\ln a}$
67. $e^{3 \ln^2 x}$
68. $7e^{5x \ln 2 + 3x \ln 9}$
69. $\frac{4 \ln y}{\ln 2} - \frac{3 \ln x}{\ln 3}$
70. $e^{\ln 35}$ or $\ln e^{35}$
71. $e^{\sqrt[3]{b} \ln a}$
72. 64
73. $\frac{7}{9}$
74. $\frac{1}{2}$
75. 7
76. $(x^2+4)^{x-3}$
77. 2
78. $32x^5 + 1$
79. $3(4x)^x + 7(5^x)$
80. -3
81. $b^a d^c$
82. $\ln 25$
83. 1
84. $\frac{9}{4}$
85. 2
86. n
87. $x \xrightarrow{\times \ln 5} \xrightarrow{\exp} \xrightarrow{\times 3} f(x)$
88. $x \xrightarrow{-5} \xrightarrow{\ln} \xrightarrow{\div \ln 7} g(x)$
89. $x \xrightarrow{\times 4} \xrightarrow{+1} \xrightarrow{\times \ln 2} \xrightarrow{\exp} \xrightarrow{\times -1} h(x)$
90. $x \xrightarrow{\wedge 2} \xrightarrow{\sin} \xrightarrow{\wedge 2} \xrightarrow{\times 3} \xrightarrow{\exp} i(x)$

91.



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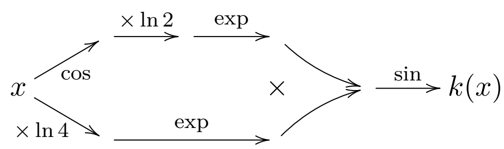

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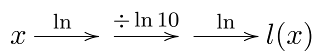

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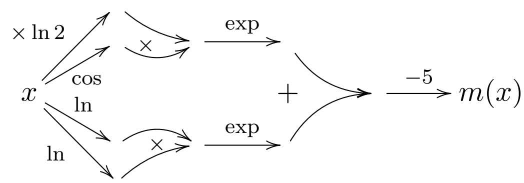


95. $F(x) = x^8$

96. $G(x) = 81x^4$

97. $H(x) = 3 \cdot 4^x$

98. $I(x) = \dfrac{x}{3} + 7$

99. $J(x) = \log_7 x$

100. $K(x) = \log(x+2)$

101. $L(x) = \log_4(1-5x)$

102. $M(x) = (2x)^{x+1}$

103. $N(x) = 2(\cos x)^{\arctan x + 1}$

104. $O(x) = 3\left(\dfrac{5}{x+4}\right)^{2x+3} - \left|\cos(x^3+1)\sin(3\log_7 x)\right|$

105. $P(x) = \log_9 \sqrt{\cos(2^x+2)^5\left[\log_2 x + 2x^5\right]}$

106. $Q(x) = x^{\sqrt{x}}$

107. $R(x) = \log_x 6$

108. $x = e^4$

109. $x = e^{-2/5}$

110. $x = e^{14}$

111. $x = \ln 17$

112. $x = e^{3/8}$

113. $x = \ln \dfrac{55}{14}$

114. DNE

115. $x = \ln 7$

116. $x = e^{-3/5}$

117. $x = e^{-7}$

118. $x = \ln 9$

119. DNE

120. $x = \ln 4$

Chapter 5

Solving equations

5.1 The big picture

What is an equation? What is "solving" an equation?

An **equation** is a mathematical statement with an equals sign in it. Usually, we write down equations with a variable in them, and the object is to **solve** the equation, which means to determine all values of the variable which make the equation true.

Equations versus identities

Consider these two "equations":

$$5x^2 + 3x - 2 = 0 \qquad\qquad \sin^2 x + \cos^2 x = 1$$

How do you tell the difference between an equation and an identity?

This is sometimes hard. Generally speaking, when you learn an identity, you are told that it is an identity, and you have to remember that it is true for any values of the variables. Otherwise, most things with = signs are equations.

Notation for solutions

Suppose we are asked to solve the equation $x^2 = 25$. What is the answer, and **how do we write that answer**?

Here are bunch of different ways to write the solution of $x^2 = 25$, any of which are OK:

- $-5, 5$
- $5, -5$
- ± 5
- $x = 5, -5$
- $x = -5, 5$
- $x = 5, x = -5$
- $x = -5, x = 5$
- $x = \pm 5$
- $\{-5, 5\}$
- $\{5, -5\}$
- $\{\pm 5\}$
- $x \in \{-5, 5\}$

Here are some ways you should avoid:

- $(-5, 5)$
- $[-5, 5]$
- $\langle -5, 5 \rangle$
- $x = \{-5, 5\}$

How I go about solving equations

Solving equations is a combination of science and art.

There's no way to write down a complete set of directions you can follow to solve every equation you might encounter.

Nonetheless, here is a procedure I follow when I have to solve an equation:

1. Classify the equation into one of three types:

SCIENCE ↗	**Type 1:**	The *variable* you're solving for *appears only once*
SCIENCE ↘	**Type 2:**	A *quadratic-type* equation;
ART ⟶	**Type 3:**	*Other*

2. (§5.2) Solve a Type 1 equation by diagramming the side with the variable and applying appropriate inverses.

3. (§5.3) Solve a Type 2 by factoring, completing the square, or the quadratic formula (and if necessary, by using a substitution to make the work more clear).

4. Solve a Type 3 equation by "trying something", even if you don't know what you're trying will work. Here are some options:

   - (§5.2 and §5.3) Combine like terms or otherwise rewrite the equation so that it becomes Type 1 or Type 2.
   - (§5.4) If the equation is polynomial, set one side $= 0$ and try to factor it.
   - (§5.4) Move all the terms to one side (making the other side 0) and try to factor the complicated side.
   - (§5.5) If the equation is a proportion, cross-multiply.
   - (§5.5) If there are fractions or rational expressions, rewrite by finding a common denominator.
   - (§5.6) Rewrite part of the equation using a trig identity.
   - (§5.6) Rewrite part of the equation using a log rule or exponent rule.
   - If you see the same complicated expression more than once, try a substitution.
   - Maybe try something else.

5.2 Equations where the variable appears once

Examples of equations where the variable appears once

$$3\cos^2 x + 5 = -2 \qquad \ln^2\left(\frac{t-3}{5}+2\right)^3 = 4$$

$$\frac{5}{7} = \sqrt{\frac{2\arctan x^4 - 3}{7}} \qquad 1 = 3^{4\sqrt{z}+2} - 5$$

These are NOT equations where the variable appears only once

$$3\cos^4 x + 5 = -2x \qquad \log_3\left(\frac{y-3}{5}+2y\right)^3 = 4$$

$$\sqrt{\frac{2\arctan s^4 - 3}{7s}} = \frac{5}{7} \qquad \sec\left(4\sqrt{x}+2\right) - 5 = x^4 - 7x + 3$$

BIG IDEA, PART 1

Any equation where the variable x occurs only once can be thought of as

$$f(x) = b$$

where f is some (perhaps complicated) function and b is a constant.
As an arrow diagram, this is

$$x \xrightarrow{f} b$$

The solution of this equation is

So to find x, all you have to do is find the inverse f^{-1} of f, and apply that inverse to b. That gets you back to x.

§5.1 EXAMPLE 1

In each equation,

i. Write the equation as $f(x) = b$ for a function f and a constant b.
ii. Write an arrow diagram representing your answer to part (i), labelling the arrow with an elementary function.
iii. Write the rule for the inverse f^{-1} of f.
iv. Apply the rule for f^{-1} to solve for x.
v. Indicate on your arrow diagram from part (ii) how the inverse f^{-1} is used.

a) $x + 5 = 9$

Solution: $f(x) = x + 5; \quad b = 9$

$$x \xrightarrow{+5} 9$$

b) $17 = 3x$

c) $x^2 = 14$

Solution: $f(x) = x^2; \quad b = 14$

$$x \xrightarrow{\wedge 2} 14$$

d) $\sqrt[11]{x} = 4$

$$x \longrightarrow 4$$

e) $\sin x = \dfrac{\sqrt{3}}{2}$

Solution: $f(x) = \sin x$; $\quad b = \dfrac{\sqrt{3}}{2}$

$$x \xrightarrow{\sin} \frac{\sqrt{3}}{2}$$

f) $e^x = 7$

Solution: $f(x) = e^x$; $\quad b = 7$

$$x \xrightarrow{\exp} 7$$

g) $5 = \ln x$

Solution: $f(x) = \ln x$; $\quad b = 5$

$$x \xrightarrow{\ln} 5$$

h) $|x| = 6$

Solution: $f(x) = |x|$; $\quad b = 6$

$$x \xrightarrow{|\cdot|} 6$$

NOTE: to implement this method, you need to know the inverse of each elementary function.

Inverses of elementary functions

Class	Function f		Inverse f^{-1}
arithmetic	$f(x) = x + c$	$\xrightarrow{+c}$	$\xleftarrow{-c}$
	$f(x) = x - c$	$\xrightarrow{-c}$	$\xleftarrow{+c}$
	$f(x) = cx$	$\xrightarrow{\times c}$	$\xleftarrow{\div c}$
	$f(x) = \frac{x}{c}$	$\xrightarrow{\div c}$	$\xleftarrow{\times c}$
even powers	$f(x) = x^2$	$\xrightarrow{\wedge 2}$	$\xleftarrow{\pm\sqrt{\ }}$
	$f(x) = x^4$	$\xrightarrow{\wedge 4}$	$\xleftarrow{\pm\sqrt[4]{\ }}$
	$f(x) = x^n$ (n even)	$\xrightarrow{\wedge n}$	$\xleftarrow{\pm\sqrt[n]{\ }}$
odd powers	$f(x) = x^3$	$\xrightarrow{\wedge 3}$	$\xleftarrow{\sqrt[3]{\ }}$
	$f(x) = x^5$	$\xrightarrow{\wedge 5}$	$\xleftarrow{\sqrt[5]{\ }}$
	$f(x) = x^n$ (n odd)	$\xrightarrow{\wedge n}$	$\xleftarrow{\sqrt[n]{\ }}$
absolute value	$f(x) = \lvert x \rvert$	$\xrightarrow{\lvert\cdot\rvert}$	$\xleftarrow{\pm}$
roots	$f(x) = \sqrt{x}$	$\xrightarrow{\sqrt{\ }}$	$\xleftarrow{\wedge 2}$
	$f(x) = \sqrt[3]{x}$	$\xrightarrow{\sqrt[3]{\ }}$	$\xleftarrow{\wedge 3}$
	$f(x) = \sqrt[n]{x}$	$\xrightarrow{\sqrt[n]{\ }}$	$\xleftarrow{\wedge n}$
reciprocal	$f(x) = x^{-1}$	$\xrightarrow{1/\cdot}$	$\xleftarrow{1/\cdot}$
exponential	$f(x) = e^x$	$\xrightarrow{\exp}$	$\xleftarrow{\ln}$
logarithmic	$f(x) = \ln x$	$\xrightarrow{\ln}$	$\xleftarrow{\exp}$
trig	$f(x) = \sin x$	$\xrightarrow{\sin}$	$\xleftarrow{2\pi n + \arcsin,\ 2\pi n + \pi - \arcsin}$
	$f(x) = \cos x$	$\xrightarrow{\cos}$	$\xleftarrow{2\pi n \pm \arccos}$
	$f(x) = \tan x$	$\xrightarrow{\tan}$	$\xleftarrow{\pi n + \arctan}$
inverse trig	$f(x) = \arctan x$	$\xrightarrow{\arctan}$	$\xleftarrow{\tan}$
	$f(x) = \arcsin x$	$\xrightarrow{\arcsin}$	$\xleftarrow{\sin}$
	$f(x) = \arccos x$	$\xrightarrow{\arccos}$	$\xleftarrow{\cos}$

§5.1 EXAMPLE 2

Solve the equation $3x + 5 = 23$.

"Traditional" solution:

$$3x + 5 = 23$$

What's hidden in this solution is "function/arrow" stuff:

BIG IDEA, PART 2

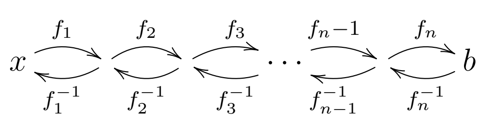
If a function is made up of a chain of compositions like

$$f = f_n \circ f_{n-1} \circ \cdots \circ f_3 \circ f_2 \circ f_1,$$

then the inverse of this function is

$$f^{-1} = f_1^{-1} \circ f_2^{-1} \circ f_3^{-1} \circ \cdots \circ f_{n-1}^{-1} \circ f_n^{-1}.$$

This comes from the arrow diagram



(If you get dressed by putting your socks on, then your shoes on, then you get undressed by taking your shoes off, then your socks off.)

§5.1 EXAMPLE 3

Find a formula for the inverse of each function:

a) $f(x) = 13 - 3x$

b) $f(x) = \dfrac{1}{x-4}$

c) $f(x) = 4\ln x$

Solution: f diagrams as $x \underset{\exp}{\overset{\ln}{\rightleftarrows}} \underset{\div 4}{\overset{\times 4}{\rightleftarrows}} f(x)$,

so $f^{-1}(y) = \exp\left(\dfrac{y}{4}\right)$, i.e. $\boxed{f^{-1}(y) = e^{y/4}}$.

d) $f(x) = 3e^{x+2}$

Solution: f diagrams as $x \underset{-2}{\overset{+2}{\rightleftarrows}} \underset{\ln}{\overset{\exp}{\rightleftarrows}} \underset{\div 3}{\overset{\times 3}{\rightleftarrows}} f(x)$,

so $f^{-1}(y) = \boxed{\ln\left(\dfrac{y}{3}\right) - 2}$.

§5.1 EXAMPLE 4

Solve each equation:

a) $\sqrt{12x+1} = 7$

b) $e^{5+x} = 19$

c) $2\ln(x-1) = 14$

Solution: This diagrams as $x \underset{+1}{\overset{-1}{\rightleftarrows}} \underset{\exp}{\overset{\ln}{\rightleftarrows}} \underset{\div 2}{\overset{\times 2}{\rightleftarrows}} 14$,

so the solution is $x = \exp\left(\frac{14}{2}\right) + 1 = \boxed{e^7 + 1}$.

d) $\sqrt[3]{2w^4 - 7} = -4$

e) $5(x^3 - 2)^2 = 45$

f) $7 \arcsin^4 t + 3 = 1$

g) $5x^{2/3} = 20$

h) $4|x - 3| = 15$

Solution: This diagrams as $x \underset{+3}{\overset{-3}{\rightleftarrows}} \underset{\pm}{\overset{|\cdot|}{\rightleftarrows}} \underset{\div 4}{\overset{\times 4}{\rightleftarrows}} 15$,

so the solutions are $x = \pm\left(\frac{15}{4}\right) + 3$. These simplify to $x = -\frac{15}{4} + 3 = \boxed{-\frac{3}{4}}$ and $x = \frac{15}{4} + 3 = \boxed{\frac{27}{4}}$.

Examples with trig

§5.1 EXAMPLE 5

Solve each equation:

a) $2\sin x + 1 = 0$

b) $\cos 3x = \dfrac{\sqrt{3}}{2}$

c) $\sin\left(x - \dfrac{\pi}{4}\right) = 1$

Solution: This diagrams as $x \underset{+\frac{\pi}{4}}{\overset{-\frac{\pi}{4}}{\rightleftarrows}} \underset{\substack{2\pi n + \arcsin, \\ 2\pi n + \pi - \arcsin}}{\overset{\sin}{\rightleftarrows}} 1$,

so the solutions are $(2\pi n + \arcsin 1) + \dfrac{\pi}{4}$ and $(2\pi n + \pi - \arcsin 1) + \dfrac{\pi}{4}$. Since $\arcsin 1 = \dfrac{\pi}{4}$, these solutions simplify to

$$\boxed{\{2\pi n + \frac{\pi}{2}, 2\pi n + \pi\}}.$$

d) $3\sec 4x = 6$

e) $7\tan^2 y = 7$

f) $7\tan x^2 = 7$

g) $2\sin(3x - \pi) = -4$

Solution: This diagrams as $x \underset{\div 3}{\overset{\times 3}{\rightleftarrows}} \quad \underset{+\pi}{\overset{-\pi}{\rightleftarrows}} \quad \underset{2\pi n + \arcsin,\ 2\pi n + \pi - \arcsin}{\overset{\sin}{\rightleftarrows}} \quad \underset{\div 2}{\overset{\times 2}{\rightleftarrows}} -4$,

so since $\arcsin -2$ DNE, there is $\boxed{\text{no solution}}$.

Solving for an exponent

§5.1 EXAMPLE 7

Solve the equation $3^x = 5$.

Example 7 generalizes:

Theorem 5.1 (Exponential equations) *The solution of* $a^x = b$ *is* $x = \dfrac{\ln b}{\ln a}$.

§5.1 EXAMPLE 8

Solve each equation:

a) $3 \cdot 4^{\sqrt{x}} = 43$

b) $7 \cdot 2^{1-x} = 56$

Equations with one type of term

QUESTION

At this point, we've learned how to solve any equation **where the variable appears only once**– we use inverses, arrows, etc.

What do you do if the variable appears more than once?

Special situation: if there is only one type of "like term" in the equation, you can combine the like terms; this will make the variable only appear once, and you can proceed as above to solve for it.

§5.1 EXAMPLE 9

Solve each equation:

a) Solve the equation $7(x^2 - 3) + 4(x^2 + 2) = 16$.

b) Find the x-coordinates of intersection points of the functions $f(x) = 5 \sin 2x - 1$ and $g(x) = 2 \sin 2x - 2$.

c) Solve the equation $\dfrac{5}{e^x} + 3 = 2e^{-x} + 7$.

d) Find all points on the graph of $F(x) = (\ln x)(\ln x)$ that have y-coordinate 15.

5.3 Quadratic-type equations

Refresher on quadratic equations

RECALL

A **quadratic equation** is any equation that contains only an x^2 term, maybe an x term, and maybe a constant term. Any such equation can be rewritten as

$$ax^2 + bx + c = 0$$

where $a \neq 0$ and b and c are constants.

To solve a quadratic equation, we learned the following methods:

- if x only appears once (i.e. there is an x^2 term but no x term), you can solve it as in the previous set of notes.

- if x appears twice (i.e. there is both x^2 and x), then you can solve the equation in any of three ways:
  1. set one side $= 0$, then factor the other side;
  2. use the quadratic formula (make sure one side $= 0$ first); or

     $$x =$$

  3. complete the square.

WARNING: Factoring is only useful if the other side of the equation is zero:

$$(x - 3)(x - 8) = 0 \qquad\qquad (x - 3)(x - 8) = 6$$

BASIC EXAMPLE

$$x^2 + 7x = 44$$

Quadratic-type equations

There are lots of commonly encountered equations that aren't technically quadratic, but can be solved as if they were quadratic. These equations are called *quadratic-type*.

MOTIVATION

In a quadratic equation $ax^2 + bx + c = 0$, the x is just a placeholder.

For instance, all these are really the same equation:

$$ax^2 + bx + c = 0 \qquad at^2 + bt + c = 0 \qquad a\heartsuit^2 + b\heartsuit + c = 0$$

with essentially the same solution:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \heartsuit = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In a quadratic-type equation, the concept is that the $\heartsuit$ in the right-hand situation above could be some potentially complicated formula of x.

Definition 5.2 *An equation is called* **quadratic-type in** $f(x)$ *if* f *is some function so that the equation can be rewritten as*

$$a\left[f(x)\right]^2 + bf(x) + c = 0$$

where a, b *and* c *are constants. In other words, a quadratic-type equation is one that has:*

- *"something" (the* $f(x)$*) in it;*
- *that same "something squared" (i.e.* $[f(x)]^2$*) in it; and*
- *a constant term (or terms) in it (the constant term might be zero); but*
- *no other terms.*

An equation is **quadratic-type** *if it is quadratic-type in* f *for some function* f*.*

EXAMPLES

- $\sin^2 x + 3\sin x + 2 = 0$ is quadratic-type in $\sin x$.
- $5\ln x + 3\ln^2 x = 7\ln x - 8$ is quadratic-type in $\ln x$.
- $\cos^3 x + 2\cos x = 8$ is not quadratic-type (contains cubed term).
- $\arctan 4x + 8 = 3\arctan 4x$ is not quadratic-type (no squared term).

§5.2 EXAMPLE 1

Given each equation in the chart below, determine if it is quadratic-type. If it is, identify the a, b, c and $f(x)$ if the equation is thought of as

$$a\,[f(x)]^2 + bf(x) + c = 0.$$

Equation	Is the equation quadratic type?	The "something" I'm calling $f(x)$	a	b	c
$2\cos^2 x + \cos x - 1 = 0$					
$x^3 - 2x^2 + x = 0$					
$x^4 - 3x^2 = -3$					
$x^{-1/3} + 3x^{-2/3} = 2$					
$4\sqrt{x} - 5x + 1 = 0$					
$x^4 = x^8$					

Equation	Is the equation quadratic type?	The "something" I'm calling $f(x)$	a	b	c
$\sin x^2 + 4\sin x - 2 = 0$					
$\sin^2 x + 4\sin x - 2 = 0$					
$e^{2x} - 3e^x - 21 = 0$					
$2x^2 - 3\lvert x\rvert - 1 = 0$					
$8\ln^6 x + 3\ln^3 x - 12 = 0$					
$8\ln^8 x + 3\ln^5 x - 12 = 0$					

Solving quadratic-type equations

1. Make sure the equation is quadratic-type.
2. Write the equation in the form

$$a\left[f(x)\right]^2 + bf(x) + c = 0$$

   and recognize what the $f(x)$, i.e. the "something" is.
3. Imagine the associated quadratic equation

$$ax^2 + bx + c = 0.$$

   - If you solve this quadratic equation by the quadratic formula, what you've done in the original quadratic-type equation is solve for $f(x)$. Then solve for x using methods from last week.
   - If you would factor this equation to solve it, factor the equation using $f(x)$ in place of x.

§5.2 EXAMPLE 3

Solve each equation, if it is quadratic-type. (If it isn't quadratic-type, you don't have to do anything.)

a) $e^{2x} + 7e^x + 12 = 0$

b) $2\sin^2 x - 3\sin x + 1 = 0$

c) $x^5 - 4x^3 + 3 = 0.$

d) $x + 50 = 15\sqrt{x}$

e) $x^8 + x^4 - 20 = 0$

Solution: This is quadratic-type in x^4, so we can solve it by factoring:

$$x^8 + x^4 - 20 = 0$$
$$(x^4 + 5)(x^4 - 4) = 0$$
$$\swarrow \qquad \searrow$$

$$\begin{aligned} x^4 + 5 &= 0 \\ x^4 &= -5 \\ x &= \pm\sqrt[4]{-5} \end{aligned} \qquad \begin{aligned} x^4 - 4 &= 0 \\ x^4 &= 4 \\ x &= \pm\sqrt[4]{4} \end{aligned}$$

no solution

So the solution is $\boxed{x = \pm\sqrt[4]{4}}$.

f) $2\ln^2 x - 12\ln x + 14 = 0$

A second approach: use a substitution

Let's look again at Example 3 (d) above. Here is another way of thinking about this problem:

$$\begin{aligned} x + 50 &= 15\sqrt{x} \\ x - 15\sqrt{x} + 50 &= 0 \\ (\sqrt{x})^2 - 15\sqrt{x} + 50 &= 0 \end{aligned}$$

Substitutions like this are nice to understand because they can be used in other equations (not just quadratic-type), and are used in other contexts in calculus and beyond.

To solve a quadratic-type equation using a substitution,

1. Make sure the equation is quadratic-type.
2. Write the equation in the form
$$a\,[f(x)]^2 + bf(x) + c = 0$$
and recognize what the $f(x)$, i.e. the "something" is.
3. Let $w = f(x)$. This makes the equation
$$aw^2 + bw + c = 0.$$
4. Solve for w.
5. Back-substitute each value of w you get in the equation you wrote down in Step 2, and solve for x.

A hidden quadratic-type equation

Here's another common class of equations you should be on the lookout for. As a prototype, consider

$$x - \frac{18}{x} + 7 = 0$$

There is a trick to rewrite this equation as a quadratic equation:

More generally, suppose you have an equation that has only a $\square$ term, a $\frac{1}{\square}$ term and a constant term. If you multiply through the equation by $\square$, you will get a quadratic-type equation in $\square$.

§5.2 EXAMPLE 4

a) Find the x-intercepts of the function $f(x) = x + 4 + 4x^{-1}$.

b) Solve the equation $\log x - 8 + \dfrac{12}{\log x} = 0$

c) Suppose $f(t) = e^t + e^{-t}$. Find all values of t so that $f(t) = 2$.

d) Find the points of intersection of the graphs of $y = 2\sin x + 1$ and $y = \csc x$.

e) Solve the equation $|v| - 3 - \dfrac{10}{|v|} = 0$.

Solution: Multiply both sides by $|v|$ to get

$$|v|^2 - 3|v| - 10 = 0,$$

then we get a quadratic-type equation in $|v|$. At this point, you can either factor this or use a substitution; in this solution, let's use a substitution (just so that an example of this method is typed up). Let $w = |v|$ to get

$$\begin{aligned} w^2 - 3w - 10 &= 0 \\ (w-5)(w+2) &= 0 \\ w = 5, w &= 2 \end{aligned}$$

Now, back-substitute these into $w = |v|$ and solve for v:

$$\begin{aligned} |v| &= 5 \qquad & |v| &= 2 \\ v &= \pm 5 \qquad & v &= \pm 2 \end{aligned}$$

Thus the solution set is $\boxed{\{-5, -2, 2, 5\}}$.

5.4 Factoring as a tool to solve general equations

RECALL

When solving quadratic-type equations, we saw that **so long as the other side of the equation was zero**, when an equation gets factored, you can set each factor equal to zero and solve separately:

$$\ln^2 x - 8\ln x + 7 = 0$$
$$(\ln x - 7)(\ln x - 1) = 0$$

§5.3 EXAMPLE 1

Solve each equation:

a) $x \cos 4x = 0$

b) $5(x+2)(e^x - 4)(\sin x - 1) = 0$

c) $(x^2 - 4)(e^x - 3) = 0$

d) $x \ln x - 3x = 0$

Solution: Factor the left-hand side to get

$$x(\ln x - 3) = 0$$

$$\swarrow \qquad \searrow$$

$$\boxed{x = 0} \qquad \begin{aligned} \ln x - 3 &= 0 \\ \ln x &= 3 \\ \boxed{x = e^3} \end{aligned}$$

e) $\tan x \ln x = \sqrt{3} \ln x$

f) $2\sin x\cos x + \sqrt{3}\sin x = 0$

g) $e^x\tan^2 x - 3e^x\tan x = 40e^x$

Polynomial equations

The idea of using factoring to solve equations is especially useful when solving polynomial equations (those that only contain whole number powers of the variable):

§5.3 EXAMPLE 2

Solve each equation:

a) $x^3 - 7x^2 + 12x = 0$

b) $3x^5 = 24x^4 - 36x^3$

5.5 Rational equations and proportions

An equation involving only rational functions is called a **rational equation**. To solve such an equation, multiply through all terms of the equation by all factors that appear in any denominators.

AN EXAMPLE WITH NO VARIABLES

$$\frac{7}{6} + \frac{5}{4} = 3 - \frac{7}{12}$$

§5.4 EXAMPLE 1

Solve each equation:

a)
$$\frac{6}{x+1} + x = \frac{8}{x+1}$$

b)
$$\frac{3}{x-1} + 2 = \frac{4}{x}$$

c)
$$\frac{18}{x} - \frac{3}{2} = \frac{6}{4x}$$

d)
$$\frac{3}{x-2} + \frac{7}{x+5} = \frac{-6}{x^2+3x-10}$$

Solving proportions by cross-multiplying

A **proportion** is a rational equation that has one term on each side (i.e. it is two fractions that are equal to one another):

EXAMPLES OF PROPORTIONS

$$\frac{3}{5} = \frac{6}{10} \qquad \frac{3}{x} = \frac{4x+3}{x-1} \qquad \frac{x^4 - 7x + 2}{3x+5} = \frac{x^2 - 3x + 7}{4} \qquad \frac{3}{x-1} = 7x$$

EXAMPLES OF EQUATIONS THAT ARE NOT PROPORTIONS

$$\frac{3}{x-2} = \frac{7x}{x+1} + 4 \qquad 3 + \frac{2}{x} = 7x - \frac{4}{x+2}$$

We can solve a proportion by cross-multiplying:

§5.4 EXAMPLE 2

Solve each equation:

a) $$\frac{x-2}{x+1} = \frac{6}{5}$$

b) $$\frac{5x-3}{x+1} = \frac{3x-5}{x-3}$$

WARNING: cross-multiplying doesn't work with equations where the left- or right-hand sides have more than one term. For example, don't "cross-multiply" something like

$$\frac{3}{x-2} = \frac{7x}{x+1} + 4$$

5.6 Tricks to solve other kinds of equations

Rewriting equations using trig identities/log rules/etc.

Often, an equation with two different types of x terms can be rewritten as an equation with only one type of x term, using a trig identity or log rule. Commonly used rules in this context (that you should know) are:

Quotient Identities: $\tan x = \dfrac{\sin x}{\cos x};\ \cot x = \dfrac{\cos x}{\sin x}$

Pythagorean Identities:
$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$\cot^2 x + 1 = \csc^2 x$$

Log rules:
$$\log_b A + \log_b B = \log_b(AB)$$
$$\log_b A - \log_b B = \log_b \frac{A}{B}$$

There are lots of other potentially useful rules here, but I expect that you would have to look those up.

§5.6 EXAMPLE 1

Solve each equation:

a) $\sin x = \sqrt{3} \cos x$

b) $\sin^2 x = \cos x + 1$

c) $\log_2 x + \log_2(3x + 4) = 5$

d) $\sin 2x = \cos x$

Note: this problem requires the identity $\sin 2x = 2 \sin x \cos x$, which you'd probably have to look up.

Exponential / logarithmic equations with different bases

Useful rules here:

$$\log_b A = \frac{\ln A}{\ln B} \qquad \log_b A^n = n \log_b A \qquad e^{b \ln A} = A^b$$

§5.6 EXAMPLE 2

Solve each equation:

a) $4^{x-1} = 3^x$

b) $\log_2 \frac{x}{4} = \log_8 x$

Equations with x and $\sqrt{Ax \pm B}$ in them

Here, there are two methods:

1. Isolate the square root term, then square both sides;
2. Use a substitution $w =$ the thing under the square root.

§5.6 EXAMPLE 3

Solve each equation:

a) $3x + 1 = \sqrt{20x + 9}$

b) $x = \sqrt{x - 3} + 23$

5.7 Chapter 5 Homework

Exercises from Section 5.2

In each of Exercises 1-6, you are given an equation.

(a) If you think of the equation as having the form $f(x) = b$, what is f?
(b) Diagram the function f you identified in part (a).
(c) Write a rule for f^{-1}.
(d) Apply the rule for f^{-1} to b to solve for x.
(e) Indicate on your diagram of f how the inverse is used.

1. $|x| = \pi$
2. $\cos x = \frac{3}{4}$
3. $\sqrt[3]{x} = -\frac{5}{3}$
4. $\ln x = 2$
5. $\frac{1}{x} = 7$
6. $19 = x^4$

In Exercises 7-25, write a rule for the inverse of each function:

7. $\alpha(x) = 8e^{x-2}$
8. $\beta(x) = 3x^2 + 8$
9. $\gamma(x) = \ln^3(x-7)^5$
10. $\delta(x) = 7\arctan\frac{e^x}{3}$
11. $\epsilon(x) = \sqrt[4]{3 + |x^2 - 2|}$
12. $\zeta(x) = \arcsin\ln\arcsin 9x^7$
13. $\eta(x) = \cos^2 4x$
14. $\theta(x) = 7^{x-3} + 8$
15. $\iota(x) = \arctan(\tan x + 1)$
16. $\kappa(x) = 5\log_7^3 x$
17. $\lambda(x) = \dfrac{2}{4x^{3/5} - 3}$
18. $\mu(x) = \sin(3x - 7)$
19. $\nu(x) = 7 + 2\sqrt{\ln(\arcsin x + 2)}$
20. $\xi(x) = \tan|x|$
21. $o(x) = (x^5 - 2)^{3/5} + 6$
22. $\pi(x) = \dfrac{2}{\cos x - 3}$
23. $\rho(x) = \cot x$
24. $\sigma(x) = (x^4 + 7)^2$
25. $\tau(x) = \ln\ln\tan^3(x^2 + 4)$

In Exercises 26-60, solve each equation:

26. $\sqrt[3]{x - 2} = 5$
27. $1 = \arctan e^t$
28. $\sqrt{\ln x - 2} = 7$
29. $4 = e^{2-5w}$
30. $(3 - y)^{2/3} = 4$
31. $5|t| - 3 = 8$

32. $3\sin^2 x + 2 = 3$

33. $5\cos x^2 - 2 = -1$

34. $\left(\frac{x^3}{4} - 3\right)^3 = 64$

35. $\ln 4\sqrt[3]{x} = 3$

36. $3 = 5^{\ln(3x+2)}$

37. $\sqrt{6 + 3\sqrt{q}} = 4$

38. $\sin e^x = \frac{3}{7}$

39. $12 = 2|6 - 3v| + 4$

40. $\log_2(\ln x) = 4$

41. $(3x^4 - 5)^3 + 4 = 12$

42. $2x^{3/4} - 6 = 11$

43. $4\cos(x - \pi) = 1$

44. $\sec x - 2 = 0$

45. $\tan\left(\theta + \frac{\pi}{12}\right) = 1$

46. $3\sin 2x = 0$

47. $7\cos\left(4x + \frac{\pi}{3}\right) = 2$

48. $\csc 4\theta = \frac{2}{\sqrt{3}}$

49. $\sin^2 x = \frac{1}{4}$

50. $3\cos\left(4x - \frac{\pi}{3}\right) = -\frac{3}{2}$

51. $\tan^3 \frac{x}{7} = \frac{1}{5}$

52. $5^x = 19$

53. $4^{3+8x} = 33$

54. $\left(\frac{3}{7}\right)^x = \frac{8}{5}$

55. $\arctan 8^{3-x} = 40$

56. $16 = 2^{4\arcsin x}$

57. $35 = (9^z - 3)^2$

58. $\left(\frac{4}{3}\right)^{|x|} = 3$

59. $-3^{\sqrt[3]{p-3}} = -15$

60. $\arctan 2\ln(\sqrt{5x-3} + 4) - 7 = -8$

61. Solve each of these similar-looking equations:

a) $4\cos^2(x - \pi) = 0$
b) $4\cos(x - \pi)^2 = 0$
c) $4\cos(x^2 - \pi) = 0$
d) $4\cos x^2 - \pi = 0$
e) $(4\cos x)^2 - \pi = 0$
f) $4\cos^2 x - \pi = 0$

62. Solve each of these similar-looking equations:

a) $\log x^3 - 3 = 4$
b) $\log(x^3 - 3) = 4$
c) $\log(x - 3)^3 = 4$
d) $\log^3(x - 3) = 4$
e) $\log^3 x - 3 = 4$
f) $\log_3 x - 3 = 4$

63. Find the x-intercept(s), if any, of the function $f(x) = 7 \cdot 2^{x-3} - 25$.

64. Find the x-coordinate(s) of all point(s) on the graph of $g(x) = \sin\cos x$ that have y-coordinate 1.

65. Find the point(s) where the functions $h(x) = \sqrt{5 + \sqrt[3]{x-2}}$ and $k(x) = 2$ intersect.

66. Find the x-intercept(s), if any, of the function $F(x) = 2\cos\frac{x}{8} + 1$.

In each of Exercises 67-76, solve each equation.

67. $4(-2\sqrt[3]{x} + 1) = 6 - 2(2\sqrt[3]{x} - 4)$

68. $9|x| + 1 = -7|x| + 11$

69. $3x^6 + 5 - 5(x^6 + 1) = 6x^6 - 7$

70. $\frac{7}{4} + \frac{1}{5}\ln x - \frac{3}{2} = \frac{4}{5}\ln x$

71. $\frac{3\arcsin 5x - 2}{14} = \frac{\arcsin 5x + 10}{10}$

72. $5e^{3v-2} + 4 = 3e^{3v-2} - 8$

73. $4 + \frac{2}{3}\cdot 7^x = \frac{1}{2}\cdot 7^x + 13$

74. $-8(3x^{-7/3} + 4) + 6x^{-7/3} = 4(x^{-7/3} - 8) + 4x^{-7/3}$

75. $\frac{2\cos 3t + 4}{3} - \frac{\cos 3t}{2} = \frac{\cos 3t}{2} + \frac{21}{18}$

76. $3(2e^{\sin x} - 4) = 7e^{\sin x} - 12$

77. Find the x-intercept(s) of the function $h(x) = 17x^5 - 5(x^5 + 2)$.

78. Find the point(s) where the graphs of the functions $f_1(x) = 3|x-3| - 6$ and $f_2(x) = 7 - \frac{|x-3|}{2}$ intersect.

    Note: "Find the point(s)" means you have to find both the x-coordinate and the y-coordinate of any such point.

79. Find the x-coordinates of any point(s) on the graph of $k(x) = 7\arctan 2x - 3(1 - 2\arctan 2x)$ which have y-coordinate 1.

80. Find the x-coordinates of any point(s) on the graph of $g(x) = 5\left(\sin\frac{x}{4} - 1\right) + 3\left(\sin\frac{x}{4} + 7\right) - 5$ that have y-coordinate 3.

81. Find the x-intercepts of the function $l(x) = \frac{1}{9}e^{x-4} + \frac{1}{3}\left(7e^{x-4} + 2\right)$.

82. Find the x-coordinates of any point(s) where the graphs of the functions $p(x) = \tan x$ and $q(x) = \frac{1}{4}\tan x - \frac{1}{2}$ intersect.

Answers

1. a) $f(x) = |x|$
   b) $x \xrightarrow{|\cdot|} f(x)$
   c) $f^{-1}(x) = \pm x$
   d) $x = \pm\pi$
   e) $\pm\pi \underset{\pm}{\overset{|\cdot|}{\rightleftarrows}} \pi$

2. a) $f(x) = \cos x$
   b) $x \xrightarrow{\cos} f(x)$
   c) $f^{-1}(x) = 2\pi n \pm \arccos x$
   d) $x = 2\pi n \pm \arccos \frac{3}{4}$
   e) $2\pi n \pm \arccos \frac{3}{4} \underset{2\pi n \pm \arccos}{\overset{\cos}{\rightleftarrows}} \frac{3}{4}$

3. a) $f(x) = \sqrt[3]{x}$
   b) $x \xrightarrow{\sqrt[3]{\ }} f(x)$
   c) $f^{-1}(x) = x^3$
   d) $x = \left(-\frac{5}{3}\right)^3 = -\frac{125}{27}$
   e) $-\frac{125}{27} \underset{\wedge 3}{\overset{\sqrt[3]{\ }}{\rightleftarrows}} -\frac{5}{3}$

4. a) $f(x) = \ln x$
   b) $x \xrightarrow{\ln} f(x)$
   c) $f^{-1}(x) = e^x$
   d) $x = e^2$
   e) $e^2 \underset{\exp}{\overset{\ln}{\rightleftarrows}} 2$

5. a) $f(x) = \frac{1}{x}$
   b) $x \xrightarrow{1/\cdot} f(x)$
   c) $f^{-1}(x) = \frac{1}{x}$
   d) $x = \frac{1}{7}$
   e) $\frac{1}{7} \underset{1/\cdot}{\overset{1/\cdot}{\rightleftarrows}} 7$

6. a) $f(x) = x^4$
   b) $x \xrightarrow{\wedge 4} f(x)$
   c) $f^{-1}(x) = \pm\sqrt[4]{x}$
   d) $x = \pm\sqrt[4]{19}$
   e) $\pm\sqrt[4]{19} \underset{\sqrt[4]{\ }}{\overset{\wedge 4}{\rightleftarrows}} 19$

7. $\alpha^{-1}(x) = 2 + \ln \frac{x}{8}$

8. $\beta^{-1}(x) = \pm\sqrt{\frac{x-8}{3}}$

9. $\gamma^{-1}(x) = \sqrt[5]{\exp(\sqrt[3]{x}) + 7}$

10. $\delta^{-1}(x) = \ln\left(3\tan\frac{x}{7}\right)$

11. $\epsilon^{-1}(x) = \pm\sqrt{\pm(x^4 - 3) + 2}$

12. $\zeta^{-1}(x) = \sqrt[7]{\frac{1}{9}\sin e^{\sin x}}$

13. $\eta^{-1}(x) = \dfrac{2\pi n \pm \arccos \pm\sqrt{x}}{4}$

14. $\theta^{-1}(x) = \log_7(x-8) + 3$

15. $\iota^{-1}(x) = n\pi \pm \arctan(\tan x - 1)$

16. $\kappa^{-1}(x) = 7^{\sqrt[3]{\frac{x}{5}}}$

17. $\lambda^{-1}(x) = \left[\dfrac{1}{4}\left(\dfrac{2}{x} + 3\right)\right]^{5/3}$

18. $\mu^{-1}(x) = 2\pi n + \arcsin \dfrac{x+7}{3}$,
$2\pi n + \pi - \arcsin \dfrac{x+7}{3}$

19. $\nu^{-1}(x) = \sin\left[\exp\left(\frac{x-7}{2}\right)^2 - 2\right]$

20. $\zeta^{-1}(x) = \pm(\pi n \pm \arctan x)$

21. $o^{-1}(x) = \sqrt[5]{(x-6)^{5/3} + 2}$

22. $\pi^{-1}(x) = 2\pi n \pm \arccos\left(\dfrac{2}{x} + 3\right)$

23. $\rho^{-1}(x) = \pi n \pm \arctan \dfrac{1}{x}$

24. $\sigma^{-1}(x) = \pm\sqrt[4]{\pm\sqrt{x} - 7}$

25. $\tau^{-1}(x) = \pm\sqrt{\pi n \pm \arctan \sqrt[3]{\exp e^x} - 4}$

26. $x = 127$

27. $t = \ln \tan 1$

28. $x = e^{51}$

29. $w = \dfrac{1}{5}(2 - \ln 4)$

30. $y = -5$

31. $t = \pm\dfrac{11}{5}$

32. $x = 2\pi n + \arcsin\sqrt{\dfrac{1}{3}}$,
$2\pi n + \pi - \arcsin\sqrt{\frac{1}{3}}$,
$2\pi n - \arcsin\sqrt{\dfrac{1}{3}}$,
$2\pi n + \pi + \arcsin\sqrt{\frac{1}{3}}$

33. $x = \pm\sqrt{2\pi n \pm \arccos \dfrac{1}{5}}$

34. $x = \sqrt[3]{28}$

35. $x = \dfrac{e^9}{64}$

36. $x = \dfrac{1}{3}\left[\exp\left(\dfrac{\ln 3}{\ln 5}\right) - 2\right]$

37. $q = \dfrac{100}{9}$

38. $x = \ln\left(2\pi n + \arcsin \dfrac{3}{7}\right)$,
$\ln\left(2\pi n + \pi - \arcsin \frac{3}{7}\right)$

39. $v = \dfrac{2}{3}, \dfrac{10}{3}$

40. $x = e^{16}$

41. $x = \pm\sqrt[4]{\dfrac{7}{3}}$

42. $x = \left(\dfrac{17}{2}\right)^{4/3}$

43. $x = \pi + 2\pi n \pm \arccos \dfrac{1}{4}$

44. $x = 2\pi n \pm \dfrac{\pi}{3}$

45. $x = \pi n + \dfrac{\pi}{6}$

46. $x = \pi n$, $x = \pi n + \dfrac{\pi}{2}$

47. $x = \dfrac{\pi n}{2} \pm \dfrac{1}{4}\arccos \dfrac{2}{7} - \dfrac{\pi}{12}$

48. $\theta = \dfrac{\pi n}{2} + \dfrac{\pi}{12}$,
$\theta = \dfrac{\pi n}{2} + \dfrac{\pi}{6}$

49. $x = 2\pi n + \frac{\pi}{6}$,
$2\pi n + \frac{5\pi}{6}$,
$2\pi n - \frac{\pi}{6}$,
$2\pi n - \frac{5\pi}{6}$

50. $x = \frac{\pi n}{2} + \pi, \frac{\pi n}{2} - \frac{\pi}{3}$

51. $x = 7\pi n + 7 \arctan \sqrt[3]{\frac{1}{5}}$

52. $x = \frac{\ln 19}{\ln 5}$

53. $x = \frac{1}{8}\left(\frac{\ln 33}{\ln 4} - 3\right)$

54. $x = \frac{\ln \frac{8}{5}}{\ln \frac{3}{7}} = \frac{\ln 8 - \ln 5}{\ln 3 - \ln 7}$

55. $x = 3 - \frac{\ln \tan 40}{\ln 8}$

56. $x = \sin 1$

57. $z = \frac{\ln 3 + \sqrt{35}}{\ln 9}$

58. $x = \pm \frac{\ln 3}{\ln \frac{4}{3}}$

59. $p = \left(\frac{\ln 15}{\ln 3}\right)^3 + 3$

60. $x = \frac{\left[\exp(\frac{1}{2}\tan 1) - 4\right]^2 + 3}{5}$

61. a) $x = 2\pi n \pm \frac{\pi}{2}$
b) $\pm\sqrt{2\pi n \pm \frac{\pi}{2}} + \pi$
c) $x = \pm\sqrt{2\pi n \pm \frac{\pi}{2} + \pi}$
d) $x = \pm\sqrt{2\pi n \pm \arccos \frac{\pi}{4}}$
e) $x = 2\pi n \pm \arccos \pm \frac{\sqrt{\pi}}{4}$
f) $x = 2\pi n \pm \arccos \pm \frac{\sqrt{\pi}}{2}$

62. a) $x = 10^{7/3}$
b) $x = \sqrt[3]{10003}$
c) $x = 10^{4/3} + 3$
d) $x = 10^{\sqrt[3]{4}} + 3$
e) $x = 10^{\sqrt[3]{7}}$
f) $x = 3^7$

63. $\left(3 + \frac{\ln \frac{25}{7}}{\ln 2}, 0\right)$

64. There are no such points.

65. The only intersection point is $(1, 2)$.

66. $\left(16\pi n \pm \frac{16\pi}{3}, 0\right)$

67. $x = \sqrt[3]{-\frac{5}{2}}$

68. $x = \pm\frac{5}{8}$

69. $x = \sqrt[6]{\frac{7}{8}}$

70. $x = e^{5/12}$

71. $x = \frac{1}{5}\sin 10$

72. $v = \frac{\ln 6 + 2}{3}$

73. $x = \frac{\ln 54}{\ln 7}$

74. $x = 14^{3/7}$

75. $t = \frac{2\pi}{3} n \pm \frac{\pi}{9}$

76. no solution

77. $\left(\sqrt[5]{\frac{5}{6}}, 0\right)$

78. $\left(-\frac{5}{7}, \frac{36}{7}\right)$ and $\left(\frac{47}{7}, \frac{36}{7}\right)$

79. $x = \frac{1}{2}\tan\frac{4}{13}$

80. $x = 2\pi n + \frac{-\pi}{2}$, $x = 2\pi n + \pi + \frac{\pi}{2}$

81. $\left(4 - \ln \frac{11}{3}, 0\right)$

82. $x = \pi n - \arctan \frac{2}{3}$

Exercises from Section 5.3

In Exercises 1-23, you are given an equation. Determine whether or not the equation is quadratic-type, meaning that it is of the form

$$a[f(x)]^2 + bf(x) + c = 0$$

for some function f and constants a, b and c. If it is, identify the function f and (in exercises 1 through 6, also identify the constants a, b and c), and then solve the equation.

1. $x^6 - 4x^3 - 21 = 0$
2. $\ln^2 x = 5 \ln x - 6$
3. $x + 2 - \frac{8}{x} = 0$
4. $x^9 + 4x^3 + 4 = 0$
5. $10x^{5/3} + 6x^{10/3} = 3(1 + x^{5/3})$
6. $2x^2 - |x| - 15 = 0$
7. $\frac{1}{4}\sin^2 x - \frac{5}{24}\sin x - \frac{1}{6} = 0$
8. $\arctan^2 x - 4 \arctan x = -3$
9. $x^5 - 36x^{-5} = 9$
10. $\ln^2(x-5) - 16\ln(x-5) + 64 = 0$
11. $e^{4x} - 4e^{2x} - 45 = 0$
12. $2\cos x = \sec x - 1$
13. $2\tan 4x = 5 - 3\tan^2 4x$
14. $e^x + 24e^{-x} = 10$
15. $x^4 + \frac{x^2}{4} - \frac{15}{8} = 0$
16. $2\left(\ln x - \frac{1}{\ln x}\right) + 3 = 0$
17. $(\sqrt[3]{x} - 4)(\sqrt[3]{x} - 8) = 2\sqrt[3]{x}(\sqrt[3]{x} - 13)$
18. $((2x-3)^3 + 3)((2x-3)^3 - 2) = (2x-3)^3 + 7$
19. $\ln x^2 + 4\ln x = 32$
20. $\tan x + 3 - 10 \arctan x = 0$
21. $(2\cos x - 3)(\cos x + 1) = -2$
22. $1 - \sin x = 2\sin^2 x$
23. $(e^x + 3)(e^x - 2) + 4 = 2e^x(e^x + 7) + 20$

In Exercises 24-27, find the x-intercept(s) of the given function:

24. $F(x) = x^8 - x^4 - 12$
25. $\widehat{F}(x) = e^{2x} - 6e^x + 5$
26. $\overline{F}(x) = x + 28x^{-1} + 11$
27. $\widetilde{F}(x) = (2x-1)^2 - 4(2x-1) - 21$

In Exercises 28-30, find the point(s) where the graphs of the given functions intersect:

28. $G(x) = 2x + 5\sqrt{x} - 7$ and $H(x) = (\sqrt{x} - 2)(\sqrt{x} - 1)$

29. $G'(x) = \ln^2 x + 3\ln x - 5$ and $H'(x) = 6\ln x + 5$

30. $G''(x) = x^6 - 3$ and $H''(x) = x^3 + 17$

31. Find the x-coordinates of all point(s) on the graph of $f(x) = 2x^{12} + 5x^6 - 7$ that have y-coordinate 35.

32. Find the x-coordinates of all point(s) on the graph of $f(x) = 4\sin^2 x + \sin x + 2$ that have y-coordinate 2.

Answers

1. quadratic-type in x^3; $a = 1$; $b = -4$; $c = -21$; solutions $x = \sqrt[3]{7}, \sqrt[3]{-3}$

2. quadratic-type in $\ln x$; $a = 1$; $b = -5$; $c = 6$; solutions $x = e^2, e^3$

3. quadratic-type in x (after multiplying through by x); $a = 1$; $b = 2$; $c = -8$; solutions $x = -4, x = 2$

4. not quadratic-type

5. quadratic-type in $x^{5/3}$; $a = 6$; $b = 7$; $c = -3$; solutions $x = \left(-\frac{3}{2}\right)^{3/5}, \left(\frac{1}{3}\right)^{3/5}$

6. quadratic-type in $|x|$; $a = 2$; $b = -1$; $c = -15$; solutions $x = \pm 3$

7. quadratic-type in $\sin x$; solutions $x = 2\pi n - \dfrac{\pi}{6}, 2\pi n + \pi + \dfrac{\pi}{6}$

8. quadratic-type in $\arctan x$; solutions $x = \tan 1, \tan 3$

9. quadratic-type in x^5; solutions $x = \sqrt[5]{-3}, \sqrt[5]{12}$

10. quadratic-type in $\ln(x - 5)$; solution $x = e^8 + 5$

11. quadratic-type in e^{2x}; solutions $x = \dfrac{1}{2}\ln 5, \dfrac{1}{2}\ln 9$

12. quadratic-type in $\cos x$; solutions $x = 2\pi n \pm \pi, 2\pi n \pm \dfrac{\pi}{3}$

13. quadratic-type in $\tan 4x$; solutions $x = \dfrac{\pi}{4}n + \dfrac{\pi}{16}, \dfrac{\pi}{4}n - \dfrac{1}{4}\arctan\dfrac{5}{3}$

14. quadratic-type in e^x; solutions $x = \ln 4, \ln 6$

15. quadratic-type in x^2; solutions $x = \pm\sqrt{\dfrac{5}{4}}$

16. quadratic-type in $\ln x$; solutions $2\left(\ln x - \dfrac{1}{\ln x}\right) + 3 = 0$

17. quadratic-type in $\sqrt[3]{x}$; solutions $x = -8, x = 16^3$
18. quadratic-type in $(2x-3)^3$; solutions $x = \frac{1}{2}\left(3 \pm \sqrt{13}\right)$
19. not quadratic-type
20. not quadratic-type
21. quadratic-type in $\cos x$; solutions $x = 2\pi n, 2\pi n \pm \frac{2\pi}{3}$
22. quadratic-type in $\sin x$; solutions $x = 2\pi n + \frac{\pi}{6}, 2\pi n + \pi - \frac{\pi}{6}, 2\pi n - \frac{\pi}{2}, 2\pi n + \pi + \frac{\pi}{2}$
23. quadratic-type in e^x; no solution
24. $\left(\pm\sqrt[4]{4}, 0\right)$
25. $(0,0)$, $(\ln 5, 0)$
26. $(-7,0)$, $(-4,0)$
27. $(-1,0)$, $(4,0)$
28. $(1,0)$
29. $(e^{-2}, -7)$, $(e^5, 35)$
30. $\left(\sqrt[3]{5}, 22\right)$, $\left(-\sqrt[3]{4}, 13\right)$
31. $x = \pm\sqrt[6]{\frac{7}{2}}$
32. $x = 2\pi n, 2\pi n + \pi, 2\pi n - \arcsin\frac{1}{4}$, $2\pi n + \pi + \arcsin\frac{1}{4}$

Exercises from Section 5.4

Solve each equation:

1. $(\cos x - 1)(\ln x + 2) = 0$
2. $\frac{2}{3}(x-3)(3x+4)(5x-7) = 0$
3. $(e^{3x-1} - 5)(2\arctan 7x - 1) = 0$
4. $w \ln w = 0$
5. $6xe^{3x} = 0$
6. $(2\sin\theta + 1)(3\tan\theta - 2) = 0$
7. $(x^2 - 9)(e^x - 5) = 0$
8. $(u^2 + 4)(e^u - 3) = 0$
9. $x^4 - 2x^4 \arcsin x = 0$
10. $e^x \sin 3x = 2e^x$
11. $2t^3 + t^3 \tan t = 0$
12. $x\ln(x-5) = 4\ln(x-5)$
13. $r^2(r+1) - 3r(r+1) = 40(r+1)$
14. $x^2e^{3x} - 5xe^{3x} + 4e^{3x} = 0$
15. $x^8 - 11x^7 + 10x^6 = 0$
16. $3x^2 - 9x - 18 = 0$
17. $4x^3 + 22x^2 = 42x$

Answers

1. $x = 2\pi n, e^{-2}$
2. $x = 3, -\frac{4}{3}, \frac{7}{5}$
3. $x = \frac{1}{3}(\ln 5 + 1), \frac{1}{7}\tan\frac{1}{2}$
4. $w = 0, 1$
5. $x = 0$
6. $x = 2\pi n - \frac{\pi}{6}, 2\pi n + \frac{7\pi}{6}, \pi n + \arctan\frac{2}{3}$
7. $x = 3, -3, \ln 5$
8. $u = \ln 3$
9. $x = 0, 2\pi n + \frac{\pi}{6}, 2\pi n + \frac{5\pi}{6}$
10. no solution
11. $t = 0, \pi n - \arctan 2$
12. $x = 4, 6$
13. $r = -1, 8, -5$
14. $x = 1, 4$
15. $x = 0, 1, 10$
16. $x = \frac{3 \pm \sqrt{33}}{2}$
17. $x = 0, -7, \frac{3}{2}$

Exercises from Section 5.5

Solve each equation:

1. $3 + \frac{2}{x-5} = \frac{4}{x-1}$
2. $\frac{4}{x} - \frac{7}{3x} = \frac{2}{5}$
3. $\frac{5}{2x-1} - \frac{7}{3x+2} = \frac{9}{6x^2+x-2}$
4. $\frac{3}{p-2} + \frac{4}{p+2} = \frac{1}{p^2-4}$
5. $y + \frac{6}{y-3} = \frac{2y}{y-3}$
6. $\frac{x}{x+4} = 3 - \frac{4}{x+4}$
7. $\frac{3x-7}{2x+5} = \frac{11}{4}$
8. $\frac{v-3}{2v-1} = \frac{v+2}{v}$
9. $\frac{4}{2x+11} = \frac{x}{x+1}$

Answers

1. $x = 3, \frac{11}{3}$
2. $x = \frac{25}{6}$
3. $x = -8$
4. $p = \frac{3}{7}$
5. $y = 2$
6. no solution
7. $x = -\frac{83}{10}$
8. $v = -3 \pm \sqrt{11}$
9. $x = -4, \frac{1}{2}$

Exercises from Section 5.6

Solve each equation:

1. $\sin x = \cos x$
2. $3\sin x - \cos x = 0$
3. $\cos^2\theta - \sin^2\theta = 0$
4. $\sec^2 x - 1 = \tan x$
5. $2\cos^2 t = 3\sin t$
6. $2\sin^2 r = 2 + \cos r$
7. $2\csc^2 x + \cot^2 x = 3$
8. $2\sin^2 t - \cos t = 1$
9. $\tan x = 2\sin x$
10. $\ln x - \ln 2 = \ln 17$
11. $\ln(x+4) - \ln x = 3$
12. $\ln y + \ln(y-2) = 3$
13. $\log(x-1) + \log(2x-3) = 1$
14. $\log_2 x + \log_2(x-4) = 5$
15. $7^x = 11^{2x-9}$
16. $5^{2z-3} = 4^{3z+1}$
17. $3^{x+2} = 2 \cdot 5^{x-1}$
18. $\log_3 x = \log_5 4x$
19. $\log_6(w+8) = \log_4(w-12)$
20. $\ln(x+2) = \log 3x$
21. $\sqrt{8x+5} = x+2$
22. $X - \sqrt{2X-7} = 1$
23. $4x = \sqrt{9x+2}$

Answers

1. $x = \pi n + \frac{\pi}{4}$
2. $x = \pi n + \arctan\frac{1}{3}$
3. $x = \pi n \pm \frac{\pi}{4}$
4. $x = \pi n, \pi n + \frac{\pi}{4}$
5. $t = 2\pi n \pm \frac{\pi}{6}$
6. $r = 2\pi n \pm \frac{\pi}{2}, 2\pi n \pm \frac{2\pi}{3}$
7. $x = \pi n \pm \frac{\pi}{3}$
8. $t = 2\pi n + \pi, 2\pi n \pm \frac{\pi}{3}$
9. $x = 2\pi n, 2\pi n \pm \pi, 2\pi n \pm \frac{\pi}{3}$
10. $x = 34$
11. $x = \frac{4}{e^3 - 1}$
12. $y = \frac{2 + \frac{1}{2}\sqrt{4e^3+4}}{2}$
13. $x = \frac{7}{2}$
14. $x = 8$
15. $x = -\frac{9\ln 11}{\ln 7 - 2\ln 11}$
16. $z = \frac{-\ln 4 - 3\ln 5}{3\ln 4 - 2\ln 5}$
17. $x = \frac{-\ln\frac{9}{2} - \ln 5}{\ln 3 - \ln 5}$

18. $x = \exp\left(\frac{\ln 4}{\frac{\ln 5}{\ln 3} - 1}\right)$
19. $w = 28$
20. $x = 1$
21. $x = 2 \pm \sqrt{5}$
22. no solution
23. $x = \frac{9 + \sqrt{209}}{32}$