Big picture

We want a method of converting from an odometer f to a speedometer f' without using graphs (i.e. if you are given $f(x) = x^4 \cos x + \sqrt{x^4+5}$, what is the formula for $f'(x)$?

First idea: The steeper the graph of f is, the greater the value of f' is. So f' is supposed to measure the slope of f.

But we only know how to measure the slope of lines. What if f has a graph that is curved?

Second idea: If f is curved, we pick an x value and approximate f by drawing a tangent line to f at x. This line passes through the point $(x, f(x))$ and "goes in the same direction" as the graph of f near x.

The slope of f at x is therefore the slope of this tangent line. To figure the slope of a line, we use the slope formula $m = \frac{y_2-y_1}{x_2-x_1}$.

But that slope formula requires two points on the line, and we only know one point on the line. What now?

Third idea: Use the same point $(x, f(x))$ twice in the slope formula. This tells us that the slope of the tangent line is

$$m = \frac{f(x)-f(x)}{x-x} = \frac{0}{0}.$$

But what do you do with $\frac{0}{0}$? This expression is technically undefined, because it has many different possible answers.

Limits: a mechanism to "evaluate" $\frac{0}{0}$

A common place where the expression $\frac{0}{0}$ occurs is where the graph of some function f has a hole in it (say at $x = a$). The "right" value of $\frac{0}{0}$ in such a situation is the value of y which fills in the hole. This value is called the limit of f as x approaches a and is denoted

$$\lim_{x\to a} f(x).$$

In Math 216, to compute a limit algebraically, use the following procedure:

1. Plug in the value of a for x in the formula for f. If you get a number, that is the answer to the limit (and reflects that the graph has no hole at a).
2. If you get $\frac{0}{0}$, this suggests the graph of f has a hole at $x = a$. Go back, factor and cancel terms in f. Then plug in a for x again. You should get a number which is the answer to the limit (this answer is the value of y which fills in the hole at $x = a$).
3. If you get $\frac{\text{nonzero}}{0}$, do something else (to be described later).

Example: Compute $\lim\limits_{x\to -2} \frac{x^2+4x+4}{x^2+7x+10}$.

Solution: First, plug in $x = -2$ to get $\frac{(-2)^2+4(-2)+4}{(-2)^2+7(-2)+10} = \frac{4-8+4}{4-14+10} = \frac{0}{0}$. Since we got $\frac{0}{0}$, factor and cancel:

$$\frac{x^2+4x+4}{x^2+7x+10} = \frac{(x+2)(x+2)}{(x+2)(x+5)} = \frac{x+2}{x+5}$$

Now plug in $x = -2$ again to get

$$\frac{-2+2}{-2+5} = \frac{0}{3} = 0.$$

(So the answer is 0.)