Calculus 1 Lecture Notes

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Contents

Chapter 1

Algebra / Precalculus Review

1.1 Exponent rules

Here is a list of exponent rules you should be familiar with. In calculus, we use these exponent rules to rewrite a given expression in a way that makes it easier to perform calculus operations on the expression.

Theorem 1.1 (Exponent rules I) *Let x, a, b and n be numbers, where $x \neq 0$. Then:*

- $x^a x^b = x^{a+b}$
- $\dfrac{x^a}{x^b} = x^{a-b}$
- $x^0 = 1$
- $x^{-a} = \frac{1}{x^a}$
- $(x^a)^b = x^{ab}$
- $\sqrt[n]{x} = x^{1/n}$ *(in particular, $\sqrt{x} = x^{1/2}$)*
- $x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$ *(this last way of writing $x^{m/n}$ is most useful)*
- $(xy)^a = x^a y^a$
- $\left(\dfrac{x}{y}\right)^a = \dfrac{x^a}{y^a}$

EXAMPLE 1

Compute each quantity:

1. $64^{2/3}$
   Solution: $64^{2/3} = \left(\sqrt[3]{64}\right)^2 = 4^2 = \boxed{16}$.

2. 2^{-3}
   Solution: $2^{-3} = \dfrac{1}{2^3} = \boxed{\dfrac{1}{8}}$.

3. $4^{-5/2}$
   Solution: $4^{-5/2} = \dfrac{1}{4^{5/2}} = \dfrac{1}{(\sqrt{4})^5} = \dfrac{1}{2^5} = \boxed{\dfrac{1}{32}}$.

EXAMPLE 2

Simplify each expression as much as possible and write the answer so that it has no radical signs (i.e. no $\sqrt{\ }$ or $\sqrt[3]{\ }$, etc.) or fractions with xs in the denominators (meaning that the answer should look like $\square x^{\square}$):

1. $3x^4x^{-2}(x^3)^3$
   Solution: $3x^4x^{-2}(x^3)^3 = 3x^4x^{-2}x^9 = 3x^{4-2+9} = \boxed{3x^{11}}$.

2. $\dfrac{1}{x^7}$
   Solution: $\dfrac{1}{x^7} = \boxed{x^{-7}}$.

3. $\dfrac{2x^2}{x^4}$
   Solution: $\dfrac{2x^2}{x^4} = 2x^{2-4} = \boxed{2x^{-2}}$.

4. $\sqrt{x}$
   Solution: $\sqrt{x} = \boxed{x^{1/2}}$.

5. $\dfrac{4}{\sqrt{x^7}}$
   Solution: $\dfrac{4}{\sqrt{x^7}} = \dfrac{4}{x^{7/2}} = \boxed{4x^{-7/2}}$.

EXAMPLE 3

Simplify each expression as much as possible, and write the answer so that it has no radical signs or fractions with xs in the denominators:

1. $\left(\frac{x}{3}\right)^{-3}$

   Solution: $\left(\frac{x}{3}\right)^{-3} = \frac{x^{-3}}{3^{-3}} = \frac{x^{-3}}{\frac{1}{27}} = \boxed{27x^{-3}}$.

2. $x^2\sqrt{\frac{x}{2}}$

   Solution: $x^2\sqrt{\frac{x}{2}} = x^2\frac{\sqrt{x}}{\sqrt{2}} = x^2\frac{x^{1/2}}{\sqrt{2}} = \frac{x^{2+1/2}}{\sqrt{2}} = \frac{x^{5/2}}{\sqrt{2}} = \boxed{\frac{1}{\sqrt{2}}x^{5/2}}$.

3. $\frac{(2x)^3x^4}{(4x)^2}$

   Solution: $\frac{(2x)^3x^4}{(4x)^2} = \frac{2^3x^3x^4}{4^2x^2} = \frac{8x^7}{16x^2} = \boxed{\frac{1}{2}x^5}$.

4. $x^0\sqrt[3]{2(2x)^2}$

   Solution: $x^0\sqrt[3]{2(2x)^2} = 1\sqrt[3]{2(2^2)(x^2)} = \sqrt[3]{8x^2} = \sqrt[3]{8}\sqrt[3]{x^2} = \boxed{2x^{2/3}}$.

Remark on existence of square roots: $\sqrt{x}$ DNE if $x < 0$, and $\sqrt{x}$ means only the nonnegative square root of x, i.e. $\sqrt{25} = 5$, not ± 5. This is so that the process of taking a square root is a function (later).

Remark on simplifying square roots: For any positive number x,

$$\left(\sqrt{x}\right)^2 = x.$$

But, if you do the square root and the squaring in the other order, the operations don't cancel:

$$\sqrt{x^2} =$$

In general, if n is even then $\sqrt[n]{x^n} = |x|$, but if n is odd, then $\sqrt[n]{x^n} = x$.

WARNING: In general,

$$(x+y)^a \neq x^a + y^a \qquad \text{and} \qquad (x-y)^a \neq x^a - y^a.$$

As a special case of this, when $a = -1$ we see that

$$\frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y} \qquad \text{and } \frac{A}{x+y} \neq \frac{A}{x} + \frac{A}{y}.$$

1.2 Functions

Definition 1.2 *Let A and B be sets. A* **function** *f from A to B is a procedure that assigns to each element of A (i.e. to each input) at most one element of B (i.e. an output).*

We denote such a function by writing "$f : A \to B$". The set A of inputs is called the **domain** *of f. The set of outputs of the function is called the* **range** *of f.*

In MATH 220, we study functions where:

- the domain is $\mathbb{R}$, the set of real numbers (sometimes the domain is a subset of $\mathbb{R}$ like an interval), and
- the outputs are also real numbers.

Such a function f is often denoted by writing "$f : \mathbb{R} \to \mathbb{R}$".

By hand, this looks like

As a first example, let f be the function $\mathbb{R} \to \mathbb{R}$ which takes the input, squares it, and then adds 3 to produce the output.

To describe this function f, we could take some example inputs and see what the outputs are, arranging the results in a table:

INPUT	OUTPUT
-2	
-1	
0	
1	
2	

Rather than continuing to list inputs and outputs like this, it is easier to take a generic input (something we call x and figure out what the generic output is. This output is called $f(x)$. Writing down a formula for $f(x)$ in terms of x is sufficient to describe any function $f : \mathbb{R} \to \mathbb{R}$; such a formula is called a **rule** for the function.

In our example above, we can describe the function by writing

$$f(x) = x^2 + 3.$$

Definition 1.3 *Let $f : A \to B$ and let $x \in A$. We write the output associated to input x as $f(x)$; this is pronounced "f* **of** *x". A formula for $f(x)$ in terms of x is called a* **rule** *for the function.*

How we use the rule for a function: Think of the x as a placeholder which represents where the input goes. Given a rule for f, you take whatever input you are given and replace all the xs in the rule with that input.

EXAMPLE 1

Let $f(x) = 2x^2 + x$. Compute and simplify the following expressions:

1. $f(2)$

   Solution: $f(2) = 2 \cdot 2^2 + 2 = 2 \cdot 4 + 2 = \boxed{10}$.

2. $f(-1)$

3. $f(x) + f(3)$

4. $f(\text{trumpet})$

5. $f(\text{hamburger})$

   Solution: $f(\text{hamburger}) = \boxed{2(\text{hamburger})^2 + \text{hamburger}}$.

6. $f(2x)$

   Solution: $f(2x) = 2(2x)^2 + (2x) = 2(4x^2) + 2x = \boxed{8x^2 + 2x}$.

7. $f(x-1)$

8. $f(x) - f(1)$

9. $f(x+h)$

   Solution:
   $$\begin{aligned} f(x+h) &= 2(x+h)^2 + (x+h) \\ &= 2(x^2 + 2xh + h^2) + x + h \\ &= \boxed{2x^2 + 4xh + 2h^2 + x + h}. \end{aligned}$$

10. $\dfrac{f(x+3) - f(x)}{3}$

WARNING: All your life you have been told that parenthesis means multiplication, i.e.

$$3(2) = 6 \quad \text{or} \quad a(b+c) = ab + ac.$$

If f is a function, the parenthesis in the definition of $f(x)$ do not mean multiplication. In particular, $f(x)$ does not mean f times x, and $f(a+b)$ is not the same thing as $f(a) + f(b)$ (in general).

$f(x)$ means, literally, this:

"the output of function f when x is the input".

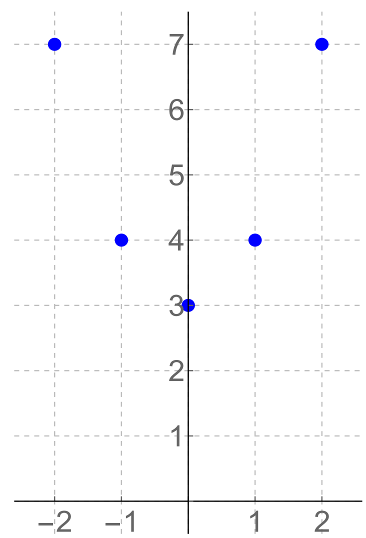
and is better understood through the diagram

$$x \xrightarrow{f} f(x).$$

The graph of a function $f : \mathbb{R} \to \mathbb{R}$

Earlier, we saw the following table of values for the function whose rule is $f(x) = x^2 + 3$:

INPUT x	OUTPUT $f(x)$
-2	7
-1	4
0	3
1	4
2	7



Turning each of the inputs and outputs to the function into an ordered pair and plotting all these points produces a picture called the **graph** of the function. Note that since every input has at most one output, functions from $\mathbb{R}$ to $\mathbb{R}$ must pass the **Vertical Line Test** (i.e. every vertical line must hit the graph in at most one point).

Operations on functions

Definition 1.4 *Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$ and let c be a constant. Then, the functions $f+g$, $f-g$, fg, cf, $\frac{f}{g}$ and $f \circ g$ are defined by*

- $(f+g)(x) = f(x) + g(x)$
- $(f-g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x)g(x)$
- $(cf)(x) = c\,f(x)$
- $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$
- $(f \circ g)(x) = f(g(x))$

$f \circ g$ is called the **composition** *of f and g.*

EXAMPLE 2

Suppose $f(x) = x + 2$ and $g(x) = x^2$. Compute a rule for each given function:

1. $(f+g)(x)$
   Solution: $(f+g)(x) = f(x) + g(x) = \boxed{x + 2 + x^2}$.
2. $(fg)(x)$
   Solution: $(fg)(x) = f(x)g(x) = \boxed{(x+2)x^2}$.
3. $(2g)(4)$
   Solution: $(2g)(4) = 2g(4) = 2(4^2) = 2 \cdot 16 = \boxed{32}$.
4. $(f-g)(3)$
   Solution: $(f-g)(3) = f(3) - g(3) = (3+2) - (3^2) = \boxed{-4}$.
5. $(f \circ g)(x)$
6. $(g \circ f)(x)$
7. $(f \circ f)(x)$

EXAMPLE 3

Given each function F, write $F = f \circ g$ where f and g are "easy" functions.

If you are familiar with diagramming functions, this means we want to identify functions f and g so that F diagrams as

$$x \xrightarrow{g} \xrightarrow{f} F(x).$$

1. $F(x) = \ln^7 x$

2. $F(x) = \ln x^7$

3. $F(x) = \sin^2 x$

4. $F(x) = e^{-x}$

5. $F(x) = 5\cos(e^x + 2x - 1)$
   Solution: $f(x) = 5\cos x$; $g(x) = e^x + 2x - 1$

6. $F(x) = (3x-2)^{12}$
   Solution: $f(x) = x^{12}$; $g(x) = 3x - 2$

Piecewise-defined functions

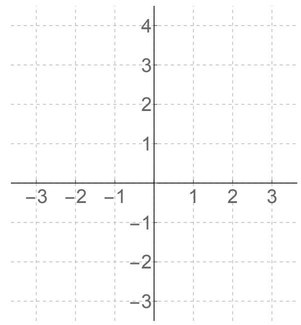
Consider the function

$$f(x) = \begin{cases} 1 - x & x < -1 \\ x^2 & x \geq -1 \end{cases}.$$

This means that to evaluate f at a number x, you look at which inequality x satisfies, then apply the corresponding formula. So a table of values for this f looks like

x	-3	-2	-1.5	-1	$-.5$	0	1	2	
$f(x)$									

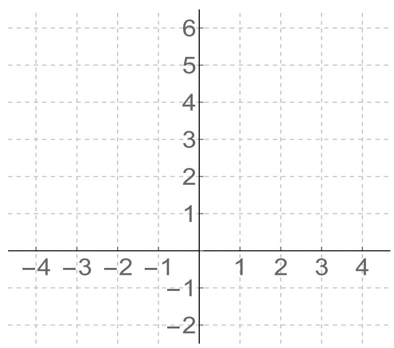
and so the graph of f looks like



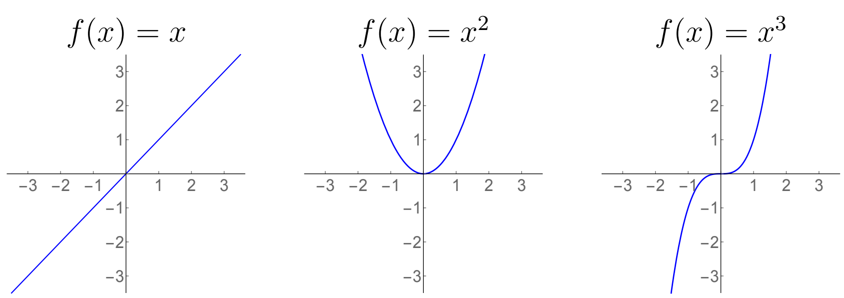
EXAMPLE 4

Graph this function:

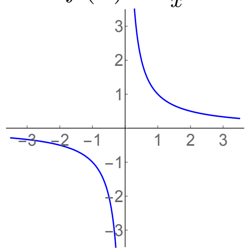
$$f(x) = \begin{cases} x^2 & x \neq 2 \\ -1 & x = 2 \end{cases}$$



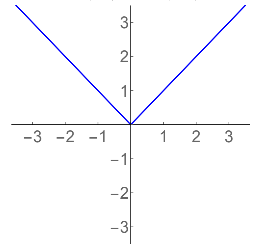
Common functions whose graphs you should know



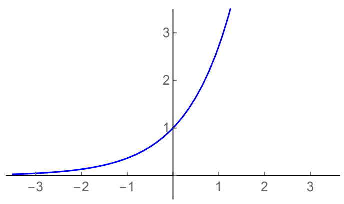
$f(x) = \frac{1}{x}$



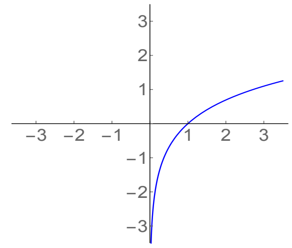
$f(x) = |x|$



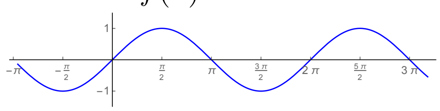
$f(x) = e^x$



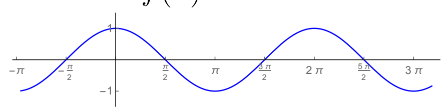
$f(x) = \ln x$



$f(x) = \sin x$



$f(x) = \cos x$



Transformations on functions

It is useful to know how the graph of a function changes if you alter the rule of the function a little bit. Suppose you know the graph of function f. Then:

Altered version of function f (all cs are positive numbers)	Corresponding transformation on the graph
$f(x) + c$	graph shifts up c units
$f(x) - c$	graph shifts down c units
$f(x + c)$	graph shifts left c units
$f(x - c)$	graph shifts right c units
$cf(x)$	graph stretched vertically by factor of c (taller if $c > 1$, shorter if $0 < c < 1$)
$f(-x)$	graph reflected through y-axis
$-f(x)$	graph reflected through x-axis

1.3 Lines

By far the most important class of functions are lines. Reasons:

1. Linear equations model a large class of real-world problems
2. Linear equations are relatively easy to work with.
3. You can often approximate the solution to hard problems (using calculus techniques) by considering something related to a linear equation.

QUESTION

What "determines" a line? That is, what makes one line different from another one?

1.

2.

Definition 1.5 *The* **slope** *of a line is the ratio of the rise of the line to its run, i.e. for any two points on the line* (x_1, y_1) *and* (x_2, y_2)*, the slope of the line is given by*

$$m = \frac{\textit{rise}}{\textit{run}} = \frac{\triangle\,\textit{output}}{\triangle\,\textit{input}} = \frac{\triangle\,y}{\triangle\,x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

If $m > 0$, then the line goes up from left to right. In this case, the greater m is, the steeper the line is.

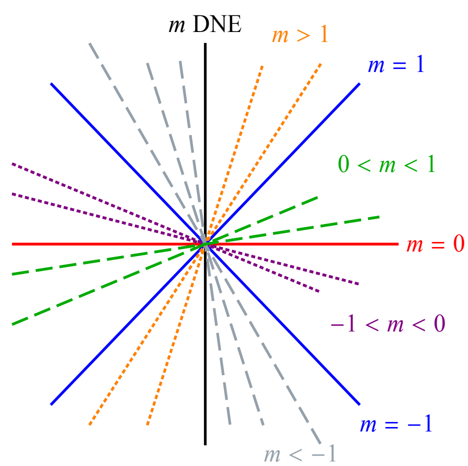
If $m = 1$, the line goes up at a $45°$ angle.

If $m = 0$, the line is horizontal.

If $m < 0$, then the line goes down from left to right. In this case, the more negative m is, the steeper the line is.

If $m = -1$, the line goes down at a $45°$ angle.

Vertical lines have undefined slope.



EXAMPLE 1

Find the slope of the line passing through the points $(2,-5)$ and $(4,11)$.

Solution: $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{11-(-5)}{4-2} = \dfrac{16}{2} = \boxed{8}$.

Given the slope m of a line, and a point (x_0, y_0) on the line, one can write the equation of the line as follows:

Definition 1.6 *The* **point-slope formula** *of a line with slope m passing through (x_0, y_0) is*

$$y = y_0 + m(x - x_0).$$

You may be familiar with the "slope-intercept" formula $y = mx + b$ for a line. The point-slope formula

$$y = y_0 + m(x - x_0)$$

is equivalent, because it can be rewritten as

It is extremely useful to know the point-slope formula, because it is easier than the $y = mx + b$ formula to apply in calculus.

EXAMPLE 2

Write the equation of the line passing through $(2,-5)$ and $(6,-7)$.

EXAMPLE 3

Write the equation of the line passing through $(-3,-2)$ with slope $\dfrac{2}{5}$.

<u>**NOTE:**</u> Vertical lines do not have a slope, so their equation cannot be written using the point-slope formula. The equation of a vertical line is $x = h$, where h is a constant. For example, the vertical line passing through $(6,-5)$ is $x = 6$.

1.4 Trigonometry

Classically, trigonometry is the study of measurements on triangles. Modern trigonometry concerns itself with three main problems:

1. converting between measurements of rotation/angle to measurements of length/distance (via the definition of the trig functions);

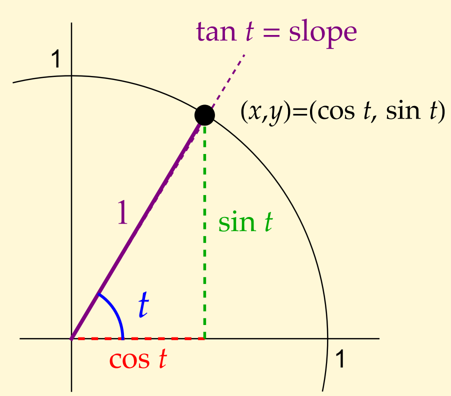
2. determining lengths and angles in diagrams via auxiliary measurements (via the Laws of Sines and Cosines; SOHCAHTOA; etc.)

3. analyzing oscillating behavior (via trig graphs).

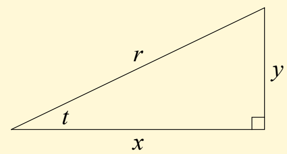
To approach these problems we use the six trigonometric functions:

Definition 1.7 (Unit circle definition of the trig functions) *Let t be a real number, or an angle in radians. Draw angle t in standard position and let (x, y) be the point on the unit circle $x^2 + y^2 = 1$ at angle t. Define*

- $\sin t = y$
- $\cos t = x$
- $\tan t = \textit{slope} = \dfrac{y}{x} = \dfrac{\sin t}{\cos t}$
- $\csc t = \dfrac{1}{y} = \dfrac{1}{\sin t}$
- $\sec t = \dfrac{1}{x} = \dfrac{1}{\cos t}$
- $\cot t = \dfrac{x}{y} = \dfrac{1}{\tan t} = \dfrac{\cos t}{\sin t}$



Definition 1.8 (Triangle definition of the trig functions) *Consider a right triangle with one angle measuring t, labelled as below:*

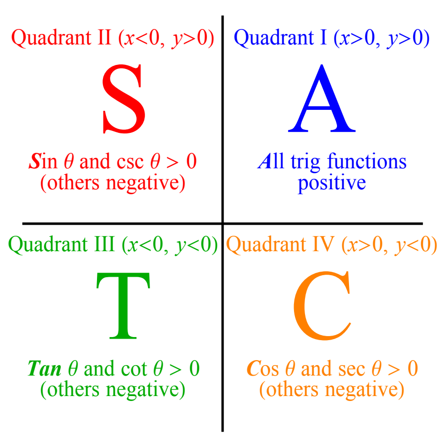


Then we define the **sine**, **cosine**, **tangent**, **cosecant**, **secant** *and* **cotangent** *functions of t by*

- $\sin t = \dfrac{\textit{opposite}}{\textit{hypotenuse}} = \dfrac{y}{r}$
- $\cos t = \dfrac{\textit{adjacent}}{\textit{hypotenuse}} = \dfrac{x}{r}$
- $\tan t = \dfrac{\textit{opposite}}{\textit{adjacent}} = \dfrac{y}{x}$
- $\csc t = \dfrac{\textit{hypotenuse}}{\textit{opposite}} = \dfrac{r}{y}$
- $\sec t = \dfrac{\textit{hypotenuse}}{\textit{adjacent}} = \dfrac{r}{x}$
- $\cot t = \dfrac{\textit{adjacent}}{\textit{opposite}} = \dfrac{x}{y}$

The two definitions of the trig functions are the same, **so long as the angle t is measured in radians**. (This is one of the many reasons why mathematicians prefer radians to degrees.) The advantage of the unit circle method is that it allows you to evaluate trig functions at angles measuring less than 0 or more than $90^\circ = \dfrac{\pi}{2}$.

Notice that we can determine the signs of the six trig functions by looking at the signs of x and y, i.e. looking at the quadrant the angle t lies in:



Either way you choose to define the trig functions, it is straightforward to deduce the following relationships:

Theorem 1.9 (Trigonometric identites) *The following identities hold for all x:*

- Quotient identities:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

- Reciprocal identities:

$$\cot x = \frac{1}{\tan x} \qquad \sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

- Pythagorean identities:

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \cot^2 x = \csc^2 x \qquad 1 + \tan^2 x = \sec^2 x$$

- Odd-even identities:

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

Using these identities and the "All Scholars Take Calculus" rules, you can find the values of other trig functions if you are given the value of one trig function, and the sign of a second trig function:

EXAMPLE 1

Find $\sec\theta$, if $\sin\theta = \dfrac{4}{7}$ and $\tan\theta < 0$.

Trig functions of special angles

You are responsible for computing any trig function at any multiple of $\frac{\pi}{6}$ or $\frac{\pi}{4}$ radians; virtually all problems in this course will use radians rather than degrees. You should especially know the following values of sine, cosine and tangent:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
x in degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	DNE	0	DNE

EXAMPLE 2

Compute each quantity (note that these are trig functions of "quadrantal angles", meaning the angle is a multiple of $\frac{\pi}{2}$):

1. $\sec \frac{3\pi}{2}$
2. $\sin \frac{5\pi}{2}$
3. $\tan 3\pi$
4. $\cos(-\pi)$

EXAMPLE 3

Compute each quantity (note that these are not trig functions of quadrantal angles, meaning the angle is not a multiple of $\frac{\pi}{2}$):

1. $\cos\frac{3\pi}{4}$

2. $\sin\frac{-5\pi}{6}$

3. $\csc\frac{5\pi}{3}$

4. $\tan\frac{-\pi}{4}$

   Solution: the reference angle is $\frac{\pi}{4}$; $\tan\frac{\pi}{4} = 1$.

   $\frac{-\pi}{4}$ is in Quadrant IV, so $\tan\frac{-\pi}{4} < 0$.

   Altogether, the answer is $\boxed{-1}$.

5. $\cos\frac{2\pi}{3}$

   Solution: the reference angle is $\frac{\pi}{3}$; $\cos\frac{\pi}{3} = \frac{1}{2}$.

   $\frac{2\pi}{3} = 2\cdot\frac{\pi}{3} = 2\cdot 60^\circ = 120^\circ$ is in Quadrant II, so $\cos\frac{2\pi}{3} < 0$.

   Altogether, the answer is $\boxed{-\frac{1}{2}}$.

6. $\tan \dfrac{4\pi}{3}$

   Solution: the reference angle is $\dfrac{\pi}{3}$; $\tan \dfrac{\pi}{3} = \sqrt{3}$.

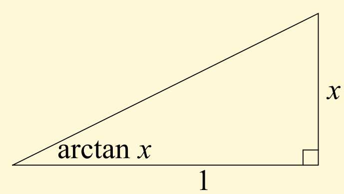
   $\dfrac{4\pi}{3} = 4 \cdot \dfrac{\pi}{3} = 4 \cdot 60^\circ = 240^\circ$ is in Quadrant III, so $\tan \dfrac{4\pi}{3} < 0$.

   Altogether, the answer is $\boxed{-\sqrt{3}}$.

Inverse trigonometric functions

Definition 1.10 *The* **arctangent** *(a.k.a.* **inverse tangent***) function is the function* $\arctan : \mathbb{R} \to \mathbb{R}$ *defined by*

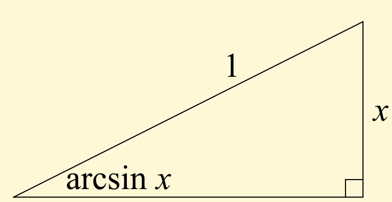
$$\arctan x = \text{ an angle (in radians) between } \frac{-\pi}{2} \text{ and } \frac{\pi}{2}\text{, whose tangent is } x.$$



EXAMPLE: $\arctan 1 = \boxed{\dfrac{\pi}{4}}$, because $\tan \dfrac{\pi}{4} = 1$.

Definition 1.11 *The* **arcsine** *(a.k.a.* **inverse sine***) function is the function* $\arcsin : [-1, 1] \to \mathbb{R}$ *defined by*

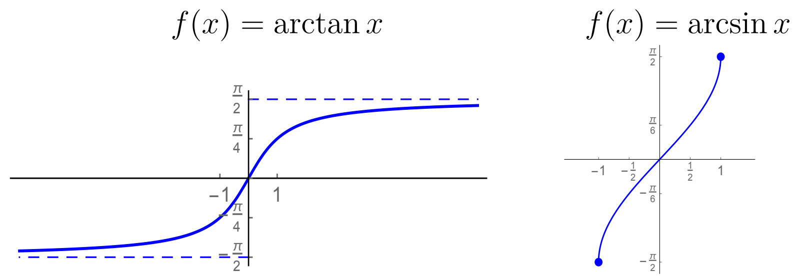
$$\arcsin x = \text{ an angle (in radians) between } \frac{-\pi}{2} \text{ and } \frac{\pi}{2}\text{, whose sine is } x.$$



EXAMPLE: $\arcsin \frac{\sqrt{3}}{2} = \boxed{\dfrac{\pi}{3}}$, because $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Notation: $\arctan x$ is sometimes written as $\tan^{-1} x$, and $\arcsin x$ is sometimes written as $\sin^{-1} x$. That said, I dislike this notation.

Graphs of arctangent and arcsine:



Theorem 1.12 (Properties of arctangent and arcsin) *These hold for all real numbers x, y:*

- $\arctan(-x) = -\arctan x$ *and* $\arcsin(-x) = -\arcsin x$.
- $y = \arctan x \iff x = \tan y$
- $y = \arcsin x \iff x = \sin y$

We will discuss inverse trig functions further in Chapter 6.

1.5 Homework exercises

Exercises from Section 1.1

1. Evaluate each expression:

   a) 4^{-2} b) $144^{1/2}$ c) $27^{5/3}$ d) $\left(\frac{1}{4}\right)^{-3/2}$

2. Write each expression in the form $\square x^{\square}$, where each of the two squares represent constants:

   a) $3\sqrt[4]{x}$ b) $\dfrac{1}{2x}$ c) $\dfrac{-2x}{3}$ d) $\dfrac{5}{9x^8}$ e) $8\sqrt{(2x)^4}$ f) $\sqrt{7x}$

3. Write each expression in the form $\square x^{\square}$, where each of the two squares represent constants:

   a) $\dfrac{-2}{\sqrt[5]{x^3}}$ b) $\sqrt{x^4}$ c) $\sqrt[5]{-32x^3}$ d) $\dfrac{8}{\sqrt[3]{8x^4}}$ e) $(3x)^4x^2$ f) $\dfrac{(3x)^2}{18x^3}$

4. Write each expression in the form $\square x^{\square}$, where each of the two squares represent constants:

   a) $4x\sqrt[3]{x}$ b) $\dfrac{(x^2)^{5/2}}{2x^3}$ c) $6\sqrt[3]{x^5}$

Exercises from Section 1.2

5. Let $f(x) = x^2 - 3$ and let $g(x) = 3 - x$. Compute and simplify:

   a) $f(-4)$ b) $g(-2)$ c) $(f - g)(1)$ d) $(fg)(4)$ e) $(f \circ g)(2)$ f) $(f \circ f)(0)$ g) $g(\text{bulldog})$ h) $g(x + 3)$ i) $\sqrt{g(1)}$ j) $g(\sqrt{x})$ k) $(f + g)(x + 1)$ l) $(fg)(2x)$

6. Let $h(x) = 2 + x^4$. Compute and simplify:

   a) $\sqrt{h(\sqrt{x})}$
   b) $h(x-1)$
   c) $h(x) - h(1)$
   d) $h(x) - 1$
   e) $4h(2x)$
   f) $h(x^2+1)$

7. Let $f(x) = x^3$. Compute and simplify $\dfrac{f(x+h) - f(x)}{h}$.

8. Let $f(x) = x^2 - x$. Compute and simplify $\frac{f(1+h)-f(1)}{h}$.

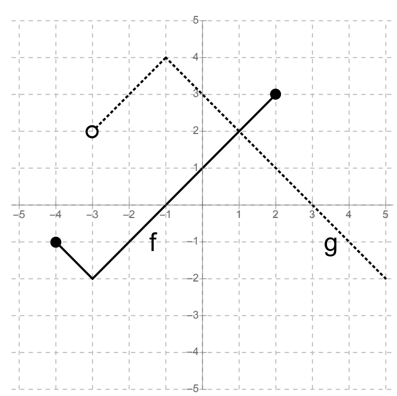
9. Let $f(x) = x + 2$. Compute and simplify $\dfrac{f(x) - f(t)}{x - t}$.

10. Let $f(x) = \begin{cases} 2x+1 & x < 1 \\ 2x+2 & x \geq 1 \end{cases}$. Evaluate $f(-1)$, $f(0)$, $f(1)$ and $f(2)$.

11. Sketch the graph of each function:

    a) $f(x) = \begin{cases} 1-x & x < 1 \\ x+1 & x \geq 1 \end{cases}$
    b) $g(x) = \begin{cases} x & x \neq 1 \\ -1 & x = 1 \end{cases}$

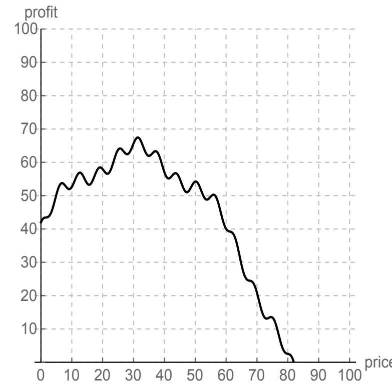
12. The graphs of unknown functions f and g are given below:



Use these graphs to estimate answers to the following questions:

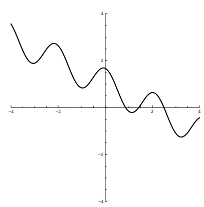
   a) Compute $f(-2)$
   b) Compute $g(3)$.
   c) Compute $g(-3)$.
   d) Compute $(f+g)(-1)$.
   e) Compute $(f \circ g)(3)$.
   f) Compute $(fg)(1)$.
   g) Find all value(s) of x (if any) such that $f(x) = g(x)$.
   h) Find all value(s) of x (if any) for which $f(x) = -1$.
   i) Find all value(s) of x (if any) for which $g(x) = 0$.

13. Suppose the graph at right is the picture of a function where the output of the function is a company's profit (in millions of dollars), and the input is the price at which the company sells its product. At (roughly) what price should the company sell its product, if its goal is to make as much money as possible? How much profit will be made at this price?

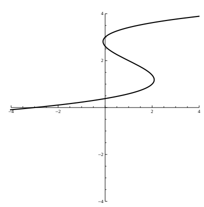


14. Determine which one or ones of the following pictures (a)-(d) depict situations where y is a function of x.

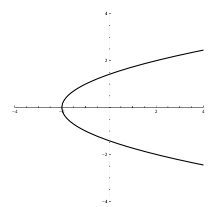
a)



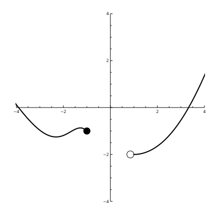
b)



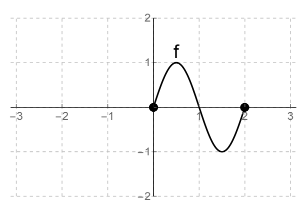
c)



d)



15. Suppose $y = f(x)$ is a function whose graph is:



Sketch the graphs of the following functions:

a) $y = f(x+5)$
b) $y = f(x) - 5$
c) $y = f(-x)$
d) $y = -f(x) + 5$
e) $y = f(x-2) + 1$
f) $y = -f(-x)$

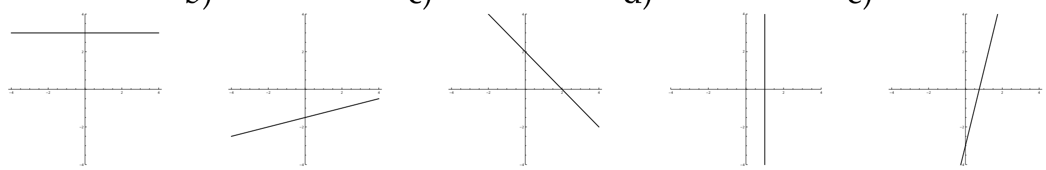
16. Sketch the graphs of the following functions:

a) $y = 2\sin x$
b) $y = (x-3)^2 + 1$
c) $y = -\ln x$
d) $y = e^{-x}$
e) $y = \cos(-x)$
f) $y = -(x+2)^3$
g) $y = -|x| + 2$
h) $y = -\dfrac{1}{x}$
i) $y = -(x+2)^2 - 4$

Exercises from Section 1.3

17. Estimate the slope of each of the following lines by looking at its graph (assume the scales on the x- and y-axes are the same):

a) b) c) d) e)



18. Find the slope of the line passing through these pairs of points:

a) $(3, -4)$ and $(5, 2)$
b) $\left(\frac{-1}{2}, \frac{2}{3}\right)$ and $\left(\frac{-3}{4}, \frac{1}{6}\right)$
c) (a, b) and $(a + s, b + r)$
d) $(2, 7)$ and $(2, -1)$
e) $(x, f(x))$ and $(x + h, f(x + h))$
f) $(-1, 4)$ and $(5, -8)$

19. Find the equation of the line with each set of properties:

a) passes through $(0, 3)$ and has slope $\frac{3}{4}$
b) passes through the origin; $m = \frac{2}{3}$
c) passes through $(2, 1)$ and $(0, -3)$
d) passes through $(-3, -2)$; $m = 4$
e) passes through $(2, 6)$ and is vertical
f) passes through $(-4, 2)$ and is horizontal
g) passes through $(5, 1)$ and $(5, 8)$
h) passes through $(-7, 3)$ and $(2, -5)$

Exercises from Section 1.4

20. Suppose $\sin x = \frac{5}{13}$. Assuming the values of the other five trig functions of x are positive, find them.

21. Suppose $\cos x = \frac{7}{25}$. If $\tan x < 0$, find the values of the other five trig functions of x.

22. Suppose $\csc x = \frac{5}{2}$. What is $\sin x$?

23. Suppose $\tan x = 2$. If $\cos x < 0$, what is $\sin x$?

24. Compute each of the following (if they are not defined, say so). Try to do these without looking anything up (to simulate how you will have to do these things on quizzes and exams).

a) $\sin\frac{\pi}{3}$
b) $\cos\frac{\pi}{2}$
c) $\tan\frac{\pi}{4}$
d) $\cos 0$
e) $\cos\frac{\pi}{6}$
f) $\cos\frac{2\pi}{3}$
g) $\sin\frac{3\pi}{2}$
h) $\cot\frac{3\pi}{4}$
i) $\sin 0$
j) $\sec\pi$
k) $\tan\frac{\pi}{6}$
l) $\sin\frac{7\pi}{4}$
m) $\sec\frac{\pi}{3}$
n) $\tan\frac{3\pi}{2}$
o) $\sin\pi$
p) $\csc\frac{5\pi}{6}$
q) $\sec\frac{-\pi}{4}$
r) $\sin\frac{-5\pi}{6}$
s) $\tan -\pi$
t) $\tan\frac{-8\pi}{3}$

25. Evaluate each of the following:

a) $\arcsin\frac{1}{2}$
b) $\arcsin 0$
c) $\arctan 1$
d) $\arctan\left(-\sqrt{3}\right)$
e) $\arctan\frac{\sqrt{3}}{3}$
f) $\arcsin\frac{-\sqrt{3}}{2}$
g) $\arcsin -1$
h) $\arctan -1$
i) $\arcsin\frac{-\sqrt{2}}{2}$

Answers

DISCLAIMER: Throughout the lecture notes, the provided answers are answers only (not complete solutions) and may contain errors and/or typos.

1. a) $\frac{1}{16}$ b) 12 c) 243 d) 8

2. a) $3x^{1/4}$ b) $2x^{-1}$ c) $\frac{-2}{3}x^1$ d) $\frac{5}{9}x^{-8}$ e) $32x^2$ f) $\sqrt{7}\,x^{1/2}$

3. a) $-2x^{3/5}$ b) x^2 c) $-2x^{3/5}$ d) $4x^{-4/3}$ e) $81x^6$ f) $\frac{1}{2}x^{-1}$

4. a) $4x^{4/3}$ b) $4x^{-4/3}$ c) $\frac{1}{2}x^{-1}$

5. a) 13 b) 5 c) -4 d) -13 e) -2 f) 6 g) $3-$ bulldog h) $-x$ i) $\sqrt{2}$ j) $3-\sqrt{x}$ k) $(x+1)^2-x-1$ l) $(4x^2-3)(3-2x)$

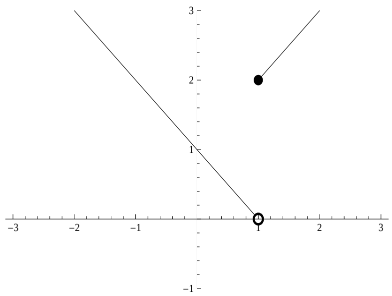
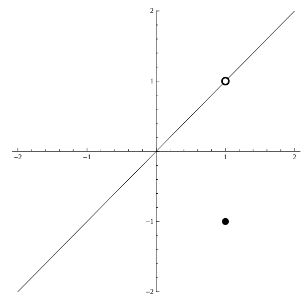
6. a) $\sqrt{2+x^2}$
   b) $2+(x-1)^4$
   c) x^4-1
   d) x^4+1
   e) $8+64x^4$
   f) $2+(x^2+1)^4$

7. $3x^2+3xh+h^2$

8. $h+1$

9. 1

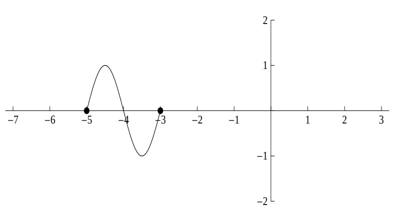
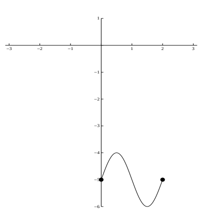
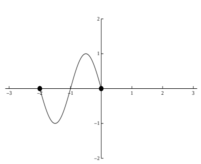
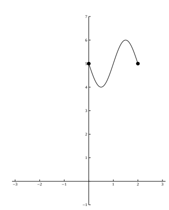
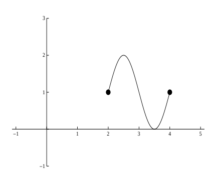
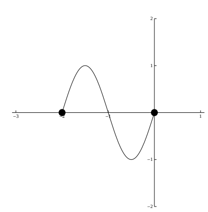
10. $f(-1)=-1$; $f(0)=1$; $f(1)=4$; $f(2)=6$.

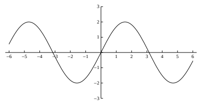
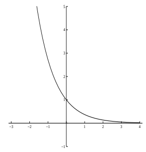
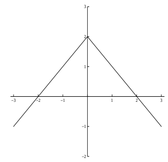
11. a) 
    b) 

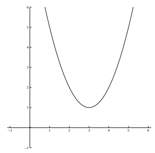
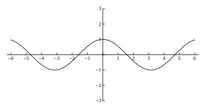
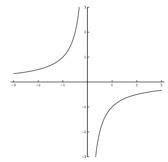
12. a) -1
    b) 0
    c) DNE
    d) 4
    e) 1
    f) 4
    g) $x=1$
    h) $x=-4, x=-2$
    i) $x=3$

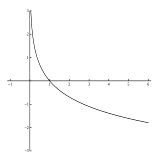
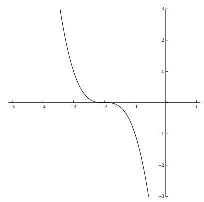
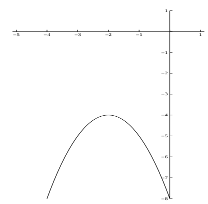
13. The price should be roughly $\$32$, and their profit will be about $\$67,000,000$.

14. (a) and (d) are functions $y=f(x)$; (b) and (c) are not.

15. a) 
    b) 
    c) 
    d) 
    e) 
    f) 

16. a)  d)  g) 

b)  e)  h) 

c)  f)  i) 

17. a) 0 b) $\approx \frac{1}{3}$ c) ≈ -1 d) DNE e) ≈ 4

18. a) 3 c) $\frac{r}{s}$ e) $\frac{f(x+h)-f(x)}{h}$

b) 2 d) DNE f) -2

19. a) $y = \frac{3}{4}x + 3$ c) $y = 2x - 3$ f) $y = 2$

d) $y = -2 + 4(x+3)$ g) $x = 5$

b) $y = \frac{2}{3}x$ e) $x = 2$ h) $y = -5 + \frac{-8}{9}(x-2)$

20. $\cos x = \frac{12}{13}$; $\tan x = \frac{5}{12}$; $\cot x = \frac{12}{5}$; $\sec x = \frac{13}{12}$; $\csc x = \frac{13}{5}$.

21. $\sin x = \frac{-24}{25}$; $\tan x = \frac{-24}{7}$; $\cot x = \frac{-7}{24}$; $\sec x = \frac{25}{7}$; $\csc x = \frac{-25}{24}$.

22. $\frac{2}{5}$

23. $\frac{-2}{\sqrt{5}}$

24. a) $\frac{\sqrt{3}}{2}$ d) 1 g) -1 k) $\frac{\sqrt{3}}{3}$ n) DNE r) $\frac{-1}{2}$

b) 0 e) $\frac{\sqrt{3}}{2}$ h) -1 l) $\frac{-\sqrt{2}}{2}$ o) 0 s) 0

c) 1 f) $\frac{-1}{2}$ i) 0 j) -1 m) 2 p) 2 q) $\sqrt{2}$ t) $\sqrt{3}$

25. a) $\frac{\pi}{6}$ d) $\frac{-\pi}{3}$ f) $\frac{-\pi}{3}$ h) $\frac{-\pi}{4}$

b) 0

c) $\frac{\pi}{4}$ e) $\frac{\pi}{6}$ g) $\frac{-\pi}{2}$ i) $\frac{-\pi}{4}$

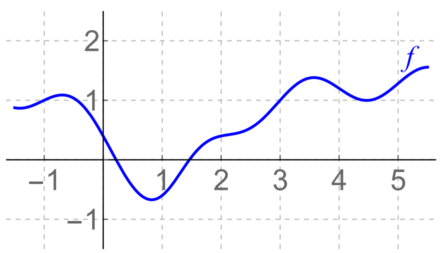
Chapter 2

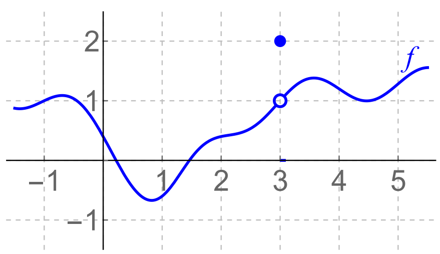
Limits

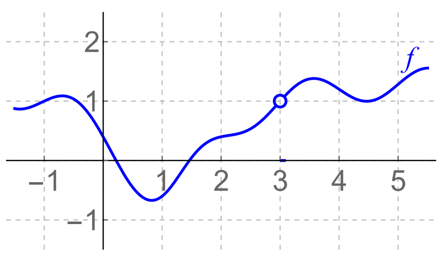
2.1 The idea of the limit

WARMUP

Given the graphs of each of these functions, tell me the value of $f(3)$:

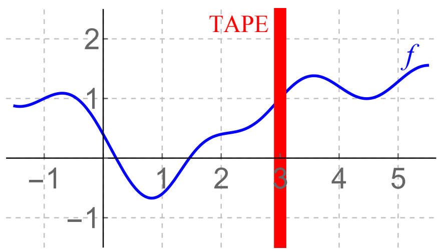






MODIFIED WARMUP

Here is the graph of some function f. The portion of the graph above $x = 3$ is "covered" (by a strip of painters tape, for example). Based only on what you see, what would you **guess** the value of $f(3)$ is?



First idea of the limit: graphical interpretation

Suppose you can see the entire graph of a function f except for the possible point on the graph sitting above (or below) $x = a$. If, based on the picture, you'd guess that $f(a) = L$, then you say

"the limit as x approaches a of $f(x)$ is L"

and you'd write

$$\lim_{x\to a} f(x) = L \quad \text{or} \quad \text{“}f(x) \to L \text{ as } x \to a\text{”}.$$

EXAMPLES: In the modified warmup above,

$$\lim_{x\to 3} f(x) =$$

In all three warmup examples,

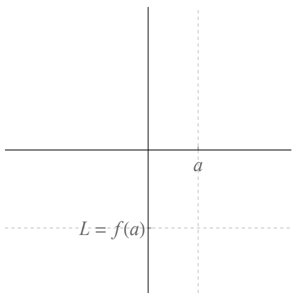
$$\lim_{x\to 3} f(x) =$$

Note: $f(3)$ is different in the three warmup examples. In one example, $f(3)$ doesn't even exist!

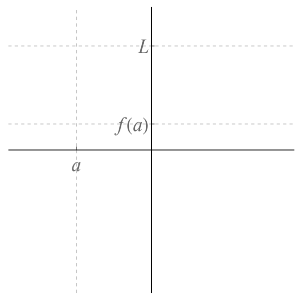
In general, if you have a function f which satisfies

$$\lim_{x \to a} f(x) = L,$$

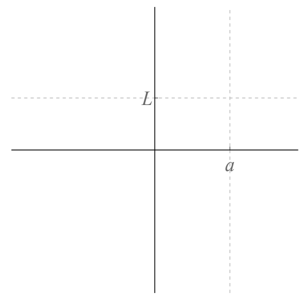
then the graph of f should look like one of these three pictures:



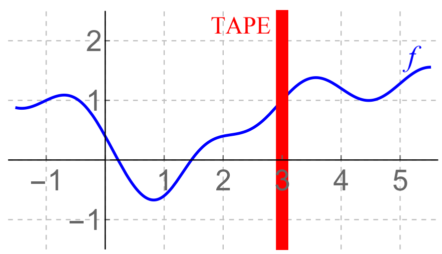
or



or



Back to the modified warmup:



In this example, we said $\lim_{x \to 3} f(x) = 1$.
Why is 1 the most reasonable guess for the value of $f(3)$?

Second idea of the limit: approximation via tables

To say

$$\lim_{x \to a} f(x) = L$$

means that as x gets closer and closer to a (without ever reaching a), the corresponding values $f(x)$ of the function get closer and closer to L.

EXAMPLE 1

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = ?$$

Solution: The **first idea of the limit** requires a graph to apply, and we don't have a graph.

To implement the **second idea of the limit**, let's take $x-$values which get closer and closer to 0 and see if the corresponding $f(x)-$values approach a number:

x	$.1$	$.05$	$.01$	$.001$	$.0000001$
$\frac{\sin x - x}{x^3}$	$-.166583$	$-.166646$	$-.166666$	$-.166666$	$-.166666$

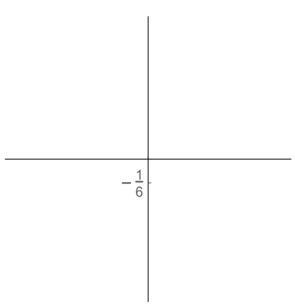
x	$-.1$	$-.05$	$-.01$	$-.001$	$-.0000001$
$\frac{\sin x - x}{x^3}$	$-.166583$	$-.166646$	$-.166666$	$-.166666$	$-.166666$

Based on this, we can conjecture that

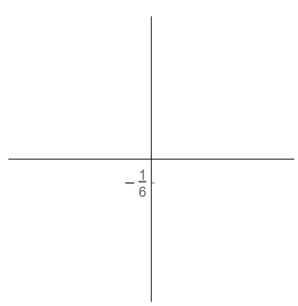
$$\lim_{x \to 0} \frac{\sin x - x}{x^3} =$$

IMPORTANT:

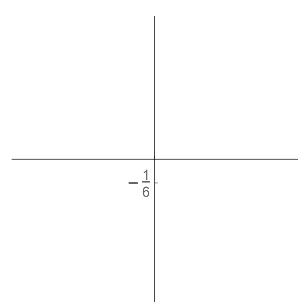
This suggests that the graph of $f(x) = \frac{\sin x - x}{x^3}$ looks like



or



or



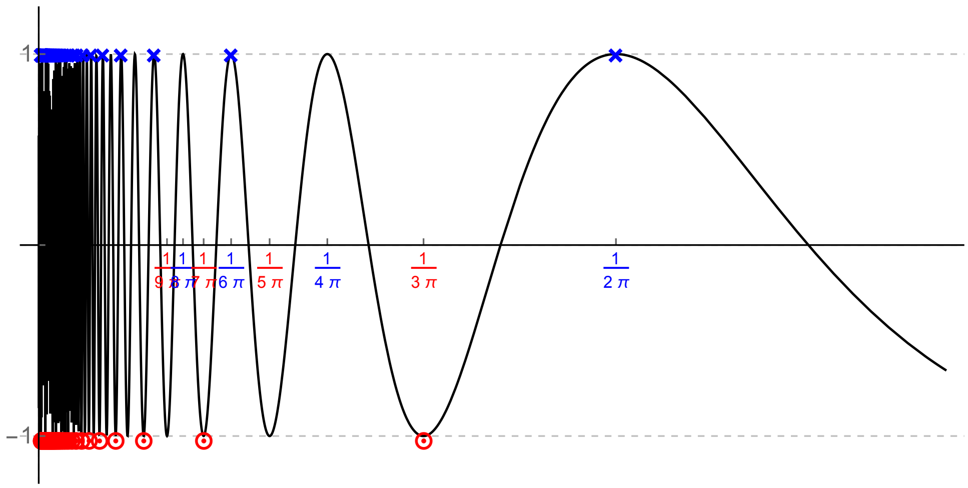
The method of the previous example sometimes works well, but it can lie:

EXAMPLE 2

Let $f(x) = \cos \frac{1}{x}$.

$$\lim_{x \to 0} f(x) = ?$$

Solution: Here's a graph of $f(x)$:



Let's try the method of Example 1:

x	$\frac{1}{2\pi}$	$\frac{1}{4\pi}$	$\frac{1}{6\pi}$	$\frac{1}{100\pi}$	$\frac{1}{1000\pi}$
$f(x)$					

x	$\frac{1}{3\pi}$	$\frac{1}{5\pi}$	$\frac{1}{7\pi}$	$\frac{1}{101\pi}$	$\frac{1}{1001\pi}$
$f(x)$					

Third idea of the limit: formal definition

Suppose $f(x)$ is defined for all x near a but possibly not at a. If $f(x)$ is as close to L as we like **for all** x sufficiently close to a (but not a itself), we say

$$\lim_{x \to a} f(x) = L.$$

In Example 2, there is no L such that $f(x) = \cos \frac{1}{x}$ stays close to L for all x near 0. Therefore

2.2 One-sided limits

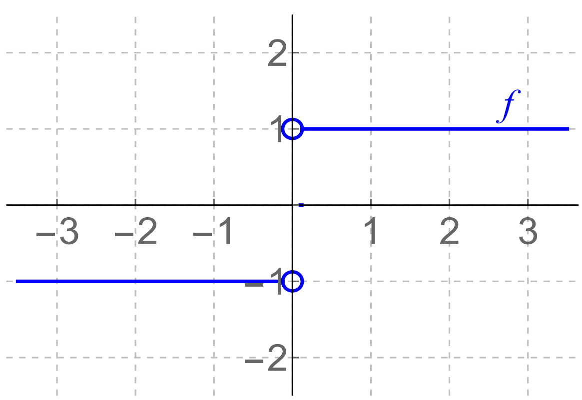
EXAMPLE 3

Let f be the **signum function**

$$f(x) = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

$$\lim_{x \to 0} f(x) = ?$$

Solution: Here is a graph of f:



Definition 2.1 *Suppose $f(x)$ is defined for all x near a with $x > a$. If (whenever x gets closer and closer to a from the right, $f(x)$ approaches L), then we say the* **limit of $f(x)$ as x approaches a from the right** *is L and we write*

$$\lim_{x \to a^+} f(x) = L.$$

Suppose $f(x)$ is defined for all x near a with $x < a$. If (whenever x gets closer and closer to a from the left, $f(x)$ approaches L), then we say **the limit of $f(x)$ as x approaches a from the left is** *L and we write*

$$\lim_{x \to a^-} f(x) = L.$$

These are also called, respectively, **left-hand limits** and **right-hand limits**. Collectively, left- and right-hand limits are referred to as **one-sided limits**.

EXAMPLE: In the previous example where $f(x) = \dfrac{|x|}{x}$,

$$\lim_{x \to 0^+} f(x) = \qquad\qquad \lim_{x \to 0^-} f(x) =$$

Theorem 2.2 $\lim_{x \to a} f(x)$ *exists **only if*** $\lim_{x \to a^+} f(x)$ *and* $\lim_{x \to a^-} f(x)$ *both exist and are equal. In this situation,*

$$\lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x).$$

EXAMPLE: For the function $f(x) = \dfrac{|x|}{x}$, since

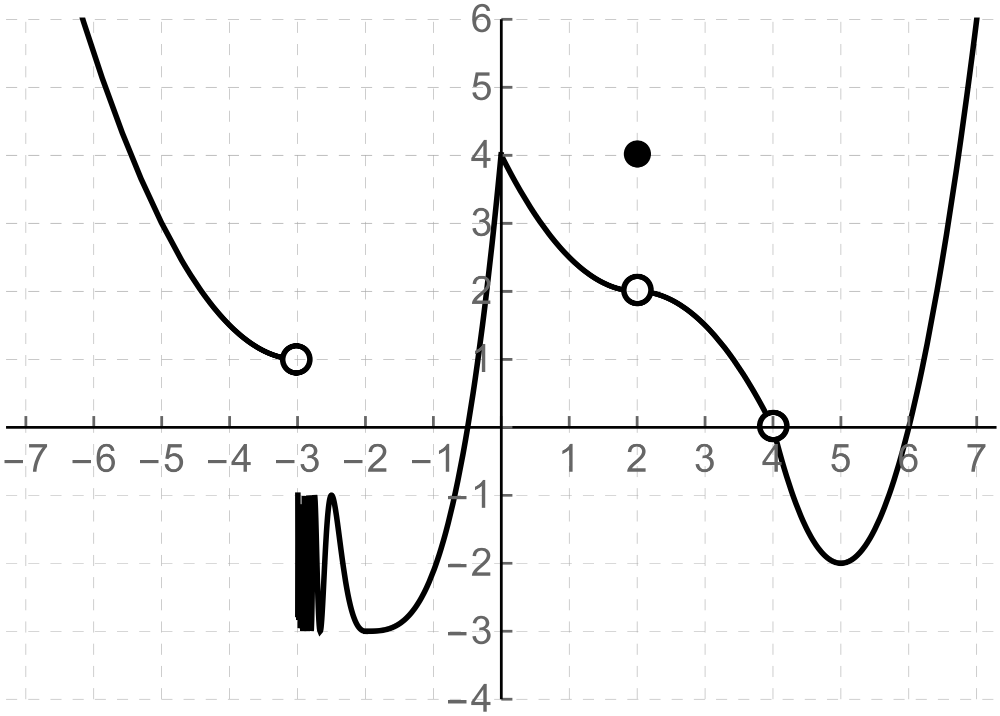
$$\lim_{x \to 0^+} f(x) = 1 \neq -1 = \lim_{x \to 0^-} f(x),$$

we see that

$$\lim_{x \to 0} f(x) \text{ DNE}.$$

EXAMPLE 4

Consider the following graph of some unknown function f:



Based on this graph, find the following:

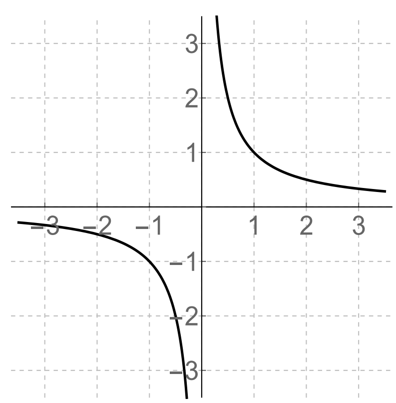
1. $\lim_{x\to 2} f(x)$
2. $\lim_{x\to 0} f(x)$
3. $f(2)$
4. $f(0)$
5. $\lim_{x\to 2^+} f(x)$
6. $\lim_{x\to -3^-} f(x)$
7. $\lim_{x\to -3^+} f(x)$
8. $\lim_{x\to -3} f(x)$
9. $f(4)$
10. $\lim_{x\to 4^+} f(x)$
11. $\lim_{x\to 4^-} f(x)$
12. $\lim_{x\to 4} f(x)$

2.3 Infinite limits and limits at infinity

Consider the reciprocal function $f(x) = \dfrac{1}{x}$. What happens to $f(x)$ as $x \to 0$?

x	1	.5	.1	.001	.0000001
$f(x)$	1	2	10	1000	1000000

x	-1	$-.5$	$-.1$	$-.001$	$-.0000001$
$f(x)$	-1	-2	-10	-1000	-1000000

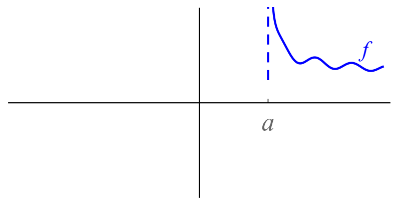


We invent new notation to describe this situation. We say

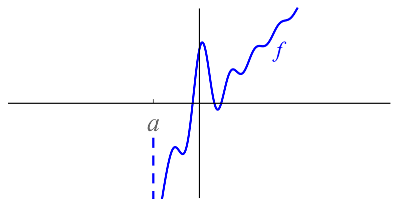
$$\lim_{x\to 0^+} \frac{1}{x} = \infty \qquad \text{and} \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

Formally:

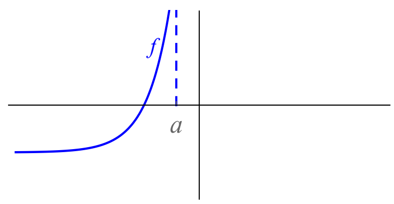
- to say $\lim_{x \to a^+} f(x) = \infty$ means that as x gets closer and closer to a from the right, the numbers $f(x)$ grow without bound.



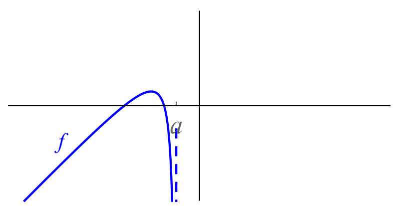
- to say $\lim_{x \to a^+} f(x) = -\infty$ means that as x gets closer and closer to a from the right, the numbers $f(x)$ become more and more negative without bound.



- to say $\lim_{x \to a^-} f(x) = \infty$ means that as x gets closer and closer to a from the left, the numbers $f(x)$ grow without bound.



- to say $\lim_{x \to a^-} f(x) = -\infty$ means that as x gets closer and closer to a from the left, the numbers $f(x)$ become more and more negative without bound.



All these situations are called **infinite limits**. The graphical description of an infinite limit is as follows:

Definition 2.3 *If* $\lim_{x\to a^+} f(x) = \pm\infty$ *or* $\lim_{x\to a^-} f(x) = \pm\infty$, *we say the vertical line* $x = a$ *is a* **vertical asymptote (VA)** *for* $f(x)$.

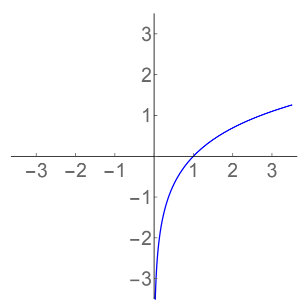
EXAMPLE: $x = 0$ is a VA for $f(x) = \dfrac{1}{x}$.

NOTE: ∞ is **not a number**. It is only a symbol. However, in the context of limits, ∞ can be manipulated in some ways as if it was a number (we'll see how in Chapter 3). For now you should remember these facts:

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

<u>**One infinite limit to memorize**</u>:

$$\lim_{x\to 0^+} \ln x = -\infty$$



Other infinite limits are computed using techniques we will study later, using some rules of arithmetic with ∞.

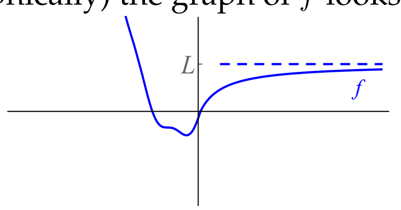
Limits at infinity

We want to consider the values of $f(x)$ when x gets larger and larger without bound. For example, suppose $f(x) = \frac{1}{x}$:

x	1	10	10000	10^{100}	10^{10000}
$f(x)$	1	$\frac{1}{10}$	$\frac{1}{10000}$	$\frac{1}{10^{100}}$	$\frac{1}{10^{10000}}$

We say $\lim_{x\to\infty} f(x) = L$ if

- (heuristically) when x grows without bound, $f(x)$ approaches L.
- (graphically) the graph of f looks like



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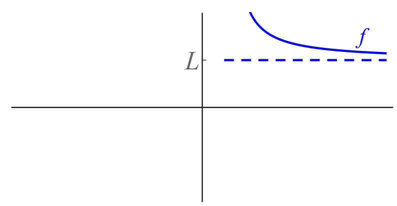
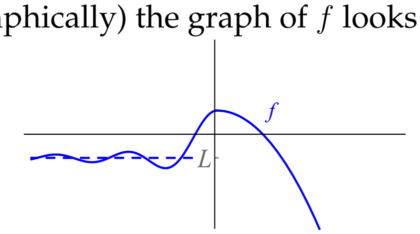


We say $\lim_{x\to-\infty} f(x) = L$ if

- (heuristically) when x becomes more and more negative without bound, $f(x)$ approaches L.
- (graphically) the graph of f looks like



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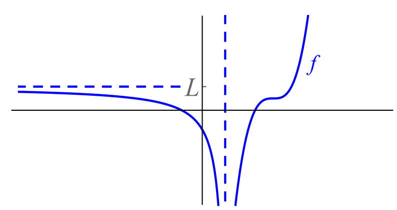
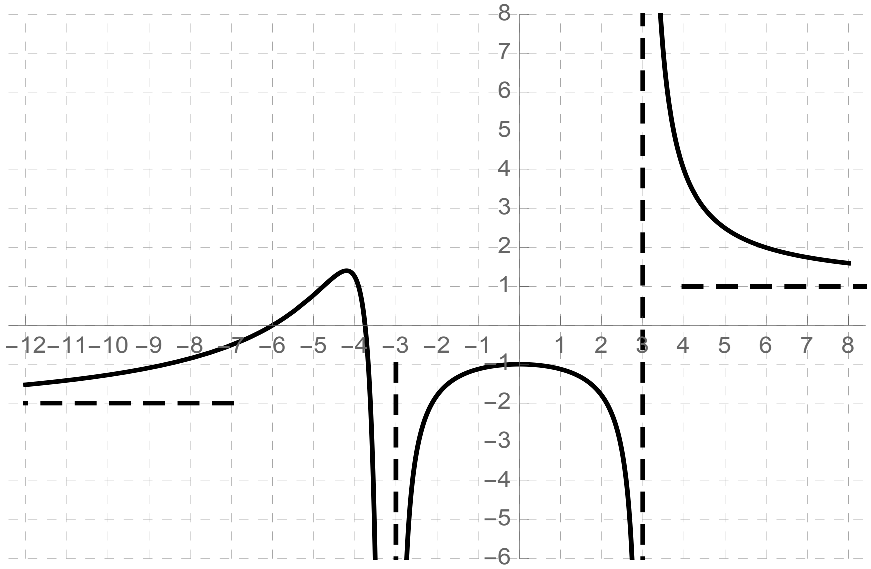


> **Definition 2.4** *If* $\lim_{x\to\infty} f(x) = L$ *or* $\lim_{x\to-\infty} f(x) = L$*, we say the horizontal line* $y = L$ *is a* **horizontal asymptote (HA)** *for* $f(x)$*.*

EXAMPLE

Consider the following graph of some unknown function f:



Based on this graph, find the following:

1. $\lim_{x \to \infty} f(x)$
2. $\lim_{x \to -\infty} f(x)$
3. $\lim_{x \to -3^+} f(x)$
4. $\lim_{x \to -3^-} f(x)$
5. $\lim_{x \to -3} f(x)$
6. $\lim_{x \to 3^+} f(x)$
7. $\lim_{x \to 3^-} f(x)$
8. $\lim_{x \to 3} f(x)$
9. the equation(s) of any vertical asymptote(s) of f
10. the equation(s) of any horizontal asymptote(s) of f

2.4 Homework exercises

Exercises from Section 2.1

In Problems 1-2 below, you are given a limit. Use a calculator or computer to complete the tables and use the results to estimate the value of the limit:

1. $\lim_{x \to 3} \frac{x-3}{x^2 - 7x + 12}$

x	2.9	2.99	2.999
$f(x)$			

x	3.1	3.01	3.001
$f(x)$			

2. $\lim_{x \to -2} \frac{\sqrt{2-x} - 2}{x+2}$

x	-2.1	-2.01	-2.001
$f(x)$			

x	-1.9	-1.99	-1.999
$f(x)$			

3. Find the value of $\lim_{x \to 0} \frac{\ln(x+1) - x}{x^2}$ using tables similar to Problems 1 and 2. (This time, you have to pick your own x values.)

4. Find the value of $\lim_{x \to 1} \frac{1 - \frac{2}{x+1}}{x-1}$ using tables similar to Problems 1 and 2. (Again, you have to pick your own x values.)

5. Complete the following charts for the function $f(x) = \frac{|x-5|}{x-5}$:

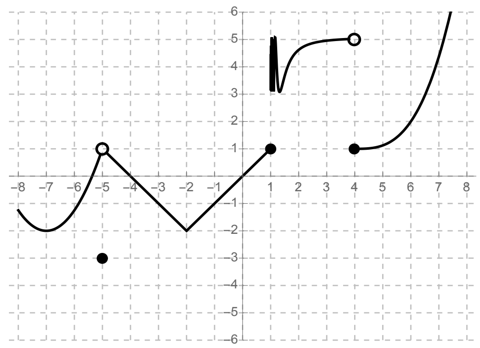
x	5.1	5.01	5.001
$f(x)$			

x	4.9	4.99	4.999
$f(x)$			

What do these charts suggest to you about $\lim_{x \to 5} \frac{|x-5|}{x-5}$?

Exercises from Section 2.2

6. Given the graph of f below, evaluate the given expressions. If the quantity does not exist, say so.



a) $\lim_{x\to -5} f(x)$

b) $f(-5)$

c) $\lim_{x\to -1} f(x)$

d) $\lim_{x\to 1^-} f(x)$

e) $\lim_{x\to 1^+} f(x)$

f) $\lim_{x\to 1} f(x)$

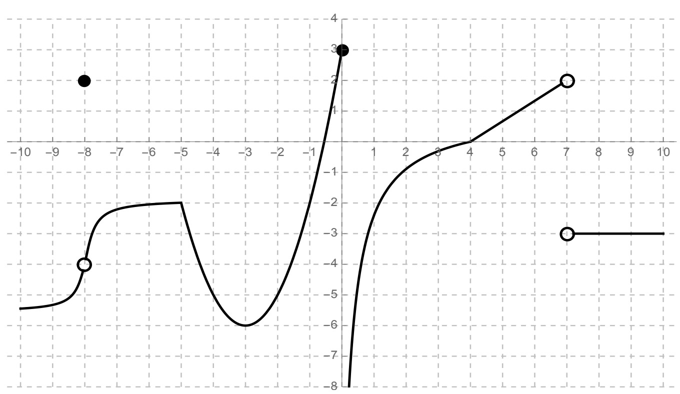
g) $\lim_{x\to 4^-} f(x)$

h) $\lim_{x\to 4^+} f(x)$

i) $\lim_{x\to 4} f(x)$

j) $f(4)$

7. Given the graph of g below, evaluate the given expressions. If the quantity does not exist, say so.



a) $\lim_{x\to -8} g(x)$

b) $f(-8)$

c) $\lim_{x\to -5} g(x)$

d) $\lim_{x\to 0^-} g(x)$

e) $\lim_{x\to 1} g(x)$

f) $\lim_{x\to 7^-} g(x)$

g) $\lim_{x\to 7^+} g(x)$

h) $\lim_{x\to 7} g(x)$

i) $g(7)$

j) $\lim_{x\to -2^-} g(x)$

8. Sketch a graph of a function f which has all of the following four properties (there are many possible correct answers):
   - $f(0)$ is not defined;
   - $\lim_{x\to 0} f(x) = 4$;
   - $f(2) = 6$;
   - $\lim_{x\to 2} f(x) = 3$.

9. Sketch a graph of a function f which has all of the following five properties (there are many possible correct answers):
   - $\lim_{x\to -1^+} f(x) = 3$;
   - $\lim_{x\to -1^-} f(x) = -2$;
   - $\lim_{x\to 2^-} f(x)$ DNE;
   - $f(2) = 0$;
   - $\lim_{x\to 2^+} f(x) = 3$.

10. Sketch a graph of a function f which has all of the following four properties (there are many possible correct answers):
    - $\lim_{x\to 3} f(x) = -1$;
    - $f(3) = 2$;
    - $\lim_{x\to -4^-} f(x) = -5$;
    - $\lim_{x\to -4^+} f(x) = -1$.

Exercises from Section 2.3

In Problems 11-12 below, you are given a limit. Complete the table (use a calculator or computer if necessary) and use the results to estimate the value of the limit:

11. $\displaystyle\lim_{x\to\infty} \frac{4x+3}{2x-1}$

x	10	100	1000	10^6	10^{10}
$f(x)$					

12. $\displaystyle\lim_{x\to\infty} \frac{-6x}{\sqrt{4x^2+5}}$

x	10	100	1000	10^6	10^{10}
$f(x)$					

In Problems 13-18, graph each function inside the limit using *Mathematica* (or a calculator) and use the graph of the function to estimate $\lim_{x\to\infty} f(x)$:

13. $f(x) = \dfrac{|x|}{x+1}$

    Hint: the *Mathematica* code to plot this function (where x ranges from -10 to 10) is

    ```
    Plot[ Abs[x] / (x+1), {x, -10, 10}]
    ```

14. $f(x) = \dfrac{\ln x}{\sqrt{x}}$

    Hint: Mathematica code to plot this function is

    ```
    Plot[ Log[x] / Sqrt[x], {x, -10, 10}]
    ```

15. $f(x) = \dfrac{\sin x}{x}$

16. $f(x) = x \arctan\left(\dfrac{1}{x}\right)$

17. $f(x) = x - \sqrt{x(x-1)}$

18. $f(x) = \dfrac{x+1}{x\sqrt{x}}$

In Problems 19-20, complete the tables (use a calculator or computer if necessary) and use the results to estimate the value of the limit:

19. $\lim\limits_{x \to 1^+} \dfrac{2+x}{1-x}$

x	2	1.1	1.01	1.0001	1.000001
$f(x)$					

20. $\lim\limits_{x \to 3^-} \dfrac{x^2+7}{x-3}$

x	2	2.9	2.99	2.9999	2.999999
$f(x)$					

In Problems 21-26, graph the function inside the limit using *Mathematica* (or a calculator) and use the graph of the function to estimate the given limit:

21. $\lim\limits_{x \to \pi^-} \sec \dfrac{x}{2}$

    Hint: Mathematica code to plot this function (where x ranges from -10 to 10) is

    ```
    Plot[ Sec[x/2], {x, -10, 10}]
    ```

22. $\lim\limits_{x \to \pi^+} \sec \dfrac{x}{2}$

23. $\lim\limits_{x \to 0} \dfrac{(x-1)^2}{x^2}$

24. $\lim\limits_{x \to \pi^-} (\cot x - \sec x)$

25. $\lim\limits_{x \to 4^-} \dfrac{3x^2-6x+5}{x^2-5x+4}$

26. $\lim\limits_{x \to 1^-} \dfrac{x-4}{e^x - e}$

27. By using *Mathematica* to graph the function, find the equation of any horizontal and/or vertical asymptotes of the function

$$f(x) = \frac{x^2+3}{x^3-5x^2+4x}.$$

    Hint: Mathematica code to plot this function (where x ranges from -10 to 10) is `Plot[ (x^2 + 3)/(x^3 - 5x^2 + 4x), {x, -10, 10}]`

    Make sure to use parentheses to surround the numerator and denominator when using *Mathematica*.

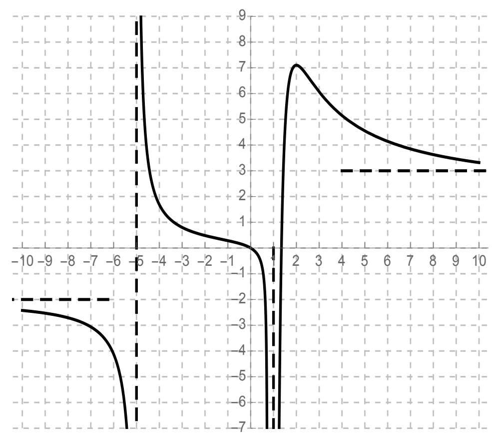
28. By using *Mathematica* to graph the function, find the equation of any horizontal and/or vertical asymptotes of the function

$$f(x) = \frac{2x^2-8x-42}{x^2-25}.$$

29. By using *Mathematica* to graph the function, find the equation of any horizontal and/or vertical asymptotes of the function

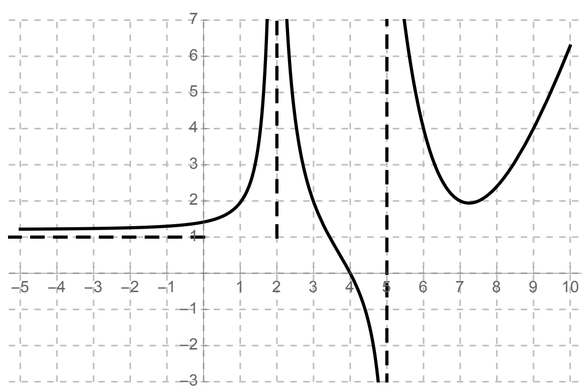
$$f(x) = \frac{(x-3)(x+4)(x-7)}{x(x-3)(x+1)}.$$

30. Given the graph of f below, evaluate each given limit.



a) $\lim_{x\to\infty} f(x)$
b) $\lim_{x\to-\infty} f(x)$
c) $\lim_{x\to-5^-} f(x)$
d) $\lim_{x\to-5^+} f(x)$
e) $\lim_{x\to-5} f(x)$
f) $\lim_{x\to1^+} f(x)$
g) $\lim_{x\to1^-} f(x)$
h) $\lim_{x\to1} f(x)$

31. Given the graph of g below, evaluate each given limit.



a) $\lim_{x\to\infty} g(x)$
b) $\lim_{x\to-\infty} g(x)$
c) $\lim_{x\to2^-} g(x)$
d) $\lim_{x\to2^+} g(x)$
e) $\lim_{x\to2} g(x)$
f) $\lim_{x\to5^+} g(x)$
g) $\lim_{x\to5^-} g(x)$
h) $\lim_{x\to5} g(x)$

32. Sketch a graph of a function f which has all of the following three properties (there are many possible correct answers):

- $\lim\limits_{x\to 1^+} f(x) = \infty$;
- $\lim\limits_{x\to 1^-} f(x) = -\infty$;
- $\lim\limits_{x\to \infty} f(x) = 3$.

33. Sketch a graph of a function f which has all of the following four properties (there are many possible correct answers):

- $\lim\limits_{x\to -4} f(x) = \infty$;
- $\lim\limits_{x\to \infty} f(x) = -2$;
- $\lim\limits_{x\to -\infty} f(x) = 5$;
- $f(0) = -3$.

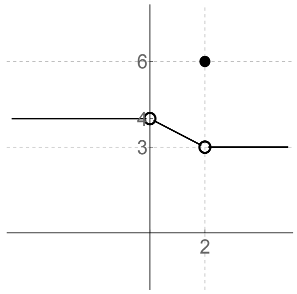
34. Sketch a graph of a function f which has all of the following four properties (there are many possible correct answers):

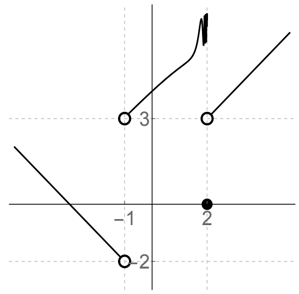
- $\lim\limits_{x\to 2^-} f(x) = 4$;
- $f(2) = -1$;
- $\lim\limits_{x\to 2^+} f(x) = \infty$;
- $\lim\limits_{x\to -\infty} f(x) = \lim\limits_{x\to \infty} f(x) = 2$.

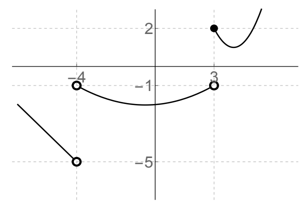
Answers

1. $\lim\limits_{x\to 3} \dfrac{x-3}{x^2-7x+12} = -1$
2. $\lim\limits_{x\to -2} \dfrac{\sqrt{2-x}-2}{x+2} = \dfrac{-1}{4}$
3. $\dfrac{-1}{2}$
4. $\dfrac{1}{2}$
5. $\lim\limits_{x\to 5} \dfrac{|x-5|}{x-5}$ DNE (the left- and right-hand limits are unequal)
6. a) 1 b) -3 c) -1 d) 1 e) DNE f) DNE g) 5 h) 1 i) DNE j) 1
7. a) -4 b) 2 c) -2 d) 3 e) about -2.5 f) 2 g) -3 h) DNE i) DNE j) -5
8. Many answers are possible; one solution is on the next page after # 10, at left.
9. Many answers are possible; one solution is on the next page after # 10, in the center.

10. Many answers are possible; one solution is below, at right.







11. $\lim_{x\to\infty} \dfrac{4x+3}{2x-1} = 2$

12. $\lim_{x\to\infty} \dfrac{-6x}{\sqrt{4x^2+5}} = -3$

13. 1

14. 0

15. 0

16. 1

17. $\dfrac{1}{2}$

18. 0

19. $\lim_{x\to 1^+} \dfrac{2+x}{1-x} = -\infty$

20. $\lim_{x\to 3^-} \dfrac{x^2+7}{x-3} = -\infty$

21. ∞

22. $-\infty$

23. ∞

24. $-\infty$

25. $-\infty$

26. ∞

27. HA: $y = 0$
VA: $x = 0, x = 1, x = 4.$

28. HA: $y = 2$
VA: $x = 5, x = -5.$

29. HA: $y = 1$
VA: $x = 0, x = -1.$

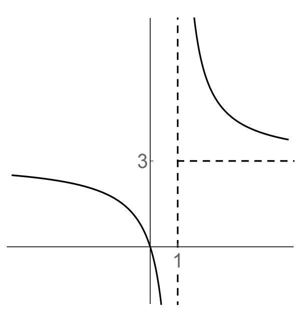
30. a) 3
b) -2
c) $-\infty$
d) ∞
e) DNE
f) $-\infty$
g) $-\infty$
h) $-\infty$

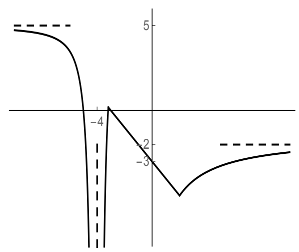
31. a) ∞
b) 1
c) ∞
d) ∞
e) ∞
f) ∞
g) $-\infty$
h) DNE

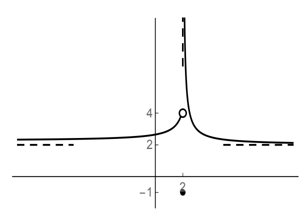
32. Many answers are possible; one solution is below at left.

33. Many answers are possible; one solution is below, in the center.

34. Many answers are possible; one solution is below, at right.





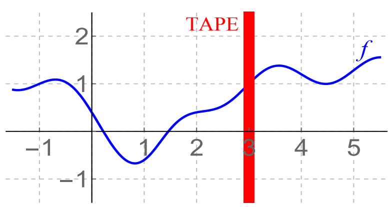


Chapter 3

Computing Limits

3.1 Continuity

Recall the modified warmup example from an earlier lecture:

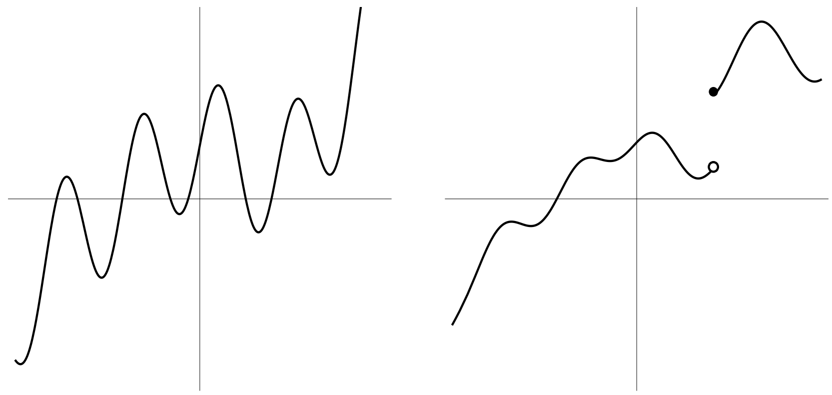


What is $f(3)$?

We don't know the answer, but we said that the most "reasonable" guess was 1.

Why was this the most "reasonable" guess?

Mathematically, this idea is described by the notion of *continuity*. For example:



Functions whose graphs have no breaks are called “continuous”:

Definition 3.1 *A function f is called* **continuous at the point** $x = a$ *if*

1. $f(a)$ *exists;*
2. $\lim_{x \to a} f(x)$ *exists (i.e.* $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$*); and*
3. $\lim_{x \to a} f(x) = f(a)$ *(i.e.* $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$*).*

Otherwise we say f is **discontinuous at** $x = a$*.*

The word continuous is abbreviated “cts”.

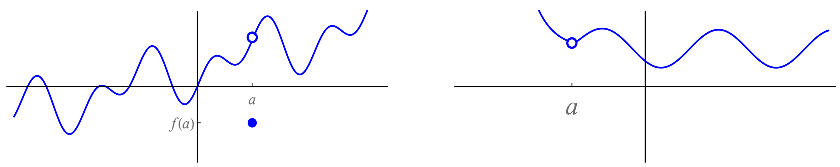
Definition 3.2 *A function f is called* **continuous on an interval** *if it is continuous at every point in that interval. A function f is called* **continuous** *if it is continuous at every point in its domain.*

Classification of discontinuities

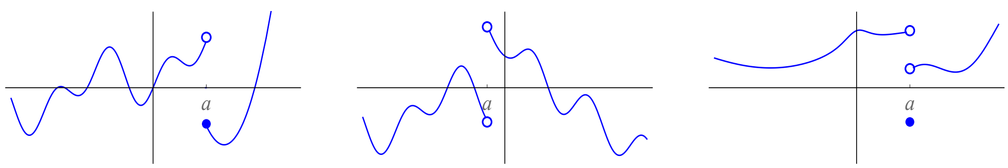
When looking at the graph of a function, its easier to tell where the function is *discontinuous* than where it is continuous, because the discontinuities in a function usually stand out.

It turns out that there are four types of discontinuities (it's not critical that you know this vocabulary):

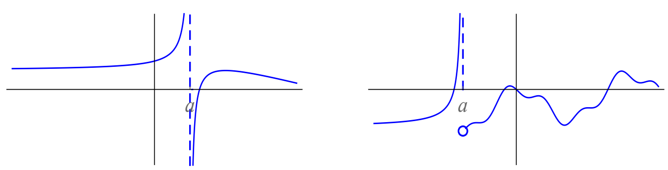
1. **removable discontinuity** (a.k.a. **hole discontinuity**): $\lim_{x \to a} f(x)$ exists but either $f(a)$ DNE or $f(a) \neq \lim_{x \to a} f(x)$:



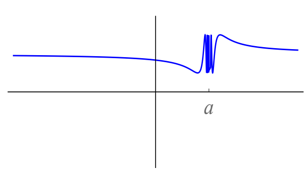
2. **jump discontinuity**: $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist but are not equal:



3. **infinite discontinuity**: $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x) = \pm\infty$:



4. **oscillating discontinuity**: $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ DNE because of too many wiggles:



Dictionary of continuous functions

What is important is that you have a working knowledge of functions which are continuous.

In particular, the following functions are continuous, because there are no breaks in their graphs:

> **Theorem 3.3** *Suppose f and g are continuous functions. Then:*
>
> 1. *$f+g$, $f-g$, fg, and $f \circ g$ are continuous; and*
> 2. *$\dfrac{f}{g}$ is continuous at all x where $g(x) \neq 0$.*

> **Theorem 3.4** *Any function made up of powers of x, sines and cosines, arcsines, arctangents, exponentials and/or logarithms using addition, subtraction, and/or multiplication is continuous (at every point of its domain).*

> **Theorem 3.5** *Any function which is the quotient of functions made up of powers of x, sines, cosines, arcsines, arctangents, exponentials and/or logarithms is continuous everywhere* ***except where the denominator is zero****.*

EXAMPLES

$$f(x) = 3\arcsin(x^2+4)\cos^5\left(\frac{3x}{x^2+4}\right) - 5e^{\sin(3x^8-5)}\ln(x^4+3)$$

is continuous everywhere on its domain.

$$g(x) = \frac{x^3 + 3\cos(2x^2-5) - 6^{x-4\sin\sqrt[3]{x}}}{x-3}$$

is continuous everywhere except $x = 3$.

3.2 Evaluation of limits: general concepts

First concept: limits behave "nicely" with respect to arithmetic

Theorem 3.6 (Main Limit Theorem) *Suppose* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *both exist and are finite, where* a *is either* $\pm\infty$ *or a finite number. Then:*

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$;
2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$;
3. $\lim_{x\to a} [f(x)g(x)] = \left[\lim_{x\to a} f(x)\right] \left[\lim_{x\to a} g(x)\right]$;
4. $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *provided the denominator is nonzero.*

Second concept: limits can be interchanged with many common operations

Theorem 3.7 (Interchange of Limit and Common Operations) *Suppose* $\lim_{x\to a} f(x)$ *exists and is finite. Then:*

1. $\lim_{x\to a} [k\ f(x)] = k \cdot \lim_{x\to a} f(x)$ *for any constant* k.
2. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$.
3. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, *provided both sides exist.*
4. $\lim_{x\to a} |f(x)| = \left|\lim_{x\to a} f(x)\right|$.
5. $\lim_{x\to a} e^{f(x)} = \exp\left(\lim_{x\to a} f(x)\right)$.
6. $\lim_{x\to a} \ln f(x) = \ln\left(\lim_{x\to a} f(x)\right)$.
7. $\lim_{x\to a} \sin f(x) = \sin\left(\lim_{x\to a} f(x)\right)$.
8. *Statements similar to (5), (6), (7) above hold for* $\cos$, $\arctan$, *and* $\arcsin$.

Third concept: evaluate limits of cts functions by plugging in

If f is continuous at a, then $\lim_{x \to a} f(x) = f(a)$.

Fourth concept: manipulate expressions with ∞ using rules

When evaluating limits, you often encounter expressions that work out to be $\pm\infty$. Although ∞ is not a number, you can sometimes work with ∞ as if it was a number (especially when computing limits).

Useful arithmetic rules with ∞

When computing a limit, and you encounter an expression of the form

$$\frac{3}{0} \text{ or } \frac{-5}{0} \text{ or } \frac{1}{0} \text{ or } \frac{\infty}{0} \text{ or anything else of the form } \frac{\text{nonzero}}{0},$$

that expression will evaluate to $\pm\infty$ (you need careful analysis to determine whether it is ∞ or $-\infty$).

Once you've encountered an expression with $\pm\infty$ in it, you can then continue evaluating the limit using the following rules (which I hope you think are intuitive):

Adding/subtracting a finite amount to $\pm\infty$ doesn't change it: For any $c \in \mathbb{R}$, $\infty \pm c = \infty$.

Multiplying/dividing $\pm\infty$ by positive constant doesn't change it: For any $c > 0$, $c \cdot \infty = \frac{\infty}{c} = \infty$. (This includes $\infty \cdot \infty = \infty$.)

Multiplying/dividing $\pm\infty$ by negative constant reverses it: For any $c < 0$, $c \cdot \infty = \frac{\infty}{c} = -\infty$. (This includes $-\infty \cdot \infty = -\infty$.)

Dividing a number by infinity gives 0: For any $c \in \mathbb{R}$, $\frac{c}{\infty} = 0$.

Natural exponentials and logs of ∞ are ∞: $e^{\infty} = \infty$ and $\ln \infty = \infty$.

Positive powers of ∞ are ∞; If $c > 0$, then $\infty^c = \infty$. (This includes $\sqrt{\infty} = \infty$ and $\sqrt[n]{\infty} = \infty$.)

Negative powers of ∞ are zero: If $c < 0$, then $\infty^c = 0$.

WARNING: Here are some expressions that we haven't covered with the rules on the previous page. They are called **indeterminate forms** because they work out to different things depending on the particular limit you are evaluating (we will see more on how to deal with these in Section 8.2).

$$\frac{0}{0} \text{ is indeterminate}$$
$$\frac{\infty}{\infty} \text{ is indeterminate}$$
$$\infty - \infty \text{ is indeterminate}$$
$$0 \cdot \infty \text{ is indeterminate}$$
$$\infty^0 \text{ is indeterminate}$$
$$0^0 \text{ is indeterminate}$$
$$1^\infty \text{ is indeterminate}$$

When you encounter an indeterminate form in a limit, that doesn't mean you are done–you have to do some work to figure out what the limit is.

EXAMPLE 1

Determine if the following expressions evaluate to anything meaningful, or whether they are indeterminate:

1. $\dfrac{12-8-4}{3+5-8}$

2. $\dfrac{6-4-2}{3+8-4}$

3. $\dfrac{10-5}{3-3}$

4. $e^{-\infty}$

5. $-5(\infty + 2^{\infty/2000})$

6. $-5(\infty - 2^{\infty/2000})$

7. $3 + \dfrac{4}{\ln \infty}$

3.3 Evaluating limits at infinity

The most important examples of limits to understand how to evaluate are those for which $x \to \infty$ (i.e. **limits at infinity**):

Limits at infinity of rational functions

EXAMPLE 1

$$\lim_{x\to\infty} \frac{4+3x^2}{8x^2+3x+2}$$

Remark: this example could have been phrased differently: suppose you were asked to find the horizontal asymptotes of $f(x) = \dfrac{4+3x^2}{8x^2+3x+2}$. In this case, you'd compute the limit as above, and identify the HA as

Rephrasing this as a story problem: Suppose the population of an endangered species in a national park at time x, in thousands, is given by $f(x) = \dfrac{4+3x^2}{8x^2+3x+2}$ (the function from Example 1). What is the long-term population of this species in this park projected to be?

EXAMPLE 2

$$\lim_{x\to\infty} \frac{-3-5x^2}{2x^4-x+\frac{5}{2}}$$

EXAMPLE 3

$$\lim_{x\to\infty} \frac{2x^2 + 7x - 2}{x - 1}$$

General principle behind examples 1-3: Suppose f is a rational function, i.e. has form

$$f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \ldots + a_2x^2 + a_1x + a_0}{b_n x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \ldots + b_2x^2 + b_1x + b_0}.$$

Then:

1. If $m < n$ (i.e. largest power in numerator $<$ largest power in denominator), then
$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0.$$

2. If $m > n$ (i.e. largest power in numerator $>$ largest power in denominator), then
$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \pm\infty.$$

3. If $m = n$ (i.e. largest powers in numerator and denominator are equal), then
$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \frac{a_m}{b_n}.$$

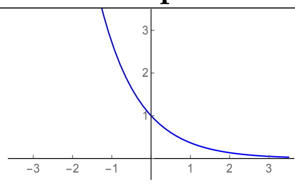
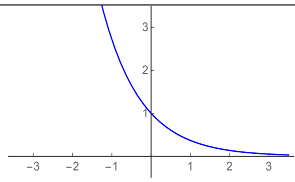
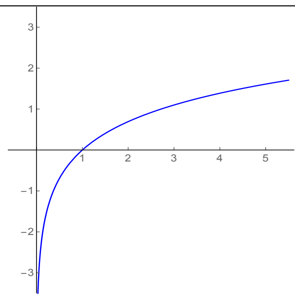
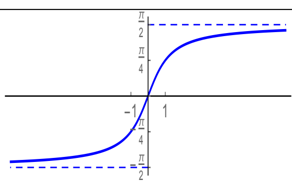
EXAMPLE 4

$$\lim_{x\to\infty} \frac{x}{\sqrt{2x^2+1}}$$

EXAMPLE 5

$$\lim_{x\to\infty} \cos\left(\frac{x}{3x^2+4}\right)$$

Limits at infinity based on graphs

Graph	Corresponding limit statement	Slang version
	$\begin{cases} \lim\limits_{x\to\infty} e^x = \infty \\ \lim\limits_{x\to-\infty} e^x = 0 \end{cases}$	$\begin{cases} e^{\infty} = \infty \\ e^{-\infty} = 0 \end{cases}$
	$\begin{cases} \lim\limits_{x\to\infty} e^{-x} = 0 \\ \lim\limits_{x\to\infty} e^{-x} = \infty \end{cases}$	$\begin{cases} e^{-\infty} = 0 \\ e^{-(-\infty)} = e^{\infty} = \infty \end{cases}$
	$\begin{cases} \lim\limits_{x\to\infty} \ln x = \infty \\ \lim\limits_{x\to0^+} \ln x = -\infty \end{cases}$	$\begin{cases} \ln\infty = \infty \\ \ln 0 = -\infty \end{cases}$
	$\begin{cases} \lim\limits_{x\to\infty} \arctan x = \dfrac{\pi}{2} \\ \lim\limits_{x\to-\infty} \arctan x = -\dfrac{\pi}{2} \end{cases}$	$\begin{cases} \arctan\infty = \dfrac{\pi}{2} \\ \arctan(-\infty) = -\dfrac{\pi}{2} \end{cases}$

EXAMPLE 6

$$\lim_{x\to\infty} \left(e^{-3x} + \arctan 2x\right)$$

EXAMPLE 7

$$\lim_{x\to\infty} \sin x$$

3.4 Evaluating limits not at infinity

Evaluation of limits of continuous functions

Key fact: If f is continuous at a, then $\lim_{x \to a} f(x) = f(a)$.

EXAMPLE 1

$$\lim_{x \to 3} \frac{x^2 + 3}{x - 1}$$

EXAMPLE 2

$$\lim_{x \to \frac{\pi}{2}} 3 \cos 2x$$

EXAMPLE 3

$$\lim_{x \to 0} e^{2x}$$

Evaluation of limits of functions which are not known to be continuous

Given limit $\lim_{x \to a} f(x)$, start by plugging in a to the function f.

1. if you get a number when you plug in, almost always this is the answer (and the function is actually continuous at a);
2. if you get $\frac{nonzero}{0}$, the limit is infinite; carefully analyze the sign of $f(x)$ to determine whether the answer is ∞ or $-\infty$;
3. if you get $\frac{0}{0}$, use an algebraic technique to rewrite f:
   a) if f can be factored, factor and cancel terms;
   b) if f contains square roots which are added or subtracted, multiply through by the conjugate (then factor and cancel);
   c) if f contains "fractions within fractions", clear the denominators of the interior fractions (then factor and cancel).

EXAMPLE 4

$$\lim_{x \to 5} \frac{x^2 - 3x - 10}{x^2 + x - 20}$$

EXAMPLE 5

$$\lim_{x \to 3} \frac{x^2 + 2x - 15}{x^2 - 7x + 12}$$

EXAMPLE 6

$$\lim_{x\to 2^+} \frac{4}{4-x^2}$$

EXAMPLE 7

$$\lim_{x\to 2^-} \frac{4}{4-x^2}$$

EXAMPLE 8

$$\lim_{x\to 2} \frac{4}{4-x^2}$$

Solution: From Examples 6 and 7, we see that

$$\lim_{x\to 2^+} \frac{4}{4-x^2} \neq \lim_{x\to 2^-} \frac{4}{4-x^2}.$$

Therefore the two-sided limit

$$\lim_{x\to 2} \frac{4}{4-x^2} \boxed{\text{DNE}}.$$

EXAMPLE 9

$$\lim_{x \to 3^+} \left(\frac{2}{(x-3)^2} + 2x^2 \right)$$

EXAMPLE 10

$$\lim_{x \to -2} \frac{x^2 - 3x - 10}{x^3 + 5x^2 + 6x}$$

EXAMPLE 11

$$\lim_{x \to 1/2} \frac{2x - 1}{2x^2 + x + 1}$$

More complicated examples

Key idea: if you get $\frac{0}{0}$ when you plug in, eventually you have to factor and cancel.

But in complicated situations, you first have to do some preliminary algebra to rewrite the function. Here are some worked-out examples which illustrate some techniques:

EXAMPLE 11

$$\lim_{x \to -1} \frac{\frac{1}{x} + 1}{\frac{1}{x+2} - 1}$$

Solution: When I look at this, I see "fractions inside fractions". In such a problem, here is the procedure:

Multiply through the top and bottom of the "big fraction" by the "small denominators".

In this example, the "small denominators" are x and $x+2$, and the "big fraction" is the entire function $\frac{\frac{1}{x}+1}{\frac{1}{x+2}-1}$. So you get

$$\lim_{x \to -1} \frac{\frac{1}{x} + 1}{\frac{1}{x+2} - 1} = \lim_{x \to -1} \frac{\left(\frac{1}{x} + 1\right)(x)(x+2)}{\left(\frac{1}{x+2} - 1\right)(x)(x+2)}$$

Distribute over the red $+$ and $-$ signs:

$$= \lim_{x \to -1} \frac{\frac{1}{x}(x)(x+2) + 1(x)(x+2)}{\frac{1}{x+2}(x)(x+2) - 1(x)(x+2)}$$

$$= \lim_{x \to -1} \frac{x + 2 + x(x+2)}{x - x(x+2)}$$

$$= \lim_{x \to -1} \frac{x + 2 + x^2 + 2x}{x - x^2 - 2x}$$

$$= \lim_{x \to -1} \frac{x^2 + 3x + 2}{-x^2 - x}$$

Now factor and cancel:

$$= \lim_{x \to -1} \frac{(x+2)(x+1)}{-x(x+1)}$$

$$= \lim_{x \to -1} \frac{x+2}{-x} = \frac{-1+2}{-(-1)} = \boxed{1}.$$

EXAMPLE 12

$$\lim_{t\to 4} \frac{\sqrt{t}-2}{t-4}$$

Solution: I see a square root term plus/minus another term in the numerator of the fraction. In such a situation, here is the procedure:

Multiply through the top and bottom by the "conjugate" of the square root term.

In this problem, the "conjugate" of $\sqrt{t}-2$ is $\sqrt{t}+2$ (see below for more on conjugates). So you get

$$\lim_{t\to 4} \frac{\sqrt{t}-2}{t-4} = \lim_{t\to 4} \frac{\left(\sqrt{t}-2\right)\left(\sqrt{t}+2\right)}{(t-4)\left(\sqrt{t}+2\right)}$$

Now notice the numerator is of the form $(A-B)(A+B)$, which becomes A^2-B^2.
Don't multiply out the bottom.

$$= \lim_{t\to 4} \frac{t-4}{(t-4)\left(\sqrt{t}+2\right)}$$

$$= \lim_{t\to 4} \frac{1}{\sqrt{t}+2} = \frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}.$$

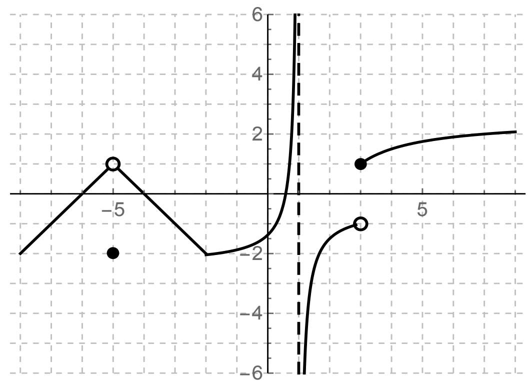
How to find conjugates:

Expression	Conjugate	Example	Conjugate of example
$\square + \sqrt{\triangle}$	$\square - \sqrt{\triangle}$	$3+\sqrt{x-1}$	$3-\sqrt{x-1}$
$\square - \sqrt{\triangle}$	$\square + \sqrt{\triangle}$	$5-\sqrt{2x}$	$5+\sqrt{2x}$
$\sqrt{\square} + \sqrt{\triangle}$	$\sqrt{\square} - \sqrt{\triangle}$	$\sqrt{t+3}+\sqrt{x-1}$	$\sqrt{t+3}-\sqrt{x+1}$
$\sqrt{\square} - \sqrt{\triangle}$	$\sqrt{\square} + \sqrt{\triangle}$	$\sqrt{u}-\sqrt{3x}$	$\sqrt{u}+\sqrt{3x}$

3.5 Homework exercises

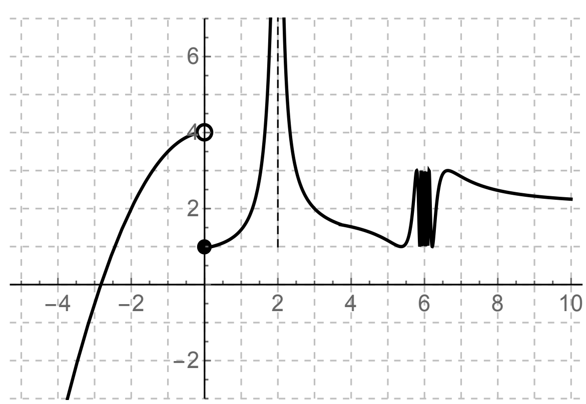
Exercises from Section 3.1

1. Consider the function f whose graph is given below:



a) At what value(s) of x, if any, is f **not** continuous?
b) At what value(s) of x, if any, does f have a removable discontinuity?
c) At what value(s) of x, if any, does f have a jump discontinuity?
d) At what value(s) of x, if any, does f have an infinite discontinuity?
e) At what value(s) of x, if any, does f have an oscillating discontinuity?

2. Consider the function g whose graph is given below:



a) At what value(s) of x, if any, does g have a removable discontinuity?
b) At what value(s) of x, if any, does g have a jump discontinuity?
c) At what value(s) of x, if any, does g have an infinite discontinuity?
d) At what value(s) of x, if any, does g have an oscillating discontinuity?

Exercises from Section 3.3

In Problems 3-14, evaluate the given limit (algebraically, by hand). If the limit does not exist, say so.

3. $\lim_{x\to\infty} \frac{x^2+3}{x^3-2}$

4. $\lim_{x\to\infty} \frac{x^2+3}{x^2-2}$

5. $\lim_{x\to\infty} \frac{x^2+3}{x-2}$

6. $\lim_{x\to\infty} \frac{3-2x^2+x}{4x(x-1)}$

7. $\lim_{x\to\infty} \frac{\sqrt{x}}{x+1}$

8. $\lim_{x\to\infty} \frac{x+1}{\sqrt{4x^2-x}}$

9. $\lim_{x\to-\infty} \frac{7x^3}{x^3+1}$

10. $\lim_{x\to\infty} \ln(4x+1)$

11. $\lim_{x\to\infty} 8\arctan x^2$

12. $\lim_{x\to\infty} \frac{4}{e^x+x}$

13. $\lim_{x\to\infty} e^{4x-5}$

14. $\lim_{x\to\infty} e^{-x^2}$

15. Suppose that the population of emperor penguins (in thousands of penguins) in Antarctica at time t (in years) is given by the function $p(t) = \frac{350}{1+\frac{3}{4}e^{-t/35}}$. Estimate the long-term population of emperor penguins in Antarctica.

16. After taking a certain antibiotic, the concentration C (in ppm) of a drug in a patient's bloodstream is given by $C(t) = \frac{t}{40t^2-80}$ where t (in hours) is the time after taking the antibiotic. What is the long-term concentration of the drug in the patient's bloodstream? (Write your answer with correct units.)

17. If you are r km from the center of a black hole, general relativity theory suggests that the velocity of a light wave at your position is given by $v(r) = \frac{300000r-7800000}{r}$ km/sec. If you are very, very far away from the black hole, what is the velocity of a light wave at your position? (Write your answer with correct units.)

Exercises from Section 3.4

In Problems 18-47, evaluate the given limit (algebraically, by hand). If the limit does not exist, say so.

18. $\lim_{x\to 2^-} \frac{x-3}{x-2}$

19. $\lim_{x\to 5^+} \frac{x^2}{x^2-25}$

20. $\lim_{x\to -2^+} \frac{x+3}{x^2+x-2}$

21. $\lim_{x\to 4} \frac{x+2}{(x-4)^2}$

22. $\lim_{x\to 0^-} \left(x^2-\frac{1}{x}\right)$

23. $\lim_{x\to 0^+} \frac{x+1}{e^x-1}$

24. $\lim_{x\to 0^+} \frac{3}{\sin x}$

25. $\lim_{x\to 0^-} \frac{3}{\sin x}$

26. $\lim_{x\to 4} \frac{x+2}{x-4}$

27. $\lim_{x\to 3^+} \ln(x-3)$

28. $\lim_{x\to -2} \left(x^2-4x\right)$

29. $\lim_{x\to 3} \frac{x+5}{x^2-1}$

30. $\lim_{x\to 0} e^{-x}$

31. $\lim_{x\to 5} \sqrt[3]{x+3}$

32. $\lim_{x\to \pi} \tan\left(\frac{x}{3}\right)$

33. $\lim_{x\to -3} \sin \pi x$

34. $\lim_{x\to e^2} \ln x^2$

35. $\lim_{x\to 5^+} \frac{x}{x^2-5}$

36. $\lim_{x\to -1} \arctan x$

37. $\lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x+3}$

38. $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} x+2 & x<2 \\ x^2 & x\geq 2 \end{cases}$ *Hint:* Consider the left- and right-hand limits at $x=2$ separately.

39. $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} 2x+1 & x<2 \\ 8 & x=2 \\ x^2-1 & x>2 \end{cases}$ *Hint:* Consider the left- and right-hand limits at $x=2$ separately.

40. $\lim_{x\to -2} \frac{x^2-3x-10}{x^2+5x+6}$

41. $\lim_{x\to 4} \frac{x^2-16}{x^2+x-20}$

42. $\lim_{x\to 1} \frac{x^3-3x^2+2x}{x-1}$

43. $\lim_{x\to 0} \frac{x^3+2x^2+x}{x^2-x}$

44. $\lim_{x\to 3} \frac{x^2-x-6}{2x^2-7x+3}$

45. $\lim_{x\to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-2}$

    Hint: Use the method of Example 16.

46. $\lim_{x\to 0} \frac{\sqrt{x+7}-\sqrt{7}}{x}$

    Hint: Use the method of Example 17.

47. $\lim_{x\to 1} \frac{1-x}{\sqrt{x+3}-2}$

    Hint: Use the method of Example 17.

In Problems 48-51, find the equations of all horizontal and vertical asymptotes of the given function.

Hint: for the VA, you need to find values of c for which $\lim_{x\to c^\pm} f(x) = \pm\infty$. This means that when you evaluate the limit, you need to get $\frac{nonzero}{0}$.

48. $f(x) = \dfrac{3-x}{x+2}$

49. $f(x) = \dfrac{x^2-4}{x+1}$

50. $f(x) = \dfrac{x+10}{x^2-8x+15}$

51. $f(x) = \dfrac{2x^2-8x+10}{x^2-11x+30}$

Answers

1. a) $x = -5$, $x = 1$, $x = 3$
   b) $x = -5$
   c) $x = 3$
   d) $x = 1$
   e) no such x
2. a) no such x
   b) $x = 0$
   c) $x = 2$
   d) $x = 6$
3. 0
4. 1
5. ∞
6. $\frac{-1}{2}$
7. 0
8. $\frac{1}{2}$
9. 7
10. ∞
11. 4π
12. 0
13. ∞
14. 0
15. 350000 penguins
16. 0 ppm
17. 300000 km/sec
18. ∞
19. ∞
20. $-\infty$
21. ∞
22. ∞
23. ∞
24. ∞
25. $-\infty$
26. DNE
27. $-\infty$
28. 12
29. 1
30. 1
31. 2
32. $\sqrt{3}$
33. 0
34. 4
35. $\frac{1}{4}$
36. $\frac{-\pi}{4}$
37. 0
38. 4
39. DNE
40. -7
41. $\frac{8}{9}$
42. -1
43. -1
44. 1
45. $\frac{-1}{4}$
46. $\frac{1}{2\sqrt{7}}$
47. -4
48. HA: $y = -1$;
    VA: $x = -2$
49. HA: none;
    VA: $x = -1$
50. HA: $y = 0$;
    VA: $x = 3, x = 5$
51. HA: $y = 2$;
    VA: $x = 6$ (not $x = 5$)

Chapter 4

Introduction to Derivatives

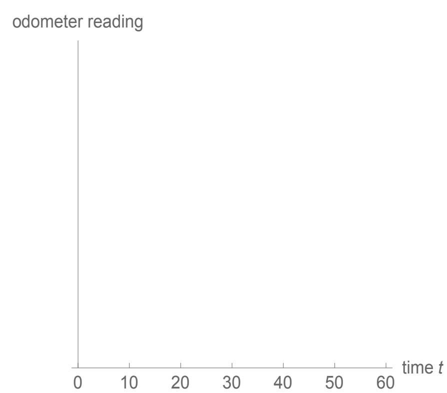
4.1 Odometers and speedometers

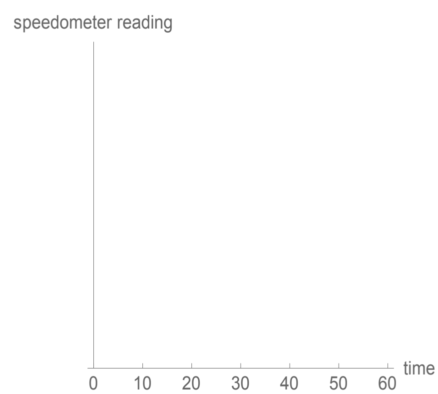
Suppose you get in your car and drive to Grand Rapids. There are two ways to record your motion as a function of elapsed time t:

1.

2.

As an example, here are two graphs representing the same trip:





Essentially, Calculus 1 centers on the conversion from one of these pictures to the other. In particular, we want to know:

1.

2.

In Chapters 4-8, we focus on the first question above and its other applications. We will turn to the second question in Chapter 9.

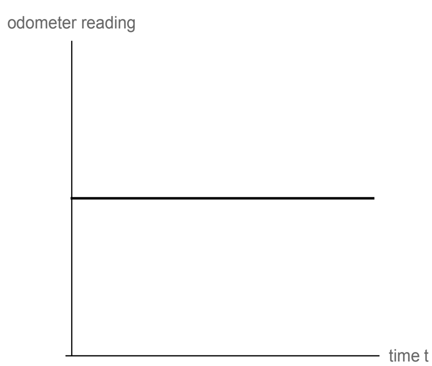
First major problem of calculus

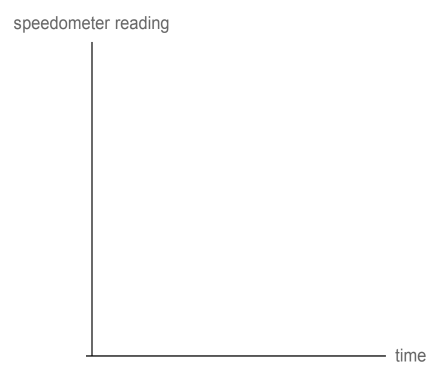
Given a function $f = f(t)$ which represents the position of an object at time t, compute the object's instantaneous velocity at time t.

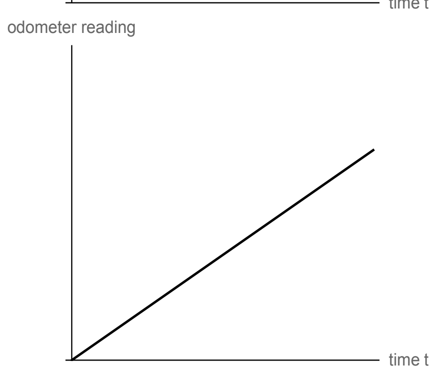
Motivation: Given the graph of a position function (i.e. a function which represents an odometer), what attribute(s) of that graph are relevant to understanding the velocity (i.e. speedometer)?

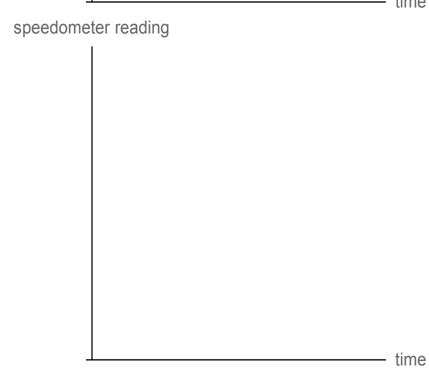
EXAMPLES

Here, you are given a series of pictures which represent odometers (that is, the $x-$axis represents time and the $y-$axis represents an odometer reading). On the blank graph to the right, sketch the graph of the corresponding **speedometer** (that is, the graph of a function where x represents elapsed time and y represents the velocity at time x).

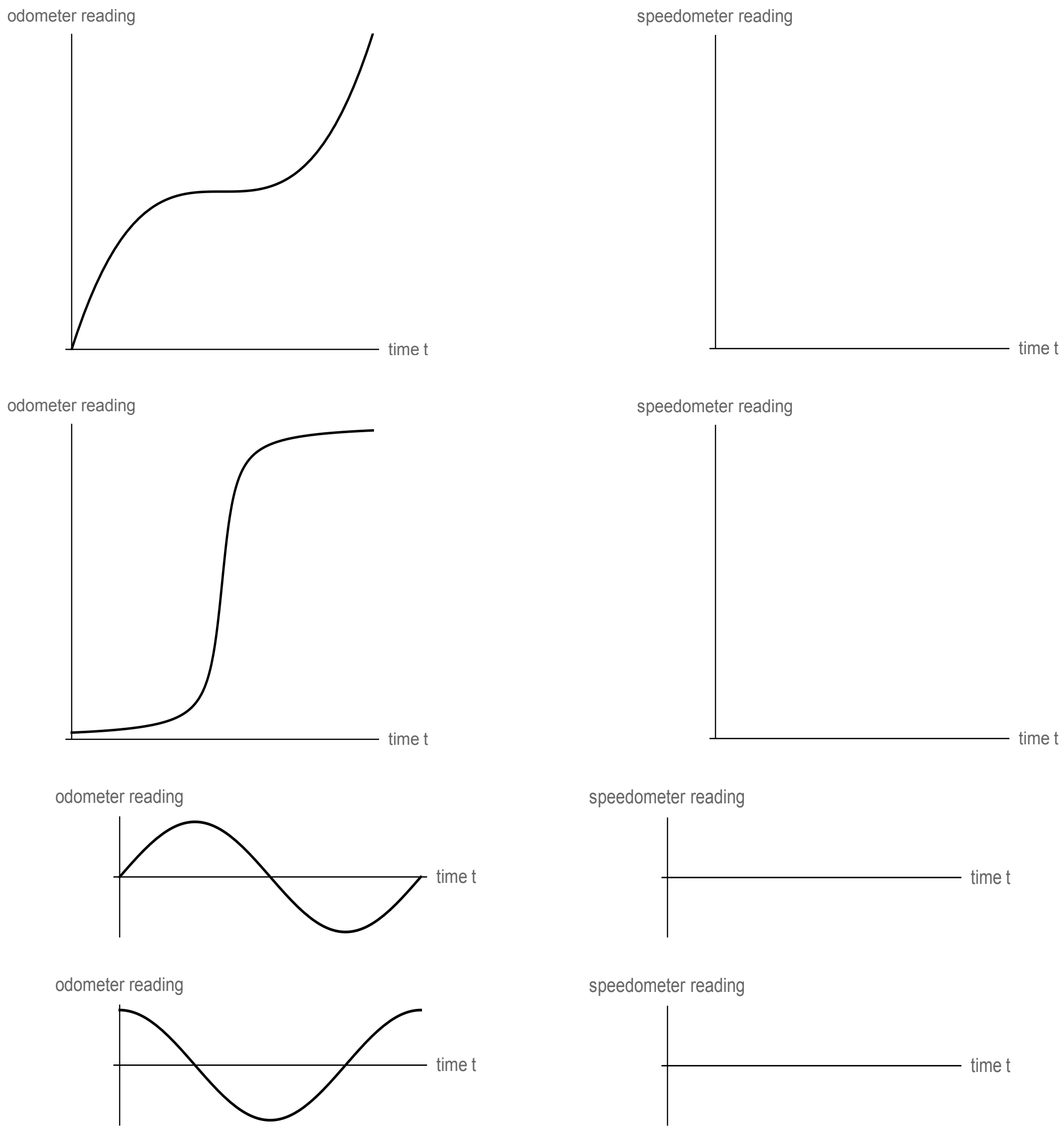








(four more graphs on the next page)



Punchline: Given a function f which measures distance traveled at time t, the corresponding velocity at time t is the slope or steepness of the graph of f at time t.

But what is meant by "slope"? We know how to find the slope of a line (from high-school algebra), but what is meant by the "slope" of a curve?

Big Ideas used to address these questions:

Tangent lines and differentiability

Definition 4.1 *Given a function f and a number x in the domain of f, the* **tangent line to f at x** *(if it exists) is the line which most closely approximates the graph of f at points very near x.*

QUESTION 1: What is meant by "most closely" approximating the graph of f? What makes one line a "better" approximation than another?

QUESTION 1 (A): Is it possible for a function f to have more than one tangent line at x?

QUESTION 2: What does it mean (conceptually) for the tangent line to f to exist at x? Why might a tangent line not exist at x?

Definition 4.2 *A function is called* **differentiable** *at x if it has a tangent line at x.*

Theorem 4.3 (Differentiability implies continuity) *Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x. Then f must be continuous at x.*

Theorem 4.4 *A function f fails to be differentiable at x if:*

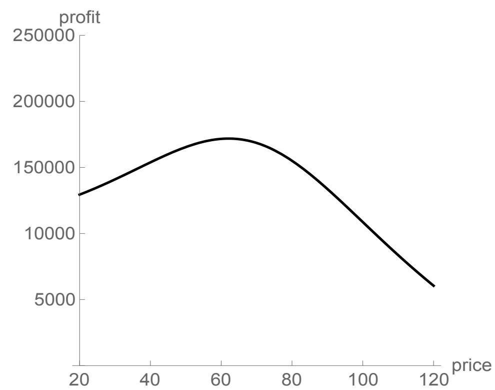
1. *f is not continuous at x; or*
2. *the tangent line to f at x is vertical; or*
3. *the graph of f has a corner or cusp at x.*

Second major problem of calculus

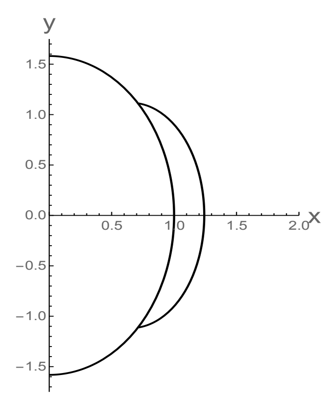
Given a function f and a particular number x
(sometimes I'll use a for the value of x),
find (if possible) the slope of the line tangent to f at x.

Why else might we care about finding the slope of a tangent line to a graph?

Business / economics:



Optometry:



(There will be other reasons coming later.)

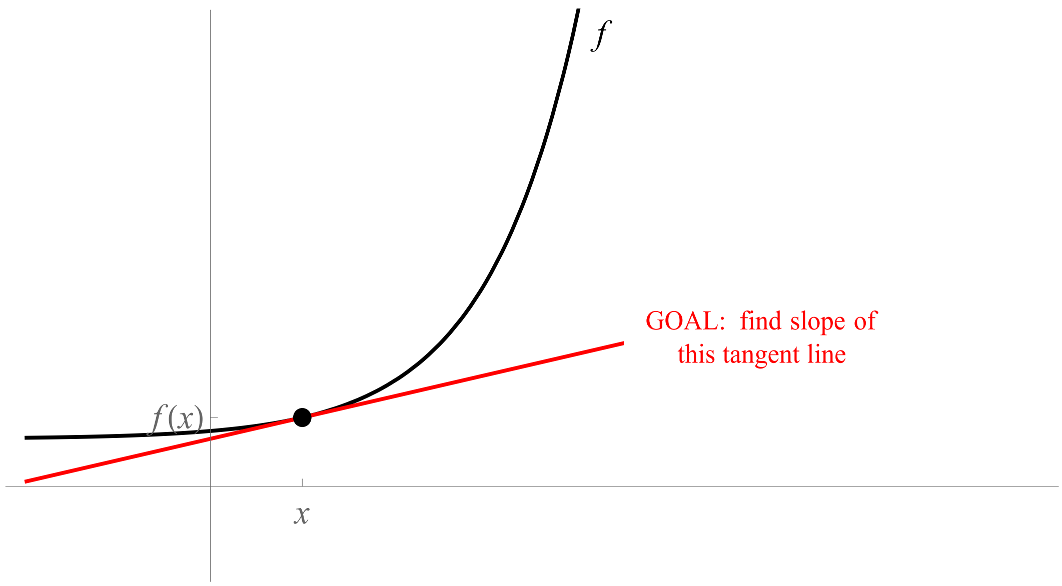
Can we find the slope of a tangent line to a graph using just algebra?

4.2 Definition of the derivative

RECALL

The second major problem of calculus is to find the slope of the line tangent to f at x.

Let's try to solve this problem theoretically, thinking of the following picture:



So we **define** the slope of the red tangent line as

Back to the first major problem
(find instantaneous velocity given position function)

An object's *average* velocity over some interval of time is given by

$$v_{avg} = \frac{\triangle\,\text{output}}{\triangle\,\text{input}} = \frac{\text{change in object's position}}{\text{elapsed time}}.$$

Therefore if the object's position at time t is given by $f(t)$, then the object's average velocity between times t_1 and t_2 is

$$v_{[t_1,t_2]} =$$

So the object's velocity over the time interval $[x, x+h]$ is

$$v_{[x,x+h]} =$$

and its instantaneous velocity at time x should therefore be

We have seen that the formula

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

solves both of the two major problems of calculus posed earlier in this chapter. This motivates the following definition:

Definition 4.5 (Limit definition of the derivative) *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let x be in the domain of f. If the limit*

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists and is finite, say that f is **differentiable at** *x. In this case, we call the value of this limit* **the derivative of** *f and denote it by $f'(x)$ or $\frac{df}{dx}$ or $\frac{dy}{dx}$.*

The word "differentiable" is abbreviated "diffble".

Some algebraic manipulation of the derivative formula:

Theorem 4.6 (Alternate limit definition of the derivative) *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let f be differentiable at x. Then*

$$f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}.$$

Notation and verbiage

- "derivative" is a noun. The verb form of this noun is "differentiate", i.e. to "differentiate" a function means to compute the derivative of that function.
- Given a function f and a particular value of x (say 4), the derivative of f at $x = 4$ is denoted

  These denote a **number**, which is the slope of the line tangent to f at $x = 4$.
- The fractional notation with "d"s above is called Leibniz notation.

The derivative as an operator

We can also think of the derivative as a **function**. But it is a different kind of function than the ones you are used to. You are used to functions like $f(x) = x^2$, where

The derivative is a new kind of function. Its inputs and outputs aren't numbers; they are *functions*. This makes differentiation into something called an *operator*:

> **Definition 4.7** *An* **operator** *is a function whose inputs and outputs are themselves functions.*

When thought of as an operator, the operation of differentiation is usually denoted $\frac{d}{dx}$ or D or just $'$. In particular,

$$\frac{d}{dx}\text{(blah)} = \text{derivative of (blah)}$$
$$\text{(blah)}' = \text{derivative of (blah)}$$

The output of the differentiation operator is itself a function, which we denote by

The function f' takes input x (a number) and produces as its output $f'(x)$ the slope of the line tangent to f at x.

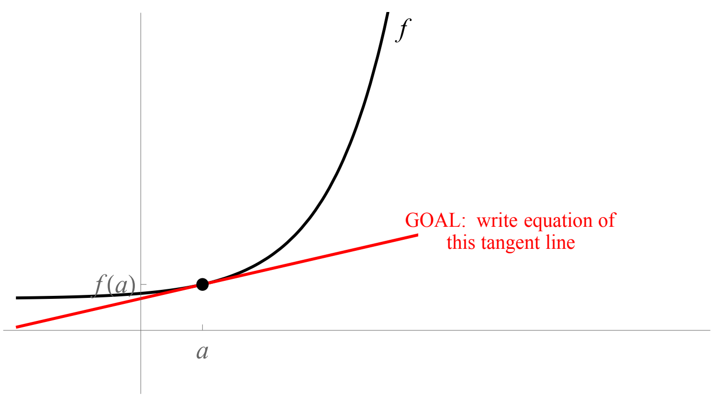
At this point, we know that the derivative is used to compute the following quantities:

1. $f'(x)$ gives the slope of the tangent line to f at the value x;
2. $f'(x)$ gives the slope of the curve f at the value x;
3. $f'(t)$ gives the instantaneous velocity of an object at time t, given that the object's position at time t is $f(t)$;
4. $f'(x)$ gives the instantaneous rate of change of $y = f(x)$ with respect to x.

Units: If $y = f(x)$ is measured in some unit U_y and x is measured in some unit U_x, then the units of $f'(x)$ are U_y/U_x. For example, if y is measured in lbs and x is measured in ft, then $f'(x)$ will be measured in lbs/ft.

QUESTION

What is the equation of the line tangent to differentiable function f at the point where $x = a$ (a is a constant)?



We will return to this formula many times, so it is good to remember it:

Theorem 4.8 (Tangent line equation) *Suppose f is differentiable at a. Then the equation of the line tangent to f at $x = a$ is*

$$y = f(a) + f'(a)(x - a).$$

EXAMPLE 1

Use the definition of derivative to compute the slope of the line tangent to $f(x) = \sqrt{x}$ at the point $(9, 3)$.

EXAMPLE 2

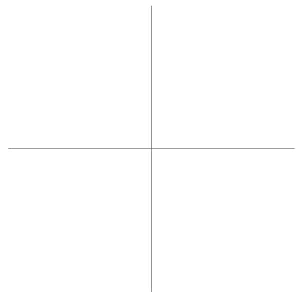
Use the definition of derivative to compute the instantaneous velocity of an object at time 4, given that the object's position (in m) at time t (in sec) is given by $f(t) = t^2 - t$.

Solution: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

EXAMPLE 3

Let $f(x) = |x|$. Find $f'(0)$.

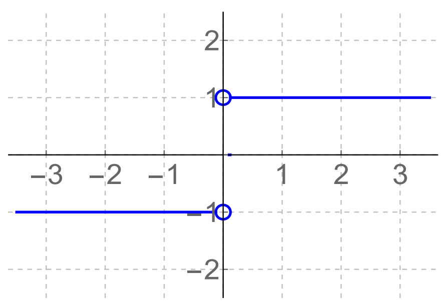
Conceptual solution: Sketch the graph of f:



Justification of this: Again, use the definition:

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| - 0}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

We studied the signum function $\dfrac{|h|}{h}$ back in Chapter 2; here is its graph:



4.3 Estimating derivatives using tables or graphs

EXAMPLE 1

A straight piece of wire is placed over a heat source, so that at various points on the wire, the temperature of the wire is different. Here is a table which gives some temperature measurements at various points on the wire:

x (cm from left end of wire)	$T(x)$ (degrees Fahrenheit)
0	76
6	94
10	110
12	102
16	85

1. Use the information in this table to estimate $T(8)$. Show the computations that lead to your answer, and write your answer with correct units.

2. What does your answer to Question 1 mean, in the context of this problem?

3. Use the information in this table to estimate $T'(8)$. Show the computations that lead to your answer, and write your answer with correct units.

4. What does your answer to Question 3 mean, in the context of this problem?

EXAMPLE 2

During a flight, an airplane crew takes periodic measurements of the distance they have travelled and the amount of fuel left in their fuel tank. Their results are described in the following chart:

t (minutes after takeoff)	0	30	60	120	200	240
x (miles travelled)	0	170	405	945	1595	1775
f (thousands of gallons)	18	14	12	7	5	1.5

1. Use the information in this table to estimate $\left.\dfrac{dx}{dt}\right|_{t=90}$. Show the computations that lead to your answer, and write your answer with correct units.

2. What does your answer to Question 1 mean, in the context of this problem?

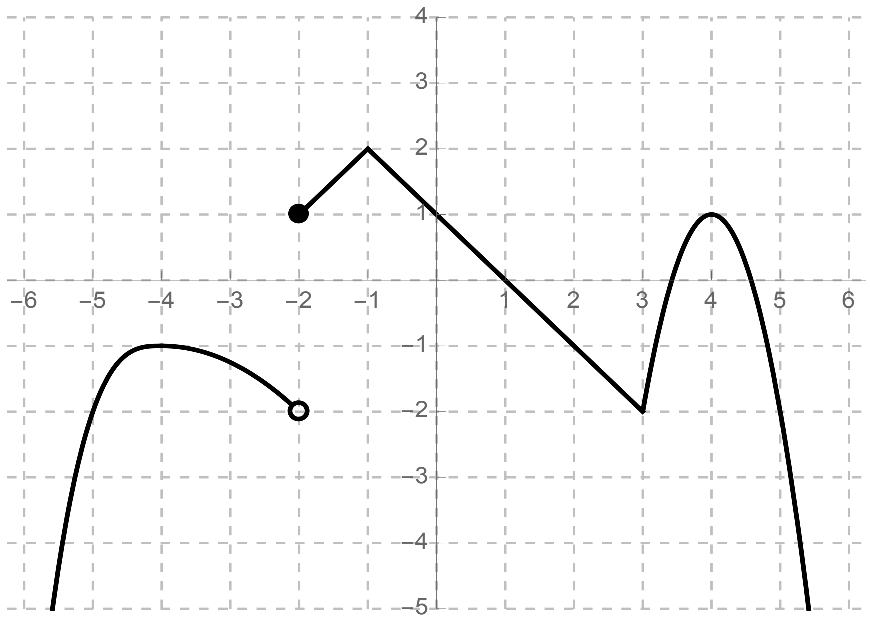
3. Use the information in this table to estimate $\left.\dfrac{df}{dt}\right|_{t=220}$. Show the computations that lead to your answer, and write your answer with correct units.

4. What does your answer to Question 1 mean, in the context of this problem?

5. What is the rate of fuel consumption of this aircraft per mile travelled, when the aircraft is at cruising speed? Show the computations that lead to your answer, and write your answer with correct units.

EXAMPLE 3

Given below is the graph of some unknown function f:

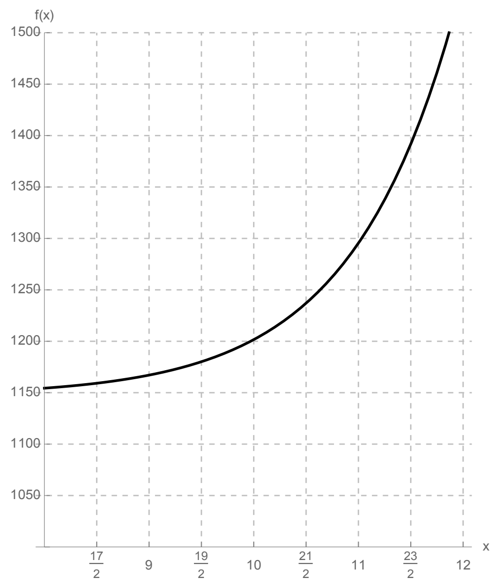


Use this graph to answer the questions below:

1. Give the values of x at which f is not continuous.
2. Give the values of x at which f is not differentiable.
3. Estimate $f(1)$.
4. Estimate $f'(1)$.
5. Estimate $f'(-5)$.
6. Find two values of x for which $f'(x) = 0$.
7. Estimate $\left.\dfrac{df}{dx}\right|_{x=5}$

EXAMPLE 4 (TRICKIER)

The graph of some unknown function f is given below.

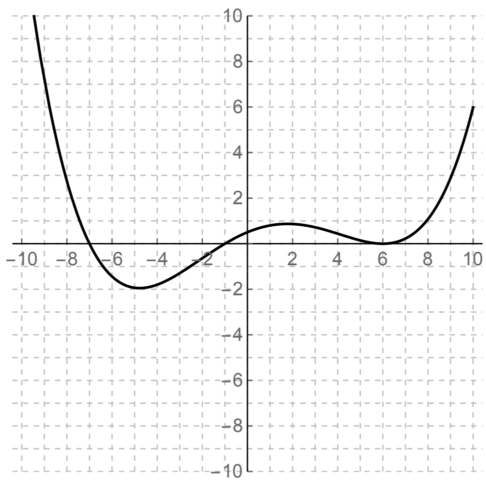


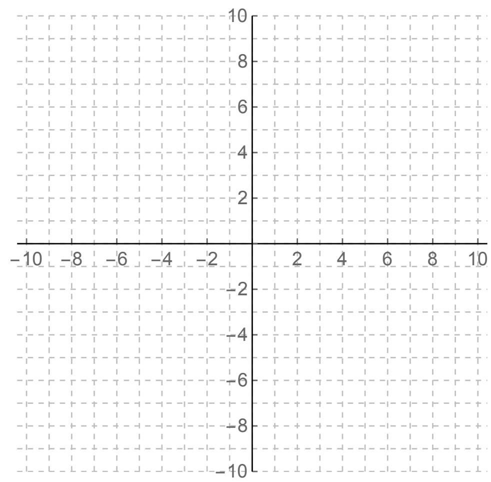
1. Use this graph to estimate $f'(10)$.

2. Use your estimate from Question 1 to write the equation of the line tangent to f at $x = 10$.

EXAMPLE 5

The graph of some unknown function f is given below at left. On the right-hand axes, sketch the graph of f'.



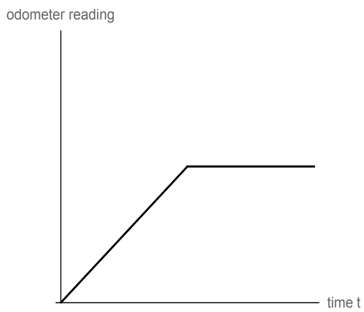


4.4 Homework exercises

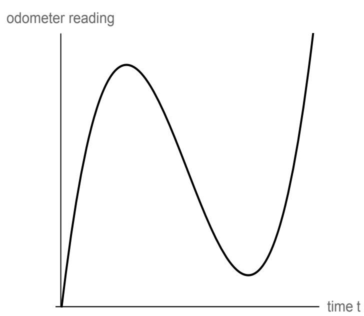
Exercises from Section 4.1

In Problems 1-4, you are given the graph of an odometer. Sketch the graph of the corresponding speedometer.

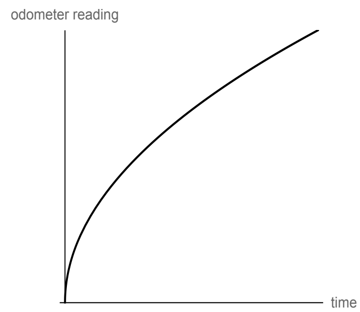
1.



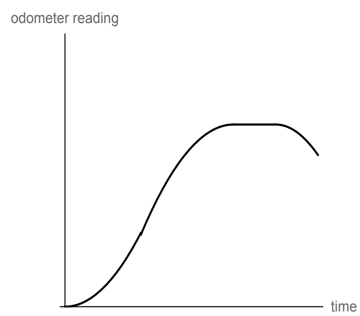
2.



3.



4.



Exercises from Section 4.2

In Problems 5-10, you must compute all derivatives using the definition of derivative (do not use any "rules" you may know if you have already taken calculus).

5. Let $f(x) = 4 - \frac{2}{3}x$. Find $f'(x)$.

6. Find the derivative of $f(x) = \dfrac{1}{x+3}$.

7. Compute $\dfrac{dy}{dx}$ if $y = \sqrt{3x-2}$.

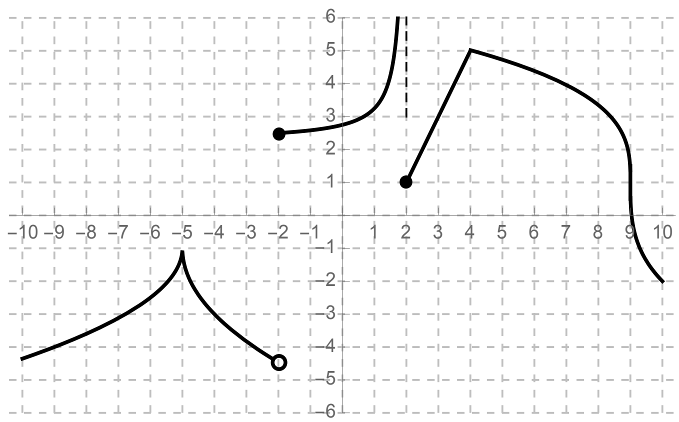
8. Find the equation of the line tangent to the function $f(x) = x^3+1$ when $x = 1$.

9. Suppose that the power supplied to a machine (in kilowatts) at time t (in hours) is $P = \sqrt{t}$. Find the instantaneous rate of change in the power supplied to the machine at time 4; write your answer with correct units.

10. Find the instantaneous velocity of an object at time 6, given that the object's position (in miles) at time t (in hours) is $f(t) = 2t^2 + 3t - 1$; write your answer with correct units.

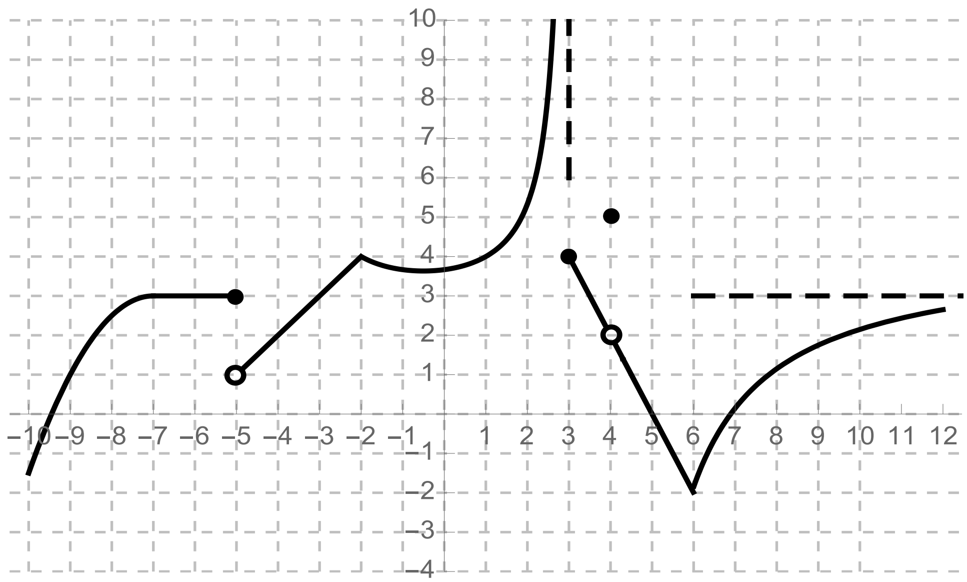
11. Use *Mathematica* to sketch a graph of the function $f(x) = |3x^2 - 15x + 12|$; use this graph to determine the values of x at which f is not differentiable.

12. Given the following graph of function f, give all the values x at which f is not differentiable:



Exercises from Section 4.3

13. Use the graph of the function f given below to answer the following questions:



a) Estimate $f(-6)$.

b) Estimate $f'(-6)$.

c) Estimate a value of x between -3 and 5 for which $f'(x) = 0$.

d) Find all values of x at which f is not continuous.

e) Find all values of x at which f is not differentiable.

f) Estimate $\left.\frac{df}{dx}\right|_{x=-3}$.

g) Is $f'(2)$ positive, negative or zero? Explain.

h) Estimate $\left.\frac{dy}{dx}\right|_{x=5}$.

i) Estimate $\lim_{x\to\infty} f(x)$.

j) Estimate $\lim_{x\to\infty} f'(x)$.

k) Find the slope of the function f when $x = -3$.

l) Find the equation of the line tangent to f when $x = -3$.

m) Find the equation of the line tangent to f when $x = 8$.

n) On the graph above, sketch the graph of the tangent line to x when $x = 2$.

14. A botanist measures the height, in inches, of a plant each day after it sprouts. Her data is gathered in the following table:

t (days)	0	1	3	4	8	10
h (height in inches)	0	2	8	9	10	10

a) Use the given data to estimate $h'(6)$. Show the computations that lead to your answer, and write your answer with correct units.

b) What does your answer to part (a) mean, in the context of this problem?

c) Use the given data to estimate $h'(1)$. Show the computations that lead to your answer, and write your answer with correct units.

d) What does your answer to part (c) mean, in the context of this problem?

15. As time passes, a scientist records the temperature and pressure of a gas inside a chamber as the chamber is heated. His data is summarized in the

following table:

time t (minutes after start of experiment)	0	1	2	4	5	6	8
pressure P (pressure in kPa)	696	764	818	891	916	935	963
temperature T (° C)	20	48	71	102	112	120	132

a) Use the given data to estimate $\left.\frac{dP}{dt}\right|_{t=1}$. Show the computations that lead to your answer, and write your answer with correct units.

b) What does your answer to part (a) mean, in the context of this problem?

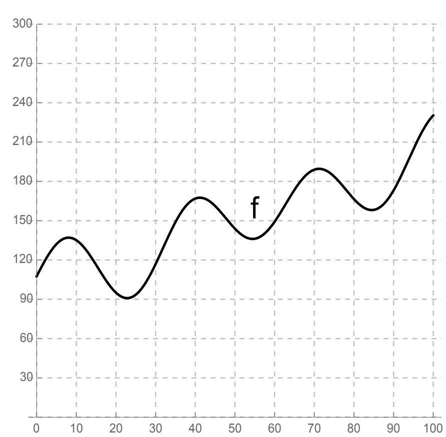
c) Use the given data to estimate $\left.\frac{dT}{dt}\right|_{t=2}$. Show the computations that lead to your answer, and write your answer with correct units.

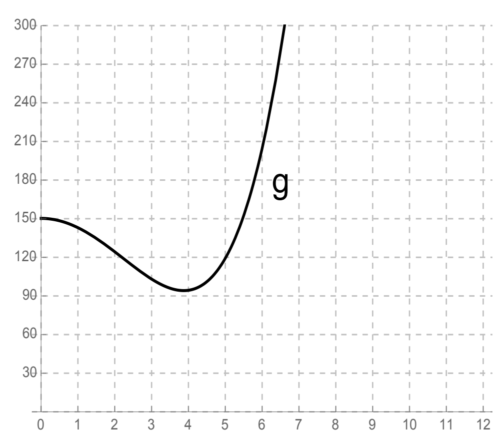
d) What does your answer to part (c) mean, in the context of this problem?

e) Use the given data to estimate the rate of change in temperature with respect to time when $t = 3$. Show the computations that lead to your answer, and write your answer with correct units.

f) Use the given data to estimate the rate of change in temperature with respect to the change in pressure when $t = 5$. Show the computations that lead to your answer, and write your answer with correct units.

16. Given the graph of f below at left, estimate $f'(30)$ and $f'(80)$.





17. Given the graph of g above at right:

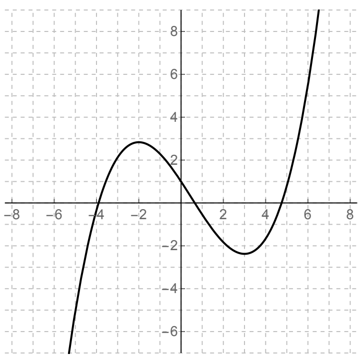
a) Estimate $g'(5)$.

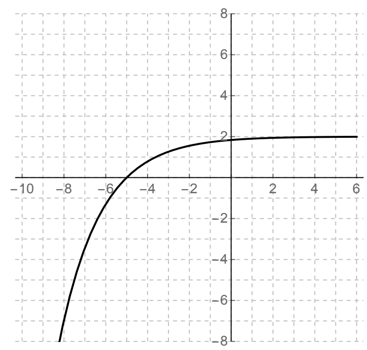
b) Write the equation of the line tangent to g when $x = 5$.

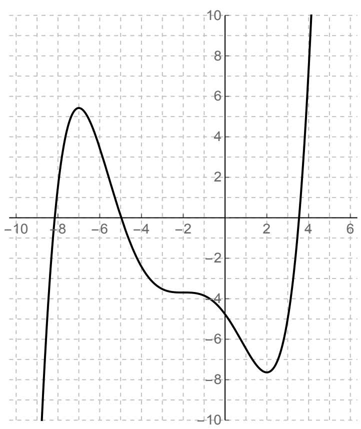
c) Estimate $\left.\dfrac{dg}{dx}\right|_{x=6}$.

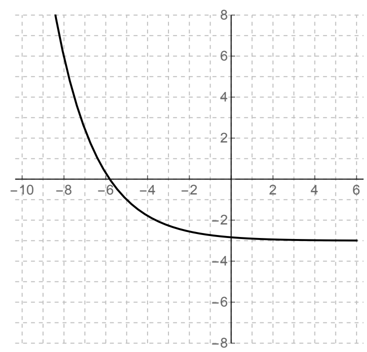
d) Sketch the graph of $g'(x)$.

In Problems 18-21, you are given the graph of an unknown function f. Sketch the graph of the function f'.

18. 

20. 

19. 

21. 

Mathematica questions (for Exam 1 review)

22. For each problem, you are given a problem that a student was trying to solve on *Mathematica*, and what the student typed in. What they typed in is WRONG. Explain why what they typed in is wrong, and write what the command should have been:

    a) The student wants to find the sine of $\pi/6$, but types in `Sin(Pi/6)`

    b) The student wants to find $\log 7$, but types in `Log[7]`

    c) The student wants to solve the equation $x^2 + 3x = 7$, but types in `Solve[x^2 + 3x = 7, x]`

    d) The student wants to define function $f(x) = x^2$, but types in `f[x] = x^2`

e) The student wants to evaluate $\frac{32+9}{63-17}$, but types in `[32+9]/[63-17]`

f) The student wants to define function $f(x) = \frac{x-1}{x+1}$, but types in `f[x_] = x-1/x+1`

23. In each problem, you are given some code in *Mathematica* (the code works). Determine what output *Mathematica* will give you.

    a) `f[x_] = x^2 + x;  f[3]`

    b) `Cos[2 Pi/3]`

    c) `g[x_] = 1/x-1; g[x+1]`

    d) `Solve[x+3 ==5, x]`

    e) `Factor[x^2 - 4, x]`

24. Suppose you typed in the following command into *Mathematica*:

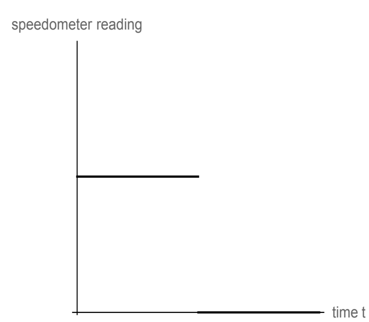
```
Plot[x^3 Log[x^2 + 1], {x, -3, 5}, PlotRange -> {0,4}]
```

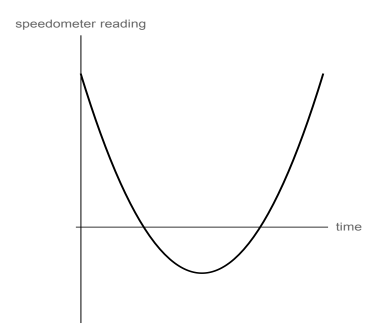
    a) What function is being plotted? (Write the function in hand-written notation, not *Mathematica* syntax.)

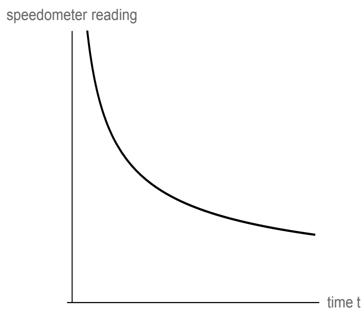
    b) What $x-$value will be at the left edge of the graph?

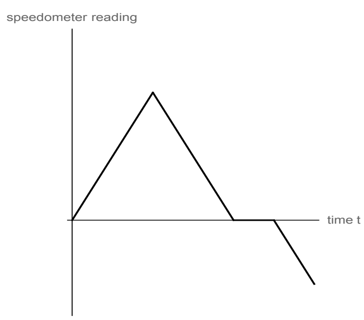
    c) What $y-$value will be at the top of the graph?

Answers

1. 

2. 

3. 

4. 

5. $\frac{-2}{3}$

6. $\frac{-1}{x^2+6x+9}$

7. $\frac{3}{2\sqrt{3x-2}}$

8. $y = 2 + 3(x-1)$

9. $P'(4) = \frac{1}{4}$ kw/hr

10. 27 mi/hr

11. $x = 1$ and $x = 4$

12. $x = -5$ (cusp),
$x = -2$ (discontinuous),
$x = 2$ (discontinuous),
$x = 4$ (corner),
$x = 9$ (vertical tangency)

13. a) 3
b) 0
c) $x \approx -\frac{1}{2}$
d) $x = -5, x = 3, x = 4$
e) $x = -5, x = -2, x = 3, x = 4, x = 6$
f) 1
g) Positive, since the graph goes up from left to right at $x = 2$.
h) -2
i) 3
j) 0
k) 1
l) $y = 1(x+3) + 3$ (a.k.a. $y = x + 6$)
m) $y \approx \frac{2}{3}(x-8) + 1.5$
n) The line should go through $(2, f(2))$ and have positive slope, lying tangent to the graph at $(2, f(2))$.

14. a) $h'(6) \approx \frac{h(8)-h(4)}{8-4} = \frac{10-9}{8-4} = \frac{1}{4}$ in/day (answer can vary somewhat)
b) On day 6, the plant is growing at a rate of $\frac{1}{4}$ inches per day.
c) $h'(1) \approx \frac{h(1)-h(0)}{1-0} = \frac{2-0}{1-0} = 2$ and $h'(1) \approx \frac{h(3)-h(1)}{3-1} = \frac{8-2}{3-1} = 3$; averaging these gives $h'(1) \approx 2.5$ in/day (this answer can vary somewhat)
d) On day 1, the plant is growing at a rate of 2.5 inches per day.

15. a) $\left.\frac{dP}{dt}\right|_{t=1} \approx \frac{P(1)-P(0)}{1-0} = \frac{764-696}{1} = 68$ and $\left.\frac{dP}{dt}\right|_{t=1} \approx \frac{P(2)-P(1)}{2-1} = \frac{818-764}{1} = 54$; averaging these gives $\left.\frac{dP}{dt}\right|_{t=1} \approx 61$ kPa/min (answer can vary somewhat)

b) 1 minute after the start of the experiment, the pressure in the chamber is increasing at a rate of 61 kPa/min.

c) $\left.\frac{dT}{dt}\right|_{t=2} \approx \frac{T(2)-T(1)}{2-1} = \frac{71-48}{1} = 23$ and $\left.\frac{dT}{dt}\right|_{t=2} \approx \frac{T(4)-T(2)}{4-2} = \frac{102-71}{2} = 15.5$; averaging these gives $\left.\frac{dT}{dt}\right|_{t=2} \approx 19.25$ ° C/min (answer can vary somewhat)

d) $\left.\frac{dT}{dt}\right|_{t=3} \approx \frac{T(4)-T(2)}{4-2} = \frac{102-71}{2} = 15.5$ ° C/min (answer can vary somewhat)

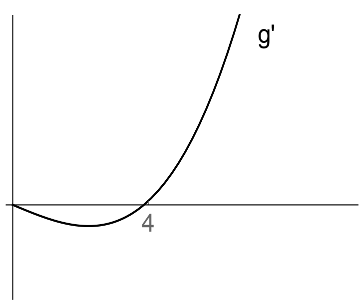
e) $\left.\frac{dT}{dP}\right|_{t=5} \approx \frac{T(5)-T(4)}{P(5)-P(4)} = \frac{112-102}{916-891} = \frac{10}{25} = .4$ and $\left.\frac{dT}{dP}\right|_{t=5} \approx \frac{T(6)-T(5)}{P(6)-P(5)} = \frac{120-112}{935-916} = \frac{8}{19} \approx .421$; averaging these gives $\left.\frac{dT}{dP}\right|_{t=5} \approx .411$ ° C/kPa (answer can vary somewhat)

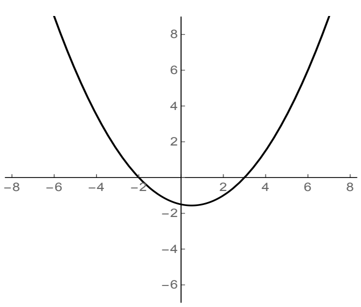
16. $f'(30) \approx 6$; $f'(80) \approx -3$

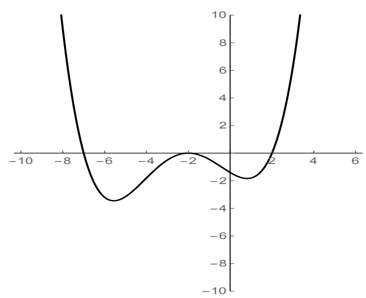
17. a) ≈ 45

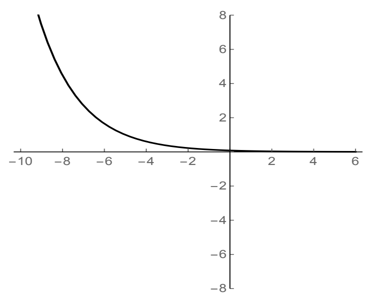
b) $y = 120 + 45(x-5)$

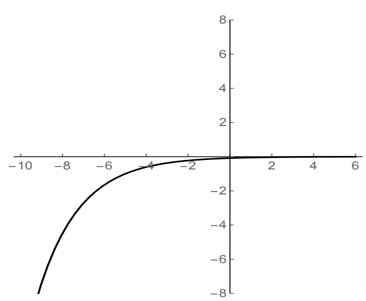
c) ≈ 120

d) 

18. 

19. 

20. 

21. 

22. a) Used parentheses instead of brackets: command should have been Sin[Pi/6]

b) Log computes natural logarithm, not logarithm base 10: command should have been Log10[7] or Log[10,7]

c) Equation inside solve command needs two equal signs, not one: should have been Solve[x^2 + 3x == 7, x]
d) Missing underscore after the x: command should have been f[x_] = x^2
e) Used brackets instead of parentheses: should have been (32+9)/(63-17)
f) Forgot parentheses: should have been f[x_] = (x-1)/(x+1)

23. a) 12
b) $-1/2$
c) $\dfrac{1}{x+1} - 1$
d) 2
e) $(x-2)(x+2)$ (the order doesn't matter)

24. a) $x^3 \ln(x^2+1)$
b) -3
c) 4

Chapter 5

Elementary Differentiation Rules

MOTIVATING EXAMPLE

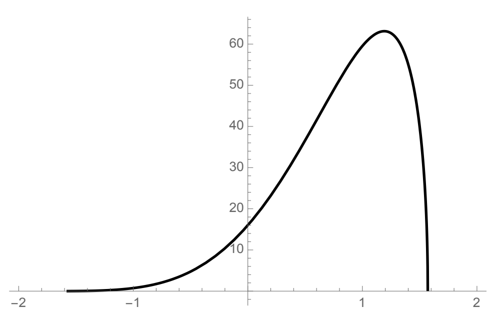
Compute the derivative $f'(0)$, given that

$$f(x) = (x+2)^4\sqrt{\cos x}.$$

Practical approach to the answer: Graph f using the *Mathematica* code

```
Plot[(x+2)^4 Sqrt[Cos[x]], {x, -2, 2}]
```

to obtain this graph of f, then estimate the value of $f'(0)$:



Problem with this practical approach:

Analytic solution: based on what we know so far, the exact answer is

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{(h+2)^4\sqrt{\cos h}-2^4}{h} =$$

GOAL

We want to figure out how to compute derivatives without using the limit definition (and without having to resort to estimates coming from graphs and/or tables).

General procedure for computing derivatives

1. Memorize the derivatives of a few basic functions
   (power, exponential, trigonometric, logarithmic, etc.)
2. Learn some rules which tell you how to compute the derivatives of more complicated functions in terms of the derivatives you have memorized.

Over the next two chapters we will develop these rules, which allow us to compute derivatives without having to resort to the limit definition. Eventually we will come to a list of rules which are given on page 165 in Section 6.7.

5.1 Constant function and power rules

EXAMPLE 1

Find the derivative of $f(x) = c$, where c is a constant.

First, what should this be? The graph of $f(x) = c$ is a ______________,

whose slope is ____. So $f'(x)$ should equal ____.

Justification of this intuition:

Theorem 5.1 (Constant Function Rule) *Let c be a constant. Then* $\frac{d}{dx}(c) = 0$.

As a reminder, $\frac{d}{dx}$(blank) means "derivative of blank".

EXAMPLE 2

Find the derivative of $f(x) = mx + b$, where m and b are constants.

First, what <u>should</u> this be? The graph of $f(x) = mx + b$ is a (straight) line, whose slope is ____. So $f'(x)$ should equal ____.

Justification of this intuition:

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{[m(x+h)+b] - [mx+b]}{h} \\
&=
\end{aligned}
$$

Theorem 5.2 (Linear Function Rule) *If $f(x) = mx + b$, then $f'(x) = m$.*

Special case: $\dfrac{d}{dx}(x) =$

EXAMPLE 3

Find the derivative of $f(x) = x^2$.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^2] - [x^2]}{h}$$

$$= \lim_{h \to 0} \frac{[x^2 + 2xh + h^2] - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$=$$

EXAMPLE 4

Find the derivative of $f(x) = x^n$, where n is a nonnegative integer.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^n] - [x^n]}{h}$$

$$= \lim_{h \to 0} \frac{[x^n + nx^{n-1}h + ... + h^n] - x^n}{h}$$

(continued on next page)

From the previous page,

$$f'(x) = \lim_{h\to 0} \frac{[x^n + nx^{n-1}h + ... + h^n] - x^n}{h}$$

$$= \lim_{h\to 0} \frac{nx^{n-1}h + ... + h^n}{h}$$

$$= \lim_{h\to 0} \left(nx^{n-1} + ... + h^{n-1}\right)$$

$$= \boxed{nx^{n-1}}.$$

EXAMPLE 5

Find the derivative of $f(x) = \sqrt{x}$.

Solution: Just to show you that you can use either definition of derivative, we'll do this example with the alternate definition:

$$f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x}$$

$$= \lim_{t\to x} \frac{\sqrt{t} - \sqrt{x}}{t - x}$$

$$= \lim_{t\to x} \frac{\left(\sqrt{t} - \sqrt{x}\right)}{(t - x)} \cdot \frac{\left(\sqrt{t} + \sqrt{x}\right)}{\left(\sqrt{t} + \sqrt{x}\right)}$$

$$= \lim_{t\to x} \frac{t - x}{(t - x)\left(\sqrt{t} + \sqrt{x}\right)}$$

$$= \lim_{t\to x} \frac{1}{\sqrt{t} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}.$$

EXAMPLE 6

Find the derivative of $f(x) = \dfrac{1}{x}$.

Solution:

$$
\begin{aligned}
f'(x) &= \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \\
&= \lim_{t \to x} \frac{\frac{1}{t} - \frac{1}{x}}{t - x} \\
&=
\end{aligned}
$$

Examples 1-6 illustrate the following general principle:

Theorem 5.3 (Power Rule) *Let $f(x) = x^n$, where $n \neq 0$. Then $f'(x) = nx^{n-1}$.*

The Power Rule can also be written this way: $\dfrac{d}{dx}(x^n) = nx^{n-1}$ whenever $n \neq 0$.

Theorem 5.4 (Special cases of the Power Rule)

$$
\begin{aligned}
\frac{d}{dx}(x) &= 1 \\
\frac{d}{dx}(mx + b) &= m \\
\frac{d}{dx}\left(\frac{1}{x}\right) &= \frac{-1}{x^2} \\
\frac{d}{dx}\left(\sqrt{x}\right) &= \frac{1}{2\sqrt{x}} \\
\frac{d}{dx}\left(x^2\right) &= 2x
\end{aligned}
$$

EXAMPLE 7

An object's position (in meters) at time t (measured in seconds) is given by $y = t^4$. Find the object's velocity at time 3.

Old solution:

$$v(3) = f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3+h)^4 - 3^4}{h} = \cdots$$

New solution:

Using exponent rules with the Power Rule

EXAMPLE 8

Compute the derivative of each function:

1. $f(x) = 2\sqrt[3]{x}$

2. $f(x) = \dfrac{2}{9x}$

3. $f(x) = \dfrac{4}{x^5}$

   Solution: Rewrite f as $f(x) = -4x^{-5}$.

   Then $f'(x) = -4(-5x^{-5-1}) = \boxed{20x^{-6}}$.

4. $f(x) = \dfrac{2}{5\sqrt[7]{x}}$

5. $f(x) = \dfrac{\sqrt{x^5}}{2}$

   Solution: Rewrite f as $f(x) = \frac{1}{2}x^{5/2}$. Then $f'(x) = \frac{1}{2} \cdot \frac{5}{2}x^{\frac{5}{2}-1} = \boxed{\frac{5}{4}x^{3/2}}$.

6. $f(x) = \dfrac{3\sqrt{x^3}}{4x^2}$

   Solution: Rewrite f as $f(x) = \frac{3}{4} \cdot \frac{x^{3/2}}{x^2} = \frac{3}{4}x^{\frac{3}{2}-2} = \frac{3}{4}x^{-1/2}$.

   Then $f'(x) = \frac{3}{4} \cdot \frac{-1}{2}x^{\frac{-1}{2}-1} = \boxed{\frac{-3}{8}x^{-3/2}}$.

5.2 Linearity rules

QUESTION

If f and g are differentiable functions,

- does $(f+g)' = f' + g'$?
- does $(f-g)' = f' - g'$?
- does $(cf)' = c \cdot f'$ when c is a constant?
- does $(fg)' = f' \cdot g'$?
- does $\left(\dfrac{f}{g}\right)' = \dfrac{f'}{g'}$?

Theorem 5.5 (Sum Rule) *If f and g are differentiable at x, then $f+g$ is differentiable at x and $(f+g)'(x) = f'(x) + g'(x)$.*

PROOF OF THE SUM RULE By definition, $(f+g)(x)$ means $f(x)+g(x)$. Now using the definition of the derivative,

$$\begin{aligned}
(f+g)'(x) &= \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h)+g(x+h)] - [f(x)+g(x)]}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= f'(x) + g'(x). \square
\end{aligned}$$

Theorem 5.6 (Difference Rule) *If f and g are differentiable at x, then $f - g$ is differentiable at x and $(f-g)'(x) = f'(x) - g'(x)$.*

PROOF OF THE DIFFERENCE RULE is similar to the proof of the Sum Rule.

Theorem 5.7 (Constant Multiple Rule) *If f is differentiable at x, then cf is differentiable at x for any constant c and $(cf)'(x) = c \cdot f'(x)$.*

PROOF OF THE CONSTANT MULTIPLE RULE:

$$\begin{aligned}
(cf)'(x) &= \lim_{h\to 0} \frac{(cf)(x+h) - (cf)(x)}{h} \\
&= \lim_{h\to 0} \frac{c\,f(x+h) - c\,f(x)}{h} \\
&= \lim_{h\to 0} \frac{c[f(x+h) - f(x)]}{h} \\
&= c \lim_{h\to 0} \frac{[f(x+h) - f(x)]}{h} \\
&= c\,f'(x).
\end{aligned}$$

Together, the Sum Rule, Difference Rule and Constant Multiple Rule are called the **linearity rules** for differentiation (for reasons that you learn in linear algebra (MATH 322)).

EXAMPLE 1

Compute the derivative of $y = 3x^2 + 2\sqrt{x} - 1$.

EXAMPLE 2

Suppose the cost of producing x units of a drug is given by $c(x) = 10x^{15} - 8x + 7$. Find the instantaneous rate of change in the cost when $x = 1$.

EXAMPLE 3

Let $y = 3\sqrt[3]{x} - \dfrac{2}{3x^3} + (3x-2)^2$. Find $\frac{dy}{dx}$.

Solution: First, rewrite y as $y = 3x^{1/3} - \dfrac{2}{3}x^{-3} + 9x^2 - 6x + 4$.

Then, $\dfrac{dy}{dx} = 3 \cdot \dfrac{1}{3}x^{\frac{1}{3}-1} - \dfrac{2}{3}(-3)x^{-3-1} + 9(2x) - 6 + 0$, which simplifies to

$$\boxed{\frac{dy}{dx} = x^{-2/3} + 2x^{-4} + 18x - 6}.$$

EXAMPLE 4

Compute $g'(x)$, if $g(x) = \dfrac{2}{5x\sqrt{x}} + \dfrac{(\sqrt{x}+1)^2}{x^2}$.

WARNING: Products do not behave nicely under differentiation. Here is an example to show why $(fg)' \neq f' \cdot g'$:

Suppose $f(x) = x^2$ and $g(x) = x^3$.

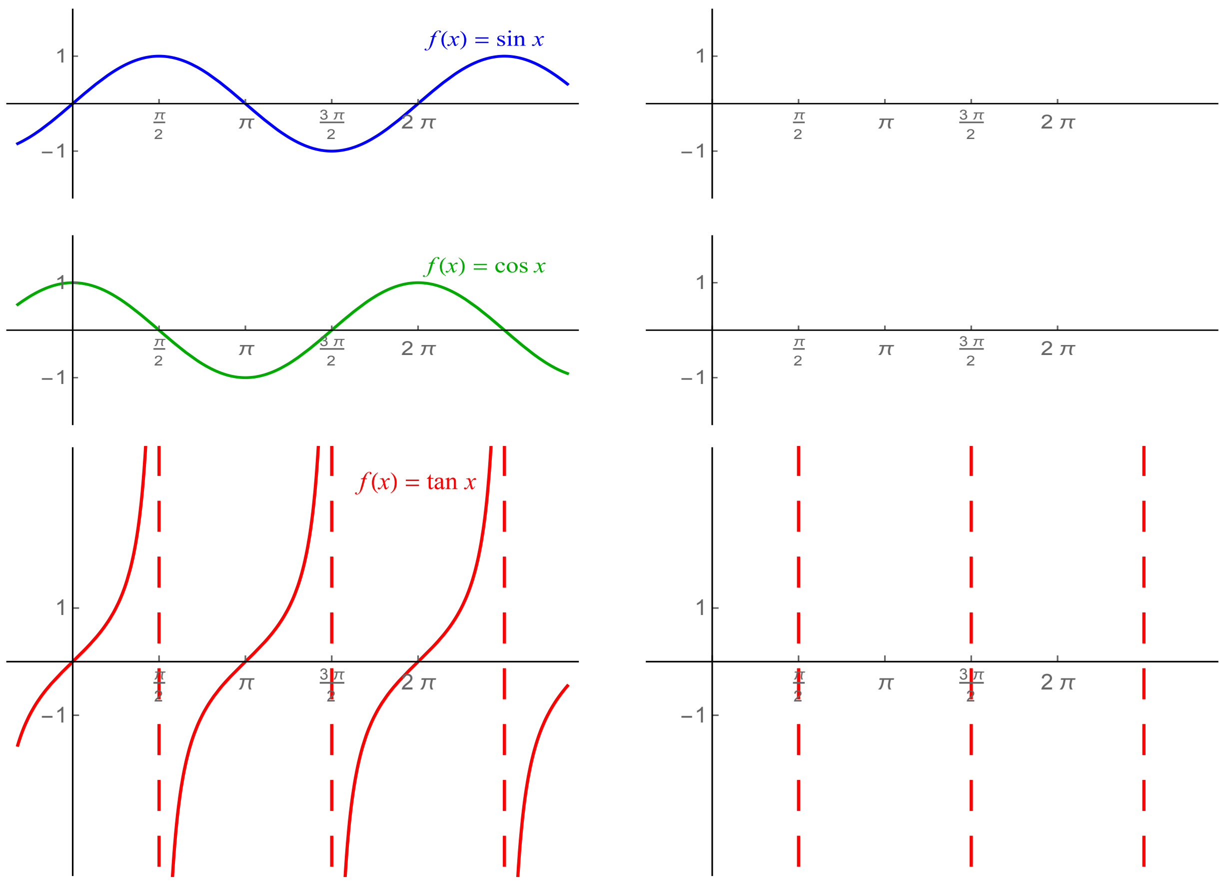
Then $f'(x) = 2x$ and $g'(x) = 3x^2$.
Therefore the product of the derivatives is $f'(x)g'(x) = (2x)(3x^2) = 6x^3$.

BUT $(fg)(x) = f(x)g(x) = x^2x^3 = x^5$.

Therefore the derivative of the product is $(fg)'(x) =$

5.3 Derivatives of sine, cosine and tangent

To figure out what the derivatives of $\sin x$, $\cos x$ and $\tan x$ are, let's first use graphs to get some intuition as to what these derivatives might be:



Theorem 5.8 (Derivatives of sine, cosine and tangent)

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\tan x) = \sec^2 x$$

The derivatives of $\cot x$, $\sec x$ and $\csc x$, as well as the derivatives of $\arctan x$ and $\arcsin x$ will be derived in the next chapter.

I have written proofs of the statements in Theorem 5.8 at the end of this section, but I probably won't go over the proofs in class. Essentially, you establish these derivative formulas rigorously by writing out the limit definition of derivative, and using a bunch of algebra and trig identities (and some other stuff).

EXAMPLE 1

Compute $f'(x)$ if $f(x) = 2\sin x - \dfrac{\cos x}{5} + 4$.

EXAMPLE 2

Let $y = 8\sqrt{x^3} + 5x - 2\tan x$. Compute $\dfrac{dy}{dx}$.

EXAMPLE 3

Find the equation of the line tangent to $y = 4\cos x - 2$ when $x = \dfrac{\pi}{3}$.

Proofs of the derivative formulas for sine, cosine and tangent

First, when computing these derivatives, we will need some trigonometric identities that are listed and numbered here for convenience:

$$\sin(x+h) = \sin x \cos h + \cos x \sin h \tag{5.1}$$

$$\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right) \tag{5.2}$$

$$\cos(x+h) = \cos x \cos h - \sin x \sin h \tag{5.3}$$

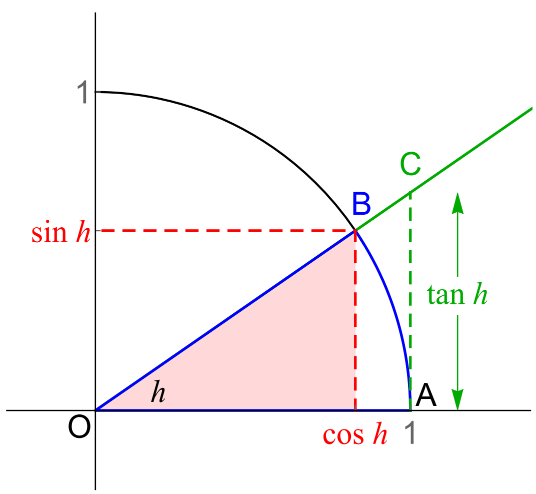
$$\tan(x+h) = \frac{\tan x + \tan h}{1 - \tan x \tan h} \tag{5.4}$$

$$1 + \tan^2 x = \sec^2 x \tag{5.5}$$

We will also need a couple of preliminary results:

Preliminary result # 1: $\lim\limits_{h \to 0} \dfrac{\sin h}{h} = 1.$

PROOF OF PRELIMINARY RESULT # 1 Consider the following picture, where the black arc is a quarter-circle and h is the angle between the x-axis and the diagonal line:



From this picture, it is clear that

area of pink triangle	$\leq$	area of blue pizza wedge with corners O, A and B	$\leq$	area of green triangle with vertices O, A and C
$\frac{1}{2}$(base)(height)	$\leq$	$\dfrac{\text{angle}}{2\pi}(\pi\ \text{radius}^2)$	$\leq$	$\dfrac{1}{2}$(base)(height)
$\frac{1}{2}(\cos h)(\sin h)$	$\leq$	$\dfrac{h}{2\pi}\pi(1)^2$	$\leq$	$\dfrac{1}{2}(1)(\tan h)$

From the previous page, we have

$$\frac{1}{2}(\cos h)(\sin h) \leq \frac{h}{2\pi}\pi(1)^2 \leq \frac{1}{2}(1)(\tan h).$$

Cancel the πs in the middle term, rewrite $\tan h$ as $\dfrac{\sin h}{\cos h}$ and multiply everything by 2 to get

$$\cos h \sin h \leq h \leq \frac{\sin h}{\cos h}.$$

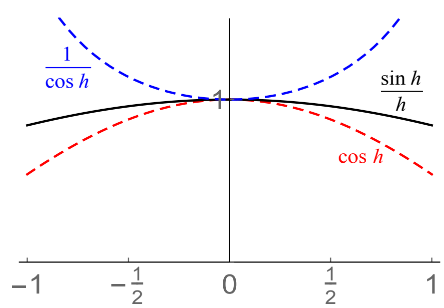
Divide everything by $\sin h$ to get

$$\cos h \leq \frac{h}{\sin h} \leq \frac{1}{\cos h}.$$

Take reciprocals (this flips all the inequality signs) to get

$$\frac{1}{\cos h} \geq \frac{\sin h}{h} \geq \cos h.$$

This proves the relationships between the graphs of $\cos h$, $\dfrac{\sin h}{h}$ and $\dfrac{1}{\cos h}$ seen below:



We can conclude that since

$$\lim_{h\to 0} \cos h = \cos 0 = 1 \qquad \text{and} \qquad \lim_{h\to 0} \frac{1}{\cos h} = \frac{1}{1} = 1,$$

that $\lim\limits_{h\to 0} \dfrac{\sin h}{h}$ must also equal 1, proving preliminary result # 1. □

Preliminary result # 2: $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0.$

PROOF OF PRELIMINARY RESULT # 2 Multiply the top and bottom by $(\cos h + 1)$:

$$\begin{aligned}
\lim_{h\to 0} \frac{\cos h - 1}{h} &= \lim_{h\to 0} \frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)} \\
&= \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} \\
&= \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h + 1)} \qquad \text{(by trig identity } \cos^2 h + \sin^2 h = 1) \\
&= \lim_{h\to 0} \left(\frac{\sin h}{h}\right) \frac{-\sin h}{\cos h + 1} \\
&= (1) \left(\frac{-\sin 0}{\cos 0 + 1}\right) = \frac{-0}{1+1} = 0. \ \square
\end{aligned}$$

PROOF THAT $\frac{d}{dx}(\sin x) = \cos x$:

Use the limit definition of derivative:

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \qquad \text{(by trig identity (5.1) above)} \\
&= \lim_{h\to 0} (\cos x) \frac{\sin h}{h} + \lim_{h\to 0} (\sin x) \frac{\cos h - 1}{h} \\
&= \cos x \left(\lim_{h\to 0} \frac{\sin h}{h}\right) + \sin x \left(\lim_{h\to 0} \frac{\cos h - 1}{h}\right) \\
&= \cos x \cdot 1 + \sin x \cdot 0 \qquad \text{(by the preliminary results)} \\
&= \cos x.
\end{aligned}$$

PROOF THAT $\frac{d}{dx}(\cos x) = -\sin x$:

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \qquad \text{(by trig identity (5.3) above)} \\
&= \lim_{h\to 0} (\cos x) \frac{\cos h - 1}{h} - \lim_{h\to 0} (\sin x) \frac{\sin h}{h} \\
&= \cos x \cdot 0 - \sin x \cdot 1 \qquad \text{(by the preliminary results)} \\
&= -\sin x.
\end{aligned}$$

PROOF THAT $\frac{d}{dx}(\tan x) = \sec^2 x$:

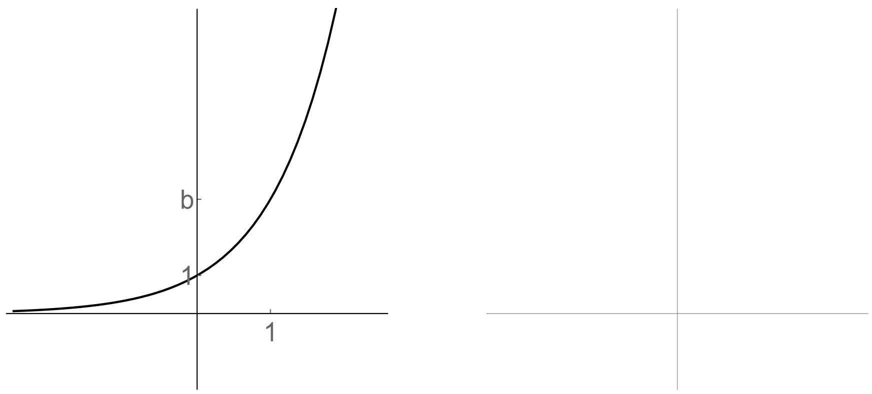
$$
\begin{aligned}
&\frac{d}{dx}(\tan x) \\
&= \lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h} \\
&= \lim_{h\to 0} \frac{\frac{\tan x + \tan h}{1 - \tan x \tan h} - \tan x}{h} && \text{(by trig identity (5.4) above)} \\
&= \lim_{h\to 0} \frac{\tan x + \tan h - \tan x(1 - \tan x \tan h)}{h(1 - \tan x \tan h)} \\
&= \lim_{h\to 0} \frac{\tan x + \tan h - \tan x + \tan^2 x \tan h}{h(1 - \tan x \tan h)} \\
&= \lim_{h\to 0} \frac{\tan h + \tan h \tan^2 x}{h(1 - \tan x \tan h)} \\
&= \lim_{h\to 0} \frac{\tan h(1 + \tan^2 x)}{h(1 - \tan x \tan h)} \\
&= \lim_{h\to 0} \frac{\tan h \sec^2 x}{h(1 - \tan x \tan h)} && \text{(by trig identity (5.5) above)} \\
&= \sec^2 x \cdot \lim_{h\to 0} \frac{\sin h}{\cos h \cdot h \cdot (1 - \tan x \tan h)} && \text{(by writing } \tan h \text{ as } \frac{\sin h}{\cos h}\text{)} \\
&= \sec^2 x \cdot \lim_{h\to 0} \left(\frac{1}{\cos h}\right)\left(\frac{\sin h}{h}\right)\frac{1}{1 - \tan x \tan h} \\
&= \sec^2 x \cdot \frac{1}{1} \cdot (1) \cdot \frac{1}{1 - \tan x \cdot 0} && \text{(by prelim. result \# 1 on the third term)} \\
&= \sec^2 x.
\end{aligned}
$$

5.4 Exponential and logarithmic functions

Question: What are the derivatives of e^x and $\ln x$?

Better question:

To get some intuition for this, let's try to compute the derivative of an exponential function with an arbitrary base $b > 0$. To do this, let $f(x) = b^x$ where b is a constant. First, some pictures to give us an idea of what to expect:



So it looks like the derivative of an exponential function is ____________________.

To check this, use the limit definition of derivative:

$$\begin{aligned} \frac{d}{dx}(b^x) &= \lim_{h\to 0} \frac{b^{x+h} - b^x}{h} \\ &= \lim_{h\to 0} \frac{b^x b^h - b^x}{h} \\ &= \lim_{h\to 0} \frac{b^x(b^h - 1)}{h} \\ &= b^x \cdot \left[\lim_{h\to 0} \frac{b^h - 1}{h}\right] \end{aligned}$$

What this tells us: the derivative of an exponential function with base b is **itself, times a constant that depends on** b. Next, we will give a name to that constant:

Definition 5.9 *Let $b > 0$. The* **natural logarithm** *of b, denoted $\ln b$ (and executed with* Log[b] *in* Mathematica*), is*

$$\ln b = \lim_{h \to 0} \frac{b^h - 1}{h}.$$

Theorem 5.10 (Derivative of exponential functions that have arbitrary base) *Let $b > 0$ be a constant. Then*

$$\frac{d}{dx}(b^x) = b^x \cdot \ln b.$$

EXAMPLE 1

Compute the derivative of $f(x) = 3 \cdot 2^x + 5^{-2x}$.

The graph of $\ln x$

Theorem 5.11 (Basic logarithm facts) *Let $a, b > 0$. Then:*

Log of 1 is 0: $\ln 1 = 0$.

Logs are increasing: *if $a < b$, then $\ln a < \ln b$.*

Logs have VA at $x = 0$: $\lim_{b \to 0^+} \ln b = -\infty$.

Logs have no HA: $\lim_{b \to \infty} \ln b = \infty$.

PROOF For the first statement, use our definition of natural log:

$\ln 1 =$

For the second statement (logs are increasing), notice that if $a < b$,

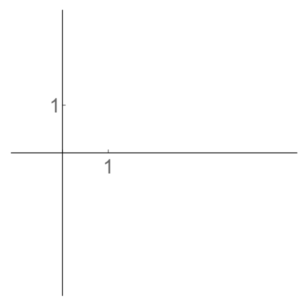
$$\frac{a^h - 1}{h} < \frac{b^h - 1}{h}.$$

Taking limits as $h \to 0$ preserves this inequality, so $\ln a < \ln b$.

For the third statement (VA at $x = 0$), notice that as $b \to 0^+$, $\frac{b^h - 1}{h} \to \frac{0-1}{h} = \frac{-1}{h}$. Take limit as $h \to 0$ of this to get $-\frac{1}{0} = -\infty$.

For the fourth statement (no HA), notice that as $b \to \infty$, $\frac{b^h - 1}{h}$ gets bigger and bigger without bound, so taking limits as $h \to 0$ means $\ln b$ will also get bigger and bigger without bound. Thus $\lim_{b\to\infty} \ln b = \infty$. □

The preceding theorem tells you about the graph of $\ln x$: it must go through $(1, 0)$, increase from left to right, have VA $x = 0$, and no HA, so it looks like:



The number e

Definition 5.12 *The number e, called* **Euler's constant***, is the number which satisfies $\ln e = 1$. (In* Mathematica*, this number is obtained by typing* E*.) The* **natural exponential function** *is the function*

$$\exp(x) = e^x.$$

e is an irrational number that is roughly $2.71828...$.

e cannot be expressed in terms of rational numbers, roots, or trig functions.

You probably have heard of e before, but you probably didn't hear why you were being told about e. Mathematicians like base e for their exponents and logarithms because they lead to easier derivatives than other bases do. In particular:

Theorem 5.13 (Derivative of the natural exponential function) $\frac{d}{dx}(e^x) = e^x$.

PROOF $\frac{d}{dx}(e^x) = e^x \cdot \ln e = e^x \cdot 1 = e^x$. □

EXAMPLE 2

Find the derivative of $y = 2e^x - 4\sin x + \cos x - 2x^6 - 1$.

More logarithm rules

In this section, I have told you the following definition of natural logarithm:

$$\ln b = \lim_{h\to 0} \frac{b^h - 1}{h}.$$

You probably have heard of logarithms before, but they were probably presented to you differently. However, **this is the same notion of logarithm** that you already knew about. In particular, we can derive algebraic rules for logarithms (that are hopefully familiar to you) using only this limit definition:

Theorem 5.14 (Algebra with logarithms) *Let $a, b > 0$ and let $n \in \mathbb{R}$. Then:*

Log of a product is the sum of the logs: $\ln(ab) = \ln a + \ln b$.

Log of a quotient is the difference of the logs: $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$.

Exponents in a log can be pulled in front: $\ln(b^n) = n \ln b$.

Cancellation laws: $\ln e^b = b$ *and* $e^{\ln b} = b$.

PROOF To prove the first statement, notice

$$\begin{aligned}
\ln a + \ln b &= \lim_{h\to 0} \frac{a^h - 1}{h} + \lim_{h\to 0} \frac{b^h - 1}{h} \\
&= 1 \cdot \lim_{h\to 0} \frac{a^h - 1}{h} + \lim_{h\to 0} \frac{b^h - 1}{h} \\
&= \lim_{h\to 0} b^h \cdot \lim_{h\to 0} \frac{a^h - 1}{h} + \lim_{h\to 0} \frac{b^h - 1}{h} \\
&= \lim_{h\to 0} \frac{b^h(a^h - 1)}{h} + \lim_{h\to 0} \frac{b^h - 1}{h} \\
&= \lim_{h\to 0} \frac{a^h b^h - b^h}{h} + \lim_{h\to 0} \frac{b^h - 1}{h}
\end{aligned}$$

Continuing from the previous page:

$$\begin{aligned}\ln a + \ln b &= \lim_{h\to 0} \frac{a^h b^h - b^h}{h} + \lim_{h\to 0} \frac{b^h - 1}{h} \\ &= \lim_{h\to 0} \left[\frac{a^h b^h - b^h}{h} + \frac{b^h - 1}{h} \right] \\ &= \lim_{h\to 0} \frac{a^h b^h - b^h + b^h - 1}{h} \\ &= \lim_{h\to 0} \frac{a^h b^h - 1}{h} \\ &= \lim_{h\to 0} \frac{(ab)^h - 1}{h} = \ln(ab).\end{aligned}$$

The second and third statements have similar proofs; these are omitted.

To prove the first cancellation laws, observe $\ln e^b = b \ln e = b(1) = b$.

That leaves the second cancellation law: suppose $b = e^x$. Then

$$e^{\ln b} = e^{\ln e^x} = e^x = b$$

(in the red equals sign, we used the first cancellation law). □

EXAMPLE 3

Evaluate each expression:

$\ln e^5 =$ $\qquad$ $e^{\ln 2} =$

$\ln \sqrt{e} =$ $\qquad$ $e^{4 \ln 3} =$

$\ln e^x =$ $\qquad$ $3e^{\frac{1}{2} \ln 16} =$

The last examples above generalize into the following fact, which is incredibly useful:

Theorem 5.15 (Change of base formula for exponentials) *Let $a > 0$ and let $b \in \mathbb{R}$. Then*

$$a^b = e^{b \ln a}.$$

In calculus, we use the formula of Theorem 5.15 to simplify expressions like the $e^{4 \ln 3}$ in Example 3, and also use it to rewrite expressions so they are easier to differentiate.

EXAMPLE 4

Rewrite each expression so that it contains no $\ln$ nor e:

$3.4e^{x \ln 1.6} =$

$e^{5 \ln x} =$

$4e^{\frac{1}{7} \ln(2x)} =$

EXAMPLE 5

Simplify each expression and then rewrite it so that it is a constant times a single exponential expression, whose base is e (in other words, reverse the technique of the previous example):

$-2 \cdot 5^{3x} =$

$3(2e^x)^4 e^{3x} e^{-2y} =$

$(e^{2x})^4 2^{-3x} =$

$5 \cdot 7^{2y+x} 4^x =$

The derivative of e^{rx}

EXAMPLE 6

Find the derivative of $y = e^{4x}$.

Example 6 generalizes into the following result, which previews something we'll learn later called the Chain Rule:

Theorem 5.16 *For any constant r, $\frac{d}{dx}\left(e^{rx}\right) = re^{rx}$.*

EXAMPLE 7

An object's position at time t (t is measured in hours), is $f(t) = \frac{2e^{7t} + 5e^{-t}}{e^{3t}}$ km. Compute the object's velocity at time t, and the object's velocity at time 0.

Logarithms with arbitrary bases

Definition 5.17 *Let $a > 0$. Define the* **logarithm base** a *of* b *by the following formula:*

$$\log_a x = \frac{\ln x}{\ln a}.$$

EXAMPLE 8

Rewrite each expression in terms of only natural logarithms:

$\log x$ (if no base is given, this means $\log_{10} x$)

$4 \log_3(4x) =$

Theorem 5.18

$\log_a x = y$ *if and only if* $a^y = x$.

PROOF This is a direct calculation, using the definition in Definition 5.17:

$$\begin{aligned} \log_a x = y &\iff \frac{\ln x}{\ln a} = y \\ &\iff \ln x = y \ln a \\ &\iff \ln x = \ln a^y \\ &\iff x = a^y. \ \square \end{aligned}$$

This rule is used to actually compute logarithms in arbitrary bases:

EXAMPLE 9

Evaluate the following expressions:

1. $\log 10000$

2. $\log_9 3$

3. $\log_3 \frac{1}{27}$

   Solution: $\frac{1}{27} = \frac{1}{3^3} = 3^{-3}$, so $\log_3 \frac{1}{27} = \boxed{-3}$.

4. $\log_6 36$

   Solution: $36 = 6^2$, so $\log_6 36 = \boxed{2}$.

5. $\log_2 64$

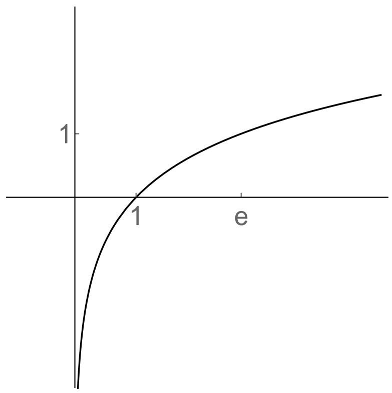
   Solution: $64 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6$, so $\log_2 64 = \boxed{6}$.

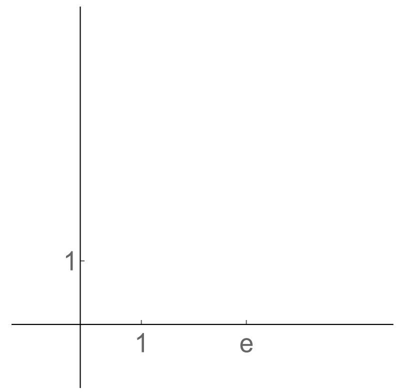
6. $\log_4 32$

   Solution: $32 = 4 \cdot 4 \cdot 2 = 4 \cdot 4 \cdot 4^{1/2} = 4^{1+1+1/2} = 4^{5/2}$ so $\log_4 32 = \boxed{\frac{5}{2}}$.

The derivative of $\ln x$

First, let's get some intuition as to what the derivative of $\ln x$ should be, using graphs:





What function do we know that has a graph that looks like the one at right?

To verify whether this guessed derivative of $\ln x$ is correct, we'll use a trick, where we compute $\frac{dy}{dx}$ (the rate of change of y with respect to x) by first computing $\frac{dx}{dy}$ (the rate of change of x with respect to y). To understand how $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are related, let's consider an example:

Suppose y is position and x is time. Then

$$\frac{dy}{dx} = \text{velocity} = \text{rate of change of position per unit of time}$$
$$\frac{dx}{dy} =$$

In particular, if $\frac{dy}{dx} = 5$ m/sec, then $\frac{dx}{dy} =$

In general, $\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{\frac{dy}{dx}}$. In other words, the Leibniz notation for derivatives works the way fractions do when taking reciprocals (even though derivatives aren't fractions).

Now, let's use this observation for the computation of the derivative of $y = \ln x$:

Theorem 5.19 (Derivative of the natural logarithm function) $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}.$

EXAMPLE 10

Find the slope of the line tangent to the function $f(x) = 3\ln x + \sqrt{x}$ at $x = 4$.

EXAMPLE 11

Find the derivative of $\lambda(z) = \dfrac{2\ln z}{3} + \log z - 2\log_4 z$.

Summary of Section 5.4

Definitions of "ln", "e", "exp" and "$\log_a$" are as follows:

- $\ln b = \lim_{h \to 0} \frac{b^h - 1}{h}$.
- e is the number such that $\ln e = 1$.
- The natural exponential function is $\exp(x) = e^x$.
- $\log_a x = \frac{\ln x}{\ln a}$.

The **graph of** $\ln x$ goes up from left to right, passes through $(1, 0)$, has VA $x = 0$ and no HA.

Derivatives and limits to know

$$\frac{d}{dx}(e^x) = e^x$$
$$\frac{d}{dx}(e^{rx}) = re^{rx}$$
$$\frac{d}{dx}(b^x) = b^x \cdot \ln b \qquad \text{(less important)}$$
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$
$$\lim_{x \to 0^+} \ln x = -\infty$$
$$\lim_{x \to \infty} \ln x = \infty$$

Rules used to manipulate expressions containing logarithms

$$\ln ab = \ln a + \ln b$$
$$\ln \frac{a}{b} = \ln a - \ln b$$
$$\ln b^n = n \ln b$$
$$\ln e^b = b$$
$$e^{\ln b} = b$$

Change of base formulas

$$\log_a x = \frac{\ln x}{\ln a} \qquad\qquad a^b = e^{b \ln a}$$

Rule used to evaluate logarithms

$\log_a x = y$ means $a^y = x$

5.5 Higher-order derivatives

We will see that many problems can be studied not just by differentiating a function once, but by repeatedly differentiating it many times. First, we establish notation to describe this procedure:

Definition 5.20 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function.*

- *The* **zeroth derivative** *of f, sometimes denoted $f^{(0)}$, is just the function f itself.*
- *The* **first derivative** *of f, sometimes denoted $f^{(1)}$ or $\dfrac{dy}{dx}$, is just f'.*
- *The* **second derivative** *of f, denoted f'' or $f^{(2)}$ or $\dfrac{d^2y}{dx^2}$, is the derivative of f'; in other words, $f'' = (f')'$. The* **third derivative** *of f, denoted f''' or $f^{(3)}$ or $\dfrac{d^3y}{dx^3}$, is the derivative of f''; in other words, $f''' = ((f')')'$.*
- *More generally, the n^{th}* **derivative** *of f, denoted $f^{(n)}$ or $\dfrac{d^ny}{dx^n}$,is the derivative of $f^{(n-1)}$; in other words $f^{(n)} = ((((f')')\cdots')'$.*

Why is the Leibniz notation $\dfrac{d^2y}{dx^2}$?

EXAMPLE 1

Let $f(x) = 2x^6$. Find $f'''(x)$.

EXAMPLE 2

If $y = \cos x + \sin x$, find $\left.\dfrac{d^2y}{dx^2}\right|_{x=\pi/4}$.

Physical interpretation of the second derivative

Suppose an object's position on a number line after t seconds of elapsed time is given by $f(t)$. Then

$$f'(t) = \text{ rate of change of position } = \text{ velocity}$$

$$f''(t) = (f')'(t) = \text{ rate of change of velocity } =$$

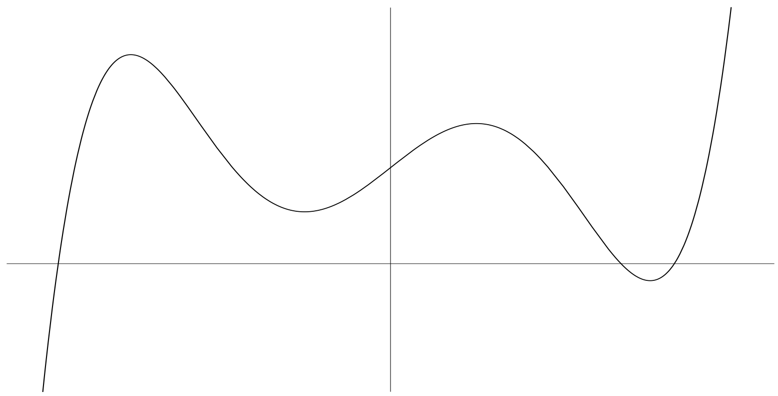
EXAMPLE 3

A bee is flying back and forth along a number line, so that its position (in ft) after t seconds of time is $f(t) = \dfrac{-1}{3}t^3 + 3t^2$. What is the velocity of the object at the instant where its acceleration is zero?

Graphical interpretation of the second derivative

Let f be a twice- differentiable function. Then

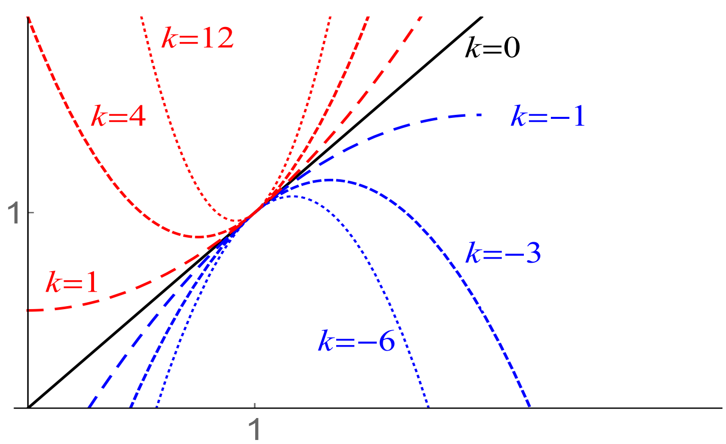
$$f'(x) = \text{ slope of graph of } f \text{ at } x$$
$$f''(x) = (f')'(x) = \text{ rate of change of slope at } x$$



EXAMPLE 4

Let k be a constant and define $f(x) = \frac{1}{2}kx^2 + (1-k)x + \frac{1}{2}k$. Examine the behavior of $f(x)$ at $x = 1$ for various k:

$$f(1) = \frac{1}{2}k + 1 - k + \frac{1}{2}k = 1 \Rightarrow \text{graph goes through } (1,1)$$
$$f'(x) = kx + 1 - k \Rightarrow f'(1) = k + 1 - k = 1 \Rightarrow \text{graph has slope } 1 \text{ at } (1,1)$$
$$f''(x) = k \Rightarrow f''(1) = k$$



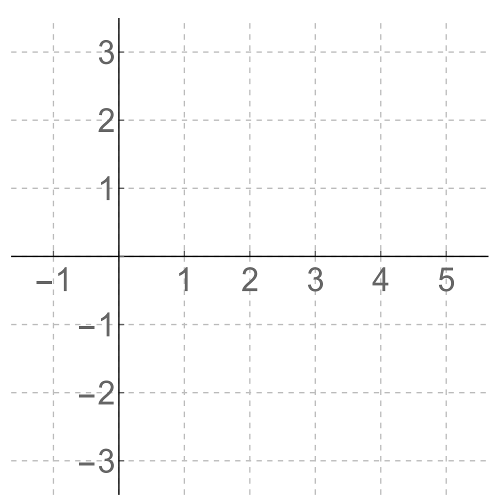
Compiling information from the first and second derivative, we can determine the general shape of a graph near a value x as follows:

	$f'(x) > 0$	$f'(x) < 0$	$f'(x) = 0$
$f''(x) > 0$			
$f''(x) < 0$			
$f''(x) = 0$			

Before the days of *Mathematica* and graphics calculators, this is how people learned to sketch the graphs of functions.

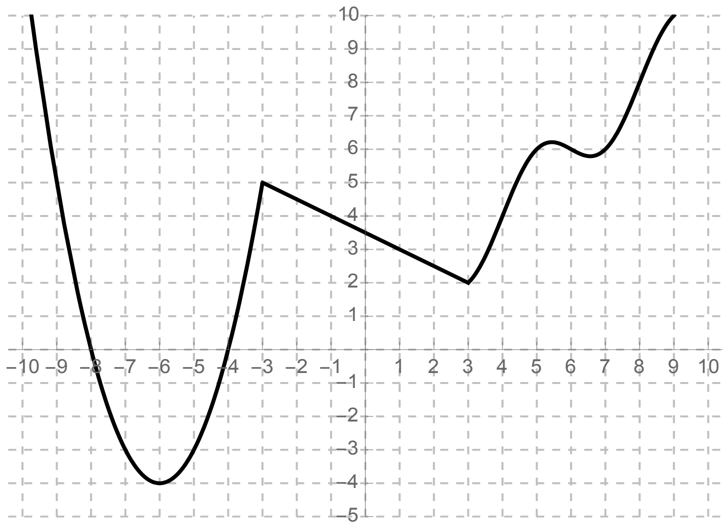
EXAMPLE 5

Suppose f is some unknown function such that $f(3) = -2$, $f'(3) = 1$ and $f''(3) = 2$. Sketch a picture of what the graph of f looks like near $x = 3$:



EXAMPLE 6

Suppose f is some function whose graph is given below:



1. Estimate $f(-6)$.

2. Estimate $f'(-6)$.

3. Estimate $f'(1)$.

4. Estimate $f''(1)$.

5. Estimate $f''(-3)$.

6. Estimate a value of x for which $f'(x) = 0$ but $f''(x) < 0$.

7. Estimate a value of x for which $f'(x) < 0$ but $f''(x) > 0$.

8. Is $f''(9)$ positive, negative, or zero? Explain.

9. Is $f''(-7)$ positive, negative, or zero? Explain.

EXAMPLE 7

Suppose that you look at your Fitbit periodically to measure the number of steps you have walked and record what you see in the following table:

time t (minutes after noon)	0	2	5	7	11	15
steps taken $f(t)$	0	35	115	147	163	191

Use the table above to estimate the answers to these questions. Show your work; use correct mathematical language and use appropriate units.

1. How fast are you walking at 12:06 PM?

   Solution: This is asking for the velocity at time 6, which is $f'(6)$. We estimate this with a difference quotient, as in Chapter 4:

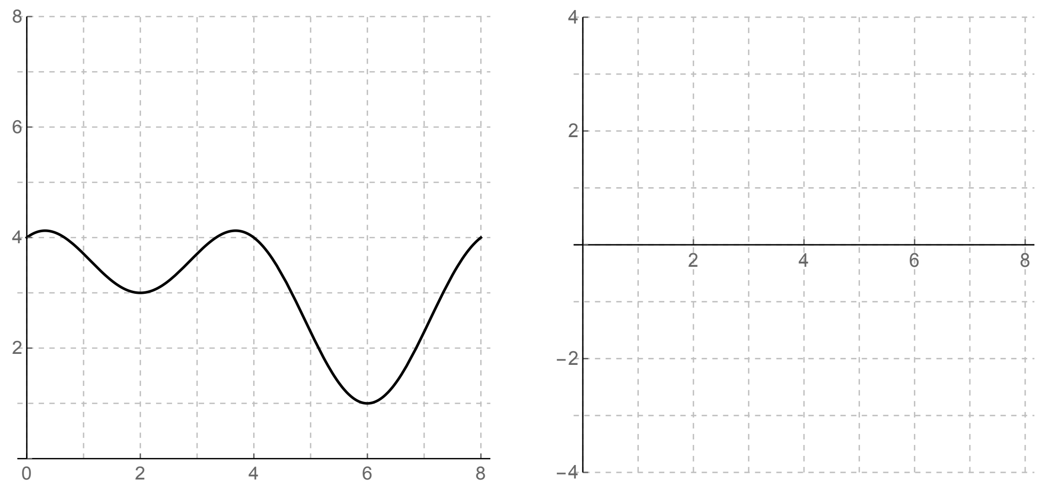
   $$f'(6) \approx \frac{f(7) - f(5)}{7 - 5} = \frac{147 - 115}{7 - 5} = \frac{32}{2} = \boxed{16 \text{ steps/min}}.$$

   Note that the concept generalizes as follows:

2. What is your acceleration at 12:07 PM? Use appropriate units.

EXAMPLE 8

The graph of some unknown function f is given below at left. Sketch the graph of f'' on the right-hand axes:



5.6 Homework exercises

In these problems (and in all future problems), you may (and should) use differentiation rules to compute any necessary derivatives (i.e. you do not have to use the limit definition).

Exercises from Sections 5.1 and 5.2

1. Find $\frac{dy}{dx}$ if $y = 3$.
2. Find $f'(x)$ if $f(x) = x^6 + 2\sqrt{x}$.
3. Find $\frac{d}{dx}\left(3x - \frac{4}{5x} + 1\right)$.
4. A business estimates that if it employs x thousands of people, then its profit, in millions of dollars, is given by the function $f(x) = 2x^3 + 2 - x^{-1}$. Find the rate of change of the business' profit relative to the change in x, when $x = 2$.
5. Find the instantaneous rate of change of y with respect to x, if $y = (x-2)(x^2 + 4)$.
6. Find the slope of the line tangent to $y = 2x^{5/2} - x^{3/2}$ when $x = 4$.
7. Find the equation of the line tangent to $f(x) = 3 - x^4 + \frac{1}{x}$ when $x = 1$.
8. Suppose an object is traveling along a number line so that its position (in meters) at time t (in seconds) is $f(t) = 2t - 4t^{-2}$. Find the object's velocity when $t = 2$.
9. Suppose an object's position at time t is given by $f(t) = 4t^2 - 5t + 2$. Find all times t where the velocity of the object is -1.

In Problems 10-21, compute the derivative of the indicated function:

10. $f(x) = 5\sqrt{x} - \frac{3}{\sqrt[3]{x}}$
11. $h(x) = \sqrt{7x}$
12. $g(t) = (t+2)(\sqrt{t} - 1)$
13. $F(x) = \frac{2}{7x} - \frac{2x}{7}$
14. $f(x) = (2x)^3$
15. $f(x) = \frac{(x+1)(x-1)}{\sqrt{x}}$
16. $g(x) = 3 + 4x - \sqrt[3]{x}$
17. $f(x) = \frac{7}{2x^4} - 2$
18. $f(x) = \frac{x}{\sqrt[3]{x^5}}$
19. $v(x) = 20\sqrt[4]{x^{11}} - 3x^2x^3$

20. $f(w) = \dfrac{(2\sqrt{w}+1)(\sqrt{w}-3)}{w}$

21. $f(t) = 2t^2 + 3t - \dfrac{1}{3t}$

Exercises from Section 5.3

22. Find $f'(x)$ if $f(x) = \dfrac{2}{3}\sin x + \dfrac{3}{4}\cos x - x^2$.

23. Find the derivative of $y = 2 - x - 4\tan x$.

24. Let $f(x) = \cos x - 3$. Find $\left.\dfrac{df}{dx}\right|_{x=\pi/4}$.

25. Find the slope of the line tangent to $y = 3\tan x - \cos x$ when $x = \dfrac{\pi}{6}$.

26. Find the instantaneous velocity of an object at time t (measured in hours), if the object's position at time t is $f(t) = 3t + \sin t$ (measured in km).

Exercises from Section 5.4

27. Simplify each of the following expressions:

a) $4\ln e^5$
b) $2e^{2\ln 3}$
c) $4\ln \sqrt[3]{e^2}$
d) $\exp\left(\dfrac{1}{2}\ln\dfrac{4}{9}\right)$
e) $\ln 1$
f) $e^{x\ln(a-1)}$
g) $e^{\ln 6}$
h) $\ln e^8$

28. Write each of the following expressions as a single exponential term, where the base of the exponent is e (meaning your answer should look like $e^{\square}$):

a) t^x
b) 4^8
c) $\dfrac{e^{4x}e^x}{e^{-2x}}$
d) $6^{x-1}2^{3x}$
e) $7^{-x}e^{4x}$
f) $(3x)^{4x}$
g) $(6x)^{t+1}e^x$
h) $\dfrac{e^{3x}}{2^x}$

29. Rewrite the following expressions in terms of <u>natural</u> logarithms:

a) $\log y$
b) $3\log_4 11$
c) $\log_{1/2}\dfrac{2}{3}$
d) $\log_2(x+4y)$

30. Evaluate each expression:

a) $\log .01$
b) $5\ln e^3$
c) $\log_9 3$
d) $\log_7 49^8$
e) $\log_4 \dfrac{1}{16}$
f) $4\log_5 625 + \log_3 27$
g) $\log_2 36 - \log_2 9$
h) $\log_{1/3} 27$

In Problems 31-44, compute the derivative of each given function.

31. $f(x) = 7^x$
32. $f(x) = 6 \cdot 8^x + 3\left(\frac{2}{5}\right)^x$
33. $f(x) = -\frac{e^x}{4}$
34. $g(t) = 4e^t - 5t(1-t) + \sin t$
35. $g(x) = 5e^{3x} - 4\cos x$
36. $f(x) = \frac{1}{6e^x} + 3\sqrt[5]{x}$
37. $h(x) = 5\ln x - 11e^x$
38. $f(x) = 5x^2 - 2\sqrt{x} + \frac{\ln x}{3}$
39. $\theta(x) = 4 - \frac{3}{x} + 2\ln x$
40. $G(z) = 4\log_7 z$
41. $f(t) = \ln(7t)$
42. $r(x) = e^x(e^{-x} - 3)$
43. $q(x) = \frac{1}{x\sqrt[3]{x}} - 4\ln x + 8e^x - e^{x/2}$
44. $f(x) = \ln x + 4\sqrt[3]{x} - \tan x + 16$
45. Suppose the volume of dirt on an ant hill at time t (in days) is $t + \ln t$ cubic inches. Find the rate at which the volume of the anthill is changing at time $t = 6$.
46. Write the equation of the line tangent to $y = 4e^{2x} + 5e^{-3x}$ when $x = 0$.

Exercises from Section 5.5

47. Find the second derivative of $f(x) = x^3 - \frac{1}{x} + 4\sin x$.
48. Find $\sigma''(x)$ if $\sigma(x) = \frac{2}{3}x^6 - \frac{2}{x} + 4$.
    Note: σ is the Greek letter sigma.
49. Let $y = 2\sin\theta$. Find $\frac{d^2y}{d\theta^2}$.
50. Find $\left.\frac{d^2f}{dx^2}\right|_{x=1}$ if $f(x) = \left(\frac{2}{x} + \sqrt{x}\right)$.
51. If $f(x) = 4e^x - 5x^4 + 3x$, find $f''(x)$.
52. Find the third derivative of $f(x) = \ln x$ when $x = 2$.
53. Find the 33rd derivative of $f(x) = e^x$.
54. Let $f(x) = \sin x$. Find $f^{(801)}(x)$.
55. Find the acceleration of an object at time 3, if the object's position (in inches) at time t (in seconds) is $f(t) = 2t^3 - t^2 + 4t$.

56. Find the acceleration of an object at time $\frac{2\pi}{3}$, if the object's velocity at time t is $v(t) = 3\sin t + 2$ mi/hr.

57. An object moves in such a fashion that its position after t units of time is $f(t) = e^t - 2t$. As time passes, is the object speeding up or slowing down?

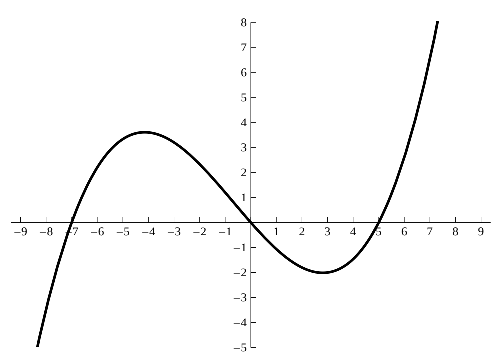
58. An object moves in such a fashion that its position at time t (measured in minutes) is $f(t) = t^3 - 9t^2$ cm. Find all times t where the acceleration of the object is zero.

59. Suppose f is some unknown function such that $f(4) = 0$, $f'(4) = -1$ and $f''(4) = 5$. Sketch a picture of what the graph of f looks like near $x = 4$.

60. Suppose g is some unknown function such that $g(-1) = 3$, $g'(-1) = 0$ and $g''(-1) = \frac{-2}{5}$. Sketch a picture of what the graph of g looks like near $x = -1$.

61. Suppose f is some unknown function such that $f(4) = 1$, $f'(4) = \frac{1}{7}$ and $f''(4) = \frac{-2}{3}$. Sketch a picture of what the graph of f looks like near $x = 4$.

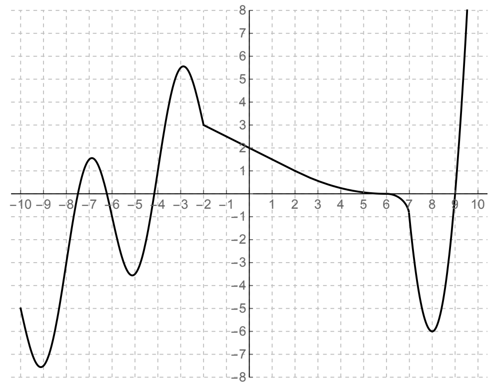
62. Pictured below is the graph of some unknown function f.



Use the graph to determine, with justification, whether each of the following quantities are positive, negative, or zero:

a) $f(5)$	d) $f(-6)$	g) $f(-1)$	j) $f(3)$
b) $f'(5)$	e) $f'(-6)$	h) $f'(-1)$	k) $f'(3)$
c) $f''(5)$	f) $f''(-6)$	i) $f''(-1)$	l) $f''(3)$

63. Pictured below is the graph of some unknown function g.



Use the graph to answer the following questions:

a) Estimate $g''(1)$.

b) Estimate $g''(5)$.

c) Estimate $g''(-7)$.

d) Find a value of x such that $g'(x) = 0$ but $g''(x) > 0$.

e) Find a value of x such that $g'(x) = 0$ but $g''(x) < 0$.

f) Find a value of x for which $g''(x)$ DNE.

64. The position of a bug which is crawling back and forth along the x-axis at various times t are given in the following chart:

time t (seconds)	0	1	4	8	12
position $x(t)$ (inches)	14	7	-5	40	220

a) Use the information in the chart to estimate $x'(3)$. Show the computations that lead to your answer, and write your answer with appropriate units.

b) In the context of this problem, what does your answer to part (a) mean?

c) In the context of this problem, what is the significance of the sign of your answer to part (a)?

d) Use the information in the chart to estimate $x''(6)$. Show the computations that lead to your answer, and write your answer with appropriate units.

e) In the context of this problem, what does your answer to part (d) mean?

f) In the context of this problem, what is the significance of the sign of your answer to part (d)?

65. During a snowstorm, you periodically measure the depth of snow that has fallen outside your house. Your observations are recorded in the following table:

time t (hours)	0	1	3	4	5	7	8
depth of snow $f(t)$ (inches)	0	6	15	18	20	21	24

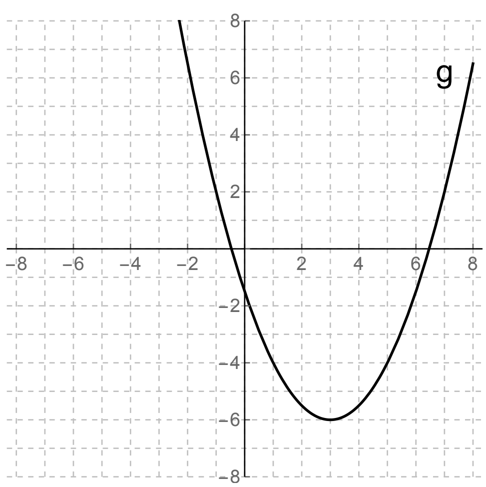
a) Use the information in the chart to estimate $f'(4)$. Show the computations that lead to your answer, and write your answer with appropriate units.

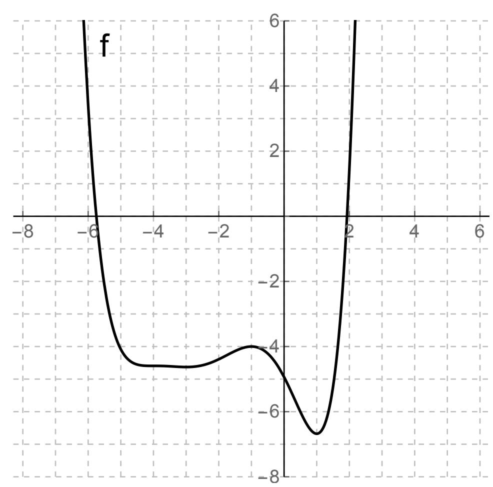
b) In the context of this problem, what does your answer to part (a) mean?

c) Use the information in the chart to estimate $f''(6)$. Show the computations that lead to your answer, and write your answer with appropriate units.

d) In the context of this problem, what does your answer to part (c) mean?

66. The graph of some unknown function g is shown below at left. Use this graph to sketch graphs of the functions g' and g''.

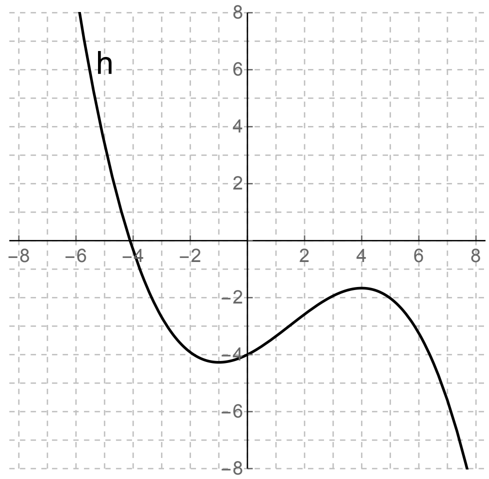




67. The graph of some unknown function f is shown above at right. Use this graph to sketch graphs of the functions f' and f''.

68. The graph of some unknown function h is shown below. Use this graph to

sketch graphs of the functions h', h'' and h'''.



69. Sketch the graph of any differentiable function f which has all of the following properties:

- $f'(3) > 0$;
- $f''(3) < 0$;
- $f'(-1) > 0$;
- $f''(-1) > 0$.

70. Sketch the graph of any differentiable function g which has all of the following properties:

- $g'(5) < 0$;
- $g''(5) < 0$;
- $g'(0) = 0$;
- $g''(0) < 0$.

71. Sketch the graph of any differentiable function h which has all of the following properties:

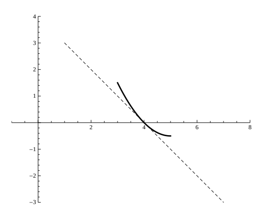
- $h'(2) = 0$;
- $h'(2) > 0$;
- $h'(-4) > 0$;
- $h''(-4) = 0$.

Answers

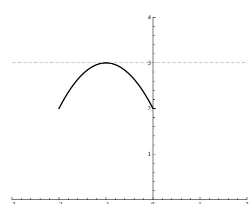
1. 0
2. $6x^5 + \dfrac{1}{\sqrt{x}}$
3. $3 + \dfrac{4}{5x^2}$
4. $\dfrac{97}{4}$ million \$ per 1000 people
5. $3x^2 - 4x + 4$
6. 37

7. $y = 3 - 5(x-1)$
8. 3 m/sec
9. $t = \frac{1}{2}$
10. $\frac{5}{2\sqrt{x}} + x^{-4/3}$
11. $\sqrt{7}\frac{1}{2\sqrt{x}}$
12. $\frac{3}{2}\sqrt{t} - 1 + \frac{1}{\sqrt{t}}$
13. $-\frac{2}{7x^2} - \frac{2}{7}$
14. $24x^2$
15. $\frac{3}{2}\sqrt{x} + \frac{1}{2}x^{-3/2}$
16. $4 - \frac{1}{3}x^{-2/3}$
17. $-14x^{-5}$
18. $-\frac{2}{3}x^{-5/3}$
19. $55x^{7/4} - 15x^4$
20. $\frac{5}{2}x^{-3/2} + 3x^{-2}$
21. $4t + 3 + \frac{1}{3t^2}$
22. $\frac{2}{3}\cos x - \frac{3}{4}\sin x - 2x$
23. $-1 - 4\sec^2 x$
24. $-\frac{\sqrt{2}}{2}$
25. $\frac{9}{2}$
26. $3 + \cos t$ km/hr
27. a) 20
    b) 18
    c) $\frac{8}{3}$
    d) $\frac{2}{3}$
    e) 0
    f) $(a-1)^x$
    g) 6
    h) 8
28. a) $e^{x \ln t}$
    b) $e^{8 \ln 4}$
    c) e^{7x}
    d) $e^{(x-1)\ln 6 + 3x \ln 2}$
    e) $e^{4x - x \ln 7}$
    f) $e^{4x \ln 3x}$
    g) $e^{x + (t+1)\ln 6x}$
    h) $e^{3x - x \ln 2}$
29. a) $\frac{\ln y}{\ln 10}$
    b) $\frac{3 \ln 11}{\ln 4}$
    c) $\frac{\ln \frac{2}{3}}{\ln \frac{1}{2}}$
    d) $\frac{\ln(x+4y)}{\ln 2}$
30. a) -2
    b) 15
    c) $\frac{1}{2}$
    d) 16
    e) -2
    f) 19
    g) 2
    h) -3
31. $7^x \cdot \ln 7$
32. $6 \cdot 8^x \cdot \ln 8 + 3\left(\frac{2}{5}\right)^x \ln \frac{2}{5}$
33. $-\frac{1}{4}e^x$
34. $4e^t - t + 10t + \cos t$

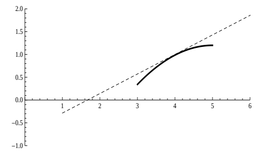
35. $15e^{3x} + 4\sin x$
36. $-\frac{1}{6}e^{-x} + \frac{3}{5}x^{-4/5}$
37. $\frac{5}{x} - 11e^{x}$
38. $10x - \frac{1}{\sqrt{x}} + \frac{1}{3x}$
39. $\frac{3}{x^2} + \frac{2}{x}$
40. $\frac{4}{\ln 7} \cdot \frac{1}{z}$
41. $\frac{1}{t}$
42. $-3e^{x}$
43. $-\frac{4}{3}x^{-7/3} - \frac{4}{x} + 8e^{x} - \frac{1}{2}e^{x/2}$
44. $\frac{1}{x} + \frac{4}{3}x^{-2/3} - \sec^2 x$
45. $\frac{7}{6}$ cubic in/day
46. $y = 9 - 7(x - 0)$
47. $6x - 2x^{-3} - 4\sin x$
48. $20x^4 - 4x^{-3}$
49. $-2\sin\theta$
50. $\frac{15}{4}$
51. $4e^{x} - 60x^2$
52. $\frac{1}{4}$
53. e^{x}.
54. $f(x) = \cos x$
55. 34 in/sec^2
56. $\frac{-3}{2}$ mi/hr^2
57. Speeding up (since acceleration is positive)
58. $t = 3$
59. Passes through $(4, 0)$, slope of tangent line is -1 and lies above the tangent line at 4:



60. Passes through $(-1, 3)$, curved downward such that the "peak" of the graph is at $(-1, 3)$:



61. Passes through $(4, 1)$, slope of tangent line is $\frac{1}{7}$, and graph lies below the tangent line:



62. a) $f(5) = 0$ (graph at $x-$axis at $x = 5$)
   b) $f'(5) > 0$ (graph going up from left to right)
   c) $f''(5) > 0$ (graph lies above tangent line)
   d) $f(-6) > 0$ (graph above $x-$axis at $x = -6$)
   e) $f'(-6) > 0$ (graph going up from left to right)
   f) $f''(-6) < 0$ (graph lies below tangent line)
   g) $f(-1) > 0$ (graph above $x-$axis at $x = -1$)
   h) $f'(-1) < 0$ (graph going down from left to right)
   i) $f''(-1) = 0$ (graph is straight at $x = -1$)
   j) $f(3) < 0$ (graph below $x-$axis at $x = 3$)
   k) $f'(3) = 0$ (tangent line horizontal)
   l) $f''(3) > 0$ (graph lies above tangent line)

63. a) 0
   b) $\approx \frac{1}{4}$ (a small positive number)
   c) ≈ -5 (a negative number)
   d) $x \approx -5.25$, $x \approx 8.1$
   e) $x \approx -6.8$, $x \approx -2.8$
   f) $x = -2$

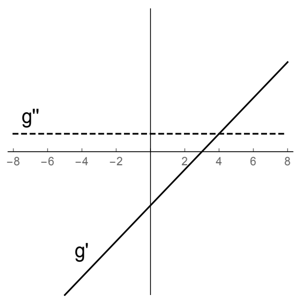
64. a) $x'(3) \approx \dfrac{x(4) - x(1)}{4 - 1} = \dfrac{-5 - 7}{4 - 1} = -4$ in/sec.
   b) The bug's velocity at time 3 is -4 in/sec.
   c) Since the velocity is negative, the bug is moving from right to left at time 3.
   d) $x'(6) \approx \dfrac{x(8) - x(4)}{8 - 4} = \dfrac{40 - (-5)}{8 - 4} = 11.25$ in/sec;
   $x'(10) \approx \dfrac{x(12) - x(8)}{12 - 8} = \dfrac{220 - 40}{12 - 8} = 45$ in/sec;
   $x''(6) \approx \dfrac{x'(10) - x'(6)}{10 - 6} = \dfrac{45 - 11.25}{10 - 6} \approx 8$ in/sec^2.
   e) The bug's acceleration at time 6 is 8 in/sec^2.
   f) Since the acceleration is positive, the bug is speeding up at time 6.

65. a) $f'(4) \approx \dfrac{f(5)-f(4)}{5-4} = 2$ and $f'(4) \approx \dfrac{x(4)-x(3)}{4-3} = 3$; averaging these we estimate $f'(4) \approx 2.5$ in/hr.

b) At time 4, the snow is falling at a rate of 2.5 inches per hour.

c) $f'(5) \approx \dfrac{f(5)-f(4)}{5-4} = 2$ and $f'(7) \approx \dfrac{f(7)-f(5)}{7-5} = \dfrac{1}{2}$. Then, $f''(6) \approx \dfrac{f'(7)-f'(5)}{7-5} = \dfrac{\frac{1}{2}-2}{2} = \dfrac{-3}{4}$ in/hr^2.

d) At time 6, since $f''(6) < 0$, the rate at which the snow is falling is decreasing (i.e. the snowstorm is "letting up").

66.


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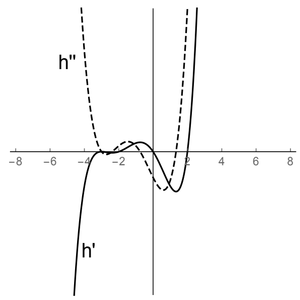
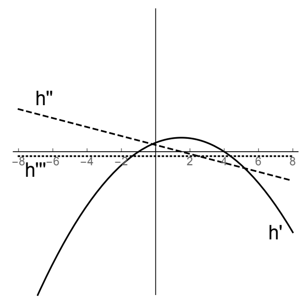
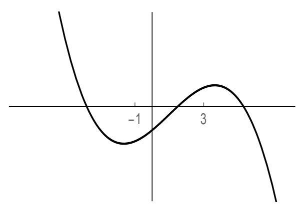
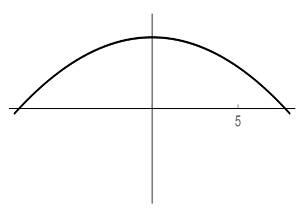


68.


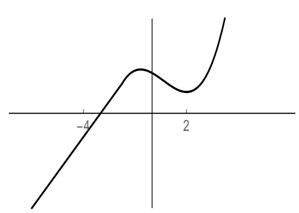
69. Answers may vary; one possible answer is



70. Answers may vary; one possible answer is



71. Answers may vary; one possible answer is



Chapter 6

Intermediate Differentiation Rules

6.1 Product rule

Question: What is $\frac{d}{dx}(fg)$ (a.k.a. $(fg)'$) in terms of f, g, f' and g'?

*First, what is $(fg)'$ **not** equal to?*

Some intiution involving units: Suppose x is time (measured in sec) and $f(x)$ and $g(x)$ are both distances (measured in meters). Then

$f'(x)$ is ________________, which is measured in ____________.

$g'(x)$ is ________________, which is measured in ____________.

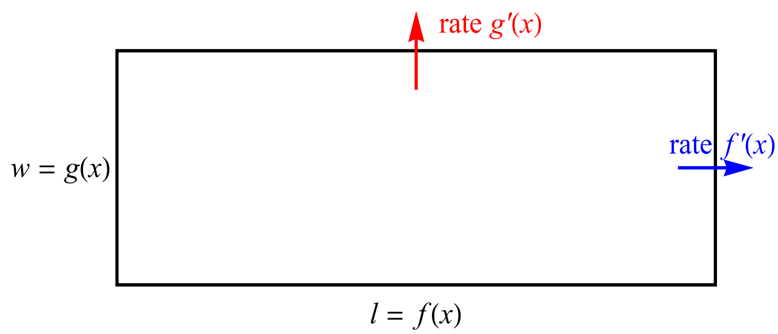
So $f'(x)g'(x)$ would be measured in ____________.

But $(fg)(x) = f(x)g(x)$ is ____________, which is measured in ____________,

which means $(fg)'(x)$ would be measured in ____________.

So this reinforces that $(fg)'(x) \neq f'(x)g'(x)$. But how do you compute $(fg)'(x)$?

More intuition: Suppose you have a rectangle whose length is $l = f(x)$ and whose width is $w = g(x)$. This makes the area $lw = (fg)(x)$. Suppose you increase l and w by a small amount. How much does the area change?



CONCLUSION:

Justification of this intuition:

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x+h) + f(x)\,g(x+h) - f(x)\,g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h)\,g(x+h) - f(x)\,g(x+h)}{h} + \frac{f(x)\,g(x+h) - f(x)\,g(x)}{h}\right] \\
&= \lim_{h\to 0} \left[g(x+h)\frac{f(x+h) - f(x)}{h}\right] + \lim_{h\to 0} \left[f(x)\frac{g(x+h) - g(x)}{h}\right] \\
&= g(x)\,f'(x) + f(x)\,g'(x).
\end{aligned}$$

This work proves the following theorem:

Theorem 6.1 (Product Rule) *Let f and g be differentiable at x. Then fg is differentiable at x and*

$$(fg)'(x) = f'(x)\,g(x) + g'(x)\,f(x).$$

The Product Rule says, in English, the following:

the derivative of a product is "*the derivative of the first times the second plus the derivative of the second times the first*".

EXAMPLE 1

Find y' if $y = 3x^2 \sin x$.

EXAMPLE 2

Find the slope of the line tangent to $f(x) = (2x^3 + 4x - 1) \tan x$, at $x = 0$.

EXAMPLE 3

Find $\dfrac{d^2y}{dx^2}$ if $y = x^4 e^x$.

Solution: First, by the Product Rule,

$$\frac{dy}{dx} = 4x^3 e^x + e^x x^4.$$

EXAMPLE 4

Find $f'(x)$ if $f(x) = \cos^2 x$.

6.2 Quotient rule

Theorem 6.2 (Quotient Rule) *Let f and g be differentiable at x, where $g(x) \neq 0$. Then $\frac{f}{g}$ is differentiable at x and*

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\,g(x) - g'(x)\,f(x)}{[g(x)]^2} = \underline{\hspace{6cm}}.$$

The proof of this is similar to the proof of the Product Rule and is omitted.

The Quotient Rule says, in English, the following:

> the derivative of a quotient is "*the derivative of the top times the bottom minus the derivative of the bottom times the top, all over the bottom squared*".

EXAMPLE 1

Find $\theta'(x)$ if $\theta(x) = \dfrac{2\sqrt{x} - 3x + 1}{5\ln x}$.

Solution: Apply the Quotient Rule:

$$\theta'(x) = \frac{\text{TOP}' \cdot \text{BOT} - \text{BOT}' \cdot \text{TOP}}{\text{BOT}^2}$$

EXAMPLE 2

Let $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$. Find the slope of the line tangent to f when $x = 0$.

EXAMPLE 3

Suppose that at time t (measured in seconds), the energy in a nuclear reaction is $\frac{3e^t}{t}$ Joules. Find the rate of change of the energy with respect to time.

EXAMPLE 4

Find $f'(x)$ if

$$f(x) = \frac{3\tan x + 6x^2 - 5x + 2}{-4\cos x - 3x^{-2/3} + 2}.$$

6.3 Derivatives of secant, cosecant and cotangent

The quotient rule can be used to compute the derivatives of $\sec x$, $\csc x$ and $\cot x$. You can either memorize the answers that are derived below, or remember how to "re-compute" them using the quotient rule, as necessary.

EXAMPLE 1

Find the derivative of $f(x) = \sec x$.

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$=$$

EXAMPLE 2

Find the derivative of $f(x) = \csc x$.

Solution:

$$\begin{aligned}\frac{d}{dx}(\csc x) &= \frac{d}{dx}\left(\frac{1}{\sin x}\right)\\ &= \frac{(1)' \cdot \sin x - (\sin x)' \cdot 1}{(\sin x)^2}\\ &= \frac{0 \cdot \sin x - \cos x \cdot 1}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = \boxed{-\csc x \cot x}.\end{aligned}$$

EXAMPLE 3

Find the derivative of $f(x) = \cot x$.

Solution:

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$

$$=$$

Theorem 6.3 (Derivatives of secant, cosecant and cotangent)

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

EXAMPLE 4

Find the instantaneous rate of change of the function $f(x) = 2x \sec x + 1$ when $x = 0$.

EXAMPLE 5

Let $y = \dfrac{\sec x + 3\cot x}{x - \sin x}$. Find $\dfrac{dy}{dx}$.

EXAMPLE 6

Find $\dfrac{d}{dt}\left(2\sqrt[5]{t^4}\csc t\right)$.

EXAMPLE 7

Suppose an object's position, measured in feet, at time t, measured in seconds, is given by $f(t) = e^t \sec t$. Find the object's velocity and acceleration at time 0.

EXAMPLE 8

Find $g'\left(\frac{\pi}{3}\right)$ if $g(t) = t^2 \sin t$.

Solution: First, by the Product Rule, $g'(t) = 2t \sin t + (\cos t)t^2$.

$$\begin{aligned} \Rightarrow g'\left(\frac{\pi}{3}\right) &= 2\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)\cdot\left(\frac{\pi}{3}\right)^2 \\ &= 2\left(\frac{\pi}{3}\right)\frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{\pi^2}{9}\right) \\ &= \boxed{\frac{\pi\sqrt{3}}{3} + \frac{\pi^2}{18}}. \end{aligned}$$

6.4 Chain rule

Goal: Compute derivatives of compositions. This means that given differentiable functions f and g, we want to find the derivative of $f \circ g$ in terms of f, f', g and g'.

Motivating example:

Suppose Mrs. Young (y) is moving 5 times as fast as Mrs. Underwood (u).

Suppose also that Mrs. Underwood is moving 3 times as fast as Mrs. Xavier (x).

What is the relationship between Mrs. Young's speed and Mrs. Xavier's speed?

Answer:

In the language of derivatives, the motivating example becomes the following question:

$$\text{“If } \frac{dy}{du} = 5 \text{ and } \frac{du}{dx} = 3\text{, what is } \frac{dy}{dx}\text{?”}$$

The answer is found as follows:

The general idea described here is what is called the Chain Rule:

Theorem 6.4 (Chain Rule, Leibniz notation)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

EXAMPLE 1

Find $\dfrac{dy}{dx}$ if $y = \sqrt{\sin x}$.

Question: What's missing here, given what we did on the previous page?

Continuing with this example, let $F(x) = \sqrt{\sin x}$. Then,

Theorem 6.5 (Chain Rule, prime notation) *If f and g are differentiable functions, then $f \circ g$ is differentiable and*

$$(f \circ g)'(x) = f'(g(x))\, g'(x).$$

I like to think of a composition as having an "outside" part (the f, which is the last thing you do in the function) and an "inside" part (the g, which is the first thing you do). If you are familiar with diagramming functions, this means the function diagrams as

$$x \xrightarrow{g} \xrightarrow{f} \qquad \text{or} \qquad x \xrightarrow{\text{IN}} \xrightarrow{\text{OUT}}$$

The Chain Rule says, in English, the following:

the derivative of a composition is "*the derivative of a composition is the derivative of the outside, with the inside plugged in, times the derivative of the inside*".

EXAMPLE 2

Find $\dfrac{d}{dx}\left[\left(\dfrac{1}{x} - \sin x\right)^4\right]$.

Alternate solution: Let $u = \dfrac{1}{x} - \sin x$. Then $y = u^4$, so by the Leibniz version of the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 4u^3 \cdot \left(\frac{-1}{x^2} - \cos x\right) = \boxed{4\left(\frac{1}{x} - \sin x\right)^3 \left(\frac{-1}{x^2} - \cos x\right)}.$$

EXAMPLE 3

Find y', if $y = \sqrt{3x+4}$.

EXAMPLE 4

Find the equation of the line tangent to $F(x) = (3x^2 - 3x - 1)^9$ when $x = 1$.

EXAMPLE 5

If an object's position, in feet, at time t (measured in minutes) is given by $f(t) = e^{-t}$, find the object's velocity and acceleration at time t.

Alternate solution: We already learned $\frac{d}{dx}(e^{rx}) = re^{rx}$, so we could just apply that rule (which is really a special case of the Chain Rule with IN $= rx$ and OUT $= e^x$).

EXAMPLE 6

Compute the second derivative of $y = \cos(x^2)$.

When to use the Product Rule, as opposed to the Chain Rule

EXAMPLE

$$\frac{d}{dx}\left(x^2 \sin x\right) \qquad \text{vs.} \qquad \frac{d}{dx}\left(\sin x^2\right)$$

Use of the Chain Rule in conjunction with other rules

EXAMPLES

Find the derivative of each of these functions:

1. $y = 2x \ln(4x^2 + 1)$

2. $f(x) = x^2 \cos^3 x - 4x \tan^2 x$

3. $y = \dfrac{(e^x + x^2 - 2)^3}{(x^{-3} - 1)^{3/2}}$

4. $f(x) = \sin\left(\dfrac{\ln x - 2}{\cos x + x}\right)$

   Solution: Start with the Chain Rule, because "$\sin$" doesn't mean anything by itself:

$$\begin{aligned} f'(x) &= \text{outside}'(\text{inside}) \cdot (\text{inside})' \\ &= \cos\left(\frac{\ln x - 2}{\cos x + x}\right) \cdot \left(\frac{\ln x - 2}{\cos x + x}\right)' \end{aligned}$$

   Now use the Quotient Rule to compute the inside′:

$$f'(x) = \boxed{\cos\left(\frac{\ln x - 2}{\cos x + x}\right) \cdot \frac{\frac{1}{x}(\cos x + x) - (-\sin x + 1)(\ln x - 2)}{(\cos x + x)^2}}.$$

5. $g(x) = \cos(\sqrt{\sec x})$

6.5 Implicit differentiation

Another application of the Chain Rule

Suppose $z = \sin y$ and $y = f(x)$, where you don't know what the function f is.

$$\frac{dz}{dx} = ?$$

Some ex:	$y = f(x)$	$z = \sin y$	$z'(x) = \frac{dz}{dx} = \frac{d}{dx}(\sin y)$
	x^6		
	$4\sqrt{x}$		
	e^x		
	$\sec x$		

General answer: By the Chain Rule,

EXAMPLE 1

Suppose that y is some unknown function of x. Find $\frac{d}{dx}\left(y^2 + 6y - 2\right)$.

In general: if y is an unknown function of x, then

$$\frac{d}{dx}\left(f(y)\right) = f'(y) \cdot \frac{dy}{dx}$$

Note: If y is a constant, rather than a function of x, then $\frac{d}{dx}\left(y^2 + 6y - 2\right) =$

EXAMPLE 2

$\frac{d}{dx}\left(x^4 - \sin y + 5\right) = ?$

EXAMPLE 3

$$\frac{d}{dx}\left(y^3 \sin x\right) = ?$$

Implicit differentiation of equations

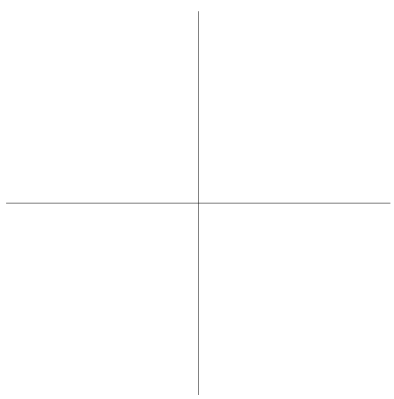
MOTIVATING EXAMPLE

Consider the equation $x^2 + y^2 = 25$.

This equation is not a function $y = f(x)$, for two reasons:

1.

2.



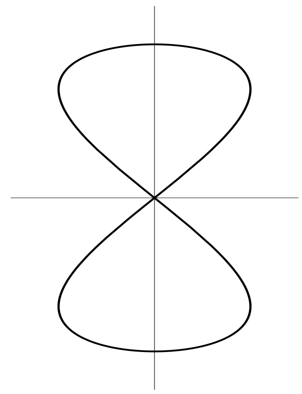
Suppose you wanted to write the equation of the tangent line to $x^2 + y^2 = 25$ at some point. You would need to compute $\frac{dy}{dx}$ at that point to get the slope. But which equation do you differentiate:

$$y = \sqrt{25 - x^2} \quad \text{or} \quad y = -\sqrt{25 - x^2}$$

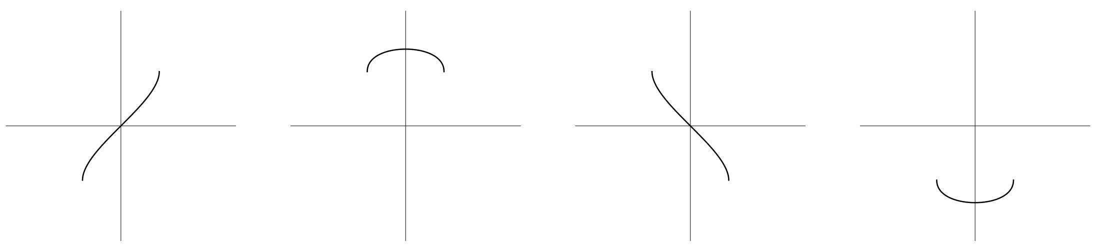
In this example, the choice is obvious:

But for a more interesting equation, there is no way to tell which equation to use. Consider the equation

$$4(y^2 - x^2) = y^4.$$



If you solve for y, you will get four different solutions:



There's no (easy) way to tell which solution goes with which graph.

Question: Is there a way to compute $\dfrac{dy}{dx}$ for some equation without solving for y in terms of x?

Answer: Yes. The method is called **implicit differentiation**. To implement it, start with the equation and differentiate both sides with respect to x (i.e. "take $\dfrac{d}{dx}$ of both sides").

General procedure to implement implicit differentiation:

1. Take $\dfrac{d}{dx}$ of both sides (as with the examples earlier).
2. If you are given x and/or y values, plug them in.
3. Solve for $\dfrac{dy}{dx}$.

EXAMPLE 4

Find the slope of the line tangent to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Follow up question # 1: What is the equation of the line tangent to the circle $x^2 + y^2 = 25$ at $(3, -4)$?

Follow up question # 2 (time permitting): In the preceding example, how would you determine the value of the second derivative at $(3, -4)$ (i.e. how would you measure the concavity of the circle)?

(If we skip this, omit problems 89 and 90 in the homework problems of Section 6.7.)

EXAMPLE 5

Compute $\left.\frac{dy}{dx}\right|_{x=3,y=3}$ for the equation $x^3 + y^3 = 6xy$.

EXAMPLE 6

Compute $\frac{dy}{dx}$ for the equation $x + e^{2xy} = 10y^3$.

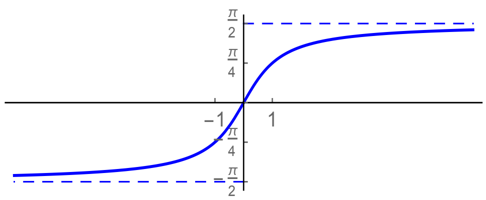
6.6 Derivatives of inverse trigonometric functions

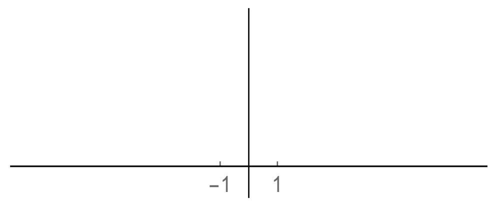
Implicit differentiation can be used to find the derivatives of $\arctan x$ and $\arcsin x$:

EXAMPLE 1

Compute the derivative of $f(x) = \arctan x$.

First, what should the graph of $\dfrac{d}{dx} \arctan x$ look like?





To figure out the derivative of $\arctan x$, we will recall that $y = \arctan x$ means $\tan y = x$. So we will find $\dfrac{dy}{dx}$ by implicitly differentiating $\tan y = x$:

EXAMPLE 2

Compute $f'(1)$ if $f(x) = \sqrt{x} \arctan x$.

EXAMPLE 3

Find the derivative of $f(x) = \arcsin x$.

Solution: As in Example 1, rewrite the function and use implicit differentiation:

$$\begin{aligned} y = \arcsin x \Leftrightarrow x &= \sin y \\ 1 &= \cos y \, \frac{dy}{dx} \\ 1 &= \sqrt{\cos^2 y} \, \frac{dy}{dx} \\ 1 &= \sqrt{1 - \sin^2 y} \, \frac{dy}{dx} \\ 1 &= \sqrt{1 - x^2} \, \frac{dy}{dx} \\ \boxed{\frac{1}{\sqrt{1 - x^2}}} &= \frac{dy}{dx} \end{aligned}$$

Theorem 6.6 (Derivatives of arctangent and arcsine)

$$\frac{d}{dx}(\arctan x) = \frac{1}{x^2 + 1} \qquad \frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}$$

EXAMPLE 4

Compute y', if $y = x^3 \arcsin x$.

Solution: Use the Product Rule:

$$\begin{aligned} y' &= (x^3)' \arcsin x + (\arcsin x)' x^3 \\ &= \boxed{3x^2 \arcsin x + \frac{1}{\sqrt{1 - x^2}} \cdot x^3}. \end{aligned}$$

EXAMPLE 5

Compute $\dfrac{d}{dx}\left(\dfrac{1}{4} \arcsin \dfrac{x}{4}\right)$.

Solution: Use the Chain Rule (outside $= \arcsin x$; inside $= \frac{x}{4}$):

$$\begin{aligned} y' &= \text{OUT}'(\text{IN}) \cdot \text{IN}' \\ &= \frac{1}{\sqrt{1 - \text{IN}^2}} \cdot \frac{1}{4} \\ &= \frac{1}{\sqrt{1 - \left(\frac{x}{4}\right)^2}} \cdot \frac{1}{4} = \boxed{\frac{1}{4\sqrt{1 - \left(\frac{x}{4}\right)^2}}}. \end{aligned}$$

6.7 Summary of differentiation rules

Derivatives of functions that you should memorize:

Constant Functions	$\frac{d}{dx}(c) = 0$
Power Rule	$\frac{d}{dx}(x^n) = nx^{n-1}$ (so long as $n \neq 0$)
Special cases of the Power Rule:	$\frac{d}{dx}(mx + b) = m$
	$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
	$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$
	$\frac{d}{dx}(x^2) = 2x$
Trigonometric Functions	$\frac{d}{dx}(\sin x) = \cos x$
	$\frac{d}{dx}(\cos x) = -\sin x$
	$\frac{d}{dx}(\tan x) = \sec^2 x$
	$\frac{d}{dx}(\cot x) = -\csc^2 x$
	$\frac{d}{dx}(\sec x) = \sec x \tan x$
	$\frac{d}{dx}(\csc x) = -\csc x \cot x$
Exponential Functions	$\frac{d}{dx}(e^x) = e^x$
	$\frac{d}{dx}(e^{rx}) = re^{rx}$
	$\frac{d}{dx}(b^x) = b^x \cdot \ln b$
Natural Log Function	$\frac{d}{dx}(\ln x) = \frac{1}{x}$
Inverse Trig Functions	$\frac{d}{dx}(\arctan x) = \frac{1}{x^2+1}$
	$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

Rules that tell you how to differentiate more complicated functions:

Sum Rule	$(f + g)'(x) = f'(x) + g'(x)$
Difference Rule	$(f - g)'(x) = f'(x) - g'(x)$
Constant Multiple Rule	$(kf)'(x) = k \cdot f'(x)$ for any constant k
Product Rule	$(fg)'(x) = f'(x)\, g(x) + g'(x)\, f(x)$
Quotient Rule	$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\, g(x) - g'(x)\, f(x)}{[g(x)]^2}$
Chain Rule	$(f \circ g)'(x) = f'(g(x))\, g'(x)$

6.8 Homework exercises

Exercises from Sections 6.1-6.3

1. Let $g(x) = (x^2+1)(x^2-3x+4)$. Find $g'(x)$.
2. Let $f(x) = 4x^2 \ln x$. Find $\dfrac{df}{dx}$.
3. Find the derivative of $f(x) = \dfrac{x}{x^2-x+1}$.
4. Find $g'(1)$ if $g(x) = \dfrac{x}{\sqrt{x}+1}$.
5. Differentiate $f(x) = \dfrac{\sin x}{x^2}$.
6. Find the derivative of $f(x) = \sqrt{x}\sin x$.
7. Find $\frac{dy}{dx}$ if $y = (2x^3 - x^{-2/3})e^x$.
8. Find the instantaneous velocity of an object at time $t = \dfrac{\pi}{3}$ seconds, if the position of the object is given by $f(t) = t^2 \sin t$ meters.
9. Find $\dfrac{d}{dx}\left[\left(\dfrac{1}{4}x^2 - 1\right)\ln x\right]$.
10. Find the second derivative of $f(x) = x\ln x$.
11. a) Find $f'(2)$ if $f(x) = 2\sin x\sqrt[5]{x}$.
    b) Explain in your own words what your answer to part (a) means.
12. Differentiate $f(x) = \dfrac{x^2+1}{x^3-1}$.
13. Find y' if $y = \dfrac{\cos x}{\sqrt{x}}$.
14. Find the slope of the line tangent to the graph of $f(x) = 4\cos x \sin x$ when $x = \dfrac{\pi}{4}$.
15. Find the acceleration of a particle at time t (measured in minutes), given that the particle's position at time t is $\dfrac{3t^2-4}{t^2+1}$ ft.
16. Let $y = \dfrac{8x^9 - \sin x}{\ln x + 5}$. Find $\dfrac{dy}{dx}$.

17. Find $f'(\pi)$ if $f(x) = x^2 \sin x$.

18. Find the equation of the line tangent to the graph of $y = \dfrac{\cos x}{x}$ when $x = \pi/2$.

19. Find the equation of the line tangent to $f(x) = (x-1)(x^2-2)$ at the point $(0, 2)$.

20. Suppose $f'''(x) = 2x \cos x$. Find $f^{(4)}(x)$.

21. Suppose f and g are functions such that $f(3) = 2$, $f'(3) = -1$, $g(3) = 4$ and $g'(3) = 2$. Find $(fg)'(3)$ and $\left(\dfrac{f}{g}\right)'(3)$.

22. Here is a table which lists of values of functions f, g, f' and g':

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	2	1	-2	-5	3	0	1	2	3
$f'(x)$	3	-1	4	2	-1	3	2	2	5
$g(x)$	2	-5	0	3	1	-4	2	0	-2
$g'(x)$	3	-2	-1	-2	4	1	0	3	7

Use this information to compute the following quantities:

a) $(fg)'(2)$

b) $(fg)'(0)$

c) $\left(\dfrac{f}{g}\right)'(4)$

d) $(f+3g)'(-1)$

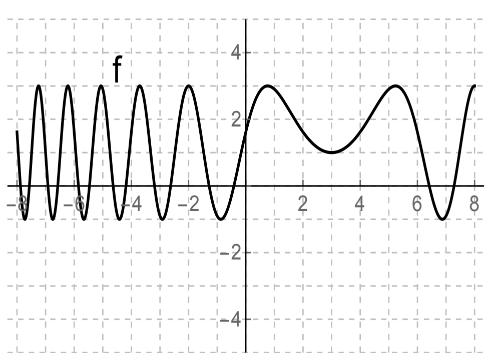
e) $\left(\dfrac{f}{f+g}\right)'(2)$

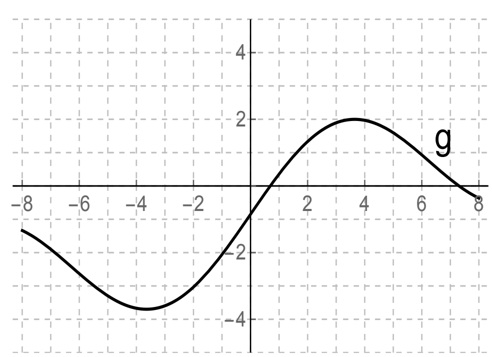
f) $h'(3)$, if $h(x) = x^2 f(x)$

g) $k'(-2)$, if $k(x) = 4x^3 g(x)$

h) $\left.\dfrac{d}{dx}\left(\dfrac{x}{g(x)}\right)\right|_{x=-1}$

23. The graphs of two functions f and g are shown below:





Use the graphs to estimate these quantities:

a) $(fg)'(0)$

b) $(fg)'(3)$

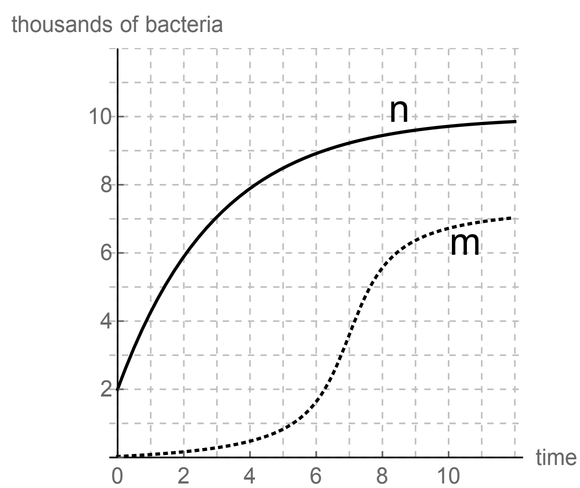
c) $\left(\frac{f}{g}\right)'(2)$

d) $\left(\frac{f}{g}\right)'(-5)$

e) $b'(6)$, if $b(x) = 5xg(x)$

f) $\left.\frac{d}{dx}\left(\frac{g(x)}{x}\right)\right|_{x=-6}$

24. A team of biologists studies the behavior of a bacteria colony under the effect of exposure to radiation as time passes. They produce graphs of functions n and m, where $n(t)$ is the bacteria population (measured in thousands of bacteria) at time t (measured in hours), and $m(t)$ is the number of mutated bacteria (measured in thousands of bacteria) at time t (measured in hours). Graphs of these functions are shown below:



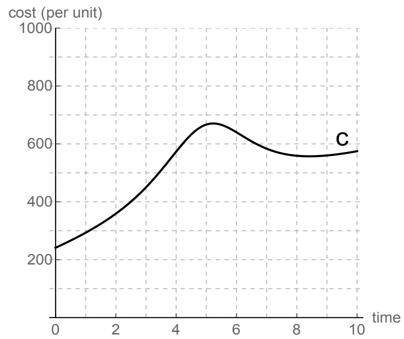
a) Let p be the proportion of bacteria that have mutated at time t. Write p as a function of m and n.

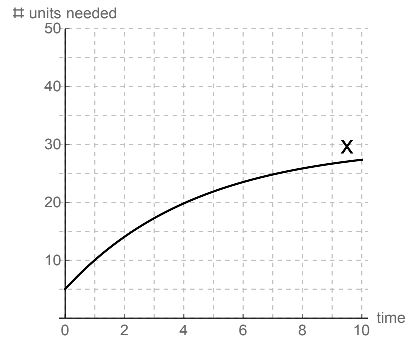
b) Estimate $p'(7)$ from the given graphs. Write your answer with appropriate units.

c) In terms of the context of this problem, what does your answer to part (b) mean?

25. Suppose that at time t (measured in months), a raw material needed by a business costs $c(t)$ dollars per unit. Suppose also that at time t (in months), the business estimates that it needs $x(t)$ units of the material. If the graphs of c and x are as given below, what is the instantaneous rate of change of the

company's total raw material costs relative to time, when $t = 7$?





In Problems 26-33, find the derivative of the given function.

26. $f(x) = 2\cot x$

27. $y = 3x^4 \csc x$

28. $f(x) = \dfrac{-1}{x^2} + \sec x - 4\sin x$

29. $f(x) = \dfrac{\sec x}{x}$

30. $y = x\sin x - \dfrac{2x}{\cot x}$

31. $f(x) = \dfrac{1}{4}\sqrt{x} + 3 - 5\csc x$

32. $y = \sqrt[4]{x} + 6\tan x - 3\cot x$

33. $f(x) = \ln x \sin x$

Exercises from Section 6.4

34. a) Find the derivative of $f(x) = (x-3)^{-3}$ using the Chain Rule.
    b) Find the derivative of $f(x) = (x-3)^{-3}$ by rewriting the function (to get rid of the negative exponent) and using the Quotient Rule.
    c) Verify that the answers you got in (a) and (b) are the same.

35. Find the derivative of $y = (2x-3)^8$.

36. Find $f'(2)$ if $f(x) = \sqrt{8-x}$.

37. Find $\dfrac{dy}{dx}$ if $y = \sqrt[3]{4x^2+5}$.

38. Differentiate $\pi(x) = \csc^2 x$.

    Note: In this problem, π is not the number π; it is just the name of the function.

39. Find the derivative of $f(x) = 4\ln(\cos x)$.

40. Suppose an object's position at time t (in seconds) is $\cos\left(\dfrac{3\pi t}{2}\right)$ mm. Find the velocity of the object at the instant $t = 1$.

41. Find $\frac{d^2y}{dx^2}$ if $y = (5x-1)^{-3}$.

42. Find the derivative of $f(x) = e^{5x}$.

43. Find the derivative of $f(x) = \sin\frac{x}{2}$.

44. Find the slope of the line tangent to $f(x) = 3\cos(x^2)$ when $x = 0$.

45. Find y' if $y = \sec\frac{1}{x} - x^2$.

46. Let $f(x) = \frac{1}{4}\sin^4(2x)$. Find $f'(x)$.

47. Find the equation of the line tangent to $f(x) = \sqrt{x^2+2x+8}$ when $x = 2$.

48. Suppose an object's position (in feet) at time t (in seconds) is given by $f(t) = (t^2+3)e^{2t}$. Find the velocity of the object when $t = 0$.

49. Suppose f and g are functions such that $f(1) = 4$, $f'(1) = -3$, $f(3) = 2$, $f'(3) = 5$, $g(3) = 1$ and $g'(3) = 2$. Find $(fg)'(3)$ and $(f \circ g)'(3)$.

50. Suppose $\frac{dy}{du} = 3$ and $\frac{du}{dx} = 6$. What is $\frac{dy}{dx}$?

51. Suppose $\frac{dy}{dx} = 8$ and $\frac{du}{dx} = 4$. What is $\frac{dy}{du}$?

52. Suppose $\frac{dy}{dv} = 5$ and $\frac{dx}{dv} = 3$. What is $\frac{dy}{dx}$?

53. Use the table of values given in Problem 22 above to compute the following quantities:

    a) $(f \circ g)'(2)$
    b) $(g \circ f)'(-3)$
    c) $(f \circ f)'(0)$
    d) $(g \circ f)'(4)$
    e) $r'(1)$, if $r(x) = g(2x)$
    f) $h'(2)$, if $h(x) = (f(x))^2$
    g) $H'(2)$, if $H(x) = f(x^2)$
    h) $k'(0)$, if $k(x) = f(g(x)\cos x)$
    i) $z'(-2)$, if $z(t) = x^2 f(g(t))$
    j) $w'(1)$, if $w(x) = g(f(x)g(x))$

54. Use the graphs given in Problem 23 above to estimate these quantities:

    a) $(f \circ g)'(0)$
    b) $(g \circ f)'(0)$
    c) $(f \circ g)'(6)$
    d) $(f \circ f)'(-5)$
    e) $r'(-1)$, if $r(x) = f(2x)$
    f) $h'(-2)$, if $h(x) = (g(x))^2$

In Problems 55-74, find the derivative of the given function.

55. $f(x) = x^2(x-2)^4$
56. $f(x) = x\sqrt{4-x^2}$
57. $f(x) = \left(\frac{1-2x}{x+1}\right)^5$
58. $f(x) = \sqrt{x\cos x}$
59. $f(x) = \frac{x^3-2}{\sqrt{x^6+1}}$
60. $f(x) = \sin\left(\frac{x+1}{x-1}\right)$
61. $f(x) = \cos(\tan x)$
62. $f(x) = \cos x \tan x$
63. $f(x) = \cos(x\tan x)$
64. $y = \cot^4(5x+1)$
65. $y = \sqrt{\frac{x}{x-1}}$
66. $f(x) = e^{\sin x}$
67. $f(x) = e^{2x-5}$
68. $g(x) = \ln(x^2+8x+5)$
69. $f(x) = \sec^2(4x)$
70. $f(x) = \frac{3}{x} - \sqrt{x} + x^2\sin x$
71. $f(x) = 2 + \ln x - x^7e^{4x}$
72. $f(x) = \frac{3+x^2\cot x}{4\sqrt{x} - \sin(e^x)}$
73. $f(x) = x^x$
74. $f(x) = x^{2x}$

Exercises from Section 6.5

75. Compute $\frac{d}{dx}\left(3y^2+5y\right)$.
76. Compute $\frac{d}{dx}\left(4y^5-3x^3\right)$.
77. Compute $\frac{d}{dx}\left(y^2e^{3x}\right)$.
78. Compute $\frac{d}{dx}\left(4x^3y^2\right)$.

In Problems 79-84, find the derivative $\frac{dy}{dx}$.

79. $x^2+y^2=49$
80. $x^3-xy+y^2=4$
81. $\sin x + 2\cos 2y = 1$
82. $x = \cos(xy)$
83. $e^x = \frac{x}{e^y}$
84. $\ln y = \cos x$
85. Find the slope of the line tangent to $x^2y - y^3 = -8$ at the point $(0,2)$.

86. Find the equation of the line tangent to $(x^2+y^2)^2 = 4x^2y$ at the point $(1,1)$.

87. Find the equation of the line tangent to the elllipse $\frac{x^2}{2}+\frac{y^2}{8}=1$ at $(1,2)$.

88. Find the slope of the line tangent to the hyperbola $\frac{y^2}{6}-\frac{x^2}{8}=1$ at the point $(-2,-3)$.

89. Find $\frac{d^2y}{dx^2}$ if $x^2+y^2=4$.

    Hint: First find $\frac{dy}{dx}$, then take the derivative of that expression implicitly.

90. Find $\frac{d^2y}{dx^2}$ if $y^2=\sin x$.

Exercises from Section 6.6

In Problems 91-96, find the derivative of the given function.

91. $f(x)=\arctan 2x$
92. $f(x)=x^3\arctan x$
93. $f(x)=\arcsin x-\frac{1}{x}+\ln x-2$
94. $f(x)=4\arcsin 3x$
95. $f(x)=x\arctan 2x$
96. $f(x)=e^{\arctan x}$

Mathematica questions (for Exam 2 review)

97. Write *Mathematica* commands which will compute the derivative of the function $f(x)=3\sin(2x^4-8)\tan(3\ln x)$ when $x=4$.

98. Write *Mathematica* commands which will compute the eighth derivative of the function $f(x)=\frac{2}{x}-\csc x$.

99. Write the output you will get (either in *Mathematica* syntax or hand-written notation) when you execute the following commands in *Mathematica*:

```
g[x_] = Log[x] +3
g''[2]
```

100. Write the output you will get (either in *Mathematica* syntax or hand-written notation) when you execute the following commands in *Mathematica*:

```
h[x_] = Cos[x] + 3x^20
D[h[x], {x, 42}]
```

Answers

1. $2x(x^2-3x+4)+(2x-3)(x^2+1)$
2. $8x\ln x+4x$
3. $\dfrac{x^2-x+1-x(2x-1)}{(x^2-x+1)^2}$.
4. $\dfrac{3}{8}$
5. $\dfrac{x^2\cos x-2x\sin x}{x^4}$
6. $\dfrac{1}{2\sqrt{x}}\sin x+\sqrt{x}\cos x$
7. $(6x^2+\frac{2}{3}x^{-5/3})e^x+(2x^3-x^{-2/3})e^x$
8. $\dfrac{\pi\sqrt{3}}{3}+\dfrac{\pi^2}{18}$ m/sec
9. $\dfrac{1}{2}x\ln x+\left(\dfrac{1}{4}x^2-1\right)\dfrac{1}{x}$
10. $\dfrac{1}{x}$
11. a) $2\sqrt[5]{2}\cos 2+\frac{2}{5}2^{-4/5}\sin 2$
    b) The answer in part (a) is some number which gives the slope of the line tangent to f at $x=2$.
12. $\dfrac{2x(x^3-1)-3x^2(x^2+1)}{(x^3-1)^2}$
13. $\dfrac{-\sqrt{x}\sin x-\frac{1}{2\sqrt{x}}\cos x}{x}$.
14. 0
15. $\dfrac{14(1-3t^2)}{(1+t^2)^3}$ ft/min^2
16. $\dfrac{(72x^8-\cos x)(\ln x+5)-\frac{1}{x}(8x^9-\sin x)}{(\ln x+5)^2}$
17. $-\pi^2$.
18. $y=-\dfrac{2}{\pi}\left(x-\dfrac{\pi}{2}\right)$
19. $y=2-2x$
20. $2\cos x-2x\sin x$
21. $(fg)'(3)=0$; $\left(\dfrac{f}{g}\right)'(3)=\dfrac{-1}{2}$.
22. a) 4
    b) 11
    c) $\dfrac{-31}{4}$
    d) -4
    e) $\dfrac{4}{9}$
    f) 30
    g) 32
    h) $\dfrac{-7}{9}$
23. Answers can vary a bit here:
    a) 0
    b) $\dfrac{1}{3}$
    c) -2
    d) $\dfrac{3}{25}$
    e) -15
    f) $\dfrac{1}{4}$
24. a) $p(t)=\dfrac{m(t)}{n(t)}$
    b) $p'(7)\approx\dfrac{1}{4}$ hr^{-1}
    c) At time 7, the proportion of mutated bacteria is increasing at a rate of $1/4$ per hour.
25. -675 dollars per month
26. $-2\csc^2 x$
27. $12x^3\csc x-3x^4\csc x\cot x$
28. $2x^{-3}+\sec x\tan x-4\cos x$

29. $\dfrac{x\sec x\tan x-\sec x}{x^2}$
30. $\sin x + x\cos x - \dfrac{2\cot x + 2x\csc^2 x}{\cot^2 x}$
31. $\dfrac{1}{8\sqrt{x}} + 5\csc x\cot x$
32. $\dfrac{1}{4}x^{-3/4} + 6\sec^2 x + 3\csc^2 x$
33. $\dfrac{1}{x}\sin x + \ln x\cos x$
34. $-3(x-3)^{-4}$
35. $16(2x-3)^7$
36. $\dfrac{-1}{2\sqrt{6}}$
37. $\dfrac{8x}{3}(4x^2+5)^{-2/3}$
38. $-2\csc^2 x\cot x$
39. $-4\tan x$
40. $\dfrac{3\pi}{2}$ mm/sec
41. $300(5x-1)^{-5}$
42. $5e^{5x}$
43. $\dfrac{1}{2}\cos\dfrac{x}{2}$
44. 0
45. $\dfrac{-1}{x^2}\sec\dfrac{1}{x}\tan\dfrac{1}{x} - 2x$
46. $2\sin^3(2x)\cos(2x)$
47. $y = 4 + \dfrac{3}{4}(x-2)$
48. 6 ft/sec
49. $(fg)'(3) = 9$; $(f\circ g)'(3) = -6$.
50. 18
51. 2
52. $\dfrac{5}{3}$
53.
    a) 0
    b) -1
    c) -2
    d) 15
    e) -4
    f) 4
    g) 20
    h) 12
    i) -8
    j) -48
54.
    a) 0
    b) $\dfrac{4}{3}$
    c) $\dfrac{1}{4}$
    d) 0
    e) 2
    f) -3
55. $2x(x-2)^4 + 4(x-2)^3x^2$
56. $\sqrt{4-x^2} - \dfrac{x^2}{\sqrt{4-x^2}}$
57. $5\left(\dfrac{1-2x}{x+1}\right)^4 \cdot \dfrac{-2(x+1)-(1-2x)}{(x+1)^2}$
58. $\dfrac{1}{2\sqrt{x\cos x}}\cdot(\cos x - x\sin x)$
59. $\dfrac{3x^2\sqrt{x^6+1} - \frac{6x^5}{2\sqrt{x^6+1}}(x^3-2)}{x^6+1}$
60. $\cos\left(\dfrac{x+1}{x-1}\right)\cdot\dfrac{-2}{(x-1)^2}$
61. $-\sin(\tan x)\sec^2 x$
62. $-\sin x\tan x + \cos x\sec^2 x$
63. $-\sin(x\tan x)\cdot(\tan x + x\sec^2 x)$
64. $-20\cot^3(5x+1)\csc^2(5x+1)$
65. $\dfrac{1}{2\sqrt{\frac{x}{x-1}}}\cdot\dfrac{-1}{(x-1)^2}$
66. $e^{\sin x}\cos x$
67. $2e^{2x-5}$
68. $\dfrac{2x+8}{x^2+8x+5}$
69. $8\sec^2(4x)\tan(4x)$
70. $\dfrac{-3}{x^2} - \dfrac{1}{2\sqrt{x}} + 2x\sin x + x^2\cos x$
71. $\dfrac{1}{x} - 7x^6e^{4x} - 4x^7e^{4x}$

72. $\dfrac{(2x\cot x - x^2\csc^2 x)(4\sqrt{x} - \sin(e^x)) - (\frac{2}{\sqrt{x}} - e^x\cos(e^x))(3 + x^2\cot x)}{(4\sqrt{x} - \sin(e^x))^2}$

73. $x^x(1 + \ln x)$

74. $x^{2x}(2\ln x + 2)$

75. $6y\dfrac{dy}{dx} + 5\dfrac{dy}{dx}$

76. $20y^4\dfrac{dy}{dx} - 9x^2$

77. $2y\dfrac{dy}{dx}e^{3x} + 3e^{3x}y^2$

78. $12x^2y^2 + 8x^3y\dfrac{dy}{dx}$

79. $\dfrac{-x}{y}$

80. $\dfrac{y - 3x^2}{2y - x}$

81. $\dfrac{\cos x}{4\sin 2y}$

82. $\dfrac{-\csc xy - y}{x}$

83. $\dfrac{e^{x+2y} - e^y}{-xe^y}$

84. $-y\sin x$

85. 0

86. $y = 1$

87. $y = 2 - 2(x - 1)$

88. $\dfrac{1}{2}$

89. $\dfrac{-4}{y^3}$

90. $\dfrac{-2y^2\sin x - \cos^2 x}{4y^3}$

91. $\dfrac{2}{1 + (2x)^2}$

92. $3x^2\arctan x + \dfrac{x^3}{x^2 + 1}$

93. $\dfrac{1}{\sqrt{1 - x^2}} + \dfrac{1}{x^2} + \dfrac{1}{x}$

94. $\dfrac{12}{\sqrt{1 - 9x^2}}$

95. $\arctan 2x + \dfrac{2x}{1 + 4x^2}$

96. $e^{\arctan x} \cdot \dfrac{1}{1 + x^2}$

97. This takes two lines as shown here:

```
f[x_] = 3 Sin[2x^4 - 8] Tan[3 Log[x]]
f'[4]
```

98. This could be done in one line:

```
D[2/x - Csc[x], {x, 8}]
```

A different (but less good) way to do this is in two lines:

```
f[x_] = 2/x - Csc[x]
f''''''''[x]
```

99. $\dfrac{-1}{4}$

100. $-\cos x$

Chapter 7

Optimization Analysis

7.1 What is an optimization problem?

There are many situations in the real world where you need to determine how to make some quantity as large or as small as possible. Here are some examples:

EXAMPLE 1

If an archer shoots an arrow into the air at angle θ from the ground, it will travel a horizontal distance of $\frac{v \sin 2\theta}{g}$, where v and g are constants. At what angle should the archer shoot the arrow to make it travel as far as possible? (Equivalently, what is the maximum range of the archer?)

EXAMPLE 2

An epidemic spreads through a population in such a way that the number of infected people, I, is a function of the number of susceptible people, x, by the formula

$$I(x) = 4 \ln \left(\frac{x}{30} \right) - x + 30.$$

What is the maximum number of people who will become infected?

EXAMPLE 3

A patient's temperature change T, when given dose d of some medicine, is given by

$$T = \left(1 - \frac{d}{3} \right) d^2$$

What dosage maximizes this temperature change?

EXAMPLE 4

A farmer has 50 feet of fence with which to build a rectangular pen. What dimensions of the pen make its area as big as possible?

EXAMPLE 5

A box with a square base and no top is to be constructed from plywood. If there is 48 square feet of plywood available, and if the length, width and height of the box must be at least 1 foot, what is the largest volume of a box that can be made?

Common characteristics of Examples 1-5

1. In each example, there is some quantity you are allowed to "choose"; this quantity is the **variable**.
2. In each example, there is a second quantity which depends on the variable. This quantity is called the **utility**; the goal of the problem is to maximize or minimize the utility.

Any problem which asks you to maximize or minimize a utility function depending on one (or more) variables is called an **optimization problem**.

Here is the variable and utility for each of the first three examples on the previous page:

	Variable	Utility
Example 1		
Example 2		
Example 3		

Constrained optimization problems

Examples 4 and 5 are a little different, because there are two variables present in the problem.

In MATH 220, we can only solve an optimization problem with two or more variables if there is some extra information which relates the variables. This extra information is called a **constraint** on the variables. (Take MATH 320 - Calculus 3 - if you want to learn how to solve general optimization problems with more than one variable.)

	Variables	Utility	Constraint
Example 4			
Example 5			

We call problems like Examples 1 to 3 *free optimization problems* and problems like Examples 4 and 5 *constrained optimization problems*.

- **Free optimization problem:**

- **Constrained optimization problem:**

Converting a constrained optimization problem to a free optimization problem

The techniques of Math 220 are best suited to solving free optimization problems. So if you are given a constrained optimization problem, you first have to convert it to a free optimization problem by

1.

2.

Let's see how this works in Examples 4 and 5:

EXAMPLE 4

(variables x and y) (utility $A = xy$) (constraint $2x + 2y = 50$)

EXAMPLE 5

(variables x and y) (utility $V = x^2y$) (constraint $x^2 + 4xy = 48$)

Henceforth we will focus on solving free optimization problems. Keep in mind that whenever you are given a constrained optimization problem, the first step is to convert it to a free optimization as above.

7.2 Theory of optimization

Our goal is to determine the maximum and minimum of some utility function $f(x)$. To understand how this is done, we first need a lot of vocabulary:

Definition 7.1 *Given a function f and a specified domain D of that function:*

1. *We say f has an* **absolute maximum** *(a.k.a.* **global maximum***) at $x = c$ if $f(x) \leq f(c)$ for all $x \in D$. In this case $f(c)$ is called the* **absolute (global) maximum value** *of f on D.*

2. *We say f has an* **absolute minimum** *(a.k.a.* **global minimum***) at $x = c$ if $f(x) \geq f(c)$ for all $x \in D$. In this case $f(c)$ is called the* **absolute (global) minimum value** *of f on D.*

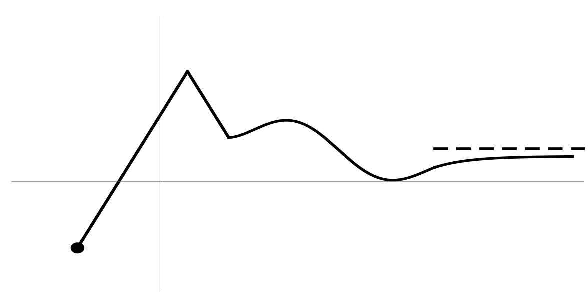
3. *We say f has a* **local maximum** *(a.k.a.* **relative maximum***) at $x = c$ if $f(x) \leq f(c)$ for all $x \in D$ sufficiently close to c. In this case $f(c)$ is called a* **local (relative) maximum value** *of f on D.*

4. *We say f has a* **local minimum** *(a.k.a.* **relative minimum***) at $x = c$ if $f(x) \geq f(c)$ for all $x \in D$ sufficiently close to c. In this case $f(c)$ is called a* **local (relative) minimum value** *of f on D.*

5. *Collectively, all maxima and minima are called* **extrema***.*

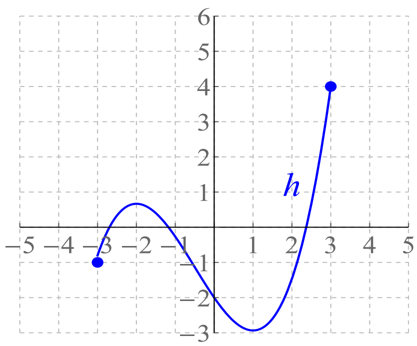
Note: If one says "f has a local maximum of 5 at 3", then one means that 5 is the $y-$value and 3 is the $x-$value, i.e. that the maximum is at the point $(3, 5)$.

Note: A function can have lots of local maxs/local mins, but has at most one global max and at most one global min. A list of all the local maxs (local mins) of a function always includes the global max (global min).



EXAMPLES

For each of the following graphs, identify all global extrema and all local extrema. At all local extrema which are **not** endpoints, find the derivative of the function at the extrema.

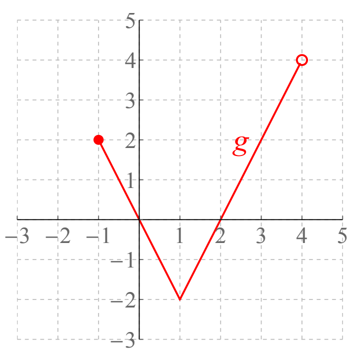


GLOBAL MAX:

GLOBAL MIN:

LOCAL MAX:

LOCAL MIN:

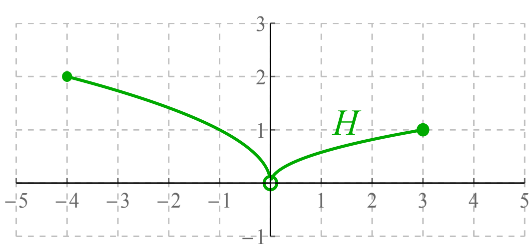


GLOBAL MAX:

GLOBAL MIN:

LOCAL MAX:

LOCAL MIN:

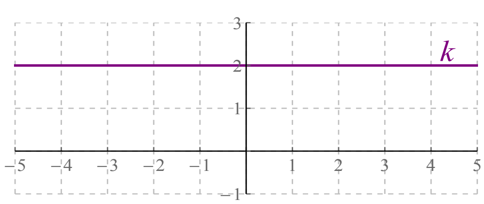


GLOBAL MAX: 2, at $x = -4$

GLOBAL MIN: none (there is no point on the graph at $(0, 0)$)

LOCAL MAX: $\begin{cases} 2 \text{ at } x = -4 \\ 1 \text{ at } x = 3 \end{cases}$

LOCAL MIN: none

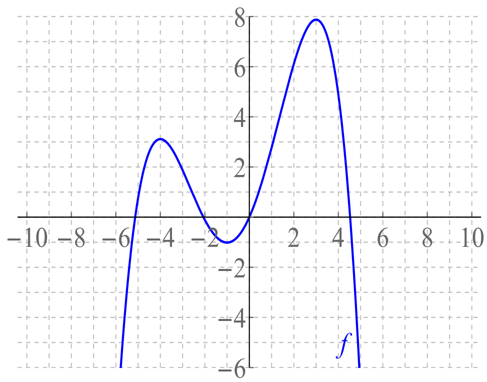


GLOBAL MAX: 2, at all x

GLOBAL MIN: 2, at all x

LOCAL MAX: 2, at all x

LOCAL MIN: 2, at all x



GLOBAL MAX: 8 at $x = 3$

GLOBAL MIN: DNE

LOCAL MAX: $\begin{cases} 3 \text{ at } x = -4 & f'(-4) = \\ 8 \text{ at } x = 3 & f'(3) = \end{cases}$

LOCAL MIN: -1 at $x = -1$ $\quad f'(-1) =$

Definition 7.2 *An* **optimization problem** *is a problem in which you are asked to find the absolute maximum and/or absolute minimum value of a function on some domain.*

Question 1: Does a function necessarily have an absolute maximum and/or absolute minimum? (In other words, does a generic optimization problem necessarily have a solution?)

Theorem 7.3 (Max-Min Existence Theorem) *If f is continuous on a closed and bounded interval $[a, b]$, then f has a global maximum value and a global minimum value on that interval.*

Note: The preceding theorem may fail if f is not cts, or if the interval is not closed, or if it is not bounded.

Question 2: How do you find the absolute maximum value and/or absolute minimum value of some function on some domain? (In other words, how do you solve an optimization problem?)

Definition 7.4 *A* **critical point** *(a.k.a.* **CP***) of a function f is a number c such that $f'(c) = 0$ or $f'(c)$ does not exist.*

Note: Critical points are numbers, not points. (They are the x-coordinates of points).

Theorem 7.5 (Critical Point Theorem) *All local extrema of a function (and therefore all global extrema) on an interval must occur at*

1. *endpoints of the interval, and/or*
2. *critical points of f lying in the interval.*

Note: Not all critical points are local extrema.

The Critical Point Theorem suggests a method of finding the global extrema of a function on an interval:

To optimize function f on interval $[a, b]$:

1. Find the critical points of f by
   (a) setting $f'(x) = 0$ and solving for x, and
   (b) finding all x for which $f'(x)$ DNE.
2. Discard any critical points which are not inside the interval $[a, b]$.
3. Plug each of the remaining critical points, as well as the two endpoints a and b, into the function f.

   The largest number you get is the absolute maximum, and the smallest number you get is the absolute minimum.

EXAMPLE A

Find the absolute extrema of the function $f(x) = 8 - x^2$ on the interval $[-4, 2]$.

EXAMPLE B

Find the absolute extrema of the function $f(x) = 2x^3 - 6x^2 + 1$ on the interval $[1, 3]$.

Solution: First, find CPs:

$f'(x) = 6x^2 - 12x$

$\underline{f'(x) = 0}$: $6x^2 - 12x = 0$
$6x(x-2) = 0$
$x = 0, x = 2$

$\underline{f'(x) \text{ DNE}}$: no such points

$\Rightarrow$ CPs: $x = 0, x = 2$

Third, test CPs and endpoints:

EXAMPLE C

Find the absolute extrema of the function $f(x) = 9\sqrt[3]{x}$ on the interval $[-1, 8]$.

Solution: First, find CPs:

$$f'(x) = 9\left(\frac{1}{3}\right)x^{-2/3} = \frac{3}{x^{2/3}}$$

$\underline{f'(x) = 0}$: $\frac{3}{x^{2/3}} = 0$
$3 = 0$
no such points

$\underline{f'(x) \text{ DNE}}$: $\frac{3}{x^{2/3}}$ DNE

Third, test CPs and endpoints:

Solving optimization word problems

General procedure to solve optimization word problems

1. Read the problem carefully, and draw a picture if necessary.
2. Identify any variable(s) and the utility (the quantity that needs to be maximized and/or minimized).
3. If there is more than one variable, find a constraint and convert the problem to a free optimization problem using the procedure outlined on p. 182.
4. Optimize the utility function using the procedure on p. 186 (find critical points, plug in critical points and endpoints to the utility, and choose the maximum and/or minimum value).
5. Make sure you answer the question that is asked.

EXAMPLE 1 (FROM PAGE 180)

If an archer shoots an arrow into the air at angle θ from the ground, it will travel a horizontal distance of $1000 \sin 2\theta$ ft. What is the maximum range of the archer?

EXAMPLE 4 (FROM PAGE 181)

A farmer has 50 feet of fence with which to build a rectangular pen. What dimensions of the pen make its area as big as possible?

EXAMPLE 5 (FROM PAGE 181)

A box with a square base and no top is to be constructed from plywood. If there is 48 square feet of plywood available, and if the length, width and height of the box must be at least 1 foot, what is the largest volume of a box that can be made?

EXAMPLE 2 (FROM PAGE 180)

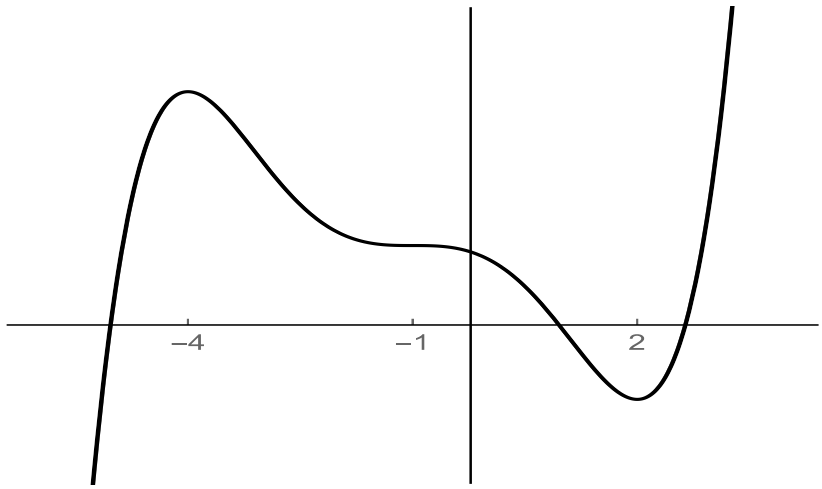
An epidemic spreads through a population in such a way that the number of infected people, I (measured in thousands), is a function of the number of susceptible people, x (measured in thousands), by the formula

$$I(x) = 4\ln\left(\frac{x}{30}\right) - x + 30.$$

What is the maximum number of people who will become infected?

7.3 Graphical analysis using derivatives

Tone



Definition 7.6

1. *A function f is called* **increasing** *on an open interval if for any x_1 and x_2 in that interval,*

$$x_1 \leq x_2 \text{ implies } f(x_1) \leq f(x_2).$$

2. *A function f is called* **decreasing** *on an open interval if for any x_1 and x_2 in that interval,*

$$x_1 \leq x_2 \text{ implies } f(x_1) \geq f(x_2).$$

3. *A function f is called* **monotone** *on an open interval if it is either increasing or decreasing on that interval.*

Note: Constant functions are both increasing and decreasing.

Note: Functions are always said to increase or decrease *on an open interval*, not at a point.

Theorem 7.7 (Monotonicity Test) *If f is differentiable on (a, b), then*

1. $f'(x) > 0$ *on* $(a, b) \Rightarrow f$ *is increasing;*
2. $f'(x) < 0$ *on* $(a, b) \Rightarrow f$ *is decreasing.*

EXAMPLE

Determine whether or not the function $f(x) = \dfrac{\ln x}{x}$ is increasing or decreasing on the interval $(0, 1)$. Determine whether or not f is increasing or decreasing on the interval $(4, 5)$.

Solution: Whether or not the function is increasing or decreasing depends on whether the derivative $f'(x)$ is positive or negative. By the Quotient Rule,

$$f'(x) = \frac{\frac{1}{x} \cdot x - 1 \cdot \ln x}{x^2} = \frac{1 - \ln x}{x^2}.$$

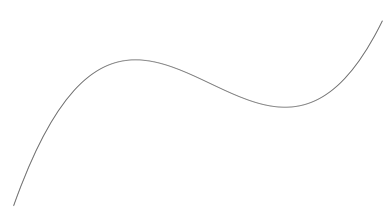
When $x \in (0, 1)$,

When $x \in (4, 5)$,

Concavity

Definition 7.8 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function.*

1. *f is called* **concave up** *(smiling) on an open interval if f' is increasing on that interval.*
2. *f is called* **concave down** *(frowning) on an open interval if f' is decreasing on that interval.*
3. *A number c is called an* **inflection point** *of f if the concavity of f changes at c.*



Theorem 7.9 (Concavity Test) *Let f be a function so that f'' exists on (a, b). Then:*

1. *if $f''(x) > 0$ for all $x \in (a, b)$, then f is concave up on (a, b);*
2. *if $f''(x) < 0$ for all $x \in (a, b)$, then f is concave down on (a, b).*
3. *c is an inflection point of f if and only if the sign of f'' changes at c.*

Remark: Based on the discussion from Chapter 5, if a function is concave up at/near x, then it will lie above the tangent line at x. If a function is concave down at/near x, then it will lie below the tangent line at x. If the function crosses its tangent line at x, then x is an inflection point of f.

EXAMPLE

Determine whether the function $f(x) = x^2e^{-x} + 2xe^{-x} - e^{-x}$ is concave up or concave down on the interval $(1, 2)$.

Solution: We need to determine whether $f''(x)$ is positive or negative on $(1, 2)$. So we compute $f''(x)$:

$$f(x) = x^2e^{-x} + 2xe^{-x} - e^{-x}$$

$$\begin{aligned}\Rightarrow f'(x) &= \left[2xe^{-x} + (-e^{-x})x^2\right] + \left[2e^{-x} + (-e^{-x})2x\right] - \left[-e^{-x}\right] \\ &= 2xe^{-x} - x^2e^{-x} + 2e^{-x} - 2xe^{-x} + e^{-x} \\ &= -x^2e^{-x} + 3e^{-x}\end{aligned}$$

$$\begin{aligned}\Rightarrow f''(x) &= \left[-2xe^{-x} - x^2(-e^{-x})\right] + \left[-3e^{-x}\right] \\ &= -2xe^{-x} + x^2e^{-x} - 3e^{-x}\end{aligned}$$

When $x \in (1, 2)$,

EXAMPLE

Find the inflection points of the function $f(x) = x^3 + 3x^2 - 2x + 1$.

Solution: Compute the second derivative of f:

$$f'(x) = 3x^2 + 6x - 2$$

$$f''(x) = 6x + 6$$

The second derivative can also be used to classify critical points as local maxima or local minima using the following test:

Theorem 7.10 (Second Derivative Test) *Suppose $f'(c) = 0$ and that f'' is continuous on an open interval containing c. Then:*

1. *if $f''(c) > 0$, then f has a local minimum at c;*
2. *if $f''(c) < 0$, then f has a local maximum at c;*
3. *if $f''(c) = 0$, then this test is inconclusive.*

More sophisticated ideas along the lines of the Second Derivative Test were developed in your lab assignment on applications of derivatives. These ideas are summarized in this theorem:

Theorem 7.11 (n^{th} Derivative Test) *Suppose f is continuous on an open interval containing c and $f'(c) = f''(c) = f'''(c) = \ldots f^{(n-1)}(c) = 0$ but $f^{(n)}(c) \neq 0$. Then:*

1. *if n is even and $f^{(n)}(c) > 0$, then f has a local minimum at c;*
2. *if n is even and $f^{(n)}(c) < 0$, then f has a local maximum at c;*
3. *if n is odd, then f has no local extremum at c.*

Before the days of *Mathematica* and graphics calculators, this is how people learned to sketch the graphs of functions. In 2024, it is more useful to use these ideas to study applied optimization problems.

7.4 More examples of optimization problems

EXAMPLE 6

A farmer grows zucchini. He has 10 acres available to plant; if he plants x acres his profit/loss will be $2x^3 - 33x^2 + 108x$ dollars. How many acres should the farmer plant (assuming he wants to make as much money as possible)?

EXAMPLE 7

In the human body, arteries must branch repeatedly to deliver blood to the entire body. Suppose a small artery branches off from a large artery at angle $\theta \in [0, \frac{\pi}{2}]$; the energy lost due to friction in this setting is approximately

$$E = \csc\theta + \frac{1-\cot\theta}{16}.$$

Find the value of θ that minimizes the energy loss.

Solution: First, write E as $E = \csc\theta + \frac{1}{16}(1-\cot\theta)$ and differentiate to get

$$E'(\theta) = -\csc\theta\cot\theta + \frac{1}{16}\csc^2\theta.$$

Next, find critical points: let $E'(\theta) = 0$ and solve for θ to get

$$\begin{aligned} 0 &= -\csc\theta\cot\theta + \frac{1}{16}\csc^2\theta \\ 0 &= \csc\theta\left(-\cot\theta + \frac{1}{16}\csc\theta\right) \end{aligned}$$

$\csc\theta = \frac{1}{\sin\theta}$ is never zero, so the only critical point is where $-\cot\theta + \frac{1}{16}\csc\theta = 0$. Rewriting with trig identities, we get

$$\frac{-\cos\theta}{\sin\theta} + \frac{1}{16\sin\theta} = 0 \;\Rightarrow\; -\cos\theta + \frac{1}{16} = 0 \;\Rightarrow\; \cos\theta = \frac{1}{16} \;\Rightarrow\; \theta = \arccos\frac{1}{16}.$$

Plug the EPs $\theta = 0$ and $\theta = \frac{\pi}{2}$ and the CP $\arccos\frac{1}{16}$ into the utility E:

$$\theta = 0: \qquad E = \csc 0 + \frac{1-\cot 0}{16} = 1 + \frac{1-\text{DNE}}{16} \text{ which DNE.}$$

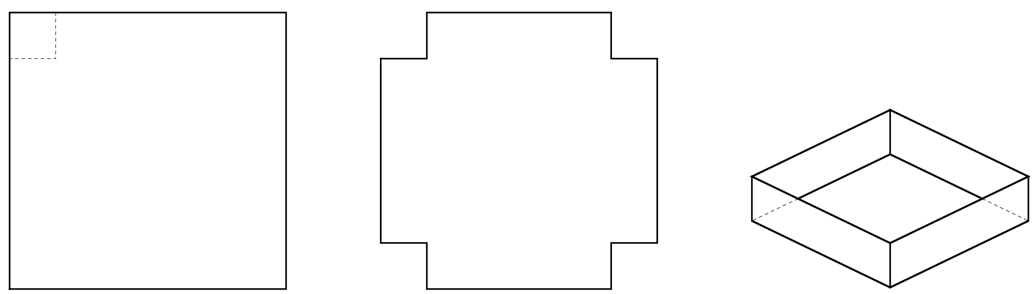
$$\begin{aligned} \theta = \arccos\frac{1}{16}: \quad E &= \csc(\arccos\frac{1}{16}) + \frac{1}{16}\left(1 - \cot(\arccos\frac{1}{16})\right) \\ &= \frac{16}{\sqrt{255}} + \frac{1}{16}\left(1 - \frac{1}{\sqrt{255}}\right) \\ &= \frac{1}{16}\left(1+\sqrt{255}\right). \end{aligned}$$

$$\theta = \frac{\pi}{2}: \qquad E = \csc\frac{\pi}{2} + \frac{1-\cot\frac{\pi}{2}}{16} = 1 + \frac{1-0}{16} = \frac{17}{16}.$$

Notice $\frac{1}{16}\left(1+\sqrt{255}\right) < \frac{1}{16}\left(1+\sqrt{256}\right) = \frac{1}{16}(1+16) = \frac{17}{16}$, so the absolute minimum is at $\boxed{\theta = \arccos\frac{1}{16}}$.

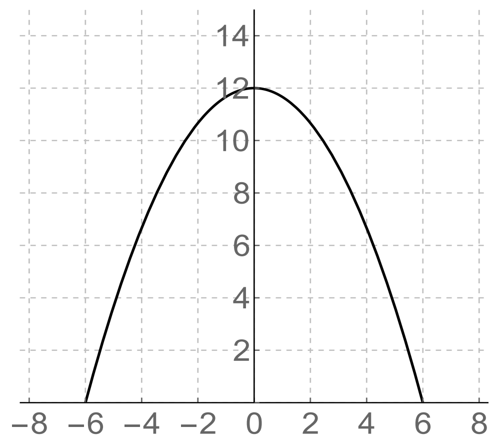
EXAMPLE 8

A 12″ by 12″ square sheet of cardboard is made into an open box (i.e. no top) by cutting squares of equal size out of each corner and folding up the sides along the dotted lines (see the pictures below). Find the dimensions of the box with the largest volume.



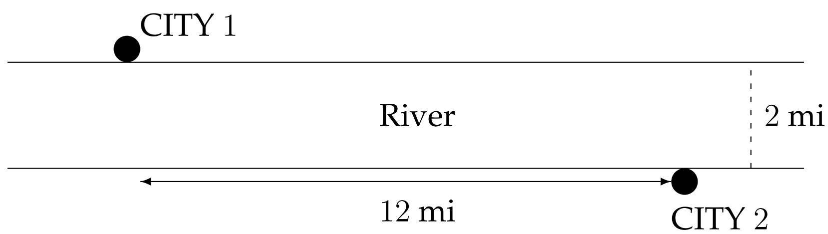
EXAMPLE 9

Find the maximum area of a rectangle if one side of the rectangle is on the $x-$axis and two corners of the rectangle are to be on the graph of $y = 12 - \frac{1}{3}x^2$ (this graph is shown below):

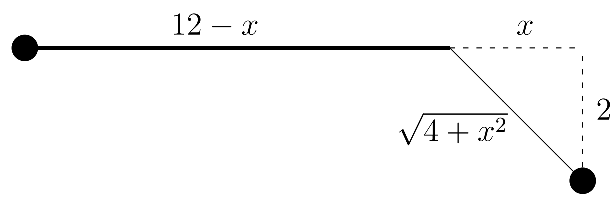


EXAMPLE 10

Michigan wants to build a new stretch of highway to link two sites on either side of a river (see the picture below) which is 2 miles wide. The second site is 12 miles downriver from the first site. It costs the state $13 million per mile to build over water and $5 million per mile to build over land. How should the state build its road to minimize costs?



Solution: First, it only makes sense to build a bridge in a straight line over the river, then to build along the riverbank to the other city. So the road goes along the solid lines shown below:



Letting x be as indicated in the picture, that means the cost of the road is

$$\begin{aligned} C(x) &= \text{cost of road along shore} + \text{cost of bridge} \\ &= 5(12-x) + 13\sqrt{4+x^2}. \end{aligned}$$

Our goal is to maximize this utility on the interval $[0, 12]$. First, differentiate (use the Chain Rule on the second term):

$$C'(x) = -5 + \frac{13}{2\sqrt{4+x^2}} \cdot (2x) = -5 + \frac{13x}{\sqrt{4+x^2}}.$$

Set this equal to zero and solve for x (details omitted, ask me if you don't follow this):

$$0 = -5 + \frac{13x}{\sqrt{4+x^2}} \Rightarrow 5 = \frac{13x}{\sqrt{4+x^2}} \Rightarrow x = \pm\frac{5}{6}$$

Plug the endpoints $x = 0$ and $x = 12$ and the critical point $x = \frac{5}{6}$ into the utility C; you will find that the minimum value of C is when $x = \frac{5}{6}$. Therefore the state should angle the bridge so that it goes $\boxed{\text{5/6 mile downastream}}$ as it crosses the river.

7.5 Homework exercises

Exercises from Section 7.1

In Problems 1-10, you are given a word problem. Identify the utility and write the utility as a function of <u>one</u> variable. (You do not need to actually solve the problem.)

1. Find the maximum product of two numbers whose sum is 12.
2. On a given day, the rate of traffic flow on a congested roadway is given by $F(v) = \dfrac{v}{24 + .01v^2}$, where v is the velocity of the traffic. Find the velocity which maximizes the rate of traffic flow.
3. A farmer will build a rectangular pen, where one side of the pen is against a river (and does not need to be fenced). If he wants the pen to enclose an area of 3 acres, what is the minimum amount of fence that he can use?
4. Find the maximum sum of two numbers, where the second number is three times the reciprocal of the first.
5. A box has a square base. Find the maximum volume of the box, if the surface area of the box is 300 square cm (assume that the box has a top and a bottom).
6. The potential energy of a particle moving along an axis (say the $x-$axis) is $E = b\left(\dfrac{a^2}{x^2} - \dfrac{a}{x}\right)$ where a, b are positive constants and $x > 0$. What value of x minimizes this potential energy?
7. A box with four sides and a bottom, but no top, has a square base. Find the minimum surface area of the box, if its volume is to be 80 cubic cm.
8. A rectangular box (with a top and bottom) has its length equal to twice its width. Find the maximum volume of the box, if the surface area of the box is 120 square inches.
9. Suppose the perimeter of a rectangle is P units, where P is a constant. Find the maximum area of such a rectangle.
10. A 6-foot tall wall runs parallel to the side of a building, 4 feet away from the building. Find the minimum length of a ladder that can lean up against the building and touch the ground, while just touching the top of the wall.

    Hint: Write the utility as a function of the angle the ladder makes with the ground.

Exercises from Section 7.2

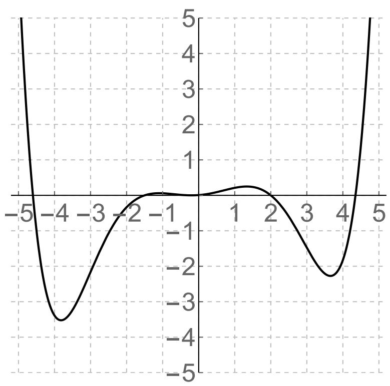
In Problems 11-16, you are given a graph of some unknown function f. In each picture, you should assume the graph continues to the left and right (i.e. that the extreme left and right ends of the graph have arrows on them). For each function:

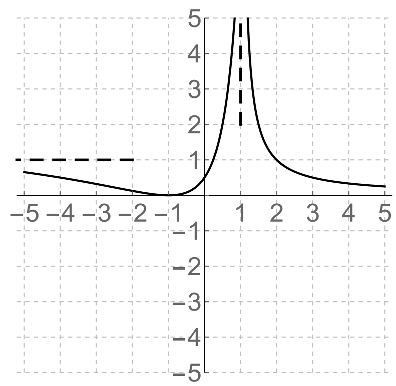
(a) Give the location of any local minima of f;

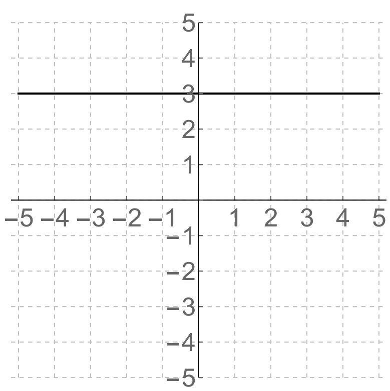
(b) Find the global minimum value of f on the interval $[-1, 4]$;

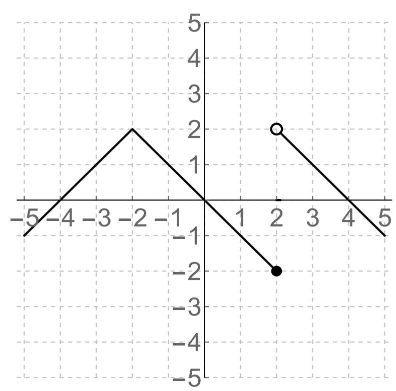
(c) Give the location of any local maxima of f;

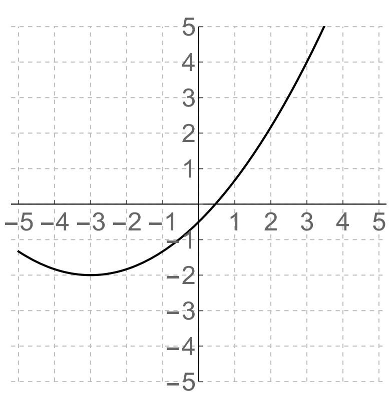
(d) Find the global maximum value of f on the interval $[-1, 4]$.

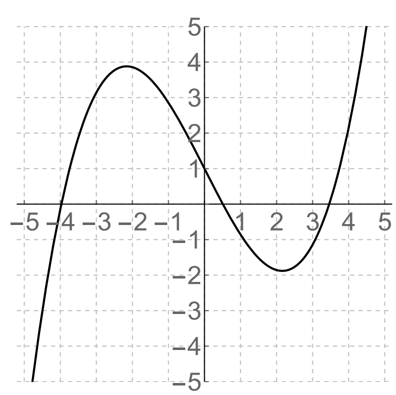
11. 

14. 

12. 

15. 

13. 

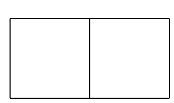
16. 

In Problems 17-24, find all critical points of the given function.

17. $f(x) = x^2(x^2 - 4)$
18. $f(x) = 3x^{1/5} + 2$
19. $f(x) = x^3 - 3x + 4$
20. $f(x) = |x|$
    Hint: consider the graph of f.
21. $f(x) = 4e^{-x}$
22. $f(x) = x^{7/3} - 28x^{1/3}$
23. $f(x) = \dfrac{3x}{x^2 - 1}$
24. $f(x) = \sin x + \cos x$
25. Show that the functions $f(x)$ and $e^{f(x)}$ have the same set of critical points.
    Hint: Let $g(x) = e^{f(x)}$. Explain why solving $g'(x) = 0$ and $f'(x) = 0$ gives the same solutions.

In Problems 26-35, find the absolute extrema of the given function on the indicated interval.

26. $f(x) = \sin x + \cos x$ on $[0, 2\pi]$
27. $f(x) = x^{2/3}$ on $[-1, 27]$
28. $f(x) = x^3 - 12x + 4$ on $[-3, 5]$
29. $f(x) = x^3 - 12x + 4$ on $[-3, 0]$
30. $f(x) = \dfrac{1}{2}e^{-x^2}$ on $[-4, 4]$
31. $f(x) = 5$ on $[-3, 4]$
32. $f(x) = \dfrac{x}{x - 2}$ on $[3, 5]$
33. $f(x) = \dfrac{3x}{x^2 - 1}$ on $[0, 2]$
34. $f(x) = 5 - x$ on $[1, 4]$
35. $f(x) = \arctan(x^2)$ on $[0, 1]$
36. If a person eats n sausages, then they will get heartburn in the amount of $h(n) = -n^3 + 12n$. If a person has the most amount of heartburn possible from eating sausages, how many sausages do they eat?
37. A farmer has 96 feet of fence with which to build a rectangular pen divided into two pieces as follows:



    What dimensions should the farmer use to build her pen, if she wants the enclosed area to be as big as possible?

38. In an endurance contest, athletes start 2 miles out to sea need to reach a location which is 2 miles inland and three miles east of their initial location (assume the seashore runs east-west). If an athlete can run 10 miles per hour and swim 5 miles per hour, what is the minimum amount of time she will need to reach the finish? (Use *Mathematica* to compute the derivative of your utility function, then use the `NSolve` command in *Mathematica* to solve for the critical point.)

39. Suppose that if a company spends x hundred dollars on advertising, then their profit will be $P(x) = -3x^3 + 225x^2 - 3600x + 18000$. How much should the company spend on advertising if they want to maximize their profit, assuming that they only have enough capital to spend $\$3000$ on advertising?

40. A box is made with a square base and no top. If the surface area of the box is 80 square units, what is the largest possible volume of the box?

Exercises from Section 7.3

In Problems 41-44, you are given a function f and an interval (a, b). Determine, with justification, the sign of f' on (a, b). Use the sign of f' to draw a conclusion about the behavior of f on (a, b).

41. $f(x) = x^2 + \dfrac{1}{x^2}$ on $(0, 1)$

42. $f(x) = e^x - e^{-x}$ on $(-1, 1)$

43. $f(x) = -2x^3 + 3x^2 - 5$ on $(2, 3)$

44. $f(x) = \ln\left(x + \dfrac{1}{x}\right)$ on $(0, 1)$

In Problems 45-49, find all the local extrema of the given function, and classify them as local maxima or local minima.

45. $y = x^4 + 4x^3 + 4x^2 - 3$

46. $f(x) = x \ln x$

47. $f(x) = x^2 - \dfrac{16}{x}$

48. $f(x) = e^{1/x^2}$

    Hint: The result of Problem 25 may be useful.

49. $f(x) = x + \dfrac{1}{x}$

In Problems 50-53, you are given a function f and an interval (a, b). Determine, with justification, the sign of f'' on (a, b). Use the sign of f'' to draw a conclusion about the behavior of f on (a, b).

50. $f(x) = e^x + e^{-x}$ on $(-\infty, \infty)$

51. $f(x) = x^4 - 16x^3 + 5$ on $(6, 7)$

52. $f(x) = -5 \sin x$ on $(\frac{\pi}{2}, \pi)$

53. $f(x) = \ln\left(x + \dfrac{1}{x}\right)$ on $(0, 1)$

In Problems 54-58, find all inflection points of the function.

54. $f(x) = x^3 - 3x^2 + 4x - 1$

55. $f(x) = x + \dfrac{1}{x}$

56. $f(x) = xe^{-2x}$

57. $f(x) = x^2 + 2x + 3$

58. $f(x) = \sin x - \cos x$

59. Suppose f is some function such that $f'(2) = f''(2) = f'''(2) = 0$ and $f^{(4)}(2) = -3$. Is $x = 2$ the location of a local maximum, local minimum, or neither?

60. Suppose f is some function such that $f'(4) = f''(4) = f'''(4) = ... = f^{(14)}(4) = 0$ and $f^{(15)}(4) = 2$. Is $x = 4$ the location of a local maximum, local minimum, or neither?

61. Suppose f is some function such that $f'(-1) = f''(-1) = f'''(-1) = ... = f^{(11)}(-1) = 0$ and $f^{(12)}(-1) = -5$. Is $x = -1$ the location of a local maximum, local minimum, or neither?

62. Suppose f is some function such that $f'(0) = f''(0) = f'''(0) = ... = f^{(100)}(0) = 0$ and $f^{(101)}(0) = -17$. Is $x = 0$ the location of a local maximum, local minimum, or neither?

63. Suppose f is some function such that $f'(4) = f''(4) = f'''(4) = ... = f^{(99)}(4) = 0$ and $f^{(100)}(4) = \frac{2}{3}$. Is $x = 4$ the location of a local maximum, local minimum, or neither?

Exercises from Section 7.4

64. Let $f(x) = \dfrac{x^2}{x^2+1}$ for $x > 0$. Determine where the graph of f is steepest (i.e. where the slope of the graph is a maximum).

65. The Gompertz growth curve, whose formula is

$$W(t) = ae^{-be^{-t}},$$

is useful in several fields (including biology and economics). Assuming a and b are positive constants, find the value of t at which the rate of change of $W(t)$ with respect to t is largest.

66. Suppose that a worker can make $Q(t) = -t^3 + 12t^2 + 60t$ items in t hours.

    a) Explain why the efficiency of the worker at time t can be measured by $Q'(t)$.

    b) Find the time at which the worker is most efficient.

67. A rectangular poster is to be made which consists of a printed region and an unprinted margin, which is 3 inches on the top and bottom but 2 inches on the left and right side. If the total area of the poster is to be 120 square inches, what dimensions of the poster maximize the area of the printed region?

68. Suppose a wire of length 4 ft is cut into two pieces. Each piece is bent to form a square; find the largest possible combined area from the two pieces.

69. The velocity of air moving through a person's windpipe is $V(r) = Cr^2(A-r)$ for constants C and A, where r is the radius of the windpipe.

    a) Find the radius which maximizes this velocity.

    b) Suppose that normally, a person's windpipe has radius A. When a person coughs, the windpipe changes radius so that air moves through the windpipe as quickly as possible. Based on your answer to (a), does a person's windpipe get wider or narrower when a person coughs?

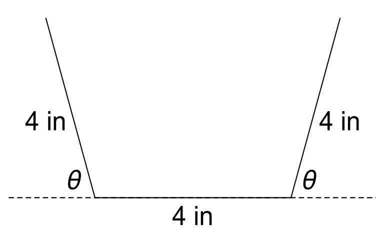
70. Find the point on the curve $y = \sqrt{x}$ which is closest to the point $(2, 0)$. *Hint:* Don't minimize the distance to the point; minimize the square of the distance to the point.

71. To transmit data (like a music file) electronically, the file has to be translated into a sequence of 0s and 1s so that it can be read by a computer or phone. An important computation related to the coding of files by 0s and 1s is the computation of a quantity called *entropy*, which is given by the following formula:

$$h(x) = x \ln x + (1-x)\ln(1-x)$$

    Find the value of $x \in (0, 1)$ which maximizes the entropy h.

72. A $12-$inch wide piece of sheet metal is bent to form a rain gutter. A cross-section of the gutter is shown in the picture below. What value of θ maximizes the volume of water that can be held by the gutter?



Answers

1. The utility is the product, denoted by $U(x) = x(12-x)$.

2. The utility is the rate of traffic flow, denoted by $F(v) = \dfrac{v}{24 + .01v^2}$.
3. The utility is the amount of fence used, denoted by $A(x) = x + \dfrac{6}{x}$ or $A(x) = 2x + \dfrac{3}{x}$, depending on your setup.
4. The utility is the sum of the numbers, denoted by $U(x) = x + \dfrac{3}{x}$.
5. The utility is the volume of the box, denoted by $V(x) = x^2 \cdot \dfrac{300 - 2x^2}{4x}$.
6. The utility is the energy, denoted by $E(x) = b\left(\dfrac{a^2}{x^2} - \dfrac{a}{x}\right)$.
7. The utility is the surface area of the box, denoted by $S(x) = x^2 + \dfrac{320}{x}$.
8. The utility is the volume of the box, denoted by $V(w) = \dfrac{2}{3}w(60 - 2w^2)$.
9. The utility is the area of the rectangle, denoted by $A(x) = x\left(\dfrac{P - 2x}{2}\right)$.
10. The utility is the length of the ladder, denoted by $L(\theta) = 6\csc\theta + 4\sec\theta$.

11.
    a) $x = -4, x = 0, x = 3.5$
    b) -2.2
    c) $x = -1.2, x = 1.4$
    d) $.4$
12.
    a) all x are local mins
    b) 3
    c) all x are local maxs
    d) 3
13.
    a) $x = -3$
    b) -1.2
    c) no local max
    d) 5
14.
    a) $x = -1$
    b) 0
    c) no local max
    d) no absolute max
15.
    a) $x = 2$
    b) -2
    c) $x = -2$
    d) no absolute max
16.
    a) $x = 2$
    b) -2
    c) $x = -2$
    d) 3
17. $x = 0, x = \sqrt{2}, x = -\sqrt{2}$
18. $x = 0$
19. $x = 1, x = -1$
20. $x = 0$
21. None
22. $x = 0, x = 2, x = -2$
23. $x = 1, x = -1$
24. $x = \dfrac{\pi}{4} + \pi n$

25. Let $g(x) = e^{f(x)}$. By the Chain Rule, the derivative of g is $g'(x) = e^{f(x)}f'(x)$. Since $e^{f(x)}$ always exists and is never zero, $g'(x) = 0$ only if $f'(x) = 0$ and $g'(x)$ DNE only if $f'(x)$ DNE. Thus $g(x)$ and $f(x)$ have the same critical points.

26. Max is $\sqrt{2}$ at $\pi/4$; min is $-\sqrt{2}$ at $5\pi/4$

27. Max is 9 at $x = 27$; min is 0 at $x = 0$

28. Max is 69 at $x = 5$; min is -12 at $x = 2$

29. Max is 20 at $x = -2$; min is 4 at $x = 0$

30. Max is $\frac{1}{2}$ at $x = 0$; min is $\frac{1}{2}e^{-16}$ at $x = \pm 4$

31. Max and min are 5 occuring at all x

32. Max is 3 at $x = 3$; min is $\frac{5}{3}$ at $x = 5$

33. No max or min because of the asymptote at $x = 1$

34. Max is 4 at $x = 1$; min is 1 at $x = 4$

35. Max is $\frac{\pi}{4}$ at $x = 1$; min is 0 at $x = 0$

36. 2 sausages

37. Relative to the picture in the homework assignment, the height should be 16 feet and the width (all the way across) should be 24 feet.

38. .728134 hours

39. $3000

40. $\frac{160}{3}\sqrt{\frac{5}{3}}$ cubic units.

41. $f'(x) = 2x - 2x^{-3} = 2x^{-3}(x^4 - 1) = (+)(-) < 0$ on $(0, 1)$, so f is decreasing on $(0, 1)$.

42. $f'(x) = e^x + e^{-x} > 0$ on $(-1, 1)$, so f is increasing on $(-1, 1)$.

43. $f'(x) = -6x^2 + 6x = -6(x)(x + 1) = (-)(+)(+) < 0$ on $(2, 3)$, so f is decreasing on $(2, 3)$.

44. $f'(x) = \frac{1}{x + \frac{1}{x}} \cdot \left(1 - \frac{1}{x^2}\right) = \frac{1}{+}(-) < 0$ on $(0, 1)$, so f is decreasing on $(0, 1)$.

45. $x = 0$ local min; $x = -1$ local max; $x = -2$ local min

46. $x = \frac{1}{e}$ local min

47. $x = -2$ local min

48. No local extrema

49. $x = -1$ local max; $x = 1$ local min

50. $f''(x) = e^x + e^(-x) > 0$ on $(-\infty, \infty)$, so f is concave up on $(-\infty, \infty)$.

51. $f''(x) = 12x^2 - 96x = 12x(x-8) = 12(+)(-) < 0$ on $(6, 7)$, so f is concave down on $(6, 7)$.

52. $f''(x) = 5\sin x > 0$ on $\left(\frac{\pi}{2}, \pi\right)$, so f is concave up on $\left(\frac{\pi}{2}, \pi\right)$.

53. $f''(x) = \frac{4x}{(x^2+1)^2} = \frac{+}{+} > 0$ on $(0, 1)$, so f is concave up on $(0, 1)$.

54. $x = 1$

55. None

56. $x = 1$

57. None

58. $x = \frac{\pi}{4} + \pi n$

59. local maximum

60. neither

61. local maximum

62. neither

63. local minimum

64. The graph is steepest at $x = \frac{1}{\sqrt{3}}$.

65. At $t = \ln b$.

66. a) The efficiency of the worker is the rate at which the worker makes items; this rate is given by the derivative $Q'(t)$.
    b) At $t = 4$.

67. The width should be $4\sqrt{5}$ inches and the height should be $6\sqrt{5}$ inches.

68. 1 sq ft. (Cut the wire into a piece of length 4 ft and a piece of length 0 ft.)

69. a) $r = 2A/3$.
    b) It gets narrower, since $2A/3$ is less than A.

70. $\left(\frac{3}{2}, \sqrt{\frac{3}{2}}\right)$

71. $x = \frac{1}{2}$

72. $\theta = \frac{\pi}{3}$

Chapter 8

Other Applications of Differentiation

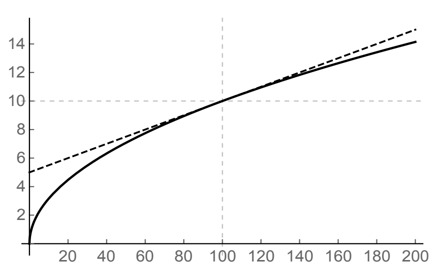
8.1 Tangent line and quadratic approximation

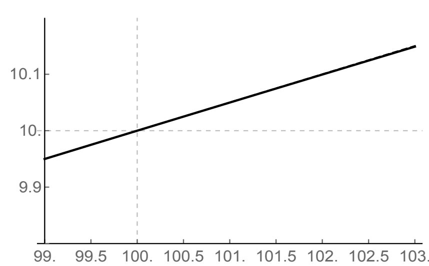
Motivation: Suppose you wanted to estimate $\sqrt{102}$ without the use of a calculator. (Put another way, how does your calculator produce an approximation of $\sqrt{102}$?)

A way of rephrasing this is as follows: let $f(x) = \sqrt{x}$. What is the approximate value of $f(102)$?

What we know is that $f(100) = \sqrt{100} = 10$, and since 102 is a little bit bigger than 100, $\sqrt{102}$ should be a bit bigger than 10. But how much bigger?

To address this issue, we use the ideas of calculus. Recall from Chapter 4 that the tangent line to a function at $x = 100$ is the line which most closely approximates the function at values near 100. Let's give a name to the tangent line at 100 and call it L.





Now from a calculation we did on page 85 of these notes, we found that the tangent line to a function f at a is

$$L(x) = f(a) + f'(a)(x - a).$$

In our setting, $f(x) = \sqrt{x}$ so $f'(x) = \frac{1}{2\sqrt{x}}$ and $a = 100$. So we have

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= f(100) + f'(100)(x - 100) \\ &= \sqrt{100} + \frac{1}{2\sqrt{100}}(x - 100) \\ &= 10 + \frac{1}{20}(x - 100). \end{aligned}$$

The whole point of this is that the tangent line **closely approximates the original function**, so

$$\sqrt{102} = f(102) \approx L(102) = 10 + \frac{1}{20}(102 - 100) = 10 + \frac{2}{20} = 10.1.$$

Note: the actual value of $\sqrt{102}$ is $10.0995...$ so our approximation of 10.1 is correct to four decimal places.

Definition 8.1 (Linear approximation) *Given a differentiable function f and a number a at which you can easily compute $f(a)$ and $f'(a)$, the values $f(x)$ for x close to a can be estimated by the formula*

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

This procedure is called **tangent line approximation** *or* **linear approximation**.

The function $L(x)$ described above (which depends on f and a) has lots of names. It is also denoted $P_1(x)$ and is called:

1. the **tangent line to f at** a;
2. the **linearization of f at** a;
3. the **standard linear approximation to f at** a; and
4. the **first Taylor polynomial of f centered at** a.

EXAMPLE 1

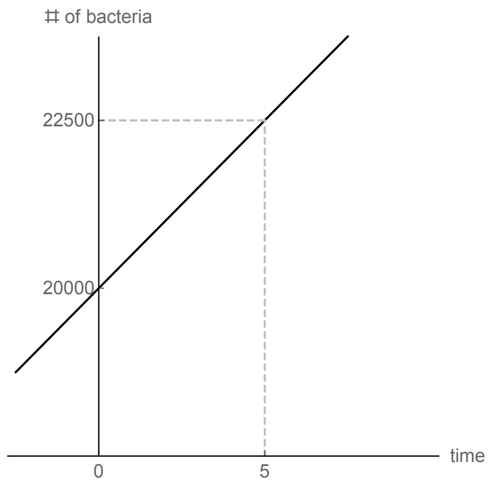
Estimate $\sqrt[4]{17}$ using tangent line approximation.

Note: You do not need to know the formula for f to perform a tangent line approximation. All you need to know are the values of $f(a)$ and $f'(a)$ (these two numbers can often be determined experimentally if f is some unknown function dealing with some experiment).

EXAMPLE 2

A biologist is growing bacteria in a petri dish. At 2:00 PM, she estimates that there are 20000 living bacteria in the dish, and that the number of bacteria is growing at a rate of 500 bacteria per minute. Use tangent line approximation to estimate the number of bacteria in the dish at 2:05 PM.

A more interesting calculus problem: In the example above will the answer overestimate, or underestimate the number of bacteria that are actually in the dish?



To get a better approximation which accounts for this kind of error, we approximate f not by a line but by a parabola which has the same slope and concavity as f at a.

Question: What would the equation of this parabola be?

Let's call this parabola $Q(x)$. Since $Q(x)$ is a parabola, we could write

$$Q(x) =$$

but it is actually easier to write the equation of this parabola "centered at a", i.e.

$$Q(x) = c_0 + c_1(x - a) + c_2(x - a)^2.$$

To find c_0, c_1 and c_2, use the concept that Q has to have the same value, slope and concavity as f at a.

The value of Q at a is

This should be the same as the value of f at a, which is

Conclusion:

The slope of Q at a is

This should be the same as the slope of f at a, which is

Conclusion:

The concavity of Q at a is $Q''(a) = 2c_2$.

This should be the same as the concavity of f at a, which is

Conclusion:

From all this, we know that

$$Q(x) = c_0 + c_1(x - a) + c_2(x - a)^2.$$

where

$$c_0 = f(a) \qquad c_1 = f'(a) \qquad c_2 = \frac{1}{2}f''(a).$$

To summarize:

Definition 8.2 (Quadratic approximation) *Given a twice-differentiable function f and a number a at which you can easily compute $f(a)$, $f'(a)$ and $f''(a)$, the values $f(x)$ for x close to a can be approximated by the formula*

$$f(x) \approx Q(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2.$$

This procedure is called **quadratic approximation**.

In general, quadratic approximation of a function is more accurate than linear approximation. In Math 230, you will learn how to approximate functions f by polynomials of larger degree which can produce highly accurate estimates to problems.

The function $Q(x)$ described above (which depends on f and a) also has lots of names. It is also denoted $P_2(x)$ and is called:

1. **parabolic approximation to** f **at** a;
2. the **standard quadratic approximation to** f **at** a; and
3. the **second Taylor polynomial of** f **at** a.

EXAMPLE 3

Approximate $\sqrt{102}$ using quadratic approximation.

EXAMPLE 4

Suppose the biologist in Example 2 assumes (in addition to what she knew in Example 2) that the number of bacteria in her petri dish at time t is given by a function whose second derivative at 2:00 PM is 10. Estimate the number of bacteria in her dish at 2:15 PM using quadratic approximation.

EXAMPLE 5

A pharmacy researcher measures a patient's blood pressure periodically after receiving a dose of an experimental medicine. His data is collected in the following table:

t (minutes after dosage)	0	1	2	3	4
$P(t)$ (blood pressure in mmHg)	230	190	162	142	128

Use quadratic approximation at $t = 3$ to estimate the patient's blood pressure at time 6.

Differentials

We will now establish some additional notation which will be used later in the course. Given a function $y = f(x)$, we create a new function with 2 inputs and one output. The two inputs are:

$$x = \text{an "initial" value of } x$$
$$dx = \text{a change in the value of } x$$

Thus, we think of x as changing from x to $x + dx$. Given these inputs, we define dy to be the estimated change in y that we would compute using tangent line approximation at x:

$$\begin{aligned} dy &= L(x + dx) - L(x) \\ &= [f(x) + f'(x)(x + dx - x)] - [f(x) + f'(x)(x - x)] \\ &= [f(x) + f'(x)\,dx] - [f(x)] \\ &= f'(x)\,dx. \end{aligned}$$

The quantities dy and dx are called *differentials*. They represent small changes in y and x, respectively and are related by the formula

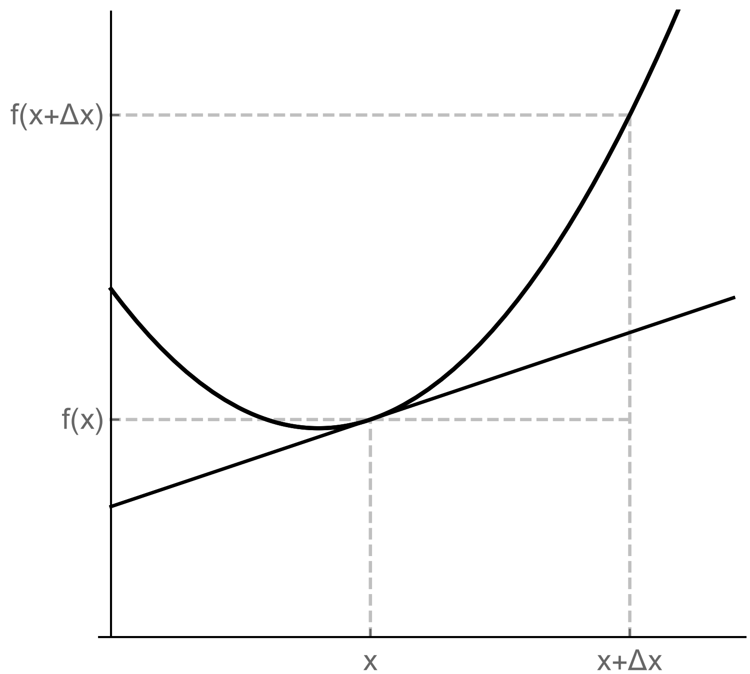
$$dy = f'(x)\,dx$$

.

EXAMPLE 6

Suppose $y = 2x^6 + \sin x - 3$. Compute dy (in terms of x and dx).

A picture associated to differentials:



In principle $dy \approx$ the actual change in y, since $L(x) \approx f(x)$.

8.2 L'Hôpital's rule

Recall that most limits are evaluated by "plugging in", i.e.

$$\lim_{x \to 5} \frac{2x+1}{x-3} = \frac{2(5)+1}{5-3} = \frac{11}{2}.$$

Other limits are not so easy:

$$\lim_{x \to 2} \frac{x^2-4}{x-2}$$

$$\lim_{x \to 0} \frac{\sin x}{x}$$

The $\frac{0}{0}$ obtained by plugging in 2 to the expression $\frac{x^2-4}{x-2}$ or by plugging in 0 to $\frac{\sin x}{x}$ is called an "indeterminate form". Note that both examples above are of the form $\frac{0}{0}$, but evaluate to different answers. More generally:

Definition 8.3 *An* **indeterminate form** *is an expression which can work out to one of many different answers, depending on the context.*

Examples of indeterminate forms:

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad \infty-\infty \qquad 1^{\infty} \qquad \infty^0 \qquad 0^0$$

Forms which are <u>not</u> indeterminate:

$$\frac{0}{\text{nonzero constant}} = 0 \qquad \frac{\text{nonzero constant}}{0} = \pm\infty$$

$$\frac{\infty}{0} = \pm\infty \qquad \frac{0}{\infty} = 0 \qquad 0\cdot 0 = 0 \qquad 0^1 = 0 \qquad 1^0 = 1$$

$$(\text{nonzero constant})^0 = 1 \qquad \frac{\infty}{\text{nonzero constant}} = \pm\infty \qquad \frac{\text{nonzero constant}}{\infty} = 0$$

In Chapter 3, we learned to evaluate limits that have indeterminate forms in them by factoring and cancelling, or performing other algebraic manipulations (like conjugating square roots and clearing fractions within fractions).

One additional, and very useful, method to evaluate indeterminate forms in limits is called L'Hôpital's Rule:

Theorem 8.4 (L'Hôpital's Rule) *Suppose f and g are differentiable functions. Suppose also that either*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$

or

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty.$$

Then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} \stackrel{L}{=} \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Application: Expressions like $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ can often be evaluated by taking derivatives of the top and bottom independently, then plugging in a.

WARNING: We are **not** differentiating $\dfrac{f}{g}$ here. To do this, use the quotient rule (but that has nothing to do with the evaluation of the limit).

WARNING: Be sure that the limit you are calculating is a common (i.e. easy) indeterminate form before using L'Hôpital's Rule.

Notation: The symbol $\stackrel{L}{=}$ is used to denote usage of L'Hôpital's Rule. It is just an equals sign, and the L tells the reader that you are using L'Hôpital's Rule.

EXAMPLE 1

$$\lim_{x\to 2} \frac{x^2-4}{x-2}$$

EXAMPLE 2

$$\lim_{x\to 0} \frac{\sin x}{x}$$

EXAMPLE 3

$$\lim_{x\to 0} \frac{3}{x^2}$$

EXAMPLE 4

$$\lim_{x\to 3} \frac{x-3}{2x+1}$$

EXAMPLE 5

$$\lim_{x\to\infty} \frac{7+2x^2}{x^2-3x+1}$$

EXAMPLE 6

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2}$$

Solution:

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \frac{''0''}{0} \stackrel{L}{=} \lim_{x\to 0} \frac{-\sin x}{2x} = \frac{''0''}{0} \stackrel{L}{=} \lim_{x\to 0} \frac{-\cos x}{2} = \frac{-1}{2}.$$

EXAMPLE 7

$$\lim_{x\to\infty} \frac{x^2}{x+1}$$

EXAMPLE 8

Evaluate this limit, where n is a positive integer:

$$\lim_{x\to\infty} \frac{x^n}{e^x}$$

Harder indeterminate forms

You can also evaluate other indeterminate forms (like $0 \cdot \infty$, $\infty - \infty$, 1^∞, ∞^0, 0^0) by first doing some algebra, then using L'Hôpital's Rule:

EXAMPLE 9

$$\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right)$$

EXAMPLE 10

$$\lim_{x \to 0^+} (\csc x - \cot x)$$

Solution:

$$\lim_{x \to 0^+} (\csc x - \cot x) ='' \infty - \infty'' \text{ which is indeterminate}$$

Rewrite as

$$\lim_{x \to 0^+} (\csc x - \cot x) = \lim_{x \to 0^+} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) = \lim_{x \to 0^+} \frac{1 - \cos x}{\sin x} = \frac{''0''}{0}$$

$$\stackrel{L}{=} \lim_{x \to 0^+} \frac{\sin x}{\cos x} = \frac{0}{1} = 0.$$

EXAMPLE 11

$$\lim_{x \to \infty} \left(1 + \frac{1}{x} \right)^x$$

Note: The answer to this problem should be memorized.

<u>WARNING</u>: L'Hôpital's Rule is a dangerous thing to rely on too much for two reasons:

(1)

EXAMPLE 12

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{e^x - 1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3}$$

(2)

EXAMPLE 13

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$$

8.3 Newton's method

Goal: use calculus to quickly and accurately approximate solutions to equations.

First, to solve any equation in one variable, it is sufficient to solve equations where one side is equal to zero (i.e. to find **roots** a.k.a. **x-intercepts** of functions). This is because if you are given an equation of the form

$$g(x) = h(x)$$

you can rewrite it as

$$g(x) - h(x) = 0 \text{ or } h(x) - g(x) = 0.$$

So our goal is: given function f, find (or at least approximate) r such that $f(r) = 0$. The procedure we will use is called **Newton's method** and works as follows:

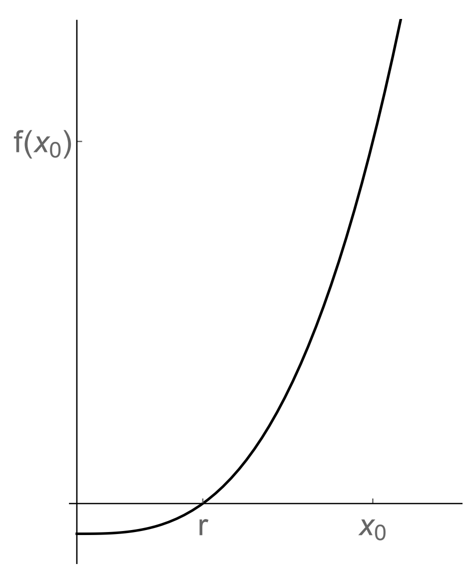
Newton's method

1. Guess the value of r. Call your guess x_0 (x_0 is called the "initial guess" or "seed").
2. Draw the tangent line to f at x_0.
3. Find the $x-$intercept of the tangent line from step (2). Call this $x-$int x_1.

   (Ideally, x_1 is closer to r than x_0 is.)
4. Draw the tangent line to f at x_1.
5. Find the $x-$intercept of the tangent line from step (2). Call this $x-$int x_2.

   (Ideally, x_2 is closer to r than x_1 is.)
6. Repeat the procedure over and over: given x_n, sketch the tangent line to f at x_n; call this $x-$int of this tangent line x_{n+1}.
7. You get a sequence of points $x_0, x_1, x_2, x_3, \ldots$

   The numbers x_n should (hopefully) get closer and closer to r, so they approximate r better and better as n gets larger.



Let's implement this procedure for an arbitrary function f and initial guess x_0:

The tangent line to f at x_n has equation

and x_{n+1}, the $x-$intercept of this line is found as follows:

EXAMPLE 1

Approximate a solution to $x^3 - x = 2$ by using Newton's method with initial guess 2 and two steps.

EXAMPLE 2

Approximate a solution to $x^3 - x = 2$ by using Newton's method and getting an approximation correct to 4 decimal places.

Newton's method on *Mathematica*

Newton's method is easy to implement on *Mathematica*. You need three lines of code, all in the same cell. For example, to implement Newton's method for the function $f(x) = x^2 - 2$ where $x_0 = 3$ and you want to perform 6 iterations (to find x_6), just type

```
f[x_] = x^2 - 2;
Newton[x_] = N[x - f[x]/f'[x]];
NestList[Newton, 3, 6]
```

and execute (all three lines at once). The first line defines the function f, the second line gives a name to the formula you iterate in Newton's method, and the last line iterates the formula and spits out the results.

The resulting output for the code listed above is:

```
{3, 1.83333, 1.46212, 1.415, 1.41421, 1.41421, 1.41421}
```

These numbers are $x_0, x_1, x_2, ..., x_6$ so for example, $x_2 = 1.46212$ and $x_4 = 1.41421...$ and $x_6 = 1.41421$ (the same as x_4 to 5 decimal places).

To implement Newton's method for a different function, different initial guess and different number of iterations, simply change the formula for f, change the 3 to the appropriate value of x_0 and the 6 to the number of times you want to iterate Newton's method.

EXAMPLE 3

Use Newton's method to approximate the solution to

$$\cos 2x + 3x = \sin x.$$

Obtain an approximation which is accurate to four decimal places.

***Mathematica* code:**

In:
```
f[x_] = Cos[2x] + 3x - Sin[x];
Newton[x_] = N[x - f[x]/f'[x]];
NestList[Newton, -1/2, 10]
```

Out:
```
{-1/2, -0.373791, -0.367115, -0.367093, -0.367093, -0.367093,
-0.367093, -0.367093, -0.367093, -0.367093, -0.367093}
```

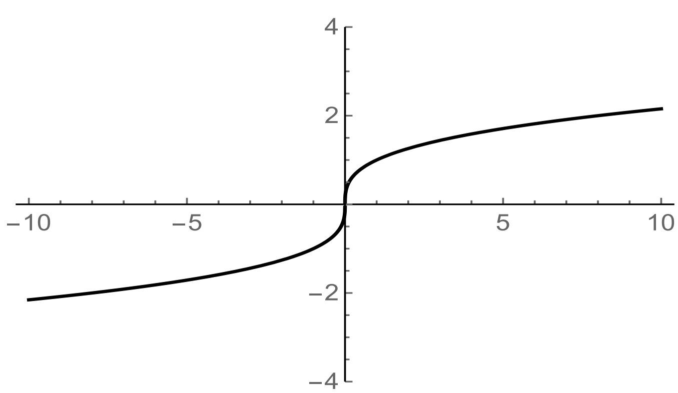
Potential problems with Newton's method

EXAMPLE 4

Use Newton's method to find a solution of

$$x^{1/3} = 0$$

using initial guess $x = 1$.

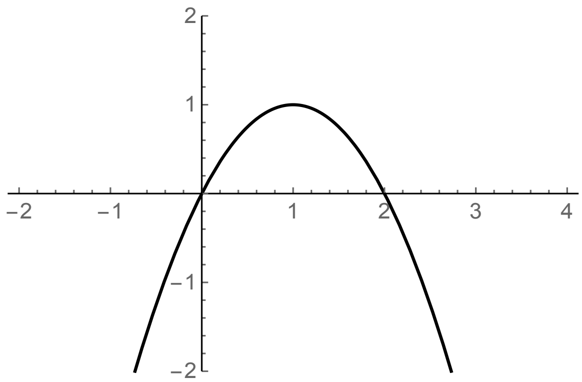


EXAMPLE 5

Use Newton's method to find a solution of

$$2x - x^2 = 0$$

using initial guess $x = 1$.

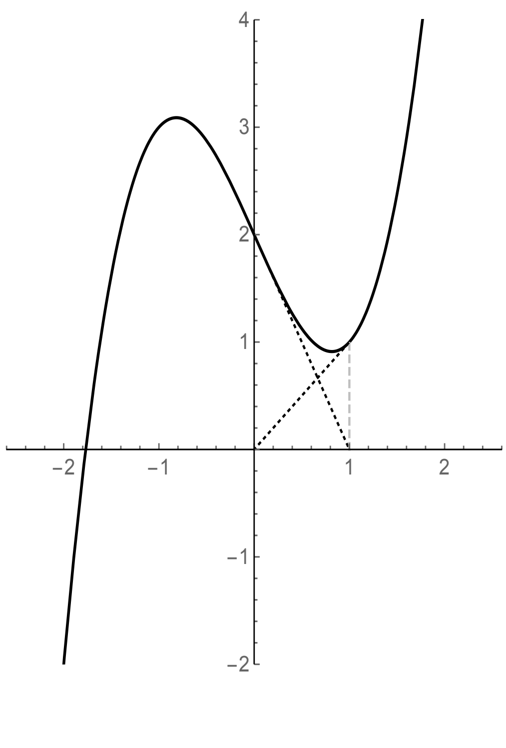


EXAMPLE 6

Use Newton's method to find a solution of

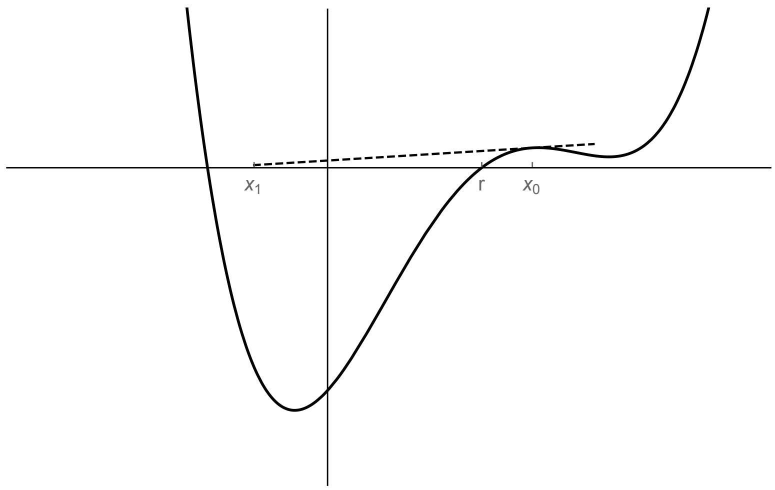
$$x^3 - 2x + 2 = 0$$

using initial guess $x = 0$.



EXAMPLE 7

Here is a graph of $f(x) = \frac{1}{2}(x+2)(x-3)(x-4)(x-5)+4$:



This function has two roots, a negative one which is about -2 and a positive one which is about 3. If you use initial guess $x_0 = 3.4$, you get the following:

$$x_1 = -8.9451; x_2 = -6.28878; x_3 = -4.38652; ...\ x_n \to -1.96114$$

which is the negative root. In other words, you get a root, but not the root you wanted.

Major reasons why Newton's method fails

1. "overshooting" (as in Example 4) - caused by vertical tangency at the root
2. $f'(x_n)$ being equal to zero for some n (as in Example 5) - caused by horizontal tangency at x_n
3. periodicity in the sequence x_n (as in Example 6) - caused by "poor" or "unlucky" initial guess
4. getting an unexpected root (as in Example 7) - caused by having a point where $f'(x)$ is small too close to the root you want

Reasons for the failure of Newton's method can always be explained graphically.

8.4 Homework exercises

Exercises from Section 8.1

In Problems 1-4 below, compute the linear approximation $L(x)$ to f at the given value of a:

1. $f(x) = \sqrt[5]{x}$, $a = 1$
2. $f(x) = \cot x$, $a = \pi/4$
3. $f(x) = \sin 3x$, $a = \pi$
4. $f(x) = xe^x$, $a = 0$

In Problems 5-8 below, compute the quadratic approximation $Q(x)$ to f at the given value of a:

5. $f(x) = x^{2/3}$, $a = 27$
6. $f(x) = 4\cos x$, $a = \pi$
7. $f(x) = \ln(x+1)$, $a = 0$
8. $f(x) = 3\sec x$, $a = 0$

In Problems 9-16 below, estimate the following quantities using tangent line approximation:

9. $\sqrt{50}$
10. $(8.1)^3$
11. $\ln(1.3)$
12. $e^{.2}$
13. $\sqrt[3]{66}$
14. $\sin(.2)$
15. $\arctan(1/3)$
16. $\cos(\frac{\pi}{2} + 1/8)$

17. Is the estimate you made in problem 9 an overestimate or an underestimate? Explain (without obtaining a decimal approximation to $\sqrt{50}$ using a computer or calculator).
18. Is the estimate you made in problem 10 an overestimate or an underestimate? Explain (without obtaining a decimal approximation to $(8.1)^3$ using a computer or calculator).

In Problems 19-22 below, estimate the following quantities using quadratic approximation:

19. $17^{3/2}$
20. $\cos \dfrac{1}{2}$
21. $e^{1/3}$
22. $\sqrt{150}$

23. After turning his gas grill on, a cook looks at the grill's internal temperature regularly, writing what he sees in the following table:

t (minutes after grill is lit)	0	1	2	4	5
$T(t)$ (temperature in °F)	70	240	320	440	475

a) Use linear approximation to estimate what the temperature of the grill will be 7 minutes after it is turned on.

b) Use quadratic approximation to estimate what the temperature of the grill will be 7 minutes after it is turned on.

c) Use the same quadratic approximation you computed in part (b) to estimate what the temperature of the grill will be 13 minutes after it is turned on.

d) Does your answer to part (c) make sense? Explain.

e) What about the procedure of quadratic approximation made our answer to part (c) so far off?

In Problems 24-27, compute the differential dy.

24. $y = 3x^2 - 4$

25. $y = x\sqrt{1 - x^2}$

26. $y = \arcsin x$

27. $y = e^{3x}$

28. a) Compute dy if $y = \frac{1}{2}x^3$, when $x = 2$ and $dx = .1$.

b) Sketch a picture representing the computation done in part (a) of this problem, labelling x, dx and dy appropriately.

29. a) Compute dy if $y = 1 - x^4$, when $x = 1$ and $dx = .1$.

b) Sketch a picture representing the computation done in part (a) of this problem, labelling x, dx and dy appropriately.

Exercises from Section 8.2

In Problems 30-43, compute the indicated limit (indicating if the limit is $\pm\infty$ or does not exist):

30. $\lim_{x \to 3} \frac{2x - 6}{x^2 - 9}$

31. $\lim_{x \to 0} \frac{\sqrt{4 - x^2} - 2}{x}$

32. $\lim_{x\to 0} \frac{2e^x - 2x - 2}{x^2}$

33. $\lim_{x\to 3} \frac{x^2 - 2x - 3}{x - 3}$

34. $\lim_{x\to 2} \frac{x^2 + 10}{x + 2}$

35. $\lim_{x\to 0} \frac{\sin 4x}{\sin 9x}$

36. $\lim_{x\to \infty} \frac{3x^2 - 2x + 1}{2x^2 + 3}$

37. $\lim_{x\to \infty} \frac{x^4}{e^{x/3}}$

38. $\lim_{x\to \infty} \frac{\ln x}{x^2}$

39. $\lim_{x\to \infty} \frac{e^x}{x^9}$

40. $\lim_{x\to \infty} \frac{2000x^{2014}}{e^x}$

41. $\lim_{[x\to 2^+} \left(\frac{8}{x^2 - 4} - \frac{x}{x - 2}\right)$

    Hint: Add the fractions by finding a common denominator.

42. $\lim_{x\to \infty} x^{1/x}$

    Hint: Follow the procedure of Example 11 on page 224.

43. $\lim_{x\to 0} x^x$

Exercises from Section 8.3

44. Approximate (by hand) a solution to $2x^3 + x^2 - x = -1$ by using Newton's method with initial guess $x = -1$ and two steps.

45. Approximate (by hand) a solution to $x^5 = 4$ by using Newton's method with initial guess $x = 1$ and two steps.

In Problems 46-49, use *Mathematica* to estimate a solution to the following equations using Newton's method; solutions should be correct to 4 decimal places:

46. $x^3 = 3$

47. $x^5 + x = 1$

48. $3\sqrt{x - 1} = x$

49. $2x^3 = \cos x$

In Problems 50-52, use *Mathematica* to estimate all solutions to the following equations using Newton's method; solutions should be correct to 4 decimal places.

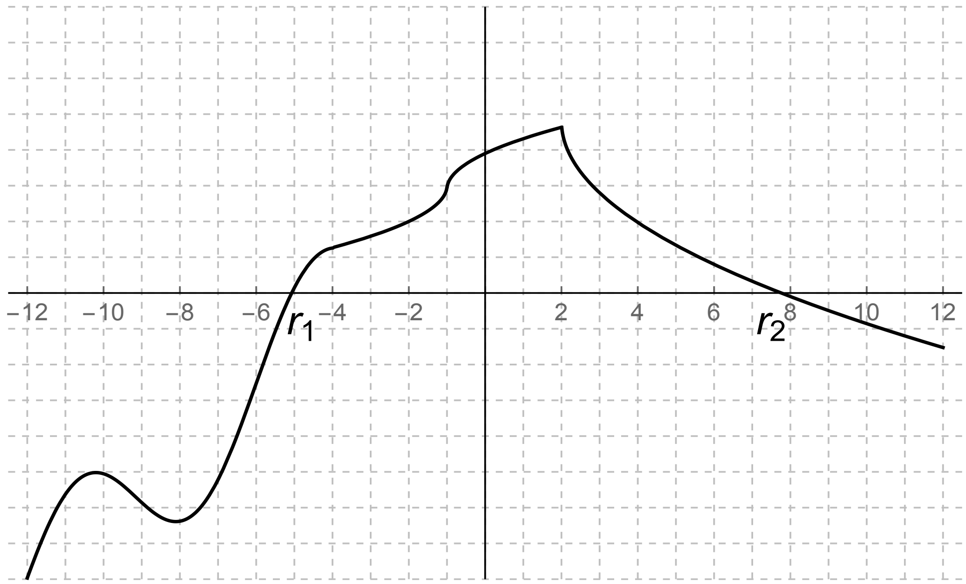
Hint: First, have *Mathematica* plot both functions on the same $xy-$plane; use the plot to determine the number of solutions to the equation. For each solution, run Newton's method with an initial guess close to the $x-$value of the appropriate solution.

50. $e^{x-5} = \ln x$

51. $\arctan 2x = x^2 - 1$

52. $6 \sin \frac{x}{6} = 8x - x^3$

In Problems 53-60, use the graph of some unknown function f shown here. As you can see, the equation $f(x) = 0$ has two solutions, r_1 (the negative one, near -5) and r_2 (the positive one, near 8).



53. Suppose you were to execute Newton's method for this function with initial guess $x = 4$. Will this produce an approximation to r_1 or r_2 (or will it not work at all)? Explain.

54. Suppose you were to execute Newton's method for this function with initial guess $x = -6$. Will this produce an approximation to r_1 or r_2 (or will it not work at all)? Explain.

55. Suppose you were to execute Newton's method for this function with initial guess $x = -10$ (assume that -10 is the $x-$coordinate of the "peak" of the function). Will this produce an approximation to r_1 or r_2 (or will it not work at all)? Explain.

56. Suppose you were to execute Newton's method for this function with initial guess $x = 3$. Will this produce an approximation to r_1 or r_2 (or will it not work at all)? Explain.

57. Suppose you were to execute Newton's method for this function with initial guess $x = -1$. Will this produce an approximation to r_1 or r_2 (or will it not work at all)? Explain.

58. Suppose you were to execute Newton's method for this function with initial guess $x = 2$. Will this produce an approximation to r_1 or r_2 (or will it not work at all)? Explain.

59. Suppose you were to execute Newton's method for this function with initial guess $x = -7.5$. Will this produce an approximation to r_1 or r_2 (assume that the slope at -7.5 is a very small positive number)? Explain.

60. Suppose you were to execute Newton's method for this function with initial guess $x = 10$. Will this produce an approximation to r_1 or r_2 (or will it not work at all)? Explain.

61. Attempt Newton's method on the function $f(x) = 4x^3 - 12x^2 + 12x - 3$ with initial guess $x = \frac{3}{2}$. Try lots of iterations.

    a) What happens?

    b) Sketch the graph of the function f using *Mathematica* and explain, via the graph, the phenomenon you observe in part (a).

Answers

1. $L(x) = 1 + \frac{1}{5}(x - 1)$
2. $L(x) = 1 - 2\left(x - \frac{\pi}{4}\right)$
3. $L(x) = -3(x - \pi)$
4. $L(x) = x$
5. $Q(x) = 9 + \frac{2}{9}(x - 27) - \frac{1}{729}(x - 27)^2$
6. $Q(x) = -4 + 2(x - \pi)^2$
7. $Q(x) = x - \frac{x^2}{2}$
8. $Q(x) = 3 + \frac{3}{2}x^2$
9. $\frac{99}{14}$
10. 531.2
11. $.3$
12. 1.2
13. $\frac{97}{24}$
14. $.2$
15. $\frac{1}{3}$
16. $-\frac{1}{8}$
17. Overestimate, because $f''(49) < 0$.
18. Underestimate, because $f''(8) > 0$.
19. $\frac{2243}{32}$
20. $\frac{7}{8}$
21. $\frac{25}{18}$
22. $\frac{4703}{384}$

23. a) $545°$ F (answers may vary)

    b) $520°$ F (answers may vary)

    c) $355°$ F (answers may vary)

d) No, because the grill should be hotter at time 13 than it was at time 5.

e) Since we approximated using a parabola Q that opens downward, eventually Q starts to decrease. But the temperature T probably continues to increase; it is just that it increases at a slower rate.

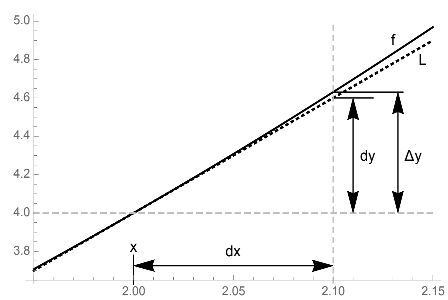
24. $6x\,dx$

25. $\dfrac{\sqrt{1-x^2} - \dfrac{-2x^2}{2\sqrt{1-x^2}}}{1-x^2}\,dx = \dfrac{1}{(1-x^2)^{3/2}}\,dx$

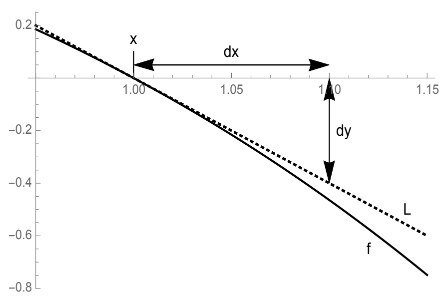
26. $\dfrac{1}{\sqrt{1-x^2}}\,dx$

27. $3e^{3x}\,dx$

28. a) .6

b) 

29. a) $-.4$

b) 

30. $\dfrac{1}{3}$

31. 0

32. 1

33. 4

34. $\dfrac{7}{2}$

35. $\dfrac{4}{9}$

36. $\dfrac{3}{2}$

37. 0

38. 0

39. ∞

40. 0

41. $\frac{-3}{2}$

42. 1

43. 1
44. $\dfrac{-235}{189}$
45. $\dfrac{35893}{25600}$
46. 1.44225
47. $.754878$
48. 7.8541
49. $.721406$
50. 1.01884 and 5.53738
51. $-.482303$ and 1.49966
52. -2.65184, 0 and 2.65184
53. r_2
54. r_1
55. won't work (since tangent line at $x = -10$ never hits $x-$axis)
56. r_2
57. won't work (tangent line at $x = -1$ is vertical)
58. won't work (function not differentiable at $x = 2$
59. r_2 (tangent line at -7.5 hits $x-$axis close to r_2)
60. r_2
61. a) Starting with the second iteration, you get infinity.
    b) If you sketch the picture associated to Newton's method, after the first iteration the tangent line is horizontal.

Chapter 9

Theory of the Definite Integral

9.1 Motivating problems: area and displacement

RECALL

To define the derivative of a function, we started with a real-world problem we wanted to solve:

Then, we approximated the solution to that problem (by finding the slope of some secant line):

Next, we observed how the approximation got better:

This told us how to define the answer to the problem (using a limit):

(In principle, we don't use this definition to compute derivatives; we use rules like the Power Rule, Product Rule, Chain Rule, etc.)

For the rest of the semester, we will consider two new classes (actually only one class) of real-world problems.

We need to define a new mathematical object which will solve these problems.

To create this new object, we will:
1. Approximate the answer to the problem.
2. Observe how the approximation gets better.
3. Define the answer to the problem using a limit.

What are the two new classes of real-world problems?

1.

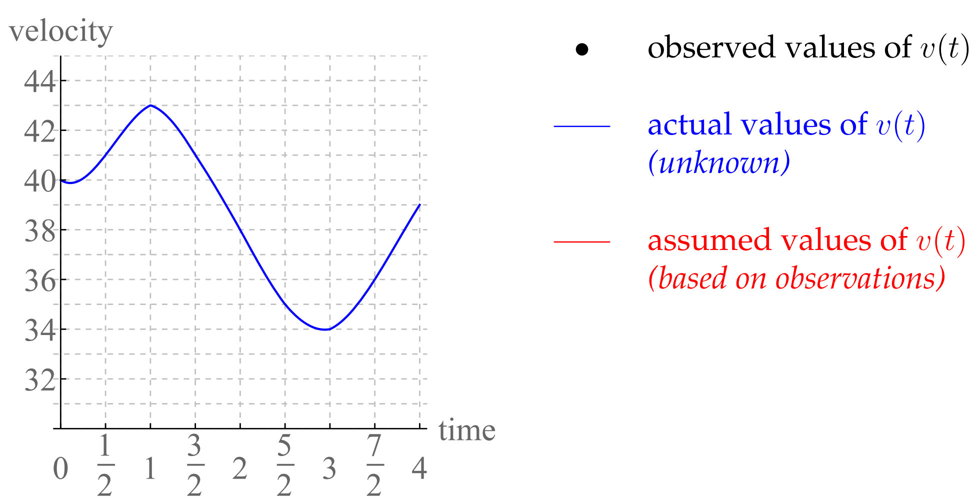
2.

First remark: Problems (1) and (2) above are really the same problem in disguise. Why?

Suppose you are in a car and you look at the speedometer once an hour:

(hr)	t	0	1	2	3
(mph)	$v(t)$	40	43	38	34

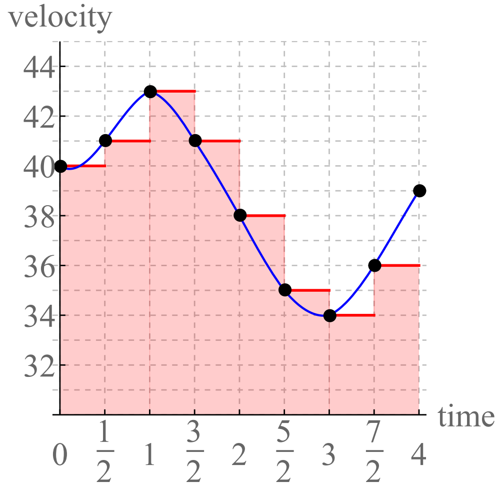
How far do you travel from $t = 0$ to $t = 4$ (i.e. what is your **displacement** from time 0 to time 4)?



This is just an approximation. How might the approximation improve?

Suppose you look at the speedometer every 30 minutes:

(hr)	t	0	$\frac{1}{2}$	1	$\cdots$
(mph)	$v(t)$	40	41	43	$\cdots$



● observed values of $v(t)$

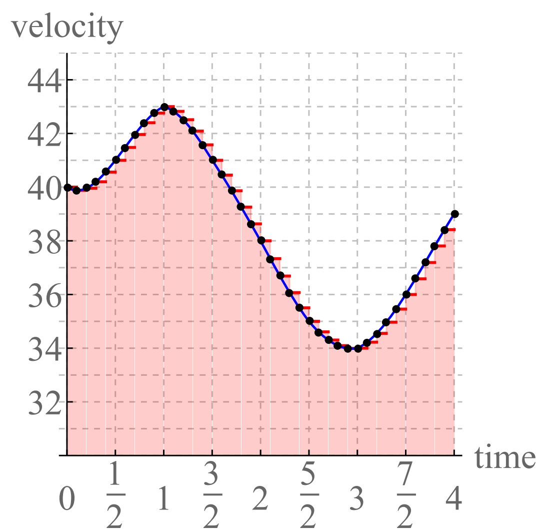
actual values of $v(t)$ *(unknown)*

assumed values of $v(t)$ *(based on observations)*

So the displacement is

$$\approx 40\left(\tfrac{1}{2}\right) + 41\left(\tfrac{1}{2}\right) + 43\left(\tfrac{1}{2}\right) + 41\left(\tfrac{1}{2}\right) + 38\left(\tfrac{1}{2}\right) + 35\left(\tfrac{1}{2}\right) + 34\left(\tfrac{1}{2}\right) + 36\left(\tfrac{1}{2}\right) = \boxed{154\text{ mi}}.$$

Take more and more measurements:



This suggests the following important principle:

displacement of an object from $t = 0$ to $t = 4$, given velocity function $v(t)$	$=$	area under the graph of v from $t = 0$ to $t = 4$

More generally: Suppose an object's position at time t is given by function $f(t)$. Then its displacement from time $t = a$ to time $t = b$ is $f(b) - f(a)$.

At the same time, its velocity at time t is given by $f'(t)$, and the displacement from time a to time b is equal to the area under the graph of f' from $t = a$ to $t = b$. Putting this together, we have the following important idea:

area under the graph of f' from $t = a$ to $t = b$	$=$	$f(b) - f(a)$

This means: the problems of finding the area between the graph of a function and the $x-$axis, and the problem of finding displacement given velocity, are really the same problem. The process that solves these problems is probably something like "differentiation in reverse".

EXAMPLE 1

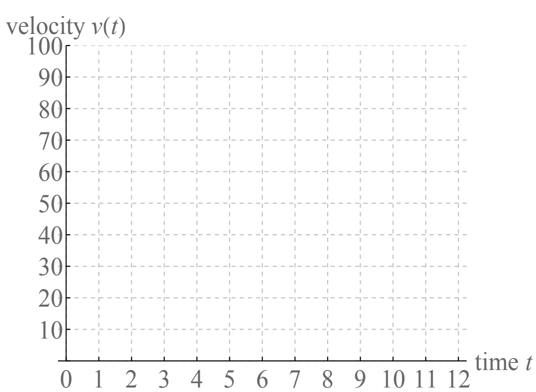
Suppose that the velocity (in m/sec) of a bird at time t (in seconds) is given by $v(t) = \frac{2}{3}t + \frac{4}{3}$. Find the distance travelled by the bird between time 0 and time 6.

EXAMPLE 2

In each situation A through D described below:

1. Based on the description given, sketch a graph of the velocity, plotted against time.
2. Determine how far you travel between time $t = 0$ and $t = 3$ (throughout this assignment, t is in hours).
3. Determine how far you travel between times $t = 5$ and $t = 9$.
4. Without being given any other information, do you know what your odometer reading is at time $t = 4$? If so, what is it?
5. Without being given any other information, do you know what your odometer reading is at time $t = 8$? If so, what is it?
6. Suppose your odometer reading at time $t = 0$ is 0. Now, do you know the odometer readings at time 8? If so, what is it?
7. Suppose your odometer reading at time $t = 0$ is 10000. Now, do you know the odometer readings at time 8? If so, what is it?
8. Suppose your odometer reading at time $t = 0$ is C, where C is an arbitrary constant. What is the odometer reading at time 4? What is the odometer reading at time 8?

Situation A: Assume that the velocity at all times is 60 miles per hour.



Displacement from $t = 0$ to $t = 3$:

Displacement from $t = 5$ to $t = 9$:

Odometer reading at $t = 4$:

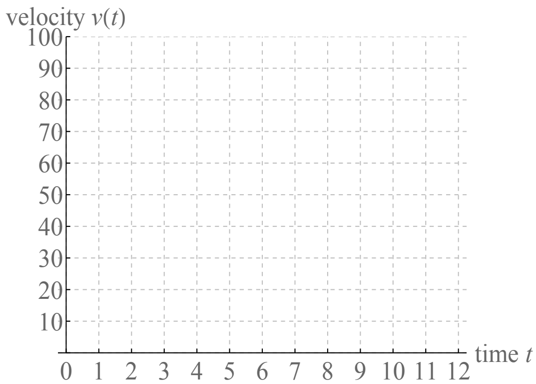
Odometer reading at $t = 8$:

If initially 0, odometer reading at $t = 8$:

If initially 10000, odometer reading at $t = 8$:

If initially C, odometer reading at $t = 8$:

Situation B: Assume that the velocity is 50 miles per hour for the first six hours, then 80 miles per hour at all times after the first six hours.



Displacement from $t = 0$ to $t = 3$:

Displacement from $t = 5$ to $t = 9$:

Odometer reading at $t = 4$:

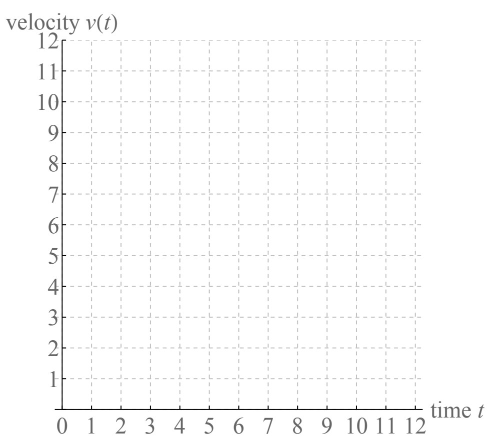
Odometer reading at $t = 8$:

If initially 0, odometer reading at $t = 8$:

If initially 10000, odometer reading at $t = 8$:

If initially C, odometer reading at $t = 8$:

Situation C: Assume that the velocity at time x is equal to x.



Displacement from $t = 0$ to $t = 3$:

Displacement from $t = 5$ to $t = 9$:

Odometer reading at $t = 4$:

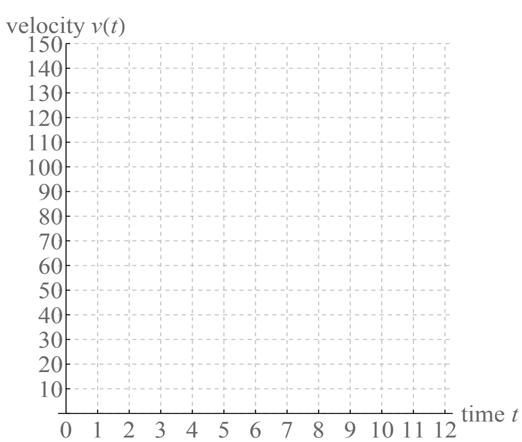
Odometer reading at $t = 8$:

If initially 0, odometer reading at $t = 8$:

If initially 10000, odometer reading at $t = 8$:

If initially C, odometer reading at $t = 8$:

Situation D: Assume that the velocity at time x is equal to x^2.



Displacement from $t = 0$ to $t = 3$:

Displacement from $t = 5$ to $t = 9$:

Odometer reading at $t = 4$:

Odometer reading at $t = 8$:

If initially 0, odometer reading at $t = 8$:

If initially 10000, odometer reading at $t = 8$:

If initially C, odometer reading at $t = 8$:

Concepts illustrated in the preceding example

- At the beginning of the semester, we discussed the "big picture" problem of converting from a function which represents an odometer to a function which represents a speedometer. The operation we eventually cooked up to do this is **differentiation**. In other words:

$$\begin{array}{rcl} \text{ODOMETER} & \xrightarrow{\text{DERIVATIVE}} & \text{SPEEDOMETER} \\ \text{POSITION} & \xrightarrow{\text{DERIVATIVE}} & \text{VELOCITY} \\ f(x) & \xrightarrow{\text{DERIVATIVE}} & f'(x) \end{array}$$

- Now, we are looking at the same problem in the other direction. That is, we want to assume we are given a speedometer (i.e. a function that represents velocity), and we want to determine the function that was the odometer:

$$\begin{array}{rcl} \text{ODOMETER} & \longleftarrow & \text{SPEEDOMETER} \\ \text{POSITION} & \longleftarrow & \text{VELOCITY} \\ ? & \xleftarrow{?} & f(x) \end{array}$$

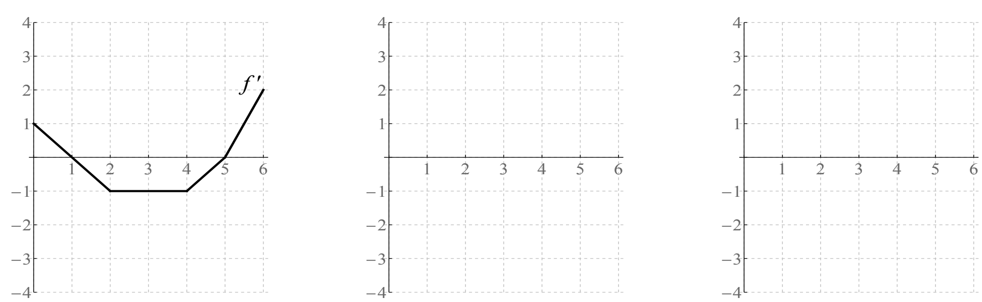
- If you are given a function f which represents your velocity, then you **cannot** use f by itself to determine your odometer reading at a certain time (because you didn't know what the original odometer reading was).

- But, if you are given a function f which represents your velocity **and you are given an initial odometer reading** (a.k.a. the odometer reading at time a), then you can determine your odometer reading at any time t by the formula

$$\begin{array}{c}\text{odometer reading}\\ \text{at time } t\end{array} = \begin{array}{c}\text{original odometer reading}\\ + \\ \text{area under velocity function}\\ \text{from time } a \text{ to time } t\end{array}$$

EXAMPLE 3

The graph of some function f' is given below at left. If $f(0) = 2$, sketch the graph of f on the middle axes. On the right-hand axes, sketch all possible graphs of f (if you don't know $f(0)$).



9.2 Riemann sums

Summation notation

Suppose a_k is some expression which can be computed in terms of k. (a_k is like $a(k)$.) For example, if $a_k = k^2 + k$, then

$$a_1 = 1^2 + 1 = 2 \qquad a_2 = 2^2 + 2 = 6 \qquad a_3 = 3^2 + 3 = 12 \qquad \text{etc.}$$

Frequently in mathematics we want to **add** together values of a_k where k ranges over some set. For example, we might want to add up

$$a_2 + a_3 + a_4 + a_5 + ... + a_{20}.$$

We use the following notation to represent this kind of addition:

Definition 9.1 *Given numbers* a_1, a_2, ... *the* **sum from** $k = 1$ **to** n **of** a_k *is*

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 ... + a_n.$$

(More generally, $\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + ... + a_n$.)

EXAMPLE 1

Write the expression $\frac{3^2}{2} + \frac{4^2}{2} + \frac{5^2}{2} + ... + \frac{17^2}{2}$ in Σ-notation.

EXAMPLE 2

Compute $\sum_{k=0}^{3} \frac{2}{k+1}$.

Approximating the area under a function

Idea: Approximate the area under a function by finding the total area of some rectangles.

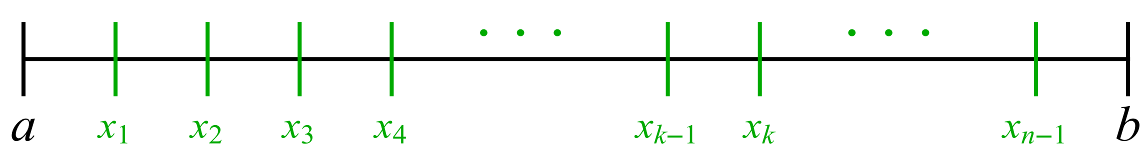
Definition 9.2 *Given an interval $[a,b]$, a* **partition** *$\mathcal{P}$ is a (finite) list of numbers $\{x_0, x_1, x_2, ..., x_n\}$ such that $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$. Such a partition divides $[a,b]$ into n* **subintervals**; *the k^{th}* **subinterval** *is $[x_{k-1}, x_k]$. For each k, set $\Delta x_k = x_k - x_{k-1}$; Δx_k is called the* **width** *of the k^{th} subinterval. Call the largest Δx_k the* **norm** *of the partition; denote the norm by $||\mathcal{P}||$.*

EXAMPLE 3

$a = 0$; $b = 1$; $\mathcal{P} = \left\{0, \frac{1}{4}, \frac{3}{4}, \frac{7}{8}, 1\right\}$.

EXAMPLE 4

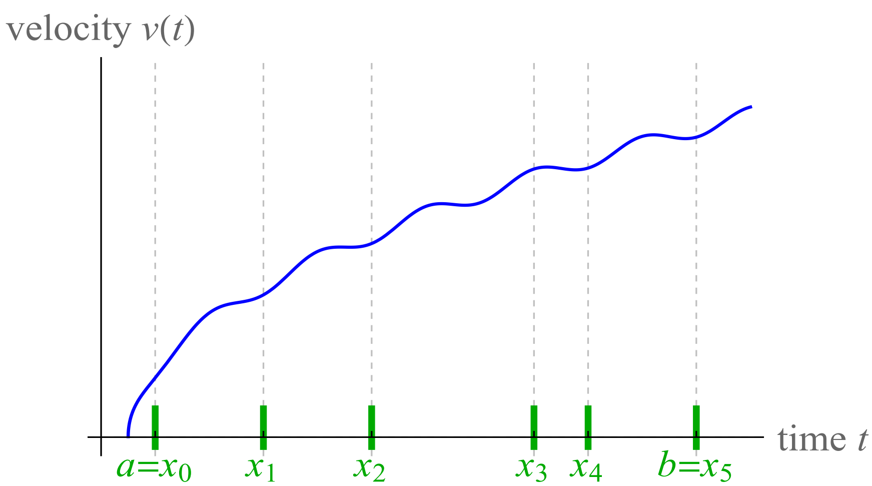
$\mathcal{P} =$ partition of $[a,b]$ into n equal-length subintervals.



Definition 9.3 *Given function* $f : [a,b] \to \mathbb{R}$ *and given partition* $\mathcal{P} = \{x_0, \ldots, x_n\}$ *of* $[a,b]$, *a* **Riemann sum** *associated to* $\mathcal{P}$ *for* f *is any expression of the form*

$$\sum_{k=1}^{n} f(c_k)\Delta x_k$$

where for all k, c_k *belongs to the* k^{th} *subinterval of* $\mathcal{P}$. *The points* $c_1, c_2, \ldots, c_n$ *are called* test points *for the Riemann sum.*

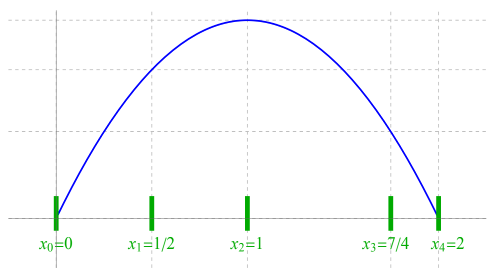


A Riemann sum approximates the area under $f(x)$ from $x = a$ to $x = b$ by adding up areas of rectangles as above. Different choices of $\mathcal{P}$ and different choices of c_k (even for the same $\mathcal{P}$) give different Riemann sums.

EXAMPLE 5

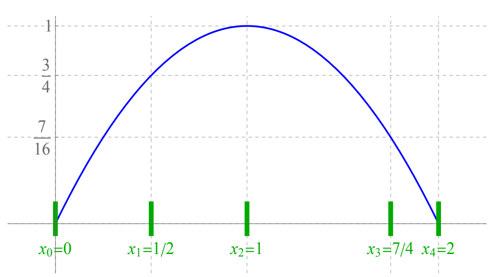
Let $f(x) = 2x - x^2$ and let $\mathcal{P} = \left\{0, \frac{1}{2}, 1, \frac{7}{4}, 2\right\}$. By choosing different test points, we get different Riemann sums for this partition. Compute each of the following specific Riemann sums associated to this f and this $\mathcal{P}$:

1. **Left sum** (this means that we choose each test point c_k to be x_{k-1}, the left endpoint of the k^{th} subinterval)



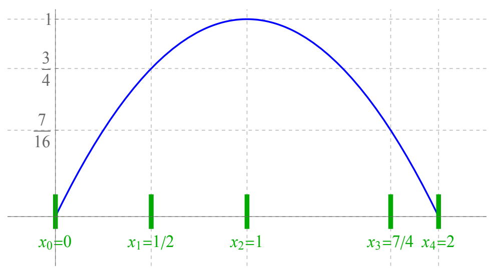
$$\sum_{k=1}^{4} f(c_k)\Delta x_k =$$

2. **Right sum** (this means we choose each c_k to be x_k, the right endpoint of the k^{th} subinterval)



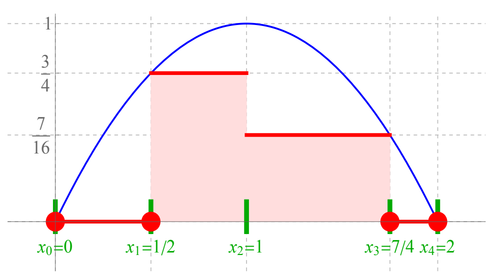
$$\begin{aligned}
&\sum_{k=1}^{4} f(c_k)\Delta x_k \\
&= f\left(\frac{1}{2}\right)\frac{1}{2} + f(1)\frac{1}{2} + f\left(\frac{7}{4}\right)\frac{3}{4} + f(2)\frac{1}{4} \\
&= \left(\frac{3}{4}\right)\frac{1}{2} + (1)\frac{1}{2} + \left(\frac{7}{16}\right)\frac{3}{4} + (0)\frac{1}{4} \\
&= \frac{3}{8} + \frac{1}{2} + \frac{21}{64} + 0 \\
&= \frac{24+32+21}{64} \\
&= \boxed{\frac{77}{64}}.
\end{aligned}$$

3. **Upper sum** (this means we choose each c_k to be the $x-$value corresponding to the absolute maximum of f on the k^{th} subinterval, making the rectangles as tall as possible)



$$\begin{aligned}
&\sum_{k=1}^{4} f(c_k)\Delta x_k \\
&= f\left(\frac{1}{2}\right)\frac{1}{2} + f(1)\frac{1}{2} + f\left(1\right)\frac{3}{4} + f\left(\frac{7}{4}\right)\frac{1}{4} \\
&= \left(\frac{3}{4}\right)\frac{1}{2} + (1)\frac{1}{2} + (1)\frac{3}{4} + \left(\frac{7}{16}\right)\frac{1}{4} \\
&= \frac{3}{8} + \frac{1}{2} + \frac{3}{4} + \frac{7}{64} \\
&= \frac{24+32+48+7}{64} \\
&= \boxed{\frac{111}{64}}.
\end{aligned}$$

4. **Lower sum** (this means we choose each c_k to be the $x-$value corresponding to the absolute minimum of f on the k^{th} subinterval, making the rectangles as short as possible)



$$\begin{aligned}
&\sum_{k=1}^{4} f(c_k)\Delta x_k \\
&= f(0)\frac{1}{2} + f\left(\frac{1}{2}\right)\frac{1}{2} + f\left(\frac{7}{4}\right)\frac{3}{4} + f(2)\frac{1}{4} \\
&= (0)\frac{1}{2} + \left(\frac{3}{4}\right)\frac{1}{2} + \left(\frac{7}{16}\right)\frac{3}{4} + (0)\frac{1}{4} \\
&= 0 + \frac{3}{8} + \frac{21}{64} + 0 \\
&= \frac{24+21}{64} \\
&= \boxed{\frac{45}{64}}.
\end{aligned}$$

The upper and lower sums associated to a partition $\mathcal{P}$ for a function f are of particular importance. Why?

Note: If f is increasing on $[a, b]$, then

Note: If f is decreasing on $[a, b]$, then

EXAMPLE 6

Estimate the area under $f(x) = x^3$ from $x = 0$ to $x = 1$ by using a lower sum for a partition into 6 subintervals of equal length.

Therefore this Riemann sum works out to

$$0\left(\frac{1}{6}\right)+\frac{1}{216}\left(\frac{1}{6}\right)+\frac{1}{27}\left(\frac{1}{6}\right)+\frac{1}{8}\left(\frac{1}{6}\right)+\frac{8}{27}\left(\frac{1}{6}\right)+\frac{125}{216}\left(\frac{1}{6}\right)=\frac{25}{144}.$$

9.3 Definition of the definite integral

In the last section: we approximated the area under f from a to b by the Riemann sum

$$\sum_{k=1}^{n} f(c_k)\Delta x_k$$

Next task:

Key observation: As $||\mathcal{P}|| \to 0$, the rectangles under the graph of f get skinnier and skinnier, so the corresponding Riemann sum estimates become more and more exact. So

$$\lim_{||\mathcal{P}||\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

should give the exact area under the graph. This motivates the following definition:

Definition 9.4 (Limit definition of the integral) *Given function $f : [a, b] \to \mathbb{R}$, the* **definite integral** *of f from a to b is*

$$\int_a^b f(x)\,dx = \lim_{||\mathcal{P}|| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

if this limit exists (in MATH 220 and MATH 230, it always will). If the limit exists, we say f is **integrable** *on $[a, b]$.*

Notation:

Two ways to think about the integral:

1. The definite integral is "continuous addition of areas of rectangles of infinitely small width".
2. The definite integral is "accumulation" of values of f from $x = a$ to $x = b$.

Some integrals can be computed without doing any sophisticated calculus:

EXAMPLE 1

Evaluate the following definite integrals:

1. $\displaystyle\int_4^7 5\,dx$

2. $\displaystyle\int_4^8 \frac{1}{4}x\,dx$

3. $\displaystyle\int_{-3}^3 \sqrt{9-x^2}\,dx$

4. $\displaystyle\int_{-2}^1 (3-|x|)\ dx$

Evaluating an integral using the definition of integral is much harder:

EXAMPLE 2

Compute

$$\int_0^1 x\,dx$$

using the limit definition of the definite integral.

Aside: The answer is

Computation using the limit definition:

$$\int_0^1 x\,dx = \lim_{||\mathcal{P}||\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

We need to choose partitions whose norm $\to 0$,
and we need to choose a type of Riemann sum.
I will choose a partition into n equal-length subintervals

(this makes all the $\Delta x_k = \frac{1-0}{n} = \frac{1}{n}$)

and compute a right-hand Riemann sum

(this makes $c_k = x_k = 0 + k\left(\frac{1-0}{n}\right) = 0 + k \cdot \frac{1}{n} = \frac{k}{n}$)

and notice that $||\mathcal{P}|| \to 0$ is the same as $n \to \infty$ in this context.
So the integral becomes

$$\lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} \qquad \text{(since } f(x) = x\text{)}$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} k$$

$$= \lim_{n\to\infty} \frac{1}{n^2}(1 + 2 + 3 + ... + n).$$

Now the question is, what is $1 + 2 + 3 + ... + n$?

From the previous page,

$$\begin{aligned}\int_0^1 x\,dx = \lim_{n\to\infty}\frac{1}{n^2}(1+2+3+\ldots+n) &= \lim_{n\to\infty}\frac{1}{n^2}\cdot\frac{n(n+1)}{2}\\ &= \lim_{n\to\infty}\frac{n+1}{2n}\\ &\stackrel{L}{=} \lim_{n\to\infty}\frac{1}{2} = \boxed{\frac{1}{2}}.\end{aligned}$$

This example was very hard (even though the integrand was very simple). This suggests that computing integrals like

$$\int_0^\pi \sin x\,dx \quad \text{or} \quad \int_1^4 x^5\,dx$$

using the definition of definite integral is impossible. We need another method, which we will discuss in Section 9.5.

9.4 Elementary properties of Riemann integrals

Theorem 9.5 *All continuous functions are integrable.*

Definition 9.6 *Let $a < b$ and let $f : [a,b] \to \mathbb{R}$ be integrable. Then*

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Definition 9.7 *Let $f : [a,b] \to \mathbb{R}$. Then $\int_a^a f(x)\,dx = 0$.*

Theorem 9.8 (Linearity properties of integrals) *Let f and g be integrable; let k be any constant. Then:*

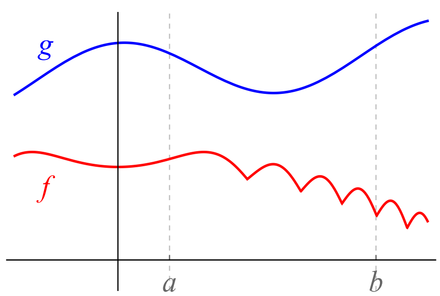
1. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$;
2. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$;
3. $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$.

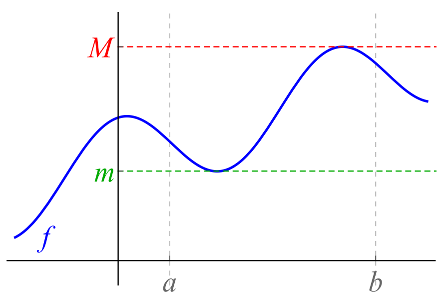
WARNING: integrals are **not** multiplicative nor divisive:

$$\int_a^b [f(x)g(x)]\, dx \neq \left[\int_a^b f(x)\, dx\right]\left[\int_a^b g(x)\, dx\right] \qquad \int_a^b \frac{f(x)}{g(x)}\, dx \neq \frac{\int_a^b f(x)\, dx}{\int_a^b g(x)\, dx}$$

Theorem 9.9 (Inequality properties of integrals) *Suppose that f and g are integrable functions. Then:*

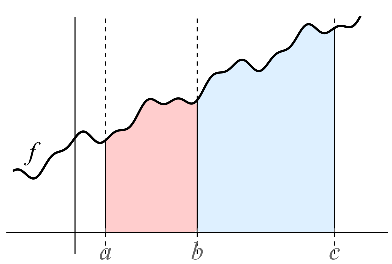
1. **(Positivity Law)** *If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\, dx \geq 0$.*
2. **(Monotonicity Law)** *If $f(x) \leq g(x)$ for all $x \in [a, b]$, then*
$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx.$$
3. **(Max-Min Inequality)** *Let m and M be the absolute min value and absolute max value of f on $[a, b]$, respectively. Then*
$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a).$$

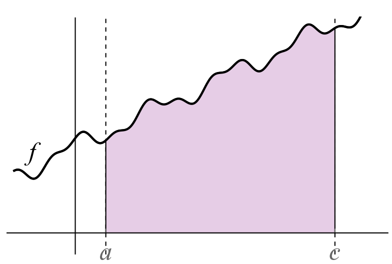




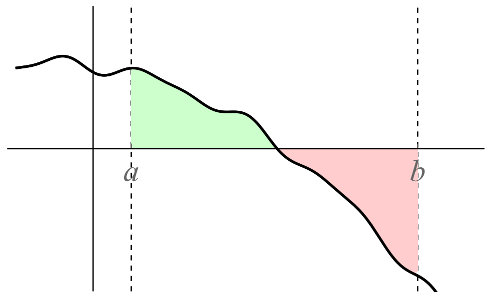
Theorem 9.10 (Additivity property of integrals) *Suppose f is integrable. Then for any numbers a, b and c,*

$$\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$$





Note: It is possible for integrals to be negative (so integrals actually compute something called "signed area"):



EXAMPLE 1

Suppose f and g are functions such that

$$\int_3^7 f(x)\,dx = 6 \quad \int_7^8 f(x)\,dx = 4 \text{ and } \int_3^7 g(x)\,dx = 2.$$

1. Compute $\int_3^8 f(x)\,dx$.

2. Compute $\int_7^3 f(x)\,dx$.

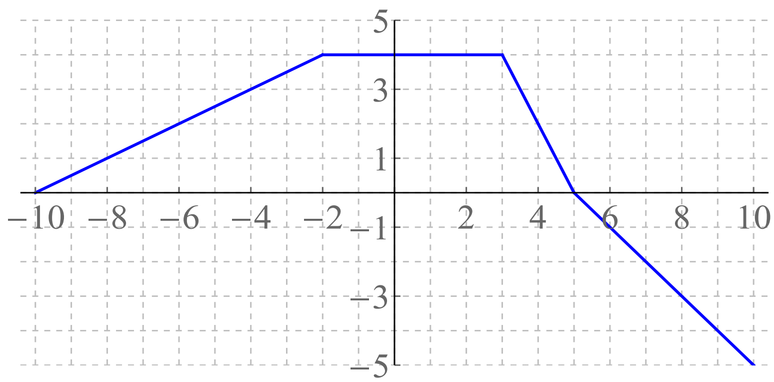
3. Compute $\int_4^4 f(x)\,dx$.

4. Compute $\int_3^7 [4f(x) + 5g(x)]\,dx$.

5. Compute $\int_3^7 [f(x) + 2x]\,dx$.

EXAMPLE 2

Here is the graph of some unknown function f:



Use the graph to estimate the answers to the following integrals:

1. $\int_{-1}^{2} f(x)\,dx$

2. $\int_{-6}^{-4} 10f(x)\,dx$

3. $\int_{5}^{3} f(x)\,dx$

4. $\int_{5}^{7} f(x)\,dx$

5. $\int_{-2}^{-2} f(x)\,dx$

9.5 Fundamental Theorem of Calculus

RECALL FROM SECTION 9.3

It is virtually impossible to compute integrals using the limit definition. So, in order to compute integrals, we need some new ideas. The theory that follows is motivated by the idea from page 237, which suggests that

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Based on this idea, to evaluate an integral like

$$\int_1^4 x^5\,dx,$$

we should think of x^5 as $f'(x)$ and try to find $f(x)$. Just by "guessing" (for now), we see that $f(x) = \frac{1}{6}x^6$ works. So if we let $f(x) = \frac{1}{6}x^6$, we have

$$\int_1^4 x^5\,dx = \int_1^4 f'(x)\,dx = f(4) - f(1) = \frac{1}{6}4^6 - \frac{1}{6}1^6 = \frac{1365}{2}.$$

In this section we justify that this idea works in general. To do this, we need some new terminology:

> **Definition 9.11** *Given function f, an* **antiderivative** *of f is a function $'f$ (read this as "f antiprime") such that $('f)' = f$.*

EXAMPLES

$'f(x) = x^2 - 3$ is an antiderivative of $f(x) = 2x$.

$'f(x) = x^2$ is an antiderivative of $f(x) = 2x$.

$'f(x) = x^2 + C$ is an antiderivative of $f(x) = 2x$ for any constant C.

QUESTION

Are there any other antiderivatives of $f(x) = 2x$?

Theorem 9.12 (Antiderivative Theorem) *Suppose $'f_1$ and $'f_2$ are both antiderivatives of the same function f. Then, for all x, $'f_1(x) =' f_2(x) + C$.*

PROOF Let $G(x) = \ 'f_1(x) -' f_2(x)$. Then

$$G'(x) = ('f_1)'(x) - ('f_2)'(x) = f(x) - f(x) = 0.$$

so G is a function whose derivative is everywhere zero. That means G has slope zero, so it must be a horizontal line, i.e. must be a constant function (this seems obvious, but is actually very deep - take MATH 430 (Advanced Calculus) to see how to prove this rigorously).

Thus $G(x) = \ 'f_1(x) -' f_2(x) = C$ so $'f_1(x) =' f_2(x) + C$ where C is a constant. □

Remark: The point of the Antiderivative Theorem is that any two antiderivatives of the same function must differ by at most a constant.
(So there are no other antiderivatives of $f(x) = 2x$ other than $F(x) = x^2 + C$.)

Restated, this means that if you have found one antiderivative of a function, you have found them all (by adding an arbitrary constant).

Theorem 9.13 (Fundamental Theorem of Calculus I) ***(Differentiation of Integrals)*** *Let f be continuous on $[a, b]$. Consider a new function*

$$'f(x) = \int_a^x f(t)\, dt.$$

Then:

1. *$'f$ is continuous and differentiable on $[a, b]$; and*
2. *$('f)'(x) = f(x)$ (i.e. $'f$ is an antiderivative of f).*

Picture:

Physical interpretation:

Mathematical significance of this part of the FTC:

1. The FTC reveals that differentiation and integration are inverse operations (because it says that if you start with a function f, take its integral (to get $'f$) and then take the derivative of that, you get back to the function f that you started with).

2. The FTC guarantees that every continuous function has an antiderivative: given function $f(x)$, the function $'f(x) = \int_a^x f(t)\,dt$ is an antiderivative of f for any choice of a.

PROOF OF FTC PART I: By the definition of derivative,

$$\begin{aligned}
('f)'(x) &= \lim_{h\to 0} \frac{'f(x+h) -' f(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right) \\
. &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt.
\end{aligned}$$

Now by the Max-Min Inequality, by letting m and M be the minimum and maximum values of f on $[x, x+h]$, we have

$$\begin{aligned}
m(x+h-x) &\le \int_x^{x+h} f(t)\,dt \le M(x+h-x) \\
\Rightarrow mh &\le \int_x^{x+h} f(t)\,dt \le Mh \\
\Rightarrow m &\le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le M
\end{aligned}$$

As $h \to 0$, m and M both go to $f(x)$, so the inside quantity must go to $f(x)$ as well, i.e.

$$('f)'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x). \;\square$$

Now for our last big theorem. Remember that the goal is to develop a method to evaluate integrals that doesn't use the limit definition. We are now able to achieve this goal:

Theorem 9.14 (Fundamental Theorem of Calculus Part II) ***(Evaluation of Integrals)*** *Let f be continuous on $[a, b]$. Suppose $'f$ is* ***any*** *antiderivative of f. Then*

$$\int_a^b f(x)\,dx = {}'f(b) -{}' f(a).$$

Notation: The expression $'f(b) -{}' f(a)$ is written $['f(x)]_a^b$ or $'f(x)|_a^b$.

Proof: Let $G(x) = \int_a^x f(t)\,dt$. $G'(x) = f(x)$ by the first part of the Fundamental Theorem of Calculus. By the Antiderivative Theorem, if $'f$ is **any** antiderivative of f, we know $'f(x) = G(x) + C$. Therefore

$$\begin{aligned}
'f(b) - {}'f(a) &= G(b) + C - (G(a) + C) \\
&= G(b) - G(a) \\
&= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\
&= \int_a^b f(t)\,dt - 0 \\
&= \int_a^b f(t)\,dt \\
&= \int_a^b f(x)\,dx \quad \text{(since the } t \text{ and } x \text{ are dummy variables).}
\end{aligned}$$

Physical interpretation of this part of the FTC: Suppose $'f(x)$ gives the position of an object at time x. Then the object's velocity is $('f)'(x) = f(x)$. This part of the FTC says that the displacement of the object from time a to time b equals the area under the velocity function f from a to b, as suggested earlier in this chapter.

More general interpretation: Suppose $'f(x)$ is any quantity. Then the rate of change of $'f$ with respect to x is $('f)'(x) = f(x)$. This part of the FTC says that the integral of the rate of change, i.e. the accumulation of the rate of change, is equal to the net change in $'f$ from $x = a$ to $x = b$.

Mathematical significance of this part of the FTC: This result provides a mechanism to evaluate definite integrals without having to compute limits of Riemann sums. In particular, if you can find any one antiderivative of f that is easy to work with (say $'f$), then you can evaluate integrals of f by subtracting values of $'f$.

You are responsible for being able to state both parts of the FTC and explain their physical interpretation and mathematical significance.

EXAMPLE 1

Evaluate the integral:

$$\int_3^4 x\,dx$$

EXAMPLE 2

Suppose an object is moving along a line so that its velocity at time t is $3\sec^2 t$. Find the distance traveled by the object between times $t = 0$ and $t = \frac{\pi}{4}$.

EXAMPLE 3

In an electrical circuit, the **current** is the instantaneous rate of change of the charge. If the current in a circuit at time t (in seconds) is $2 + \frac{1}{4}\sin t$ amperes, find the net change in the charge from time $\frac{\pi}{2}$ to time π. (P.S. An ampere times a second is a coulomb, a unit of charge.)

EXAMPLE 4

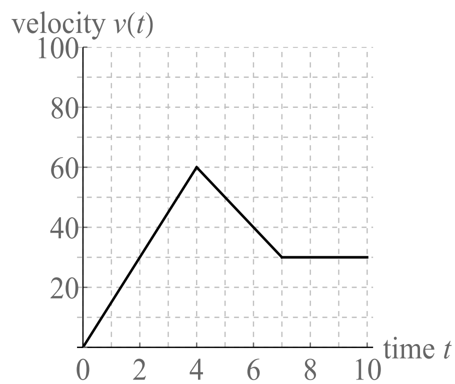
A tank is being filled with fluid at a non-constant rate: at time t (in seconds), the rate at which the tank is being filled is $2t(4-t)$ L/sec. Find the amount of fluid that is poured in the tank during the first 3 seconds.

The Fundamental Theorem of Calculus reduces the problem of computing integrals to the problem of finding antiderivatives. Thus it is important to be able to find antiderivatives of functions, and we address this task in the next chapter.

9.6 Homework exercises

Exercises from Chapter 9.1

1. During a car trip, let $v(t)$ represent the car's speedometer reading (in miles per hour) at time t (measured in hours after the beginning of the car trip). Suppose that the graph of $v(t)$ for $0 \leq t \leq 10$ is as given below:



Use this graph to estimate the answers to the following questions (answer with appropriate units):

a) What is the speedometer reading 2 hours after the trip starts?

b) What is the acceleration of the car at time 6?

c) Is the car speeding up, or slowing down at time 3? Explain.

d) Is the car moving forward or backward at time 6? Explain.

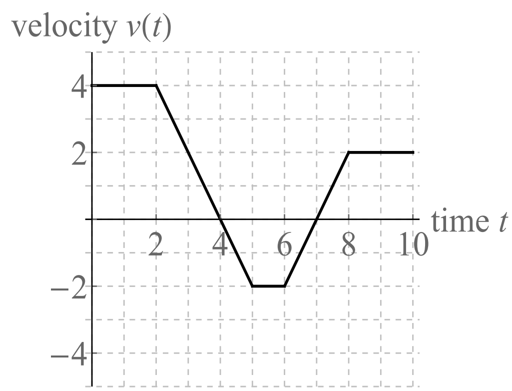
e) Find the distance the car travels during the first 3 hours of the trip.

f) Find the distance the car travels between times 4 and 9.

g) If the odometer reading of the car at the beginning of the trip is 1000, find the odometer reading six hours later.

h) If the odometer reading of the car at time 5 is 2000, what was the odometer reading three hours earlier?

2. On Mars, a rover is moving back and forth along a dirt track so that at time t (measured in seconds), its velocity (measured in cm/sec) is given by the function v whose graph is given below for $0 \leq t \leq 10$:



Use this graph to estimate the answers to the following questions (answer with appropriate units):

a) What is the velocity of the rover at time 8?

b) At what time(s) is the velocity of the rover equal to 1 cm/sec?

c) What is the acceleration of the rover at time 7?

d) Is the rover moving forward or backward at time 6? Explain.

e) Find the displacement of the rover from time 0 to time 3.

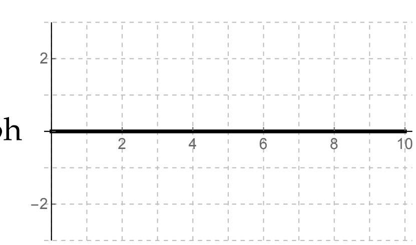
f) Find the displacement of the rover from time 6 to time 10.

g) Suppose the initial position of the rover is 0. Find all times when the position of the rover is 8.

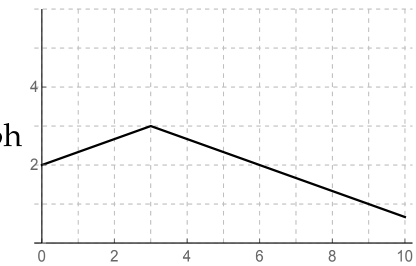
h) Suppose the initial position of the rover is 4. Sketch a crude graph of the position of the rover, as a function of t.

3. In each part of this problem, you are given the graph of the derivative f' of some function f for $0 \leq x \leq 10$, and the value of f at one value of x. Use this information to sketch the graph of f for $0 \leq x \leq 10$.

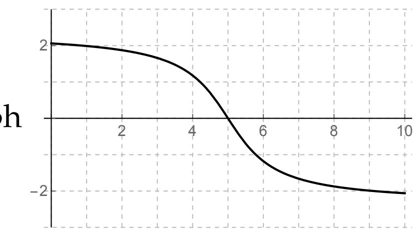
a) $f(3) = 2$; f' has graph



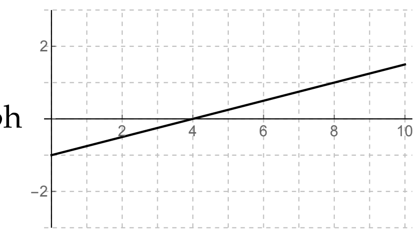
b) $f(0) = 4$; f' has graph



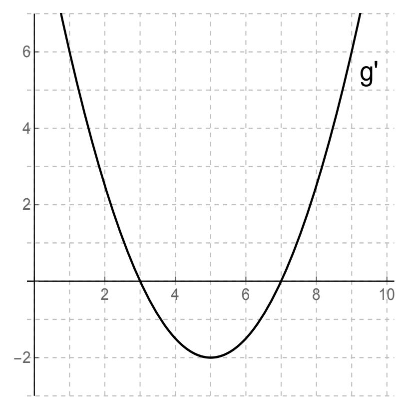
c) $f(0) = -3$; f' has graph



d) $f(0) = 5$; f' has graph



4. Suppose the graph of some derivative g' is as given below. On a single set of axes, sketch all possible graphs of g:



Exercises from Section 9.2

In Problems 5-8, write the following sums in $\Sigma-$notation:

5. $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + ... + \frac{1}{19}$

6. $\frac{5}{7^2} + \frac{5}{8^2} + \frac{5}{9^2} + ... + \frac{5}{26^2}$

7. $\frac{2^4}{4} + \frac{2^5}{5} + \frac{2^6}{6} + ... + \frac{2^{14}}{14}$

8. $\frac{3}{8^2}\sqrt{2} + \frac{3}{8^3}\sqrt{3} + \frac{3}{8^4}\sqrt{4} + ... + \frac{3}{8^{25}}\sqrt{25}$

In Problems 9-11, evaluate the given sum by hand (simplify your answer):

9. $\sum_{n=1}^{7} 2n$

10. $\sum_{n=0}^{4} \cos \pi n$

11. $\sum_{n=3}^{6} n^2$

In Problems 12-14, evaluate each of the following sums using *Mathematica*. *Note:* to evaluate a sum of the form $\sum_{n=M}^{N} a_n$ in *Mathematica*, use the following syntax:

Sum[a_n, {n, M, N}]

For example, to evaluate $\sum_{n=2}^{9} n^2$, execute `Sum[n^2, {n, 2, 9}]`. (You can also get a Σ on the Basic Math Assistant Pallette.)

12. $\sum_{n=2}^{13} \frac{1}{n}$

13. $\sum_{n=1}^{35} \frac{12n + 4n^2 + n^3}{6400}$

14. $\sum_{n=1}^{17} \cos\left(\frac{\pi}{2}n\right) 3n^2$

15. Consider the partition $\mathcal{P} = \{2, 3, 8, 10, 13\}$.
    a) Sketch a picture of this partition.
    b) What interval is this a partition of?
    c) How many subintervals comprise this partition?
    d) What is x_3 for this partition?
    e) What is the second subinterval of the partition?
    f) What is Δx_1?
    g) What is $||\mathcal{P}||$?

16. Consider the partition $\mathcal{P}$ of $[5, 12]$ into 70 equal-length subintervals.
    a) What is x_{20} for this partition?
    b) What is the twelfth subinterval of the partition?
    c) What is Δx_{32}?
    d) What is $||\mathcal{P}||$?

17. Let $f(x) = 1 + 2x - x^2$. Consider the partition $\mathcal{P} = \left\{0, \frac{1}{4}, \frac{1}{2}, 1\right\}$ of the interval $[0, 1]$.
    a) Calculate the value of the Riemann sum associated to $\mathcal{P}$ where the test points c_j are chosen to be the midpoints of their respective subintervals.

b) Sketch a picture which reflects the area being calculated in the Riemann sum you computed in part (a).

c) What is the smallest possible value of any Riemann sum associated to the partition $\mathcal{P}$? Explain your answer.

d) What is the largest possible value of any Riemann sum associated to the partition $\mathcal{P}$? Explain your answer.

e) What do your answers to parts (c) and (d) of this question tell you about the possible value of the area under f from $x = 0$ to $x = 1$?

18. Let $f(x) = 4 \sin x$ and let $\mathcal{P} = \left\{0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi\right\}$.

a) Calculate the right-hand Riemann sum associated to this partition.

b) Sketch a picture which reflects the area being calculated in part (a).

c) Calculate the lower Riemann sum associated to this partition.

d) Sketch a picture which reflects the area being calculated in part (c).

19. Let $f(x) = 2x^2 + 1$.

a) Compute the left-hand Riemann sum associated to the partition of $[1, 4]$ into three equal-length subintervals.

b) Sketch a picture which reflects the area being calculated in part (a).

c) Compute the upper sum associated to the partition of $[1, 4]$ into six equal-length subintervals.

d) Sketch a picture which reflects the area being calculated in part (c).

20. Let f be an unknown function with the following table of values:

x	-3	-1	1	4	10	11
$f(x)$	2	1	2	0	3	5

a) Use a left-hand Riemann sum to estimate the area under the graph of f from $x = -1$ to $x = 4$.

b) Use a right-hand Riemann sum to estimate the area under the graph of f from $x = 1$ to $x = 11$.

c) Can you compute an upper Riemann sum for f associated to the partition $\mathcal{P} = \{-3, -1, 1, 4\}$? If so, explain why and compute it. If not, explain why you do not have enough information to compute this Riemann sum.

21. Suppose that the velocity of a rocket t seconds after it is launched is given by function v, some of whose values are given in the following table:

t (seconds after launch)	0	1	2	4	8	9	10	12
$v(t)$ (m/sec)	0	2	5	13	30	75	110	240

Suppose also that the acceleration of the rocket is positive at all times between $t = 0$ and $t = 12$.

a) Use a left-hand Riemann sum to estimate the distance the rocket travels in the first 8 seconds after it is launched.

b) Use a right-hand Riemann sum to estimate the distance the rocket travels between times $t = 4$ and $t = 10$.

c) Can you compute an upper Riemann sum for f associated to the partition $\mathcal{P} = \{0, 2, 4, 8, 12\}$? If so, explain why and compute it. If not, explain why you do not have enough information to compute this Riemann sum.

Exercises from Section 9.3

In Problems 22-24, write a definite integral which computes the desired area. (You do not actually need to compute the integral.)

22. The area between the graph of the function $f(x) = \sin x$ and the $x-$axis from $x = 0$ to $x = \pi/2$.

23. The area between the graph of the function $f(x) = x^6$ and the $x-$axis from $x = -3$ to $x = 4$

24. The area between the graph of the function $f(x) = \arctan x$ and the $x-$axis from $x = 0$ to $x = 1$.

In Problems 25-32, evaluate each definite integral:

25. $\int_2^6 3\,dx$

26. $\int_8^{11} 0\,dx$

27. $\int_0^7 5x\,dx$

28. $\int_0^2 (2x+3)\,dx$

29. $\int_5^8 (20-2x)\,dx$

30. $\int_{-4}^4 \sqrt{16-x^2}\,dx$

31. $\int_0^4 |x-3|\,dx$

32. $\int_0^2 \sqrt{4-x^2}\,dx$

Exercises from Section 9.4

33. Assuming the following two statements,

$$\int_0^5 f(x)\,dx = 10 \quad \text{and} \quad \int_5^9 f(x)\,dx = 2.$$

compute each of the following:

(a) $\int_0^9 f(x)\,dx$ (b) $\int_0^5 2f(x)\,dx$ (c) $\int_5^0 f(x)\,dx$ (d) $\int_3^3 f(x)\,dx$

34. Assuming the following two statements,

$$\int_0^4 f(x)\,dx = 7 \quad \text{and} \quad \int_2^4 f(x)\,dx = 6.$$

compute each of the following:

(a) $\int_0^2 f(x)\,dx$ (b) $\int_0^4 7f(x)\,dx$ (c) $\int_0^4 [f(x)+9]\,dx$

35. Assuming the following two statements,

$$\int_5^8 f(x)\,dx = 4 \quad \text{and} \quad \int_5^8 g(x)\,dx = 7.$$

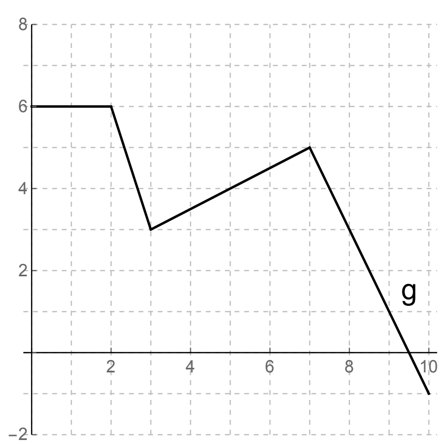
compute each of the following:

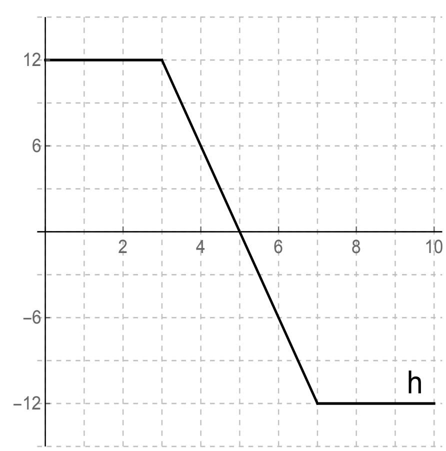
(a) $\int_5^8 [2f(x)+3g(x)]\,dx$ (b) $\int_5^8 [g(x)-f(x)]\,dx$ (c) $\int_3^7 f(x)\,dx + \int_7^3 f(x)\,dx$

36. Assume that f is an unknown function with the following properties:

$$\int_0^3 f(x)\,dx = 7 \qquad \int_3^5 f(x)\,dx = -3 \qquad \int_5^8 f(x)\,dx = 2$$

Also, assume g and h are unknown functions whose graphs are given below:





Use this information to compute the folllowing quantities:

a) $\int_0^3 [f(x) + g(x)]\ dx$

b) $\int_3^7 3g(x)\, dx$

c) $\int_8^5 [f(x) - h(x)]\ dx$

d) $\int_3^8 [2g(x) + 4f(x)]\ dx$

e) $\int_0^8 [f(x) + g(x) + h(x)]\ dx$

f) $\int_5^3 [h(x) - 2f(x)]\ dx$

g) $\int_0^5 (h(x) - 2)\ dx$

h) $\int_0^2 (g(x) + 3x)\ dx$

Exercises from Section 9.5

In Problems 37-44, classify each statement as TRUE or FALSE:

37. $'f(x) = \sin(x^2)$ is an antiderivative of $f(x) = \cos(x^2)$.

38. $'f(x) = 3x^2$ is an antiderivative of $f(x) = 6x$.

39. $'f(x) = 3x^2$ is the only antiderivative of $f(x) = 6x$.

40. If $'f$ is an antiderivative of f, then for any constant C, $'f(x) - C$ is an antiderivative of f as well.

41. If $'f$ is an antiderivative of f, then for any constant C, $C\,['f(x)]$ is an antiderivative of f as well.

42. All antiderivatives of $f(x) = \sec^2 x$ are of the form $\tan x + C$.

43. If $'f$ is an antiderivative of some continuous function f, then $\int f(x)\, dx = {'f(x)}$.

44. If $'f$ is an antiderivative of some continuous function f, then $\int_a^b f(x)\, dx = {'f(b)} - {'f(a)}$.

In Problems 45-49, compute the indicated definite integral by using the Fundamental Theorem of Calculus:

45. $\int_3^7 4x^3\, dx$

46. $\int_{\pi/3}^{\pi/2} \cos x\, dx$

47. $\int_0^3 (12t^2 - 6t)\, dt$

48. $\int_0^{\ln 6} \frac{1}{2} e^x\, dx$

49. $\displaystyle\int_7^{10} \frac{1}{x}\,dx$

50. A syringe is being emptied at a non-constant rate: at time t (in seconds), the rate at which the syringe is being emptied is $4\sin t + 2\cos t$ mL/sec. Find the amount of liquid drained from the syringe in the first $\frac{\pi}{4}$ seconds.

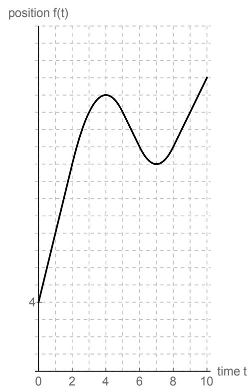
51. If the current in an electrical current at time t (in seconds) is $t - \frac{2}{t}$ amperes, find the net change in the charge in the circuit from time 1 to time 3.

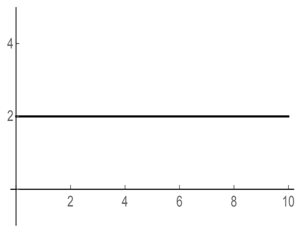
52. A truck's velocity at time t (in hours) is $v(t) = 40t(t+1)$ miles per hour. How far does the truck travel in the first 30 minutes of its journey?

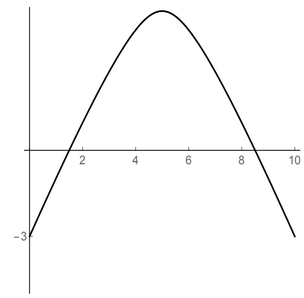
Answers

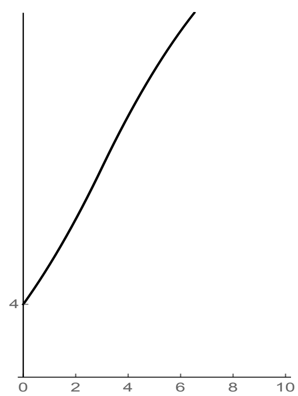
1.
   a) 30 mi/hr
   b) -10 mi/hr^2
   c) The car is speeding up, because the acceleration (i.e. the slope of v) is positive at $t = 3$.
   d) The car is moving forward, because the velocity (i.e. the height of the graph of v) is positive at $t = 6$.
   e) $\frac{135}{2} = 67.5$ miles
   f) 195 miles
   g) 1220
   h) 1855

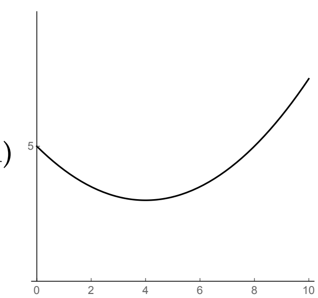
2.
   a) $v(8) = 2$ cm/sec
   b) $t = 3.5$ sec, $t = 7.5$ sec
   c) 2 cm/sec^2
   d) The rover is moving backward, because the velocity is negative at $t = 6$.
   e) 11 cm
   f) 4 cm
   g) $t = 2$, $t = 7$

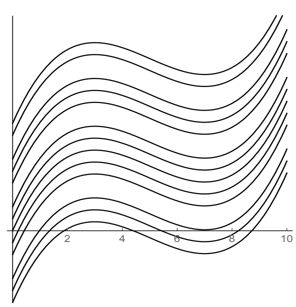
h) 

3. a) 

c) 

b) 

d) 

4. 

5. $\displaystyle\sum_{n=3}^{19} \frac{1}{n}$

6. $\displaystyle\sum_{n=7}^{26} \frac{5}{n^2}$

7. $\displaystyle\sum_{n=4}^{14} \frac{2^n}{n}$

8. $\displaystyle\sum_{n=2}^{25} \frac{3}{8^n}\sqrt{n}$

9. 56

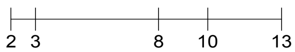
10. 1

11. 86

12. $\dfrac{785633}{360360}$

13. $\dfrac{4641}{64}$

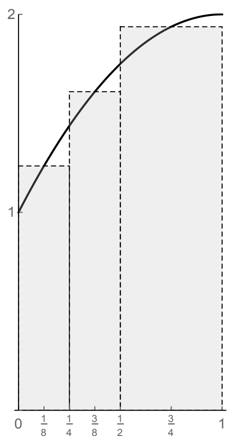
14. 432

15. a) 

b) $[2, 13]$
c) 4
d) 10
e) $[3, 8]$
f) 1
g) 5

16. a) $[6.9, 7]$
b) $[6.1, 6.2]$
c) $.1$
d) $.1$

17. a) $\dfrac{215}{128}$

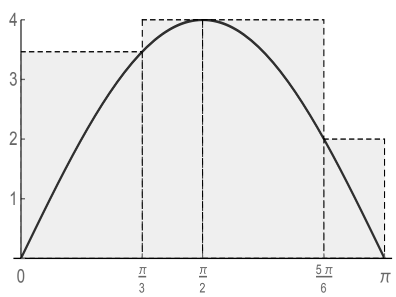
b) 

c) The smallest possible value is the lower Riemann sum associated to $\mathcal{P}$, which is $\dfrac{95}{64}$.

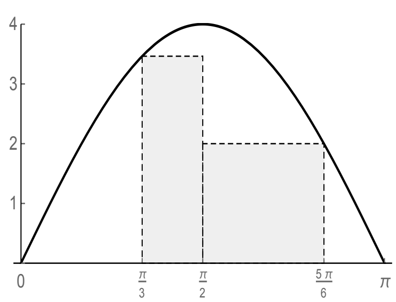
d) The largest possible value is the upper Riemann sum associated to $\mathcal{P}$, which is $\dfrac{115}{64}$.

e) The actual area under the function f must be greater than the lower sum (i.e. greater than $\frac{95}{64}$ and less than the upper sum (i.e. less than $\dfrac{115}{64}$).

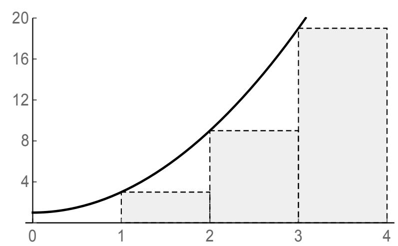
18. a) $\dfrac{(4+2\sqrt{3})\pi}{3}$

b) 

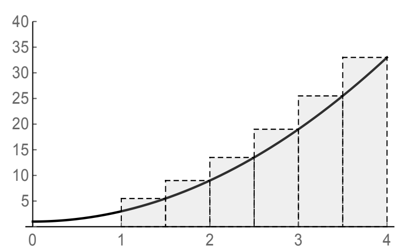
c) $\dfrac{\pi}{3}\left(\sqrt{3}+2\right)$

d) 

19. a) 31

b) 

c) $\dfrac{211}{4} = 52.75$

d) 

20. a) 8
b) 23
c) You do not have enough information, because you do not know the maximum value f achieves on each subinterval of $\mathcal{P}$.

21. a) 64 m

b) 305 m

c) Since the acceleration is positive, v is increasing. This means that the upper sum coincides with the right-hand sum, which is 1116 m.

22. $\displaystyle\int_0^{\pi/2} \sin x \, dx$

23. $\displaystyle\int_{-3}^{4} x^6 \, dx$

24. $\displaystyle\int_0^{1} \arctan x \, dx$

25. 12

26. 0

27. $\dfrac{245}{2}$

28. 10

29. 21

30. 8π

31. 5

32. π

33. a) 12 b) 20 c) -10 d) 0

34. a) 1 b) 49 c) 43

35. a) 29 b) 3 c) 0

36. a) 23.5 b) 48 c) -26 d) 36 e) 66.5 f) -18 g) 38 h) 18

37. FALSE (the derivative of $\sin(x^2)$ is $\cos(x^2) \cdot 2x$)

38. TRUE (the derivative of $3x^2$ is $6x$)

39. FALSE ($3x^2 + 1$ is also an antiderivative)

40. TRUE (the derivative of $F(x) - c$ is also $f(x)$)

41. FALSE (the derivative of $2F(x)$ is $2f(x)$, not $f(x)$)

42. TRUE (by the Antiderivative Theorem)

43. FALSE ($\int f(x) \, dx = F(x) + C$)

44. TRUE (this is the Fund. Thm. of Calculus Part 2)

45. $7^4 - 3^4$

46. $1 - \dfrac{\sqrt{3}}{2}$

47. 81

48. $\dfrac{5}{2}$

49. $\ln 10 - \ln 7$

50. $4 - \sqrt{2}$ mL

51. $4 - \ln 9$ coulombs

52. $\dfrac{20}{3}$ miles

Chapter 10

Integration Rules

10.1 General integration concepts

In Chapter 9, we formally defined the definite integral as a limit of Riemann sums:

$$\int_a^b f(x)\,dx = \lim_{||\mathcal{P}||\to 0} \sum_{j=1}^{n} f(c_j)\,\Delta x_j$$

This expression gives the area under function f from $x = a$ to $x = b$.

In practice, it is mostly impossible to compute integrals by evaluating these limits; rather, we use:

> **Theorem 10.1 (Fundamental Theorem of Calculus Part II)** ***(Evaluation of Integrals)*** *Let f be continuous on $[a, b]$. Suppose $'f$ is any antiderivative of f. Then*
>
> $$\int_a^b f(x)\,dx = {'f}(b) - {'f}(a) = {'f}(x)\Big|_a^b .$$

This suggests that it is important to find antiderivatives of functions.

> **Definition 10.2** *Given a function f, an* **antiderivative** *of f is another function $'f$ such that $('f)' = f$.*

> **Definition 10.3** *Given function f, the* **indefinite integral** *of f, denoted*
>
> $$\int f(x)\,dx,$$

is the set of all antiderivatives of f.

At this point, we have two objects which look the same but are very different:

Definite Integral	Indefinite Integral
$\int_a^b f(x)\,dx$	$\int f(x)\,dx$

EXAMPLE 1

If $f(x) = 2x$, then

$$\int f(x)\,dx =$$

$$\int_{-1}^{4} f(x)\,dx =$$

EXAMPLE 2

Suppose $\int f(x)\,dx = \cos x + C$. Compute

$$\int_{\pi/3}^{\pi/2} f(x)\,dx$$

and find $f(x)$.

General principle illustrated by the previous example:

This means that each of the derivatives we learned earlier in the semester turns into an integral that we know now:

$$\frac{d}{dx}(C) = 0 \qquad \Rightarrow$$

$$\frac{d}{dx}(x^n) = nx^{n-1}(\text{if } n \neq 0) \qquad \Rightarrow$$

$$\frac{d}{dx}(e^x) = e^x \qquad \Rightarrow \qquad \int e^x \, dx = e^x + C$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \Rightarrow \qquad \int \frac{1}{x} \, dx = \ln x + C$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{x^2+1} \qquad \Rightarrow \qquad \int \frac{1}{x^2+1} \, dx = \arctan x + C$$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \Rightarrow \qquad \int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \Rightarrow \qquad \int \cos x \, dx = \sin x + C$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \Rightarrow \qquad \int (-\sin x) \, dx = \cos x + C$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \Rightarrow \qquad \int \sec^2 x \, dx = \tan x + C$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \qquad \Rightarrow \qquad \int \csc^2 x \, dx = -\cot x + C$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \Rightarrow \qquad \int \sec x \tan x \, dx = \sec x + C$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x \qquad \Rightarrow \qquad \int \csc x \cot x \, dx = -\csc x + C$$

Furthermore, since differentiation is linear, so is integration. We have:

Theorem 10.4 (Linearity of Definite Integration) *Suppose f and g are integrable functions. Then:*

1. $\int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$;
2. $\int_a^b [f(x) - g(x)] \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx$;
3. $\int_a^b [k \cdot f(x)] \, dx = k \int_a^b f(x) \, dx$ *for any constant k.*

Theorem 10.5 (Linearity of Indefinite Integration) *Suppose f and g are integrable functions. Then:*

1. $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$;
2. $\int [f(x) - g(x)]\, dx = \int f(x)\, dx - \int g(x)\, dx$;
3. $\int [k \cdot f(x)]\, dx = k \int f(x)\, dx$ *for any constant* k.

NOTE: Integration is not multiplicative nor divisive:

$$\int f(x)g(x)\, dx \neq \left(\int f(x)\, dx\right) \cdot \left(\int g(x)\, dx\right)$$

$$\int \left(\frac{f(x)}{g(x)}\right)\, dx \neq \frac{\int f(x)\, dx}{\int g(x)\, dx}$$

EXAMPLE 3

Compute

$$\int -\frac{1}{4} \csc^2 x\, dx.$$

EXAMPLE 4

Suppose a bicyclist is driving down a road so that her velocity at time t is $3-2t+t^9$.

(a) Find the displacement of the bicyclist from time 1 to time 2.

(b) If the position of the bicyclist at time 0 is 4, find the position at time 1.

EXAMPLE 5

Compute

$$\int \left(\frac{2}{\sqrt[3]{x}} + \frac{5}{x}\right)\, dx.$$

Example 6

Compute
$$\int (4\cos x - 3x^5 + 2e^x)\,dx.$$

Example 7

Compute
$$\int \left(\frac{\sin x}{7} + \frac{4}{1+x^2} - 2\right)\,dx.$$

Example 8

A MATH 230 student is asked to compute this integral:
$$\int x\sec^2 x\,dx$$

After some substantial work, the student obtains the answer
$$\ln(\cos x) + x\tan x + C.$$

Is the student's answer correct? Why or why not?

Here is a list of integration rules which, together with linearity, allows you to do most easy integrals:

Theorem 10.6 (Integrals that we memorize)

CONSTANTS: $$\int 0\,dx = C$$

$$\int M\,dx = Mx + C$$

POWERS: $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \text{ whenever } n \neq -1$$

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$

(I don't care so much about the | | here)

TRIG: $$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

EXPONENTIALS: $$\int e^x\,dx = e^x + C$$

$$\int e^{rx}\,dx = \frac{1}{r}e^{rx} + C \text{ whenever } r \neq 0$$

$$\int b^x\,dx = \frac{1}{\ln b}b^x + C$$

INVERSE TRIG: $$\int \frac{1}{x^2+1}\,dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

Note: There are some integrals which we don't know yet. They include:

$$\int \tan x \, dx \qquad \int \cot x \, dx \qquad \int \sec x \, dx \qquad \int \sin(x^2) \, dx \qquad \int \ln x \, dx, \text{ etc.}$$

Some (most) of these integrals will be discussed in MATH 230 (alternatively, some of them can be computed using *Mathematica*, but some integrals are known to be impossible to compute, even with an infinitely powerful computer!).

10.2 Rewriting the integrand

Sometimes it is useful to use algebra, or a trigonometric identity, or a logarithm rule, to rewrite the integrand before computing an integral.

EXAMPLE 1

Find the area under the graph of $f(x) = \dfrac{(x^2-1)^2}{x}$ between $x = 1$ and $x = 2$.

EXAMPLE 2

$$\int \tan^2 x \, dx =$$

EXAMPLE 3

$$\int \ln\left(2^x\right)\, dx =$$

10.3 Elementary u-substitutions in indefinite integrals

MOTIVATING EXAMPLE

Let $f(x) = \sin(x^3)$.

Goal: Recognize integrands which arise as the result of the Chain Rule.

Idea: Identify the presence of a function and its derivative in the integrand.

Generalization of the Motivating Example

Consider the function $F(g(x))$, where $F' = f$. Then

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$$

Theorem 10.7 (Integration by $u-$substitution - Indefinite Integrals)

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du \text{ by setting } u = g(x).$$

Procedure for *indefinite* integration by $u-$substitution:

1. Make sure you can't just "write the answer" to the integral without a substitution.
2. Check the integrand to make sure a $u-$substitution is appropriate:
   - The integral should not be one you have memorized.
   - The integrand should have two terms multiplied together.
   - One of the terms being multiplied should be (up to a constant) the derivative of part of the other term (i.e. the terms should be "related").

   Schematically, the integral should look like this:
3. Let $u =$ the term whose derivative stands by itself.
4. Write the derivative of u in Leibniz notation, then multiply through by an appropriate constant to match what is in the integral.
5. Substitute in the integral so that all xs are replaced with us as appropriate.
6. Integrate with respect to u.
7. Back-substitute to get an answer in terms of x.

A picture to explain the logic:

integral with respect to x answer in terms of x

EXAMPLE 1

$$\int \left(6x^2+3\right)^4 x\,dx$$

EXAMPLE 2

$$\int 27z^2(z^3+1)^9\,dz$$

EXAMPLE 3

$$\int \tan^3(3x+1)\sec^2(3x+1)\,dx$$

Solution:

$$\begin{aligned} \text{Let } u &= \tan(3x+1). \\ \Rightarrow \frac{du}{dx} &= \sec^2(3x+1)\cdot 3 \\ \Rightarrow du &= 3\sec^2(3x+1)\,dx \\ \Rightarrow \frac{1}{3}du &= \sec^2(3x+1)\,dx \end{aligned}$$

Some integrals require rewriting before performing a substitution:

EXAMPLE 4

$$\int \tan x \, dx =$$

EXAMPLE 5

Find all functions g whose derivative is $g'(x) = e^{x+e^x}$.

The Linear Replacement Principle

Let's suppose that you start with some integral you "know". Call this integral a *prototype* integral:

Specific example	General situation
$\int \cos x \, dx =$	$\int f(x) \, dx = {}'f(x) + C$

In this section, we want to look at what happens when you replace the x in the above prototype integrals with a linear expression of the form $mx + b$:

Specific example	General situation
$\int \cos(3x + 2) \, dx$	$\int f(mx + b) \, dx =$

The big idea here is that if you remember how this general situation works, you can quickly integrate lots of functions of the form $f(mx + b)$. These integrals come up often in applications and in advanced math courses, so it is useful to integrate them without actually doing the u-substitution.

Theorem 10.8 (Linear Replacement Principle) *Suppose you know the "prototype" integral*

$$\int f(x) \, dx = {}'f(x) + C.$$

Then for any constants m and b ($m \neq 0$),

$$\int f(mx + b) \, dx = \frac{1}{m} {}'f(mx + b) + C.$$

EXAMPLE 6

$$\int (5x-2)^{12}\,dx =$$

EXAMPLE 7

$$\int 2e^{5x}\,dx =$$

EXAMPLE 8

$$\int e^{2-x/4}\,dx =$$

EXAMPLE 9

$$\int \frac{4}{5+3x}\,dx =$$

EXAMPLE 10

$$\int \frac{3\sec^2 4x}{5}\,dx = \frac{3}{5}\cdot\frac{1}{4}\tan 4x + C = \boxed{\frac{3}{20}\tan 4x + C}.$$

EXAMPLE 11

$$\int_{\pi/3}^{\pi/2} \sin 3x\,dx = \frac{1}{3}(-\cos 3x)\Big|_{\pi/3}^{\pi/2} = -\frac{1}{3}\cos\frac{3\pi}{2} + \frac{1}{3}\cos\pi = 0 - \frac{1}{3} = \boxed{-\frac{1}{3}}.$$

10.4 Elementary u-substitutions in definite integrals

EXAMPLE 1

Compute:

$$\int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)^3}\, dx$$

What's the same in a definite integral u-sub:

1.

2.

3.

What's different:

1.

2.

Theorem 10.9 (Integration by $u-$substitution - Definite Integrals) *By way of the u-substitution $u = g(x)$,*

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Procedure for *definite* integration by $u-$substitution:

1. Make sure you can't just "write the answer" to the integral without a substitution.
2. Check the integrand to make sure a $u-$substitution is appropriate:
   - The integral should not be one you have memorized.
   - The integrand should have two terms multiplied together.
   - One of the terms being multiplied should be, up to a constant, the derivative of part of the other term (i.e. the terms should be "related").

   Schematically, the integral should look like this:
3. Let $u =$ the term whose derivative stands by itself.
4. Write the derivative of u in Leibniz notation, then multiply through by an appropriate constant to match what is in the integral.
5. Substitute in the integral so that all xs are replaced with us as appropriate.
6. Change the limits of integration to values of u using the formula from Step 3.
7. Integrate with respect to u (don't back-substitute for x).

EXAMPLE 2

$$\int_0^{\ln 3} \frac{e^x}{e^x + 1}\, dx$$

EXAMPLE 3

Find the distance travelled by an object between times 0 and $\frac{\pi}{2}$, if its velocity at time t is $e^{\cos t}\sin t$.

EXAMPLE 4

Compute the area under the function $f(x) = x^{-3}(x^{-2}+1)^2$ between $x = 1$ and $x = 2$.

Solution: The area is

$$\int_1^2 x^{-3}(x^{-2}+1)^2\,dx.$$

To evaluate this, use the u-substitution

$$\begin{aligned} u &= x^{-2}+1 \\ du &= -2x^{-3}\,dx \\ \frac{-1}{2}du &= x^{-3}\,dx \end{aligned}$$

and change the limits:

$$x = 1 : u = 1^{-2}+1 = 2 \qquad x = 2 : u = 2^{-2}+1 = \frac{1}{4}+1 = \frac{5}{4}$$

so the integral becomes

$$\begin{aligned} \int_2^{5/4} \frac{-1}{2}u^2\,du &= \int_{5/4}^2 \frac{1}{2}u^2\,du \\ &= \frac{1}{6}u^3\Big|_{5/4}^2 \\ &= \frac{1}{6}(2)^3 - \frac{1}{6}\left(\frac{5}{4}\right)^3 = \frac{8}{6} - \frac{1}{6}\left(\frac{125}{64}\right) = \boxed{\frac{129}{128}}. \end{aligned}$$

10.5 Homework exercises

Exercises from Section 10.1

In Problems 1-6, an advanced student was asked to compute the given integral. Determine, in part by taking an appropriate derivative, whether or not the student's answer was correct:

1. Problem: $\int \cos x\, dx$; proposed answer: $\sin x$
2. Problem: $\int \ln x\, dx$; proposed answer: $\frac{1}{x} + C$
3. Problem: $\int \frac{1}{x+3}\, dx$; proposed answer: $\ln(x+3) + C$
4. Problem: $\int \frac{1}{x^2+25}\, dx$; proposed answer: $\ln(x^2+25) + C$
5. Problem: $\int \frac{1}{x^2+25}\, dx$; proposed answer: $\frac{1}{5} \arctan \frac{x}{5} + C$
6. Problem: $\int \frac{4}{x^2-1}\, dx$; proposed answer: $2\ln(1-x) + 2\ln(1+x) + C$

In Problems 7-12, an advanced student was asked to compute the given integral, and got an answer which is close, but wrong. After taking a derivative of the student's answer, use the derivative you get to "fix" the student's answer, making it correct.

7. Problem: $\int \cos 2x\, dx$; wrong answer: $\sin 2x + C$
8. Problem: $\int \frac{1}{3x-4}\, dx$; wrong answer: $\ln(3x-4) + C$
9. Problem: $\int \csc^2 x\, dx$; wrong answer: $\cot x + C$
10. Problem: $\int e^{3x}\, dx$; wrong answer: $e^{3x} + C$
11. Problem: $\int 2e^{-x/4}\, dx$; wrong answer: $2e^{-x/4} + C$
12. Problem: $\int \sin^3 2x \cos 2x\, dx$; wrong answer: $\sin^4 2x + C$
13. Compute $\int 0\, dx$.

14. Compute $\int_4^6 5\,dx$.
15. Evaluate $\int_3^5 x\,dx$.
16. Find $\int 4x\,dx$.
17. Find all antiderivatives of $f(x) = x + 3$.
18. Compute $\int_0^1 \sqrt[3]{x^2}\,dx$
19. Suppose that the rate at which a tank is being filled with water at time t is $5e^t$ gal/min. Find the amount of water put in the tank between times 0 and 4.
20. Find the area under the graph of $f(x) = \dfrac{4}{x}$ between $x = 2$ and $x = 9$.
21. Compute $\int (2x^3 - x)\,dx$.
22. Compute the integral $\int (\sec^2 x - 7\sin x)\,dx$.
23. Compute the indefinite integral of $\dfrac{3}{x^2} + \csc^2 x$ with respect to x.
24. Suppose that the rate at which energy is used by a machine at time t is given by $2\sec t \tan t$ J/sec. Find the energy consumption between times $\dfrac{\pi}{4}$ and $\dfrac{\pi}{3}$.
25. Find the area under the graph of $y = 1 + \dfrac{4}{x^2}$ from $x = 1$ to $x = 2$.
26. Compute $\int \dfrac{e^x}{4}\,dx$.
27. Evaluate $\int \left(\dfrac{6}{\sqrt{x}} + \dfrac{1}{x}\right)\,dx$.
28. Find all antiderivatives of $f(x) = x^{3/2} + 4x + 2$.
29. Compute $\int_{-1}^1 (x^3 - x^2)\,dx$.
30. Find $\int dx$.
31. Compute $\int (2 - \csc x \cot x)\,dx$.
32. Find the indefinite integral of $f(x) = x^3 + 4\cos x$ with respect to x.

33. Suppose an object is moving back and forth along a line so that its acceleration at time t is $a(t) = -5t$ in/sec^2. If the object's velocity at time 2 is 3 in/sec, what is its velocity at time 5?

34. Suppose a bee is moving along a number line so that its velocity at time t is $v(t) = t^2 + 3$ cm/sec. If at time 1 the bee is at position -4, what is its position at time 4?

35. Suppose a bug is crawling along a number line so that its acceleration at time t is $a(t) = \frac{1}{10}\cos t$ meters per hour squared.

    a) If its velocity at time 0 is $\frac{1}{5}$ meters per hour and its position at time 0 is 1, what is its position at time π?

    b) If its velocity at time 0 is 1 meter per hour and its position at time 0 is 0, what is its position at time $\frac{\pi}{3}$?

36. Suppose f is a function such that the slope of the line tangent to f at x is $4x - 1$. If f passes through the point $(4, 0)$, what is $f(-2)$?

In Problems 37-42, use *Mathematica* to compute the indicated integrals (write the answers as you would write them by hand).

Note: *Mathematica* computes integrals using the `Integrate` command. For example, to compute the definite integral $\int_2^4 x^2\,dx$ using *Mathematica*, execute

```
Integrate[x^2, {x, 2, 4}]
```

and to compute the indefinite integral $\int x^2\,dx$ using *Mathematica*, execute

```
Integrate[x^2, x]
```

(The `x` in the command is necessary and corresponds to the dx in the integral.) You can also get an integral sign on the Basic Math Assistant.

37. $\int \sec x\,dx$

    Note: When you compute an indefinite integral using *Mathematica*, something important is missing from its answer.

38. $\int \ln x\,dx$

39. $\int_0^{\pi/4} \tan x\,dx$

40. $\int_0^1 \arctan x\,dx$

41. $\int x^3 e^{-x}\,dx$

42. $\int \frac{3}{x^2 - x}\,dx$

Exercises from Sections 10.2 to 10.4

In Problems 43-47, you are given a definite integral and a u-substitution. Perform the u-substitution to rewrite the integral as a simpler integral (be sure to change the limits from x-values to u-values). You do **not** need to evaluate the integral.

43. $\int_{-2}^{3} \frac{2x}{x^2+5}\,dx;\ u = x^2 + 5$

44. $\int_3^7 e^{8x}\,dx;\ u = 8x$

45. $\int_0^1 20(x^7+3)x^6\,dx;\ u = x^7 + 3$

46. $\int_0^{\pi/4} \sin^3 x \cos x\,dx;\ u = \sin x$

47. $\int_0^{\ln 4} \frac{e^x}{e^x+1}\,dx;\ u = e^x + 1$

In Problems 48-66, compute the indicated integral:

48. $\int_1^8 \sqrt{\frac{2}{x}}\,dx$

49. $\int (2-x)\sqrt{x}\,dx$

50. $\int \frac{x^2+2-3x^3+1}{x^4}\,dx$

51. $\int_0^2 (x+1)(3x-2)\,dx$

52. $\int \frac{5-e^x}{e^{2x}}\,dx$

53. $\int \frac{(\ln x)^2}{x}\,dx$

54. $\int \sqrt{3-x^2}(-2x)\,dx$

55. $\int x^3(x^4-1)^5\,dx$

56. $\int \frac{3}{2+7x}\,dx$

57. $\int 5x\sqrt[3]{1-x^2}\,dx$

58. $\int \frac{6x^2}{1+x^3}\,dx$

59. $\int_2^3 \frac{6x^2}{(1+x^3)^3}\,dx$

60. $\int \sin \pi x\,dx$

61. $\int \cos 2x\,dx$

62. $\int \frac{3}{4}\cos\frac{x}{2}\,dx$

63. $\int \tan^4 x \sec^2 x\,dx$

64. $\int_{\pi/6}^{\pi/2} \cot x\,dx$

65. $\int 3e^{2x}\,dx$

66. $\int_1^9 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

67. Suppose that the rate of fuel consumption of a motor at time t is te^{-t^2} L/min. Compute the amount of fuel consumed by the motor in the first minute of operation.

68. Find the area under the graph of $y = x(x^2+1)^3$ from $x = 0$ to $x = 1$.

69. Suppose that an object is moving back and forth along a number line so that its velocity at time t is $v(t) = 4t^2\sqrt{t^3+1}$ ft/sec. What is the object's displacement from time 1 sec to time 2 sec?

Answers

1. Wrong (missing the $+C$)
2. Wrong
3. Correct
4. Wrong
5. Correct
6. Correct
7. $\frac{1}{2}\sin 2x + C$
8. $\frac{1}{3}\ln(3x-4) + C$
9. $-\cot x + C$
10. $\frac{1}{3}e^{3x} + C$
11. $-8e^{-x/4} + C$
12. $\frac{1}{8}\sin^4 2x + C$
13. C
14. 10
15. 8
16. $2x^2 + C$
17. $\frac{1}{2}x^2 + 3x + C$
18. $\frac{3}{5}$
19. $5e^4 - 5$ gal
20. $4\ln 9 - 4\ln 2$
21. $\frac{1}{2}x^4 - \frac{1}{2}x^2 + C$
22. $\tan x + 7\cos x + C$
23. $-\frac{3}{x} - \cot x + C$
24. $4 - 2\sqrt{2}$ J
25. 3
26. $\frac{e^x}{4} + C$
27. $12\sqrt{x} + \ln|x| + C$
28. $\frac{2}{5}x^{5/2} + 2x^2 + 2x + C$
29. $-\frac{2}{3}$
30. $x + C$
31. $2x + \csc x + C$
32. $\frac{1}{4}x^4 + 4\sin x + C$
33. $\frac{-99}{2}$ in/sec
34. 26
35. a) $\frac{\pi+6}{5}$
    b) $\frac{1}{20} + \frac{\pi}{3}$

36. -18

37. $-\ln[\cos(x/2) - \sin(x/2)]$
$+ \ln[\cos(x/2) + \sin(x/2)] + C$

38. $-x + x\ln x + C$

39. $\dfrac{\ln 2}{2}$

40. $\dfrac{1}{4}(\pi - \ln 4)$

41. $e^{-x}(-6 - 6x - 3x^2 - x^3) + C$

42. $3(\ln(1-x) - \ln x) + C$

43. $\displaystyle\int_9^{14} \frac{1}{u}\,du$

44. $\displaystyle\int_{24}^{56} \frac{1}{8}e^u\,du$

45. $\displaystyle\int_3^4 \frac{20}{7}u\,du$

46. $\displaystyle\int_0^{\sqrt{2}/2} u^3\,du$

47. $\displaystyle\int_2^5 \frac{1}{u}\,du$

48. $8 - \sqrt{8}$

49. $\dfrac{4}{3}x^{3/2} - \dfrac{2}{5}x^{5/2} + C$

50. $-x^{-3} - x^{-1} - 3\ln x + C$

51. 6

52. $\dfrac{-5}{2}e^{-2x} + e^{-x} + C$

53. $\dfrac{1}{3}(\ln x)^3 + C$

54. $\dfrac{2}{3}(3 - x^2)^{3/2} + C$

55. $\dfrac{1}{24}(x^4 - 1)^6 + C$

56. $\dfrac{3}{7}\ln(2 + 7x) + C$

57. $\dfrac{-15}{8}(1 - x^2)^{4/3} + C$

58. $2\ln(x^3 + 1) + C$

59. $\dfrac{-1}{28^2} + \dfrac{1}{81}$

60. $\dfrac{-1}{\pi}\cos \pi x + C$

61. $\dfrac{1}{2}\sin 2x + C$

62. $\dfrac{3}{2}\sin\dfrac{x}{2} + C$

63. $\dfrac{1}{5}\tan^5 x + C$

64. $\ln 2$

65. $\dfrac{3}{2}e^{2x} + C$

66. $\dfrac{1}{2}$

67. $\dfrac{e-1}{2e}$ L

68. $\dfrac{15}{8}$

69. $24 - \dfrac{16}{9}\sqrt{2}$ ft

Appendix A

Mathematica reference

A.1 What is *Mathematica*?

Mathematica is an extremely useful and powerful software package / programming language invented by a mathematician named Stephen Wolfram. Early versions of *Mathematica* came out in the late 1980s and early 1990s; as of 2023, the most recent version available to you is *Mathematica* 13.

Mathematica does symbolic manipulation of mathematical expressions; it solves all kinds of equations; it has a library of important functions from mathematics which it recognizes while doing computations; it does $2-$ and $3-$dimensional graphics; it has a built-in word processor tool; it works well with Java and C++; etc. One thing it doesn't do is prove theorems, so it is less useful for a theoretical mathematician than it is for an engineer or college student.

A bit about how *Mathematica* works: When you use the *Mathematica* program, you are actually running two programs. The "front end" of *Mathematica* is the part that you type on and the part you see. The "kernel" is the part of *Mathematica* that actually does the calculations. If you type in 2 + 2 and hit [ENTER] (actually [SHIFT]+[ENTER]; see below), the front end "sends" that information to the kernel which actually does the computation. The kernel then "sends" the result back to the front end, which displays 4 on the screen.

About *Mathematica* notebooks and cells: The actual files that *Mathematica* produces that you can edit and save are called *notebooks* and carry the file designation *.nb; they take up little space and can easily be saved to Google docs or on a flash drive, or emailed to yourself if you want them somewhere you can retrieve them.

Suggestion: when saving any file, include the date in the file name (so it is easier to remember which file you are supposed to be open).

A *Mathematica* notebook is broken into *cells*. A cell can contain text, input, or output. A cell is indicated by a dark blue, right bracket (a "]") on the right-hand side of the notebook. To select a cell, click that bracket. This highlights the "]" in blue. Once selected, you can cut/copy/paste/delete cells as you would highlighted blocks of text in a Word document.

To change the formatting of a cell, select the cell, then click "Format / Style" and select the style you want. You may want to play around with this to see what the various styles look like. There are three particularly important styles:

- **input:** this is the default style for new cells you type
- **output:** this is the default style for cells the kernel produces from your commands
- **text:** changing a cell to text style allows you to make comments in between the calculations

To execute an input cell, put the cursor anywhere in the cell and hit [SHIFT]+[ENTER] (or the [ENTER] on the numeric keypad at the far-right edge of the keyboard). The [ENTER] next to the apostrophe key (a.k.a. [RETURN]) gives you only a carriage return.

A.2 Important general concepts re: *Mathematica* syntax

Executing mathematical commands: To execute an input cell, put the cursor anywhere in the cell and hit [SHIFT]+[ENTER] (or the [ENTER] on the numeric keypad at the far-right edge of the keyboard). The [ENTER] next to the apostrophe key (a.k.a. [RETURN]) gives you only a carriage return.

Multiplication: use a star or a space: 2 * 3 or 2 3 will multiply numbers; a x means a times x; ax means the variable ax (in *Mathematica*, variables do not have to be named after one letter; they can be named by words or other strings of characters as well).

Parentheses: used for grouping only. Parentheses mean "times" in *Mathematica*.

Brackets: used to enclose all functions and *Mathematica* commands. For example, to evaluate a function $f(x)$, you would type f[x]; for $\sin x$ you type Sin[x]; etc.. Brackets mean "of" in *Mathematica* and <u>cannot be used for multiplication</u>.

Capitalization: All *Mathematica* commands and built-in functions begin with capital letters. For example, to find the sine of π, typing sin(pi) does you no good (this would be the variable "sin" times the variable "pi"). The correct syntax is Sin[Pi].

Spaces: *Mathematica* commands do not have spaces in them; for example, the inverse function of sine is ArcSin, not Arc Sin or Arcsin.

Pallettes: Lots of useful commands are available on the Basic Math Assistant Pallette, which can be brought up by clicking "Pallettes / Basic Math Assistant" on the toolbar. If you click on a button in the pallette, what you see appears in the cell.

Commands *Mathematica* knows: Sqrt, Sin, Cos, Tan, Csc, Cot, Sec, ArcSin, ArcCos, ArcTan, ArcCsc, ArcSec, ArcCot, ! (for factorial). It knows what Pi and E are (but not pi or e).

Logarithms: Log[ ] means natural logarithm (base e); Log10[ ] means common logarithm (base 10).

% refers to the last output (like ANS on a TI-calculator).

Exact answers versus decimal approximations: *Mathematica* gives exact answers for everything if possible. If you need a decimal approximation, click "numerical value" or use the command N[ ]. For example, N[Pi] spits out 3.14159...

To solve an equation: make sure there are two equals signs ("==") in your equation.

Getting help from the program: To get help on a command, type ? followed by the command you don't understand (with no space between the ? and the command).

To export graphics: Once *Mathematica* produces a graphic, you can right-click the graphic, and select "Copy Graphic". Then you can go in a Word document or a PowerPoint, and paste the graphic. You can subsequently resize it and/or move it around as you see fit.

Troubleshooting: For a command to run correctly, you usually want everything in your command to be black. If anything is purple or red, that suggests where the problem is. Variables that don't have values should be blue. Next, check that everything is capitalized appropriately. Next, check that you aren't missing a space if you are trying to multiply two variables. Next, if you are using variables in your code, try clearing the variables by executing something like Clear[x] (if your variable is x). Then re-run the command that is giving you trouble.

If *Mathematica* freezes up in the middle of a calculation and you see "Running..." at the top of your screen, click "Evaluation / Abort Evaluation" on the toolbar. If this doesn't help, kill the program and restart it.

To get help: Email me, and attach your *Mathematica* file to your email. I can troubleshoot things pretty quickly if the file is attached. If the file isn't attached, it is hard for me to figure out what you are doing wrong. Alternatively, seek assistance from another math major who has experience with *Mathematica*.

A.3 *Mathematica* quick reference guides

General tasks

TASK	MATHEMATICA SYNTAX
To call the preceding output	%
To get a decimal approximation to the preceding output	N[%] (or click numerical value)

Algebraic manipulations

TASK	MATHEMATICA SYNTAX
To factor an expression	Factor[ ]
To multiply out an expression (i.e. FOIL an expression)	Expand[ ]
Partial fraction decomposition	Apart[ ]
To combine rational terms (i.e. "undo" a partial fraction decomp)	Together[ ]
To simplify an answer	Simplify[ ] (or FullSimplify[ ])

Solving equations

GOAL	MATHEMATICA SYNTAX
Find exact solution(s) to equation of form $lhs = rhs$ (assuming the variable is x)	Solve[lhs == rhs, x] (two equals signs) (works only with polynomials or other relatively "easy" equations)
Find decimal approx. to solutions of equation $lhs = rhs$	NSolve[lhs == rhs, x] (two equals signs) (works only with "easy" equations)
Find decimal approx. to solutions of equation $lhs = rhs$	FindRoot[lhs == rhs, {x, $guess$}] (two equals signs)
Solve two (or more) equations together, like $\begin{cases} lhs_1 = rhs_1 \\ lhs_2 = rhs_2 \end{cases}$ (assuming variables are x and y)	Solve[{lhs_1==rhs_1, lhs_2==rhs_2}, {x,y}]

Precalculus operations

	Expression	Mathematica syntax
Special Symbols	e	E (not e) (or use Basic Math Assistant pallette)
	π	Pi (or use Basic Math Assistant)
	∞	Infinity (or use Basic Math Assistant) (or type [Esc] inf [Esc])
	$i = \sqrt{-1}$	I (not i) (or use Basic Math Assistant)
Arithmetic	$3 + 4x$	3 + 4x
	$5 - 27$	5 - 27
	$12x$	12x or 12 x or 12 * x
	xy	x y (don't forget the space)
	$\frac{x}{y}$	x/y (or use Basic Math Assistant pallette) (or type [CTRL]+/ to get $\frac{\square}{\square}$)
	$\sqrt{32}$	Sqrt[32] (or use Basic Math Assistant) (or type [CTRL]+2 for the $\sqrt{\ }$ sign)
	$\sqrt[4]{40}$	40^(1/4) (or use Basic Math Assistant)
	$\lvert x-3 \rvert$	Abs[x-3]
	30! (factorial)	30!
Trig	$\sin \pi$	Sin[Pi]
	$\cos(x(y+1))$	Cos[x(y+1)]
	$\cos 60^\circ$	Cos[60 Degree] (or use Basic Math Assistant)
	$\cot\left(\frac{2\pi}{3} + \frac{3\pi}{4}\right)$	Cot[2 Pi/3 + 3 Pi/4]
	$\sin^2 x$	Sin[x]^2 (not Sin^2[x])
	$\arctan 1$	ArcTan[1]
Exps / Logs	$\ln 3$	Log[3]
	$\log_6 63$	Log[6,63]
	$\log 18$	Log10[18] or Log[10, 18]
	2^{7y}	2^(7y) (or use Basic Math Assistant) (or type [CTRL]+6 to get $\square^{\square}$)
	e^{x-5+x^2}	E^(x-5+x^2) or Exp[x-5+x^2] (or use Basic Math Assistant)

Defining functions

CLASS OF FUNCTION	SYNTAX TO DEFINE FUNCTION
Calculus 1 function $f : \mathbb{R} \to \mathbb{R}$ $x \overset{f}{\longmapsto} y$	f[x_] = *formula* (one equals sign, underscore after the x)
Ex: $f(x) = 3\cos(x^{2-x})$	f[x_] = 3 Cos[x^(2-x)]

Algebraic operations on functions

All these commands assume you have previously defined the function(s) as outlined above.

EXPRESSION	MATHEMATICA SYNTAX
Generate table of values for f	Table[{x, f[x]}, {x, *xmin*, *xmax*, *step*}] (put //TableForm after this command to format the output in a table)
$f(x+3)$	f[x+3]
$xf(2x) - x^2 f(x)$	x f[2x] - x^2 f[x] (spaces important)
Composition $(f \circ g)(x)$	f[g[x]]
Addition $(f+g)(x)$	f[x] + g[x]
Multiplication $(fg)(x)$	f[x] g[x]
Powers $f^n(x)$ **Ex:** $\sin^2 x$	(f[x])^n (or just f[x]^n) Sin[x]^2

Graphs

The basic command to graph a function is Plot[$f(x)$, {x, $xmin$, $xmax$}]; the examples below describe how to adapt the Plot[ ] command:

GOAL	HOW TO ADAPT THE Plot[ ] COMMAND
Plot multiple graphs at once	Plot[{$formula$, $formula$, ..., $formula$}, {x, $xmin$, $xmax$}]
Plot the graph of $f(x) = formula$ with range of $y-$values specified	Plot[$formula$, {x, $xmin$, $xmax$}, PlotRange -> {$ymin$, $ymax$}]
Plot the graph of $f(x) = formula$ with x- and y-axes on same scale	Plot[$formula$, {x,$xmin$,$xmax$}, PlotRange -> $ymin$,$ymax$, AspectRatio -> Automatic]
Plot the graph of $f(x) = formula$ with a red, dashed curve	Plot[$formula$, {x,$xmin$,$xmax$}, PlotStyle -> {Red, Dashed}]

Single-variable calculus

EXPRESSION	MATHEMATICA SYNTAX
$\lim_{x \to 4} f(x)$	Limit[f[x], x -> 4]
$f'(3)$ $h'(x)$ $\frac{d}{dx}(\cos x)$ $g'''(x)$	f'[3] D[h[x], x] D[Cos[x], x] g' ' '[x] or D[g[x], {x,3}]
$\int x^2\,dx$	Integrate[x^2, x] (or use Basic Math Assistant pallette) **Note:** answer will be missing the "$+C$"
$\int_2^5 \cos x\,dx$	For an exact answer: Integrate[Cos[x], {x, 2, 5}] (or use Basic Math Assistant) For a decimal approximation: NIntegrate[Cos[x], {x, 2, 5}]
$\sum_{k=1}^{12} f(k)$	Sum[f[k], {k, 1, 12}] (or use Basic Math Assistant)
$\sum_{n=3}^{\infty} blah$	Sum[$blah$, {n, 3, Infinity}] (or use Basic Math Assistant)

A.4 More on solving equations with *Mathematica*

There are three methods to solve an equation using *Mathematica*. They have something in common: to solve an equation, the equation **must be typed with two equals signs** where the $=$ is. (A single equal sign is used in *Mathematica* to assign values to variables, which doesn't apply in the context of solving equations.)

The Solve command

To solve an equation of the form $lhs = rhs$, execute

Solve[lhs == rhs, $variable$]

where $variable$ is the name of the variable you want to solve for. For example, to solve $x^2 - 2x - 7 = 0$ for x, execute Solve[x^2 - 2x - 7 == 0, x].

You can solve an equation for one variable in terms of others: for example, Solve[a x + b == c, x] solves for x in terms of a, b and c.

WARNING: The advantage of the Solve command is that it gives exact answers (no decimals); this can be a pro or con (as sometimes the exact answers are horrible to write down). The disadvantage is that it only works on polynomial, rational and other "easy" equations. It won't work on equations that mix-and-match trigonometry and powers of x like $x^2 = \cos x$.

The NSolve command

NSolve works exactly like Solve, except that it gives decimal approximations to the solutions. It has the same drawback as Solve in that it only works on reasonably "easy" equations. The syntax is

NSolve[lhs == rhs, $variable$]

The FindRoot command

To find decimal approximations to equations that are too hard for the Solve and NSolve commands, use FindRoot. This executes a numerical algorithm to estimate a solution to an equation. The good news is that this command always works; the bad news is that it requires an initial "guess" as to what the solution is (usually you determine the initial guess by graphing both sides of the equation and seeing roughly where the graphs cross). For example, to find a solution to $x^2 = \cos x$ near $x = 1$, execute

```
FindRoot[x^2 == Cos[x], {x, 1}]
```

and to find a solution to the same equation near $x = -1$, execute

```
FindRoot[x^2 == Cos[x], {x, -1}]
```

(these probably won't give the same solution). The general syntax for this command is

FindRoot[lhs == rhs,{$variable$, $guess$}]

A.5 More on graphing with *Mathematica*

Defining a function in *Mathematica*

To graph a function $y = f(x)$ on *Mathematica*, you usually start by defining the function. For example, to define a function like $f(x) = 3\cos 4x - x$, execute

```
f[x_] = 3 Cos[4x] - x
```

You could just as well use a different letter for the independent variable. For example, typing

```
f[t_] = 3 Cos[4t] - t
```

would accomplish the same thing as above. However, don't mix and match! Typing

```
f[x_] = 3 Cos[4t] - t
```

doesn't accomplish anything, because there is a x on the left-hand side, and a t on the right-hand side.

The general syntax for defining a function is

$$function\,name\,[variable\_] \;=\; formula$$

it is important to include the underscore after the variable to tell *Mathematica* you are defining a function.

The basic Plot command

Immediately after defining a function as above, you will get (underneath your output) a list of suggested follow-up commands. One of these is plot. If you click the word plot, you will get a graph of the function you just defined. Here, *Mathematica* picks a range of x- and y-values it thinks will work well for the function you defined. It is useful to remember the syntax of this Plot command:

$$\text{Plot}[formula,\ \{variable,\ xmin,\ xmax\}]$$

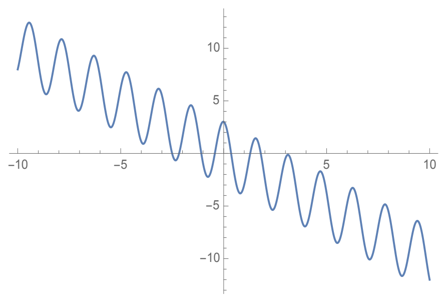
In this command:

- $formula$ is the function you want the graph of. It could be an expression like f[x] or f[t], or a typed-out formula like 3 Cos[4x] - x.
- $variable$ is the name of the independent variable (usually x or t); this must match the variable in the formula.

- *xmin* and *xmax* are numbers which represent, respectively, the left-most and right-most values of the independent variable shown on the graph. For example, if your Plot command has {x,-3,5} in it, then the graph will go from $x = -3$ to $x = 5$.

Here is an example, which plots $f(x) = 3\cos 4x - x$ from $x = -10$ to $x = 10$:

Plot[3 Cos[4x] - x, {x, -10,10}]



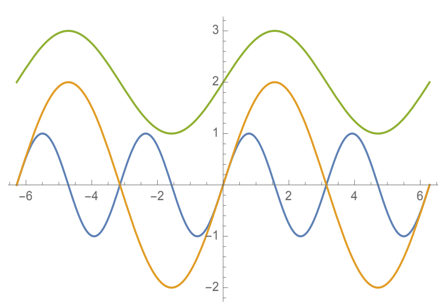
Plotting multiple functions at once

Suppose you want to plot more than one function on the same set of axes. To do this, you tweak the earlier Plot command by replacing the formula with a list of formulas inside squiggly braces, separated by commas. Thus the command you execute looks something like this:

Plot[{*formula1*,*formula2*,... }, {*variable*, *xmin*, *xmax*}]

For example, the following command plots $\sin 2x$, $2\sin x$ and $\sin x + 2$ on the same set of axes:

Plot[{Sin[2x], 2 Sin[x], Sin[x] + 2}, {x, -2 Pi, 2 Pi}]



In *Mathematica* 10, the first graph you type will be blue; the second graph you type will be orange; the third graph you type is green; other graphs are in other colors. To change the way the graphs look, consult the end of this section.

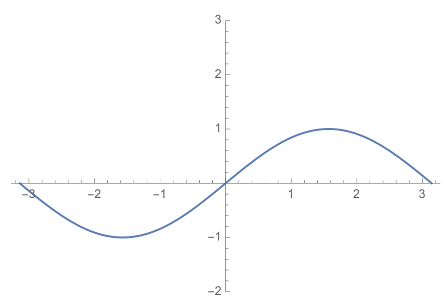
Specifying a range of y-values

By default, *Mathematica* just chooses a range of y-values it thinks will make the graph look good. If you want to force *Mathematica* to use a particular range of y-values, then you have to insert a phrase in the Plot command called PlotRange . This goes after the {x,$xmin$,$xmax$} and after another comma, but before the closing square bracket. The general command is

Plot[{$formulas$}, {var,$xmin$,$xmax$}, PlotRange -> {$ymin$,$ymax$}]

and an example of the code, which plots $\sin x$ on the viewing window $[-\pi, \pi] \times [-2, 3]$ is

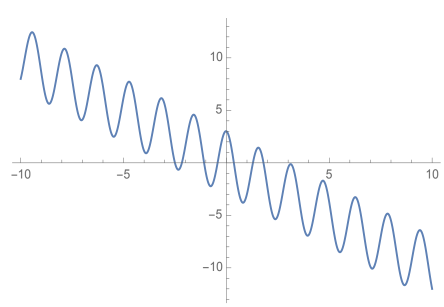
```
Plot[Sin[x], {x, -Pi, Pi}, PlotRange -> {-2,3}]
```



Making the x- and y-axes have the same scale on the screen

Here is the graph of $f(x) = 3\cos 4x - x$ that *Mathematica* produces with the command

```
Plot[3 Cos[4x] - x, {x, -10,10}]
```



If you look at this graph, the distance from the origin to $(5, 0)$ looks a lot longer than the distance from the origin to $(0, 5)$. But in actuality, both these distances are 5 units. The graph is distorted so that it fits nicely on your screen. To fix the distortion (you might want to do this if you needed to estimate the slope of a graph accurately), insert the command AspectRatio -> Automatic into the Plot command (similar to how you would insert a PlotRange command). This forces the number of pixels on your screen representing one unit in the x direction to be equal to the number of pixels on your screen representing one unit in the y direction. Here is the general syntax:

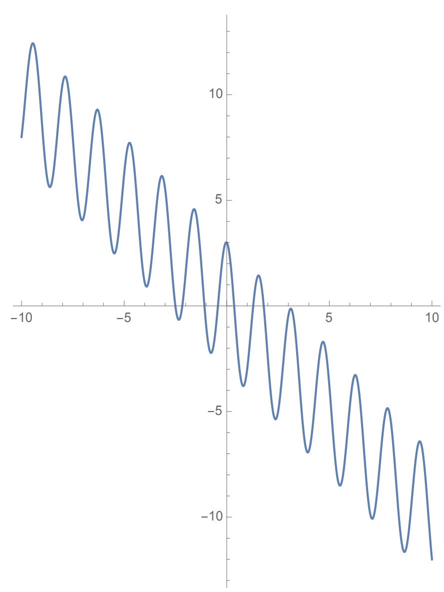
Plot[{*formulas*}, {*var*,*xmin*,*xmax*}, AspectRatio -> Automatic]

This command can also be used with the PlotRange command:

Plot[{formulas,var,xmin,xmax}, PlotRange -> {ymin,ymax}]
AspectRatio -> Automatic]

Here is an example command:

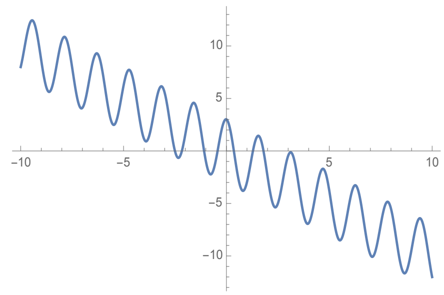
Plot[3 Cos[4x] - x, {x, -10,10}, AspectRatio -> Automatic]



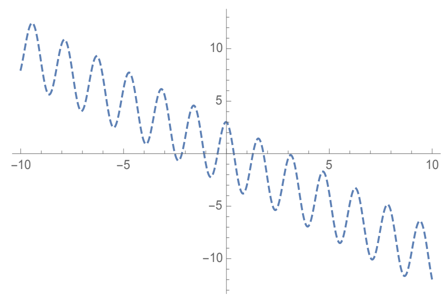
Changing the appearance of the curves

As mentioned earlier, by default *Mathematica* graphs all the functions with solid lines, using different colors for different formulas on the same picture. To change this, insert various directives into the Plot command using PlotStyle. Here are some examples:

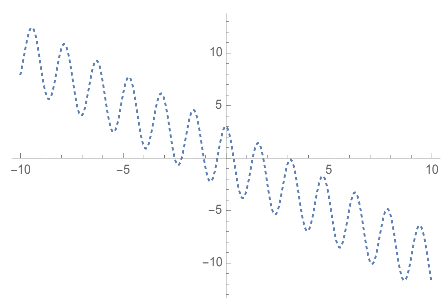
Plot[3 Cos[4x] - x, {x, -10,10}, PlotStyle -> Thick]



Plot[3 Cos[4x] - x, {x, -10,10}, PlotStyle -> Dashed]



Plot[3 Cos[4x] - x, {x, -10,10}, PlotStyle -> Dotted]



If you are plotting more than one function at once, then after the PlotStyle ->, you can type a list of graphics directives, separated by commas, enclosed by a set of squiggly braces. The directives will be applied to each function you are graphing, in the same order as they are typed after the PlotStyle ->. For example, this command plots $x, 2x$ and $3x$, where x is thick and black, $2x$ is red and dotted, and $3x$ is blue and dashed:

```
Plot[{x,2x,3x}, {x, -3,3},
 PlotStyle -> {{Thick, Black}, {Dotted, Red}, {Blue, Dashed}}]
```

A.6 Code for Newton's method

You need three lines of code, all in the same cell. For example, to implement Newton's method for the function $f(x) = x^2 - 2$ where $x_0 = 3$ and you want to perform 6 iterations (to find x_6), just type

```
f[x_] = x^2 - 2;
Newton[x_] = N[x - f[x]/f'[x]];
NestList[Newton, 3, 6]
```

and execute (all three lines at once). The first line defines the function f, the second line gives a name to the formula you iterate in Newton's Method, and the last line iterates the formula and spits out the results.

The resulting output for the code listed above is:

```
3, 1.83333, 1.46212, 1.415, 1.41421, 1.41421, 1.41421
```

These numbers are $x_0, x_1, x_2, ..., x_6$ so for example, $x_2 = 1.46212$ and $x_4 = 1.41421...$ and $x_6 = 1.41421$ (the same as x_4 to 5 decimal places).

To implement Newton's method for a different function, different initial guess and different number of iterations, simply change the formula for f, change the 3 to the appropriate value of x_0 and the 6 to the number of times you want to iterate Newton's method.

A.7 Code for Riemann sums

In this section we discuss how to compute left- and right- Riemann sums using *Mathematica*. Ultimately, to do a Riemann sum you need to execute three commands found later in this section; for now we explain where these commands come from.

1. Defining the function f

First, recall that to define a function you use an underscore. For example, the following command defines f to be the function $f(x) = x^2$:

```
f[x_] = x^2
```

2. Defining the partition $\mathcal{P}$

Defining a partition in *Mathematica* is easy. Just use braces, and list the numbers from smallest to largest. For example, to define the partition $\mathcal{P} = \left\{0, 1, \frac{5}{2}, 4, 7\right\}$, just execute

```
P = {0, 1, 5/2, 4, 7}
```

We often use partitions which divide $[a, b]$ into n equal-length subintervals. To create such a partition in *Mathematica*, use the `Range` command. For example, to define a partition of $[0, 2]$ into 10 equal-length subintervals, execute the following:

```
P = Range[0, 2, (2-0)/10]
```

The `0` is a, the `2` is b, and the last number `(2-0)/10` is $\frac{b-a}{n}$, the width of each subinterval. In general, to split $[a, b]$ into n equal-length subintervals, execute

```
P = Range[a,b,(b − a)/n]
```

3. How to get to the individual numbers in a partition $\mathcal{P}$

Suppose you have defined a partition $\mathcal{P} = \{x_0, x_1, ..., x_n\}$ in *Mathematica*. To call one of the elements of $\mathcal{P}$, use double brackets as shown below. **There is a catch:** in handwritten math notation, we write our partitions starting with index 0. But *Mathematica* starts its partitions with index 1. So if $\mathcal{P} = \{0, 1, 5/2, 4, 7\}$ has been defined in *Mathematica*, executing

```
P[[3]]
```

generates the output $\frac{5}{2}$, which we think of as x_2, not x_3.

In general, once you have typed in a partition $\mathcal{P}$,

- execute `P[[j]]` to get the $(j-1)^{th}$ term x_{j-1}, and
- execute `P[[j+1]]` to get the j^{th} term x_j.

4. How to do sums (not necessarily Riemann sums) in *Mathematica*

Suppose you want to compute some sum which is written in $\Sigma-$notation. To do this, open the Basic Math Assistant pallette and click the $\boxed{d \int \sum}$ button (located under the phrase "Basic Commands"). In the first column of buttons, you will see a $\boxed{\sum}$ which you can click on to put a $\sum$ in your cell. You will get boxes to type all the pieces of the sum in.

5. An explanation of how to generate a Riemann sum for a function

First, remember that in any Riemann sum, $\Delta x_j = x_j - x_{j-1}$. From the remarks earlier in this section, we know that in *Mathematica* this expression is P[[j+1]] - P[[j]].

Next, suppose we are doing a left-hand sum. Then the test points c_j satisfy

$$\begin{aligned} c_j &= \text{ left endpoint of the } j^{th} \text{ subinterval} \\ &= \text{ left endpoint of } [x_{j-1}, x_j] \\ &= x_{j-1}. \end{aligned}$$

Therefore, $c_j = x_{j-1}$ should be P[[j]] in *Mathematica* code, and $f(c_j)$ is f[P[[j]]].

Putting this together, the right *Mathematica* code for a left-hand Riemann sum (assuming you have defined your function `f` and your partition `P`) is

$$\sum_{j=1}^{n} \text{f[P[[j]]] (P[[j + 1]] - P[[j]])}$$

6. The final commands for left- and right-hand Riemann sums

From above, we came up with the following sequence of commands for computing a left-hand Riemann sum:

Syntax to compute a left-hand Riemann sum

To evaluate a left-hand Riemann sum, execute the following commands:

```
f[x_] = x^2
```
(or whatever your function is)

```
P = {0, 1/2, 3/4, 1}
```
(or whatever your partition is)

$$\sum_{j=1}^{n} \texttt{f[P[[j]]] (P[[j+1]] - P[[j]])}$$
(n is the number of subintervals)

To evaluate a right-hand sum, the only thing that changes is the test point c_j, which goes from the left endpoint x_{j-1} (i.e. P[[j]]) to the right endpoint x_j (i.e. P[[j+1]]). Thus the commands for computing a right-hand Riemann sum are similar:

Syntax to compute a right-hand Riemann sum

To evaluate a right-hand Riemann sum, execute the following commands:

```
f[x_] = x^2
```
(or whatever your function is)

```
P = {0, 1/2, 3/4, 1}
```
(or whatever your partition is)

$$\sum_{j=1}^{n} \texttt{f[P[[j+1]]] (P[[j+1]] - P[[j]])}$$
(n is the number of subintervals)

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