Big picture: We are considering the problem of adding up an infinite list of numbers, i.e. we look at infinite series of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + ...$$

The main "big-picture" questions are:
1. Do you get a finite sum (i.e. does the series converge)?
2. If so, what is that finite sum?

The reason these questions are hard (and require calculus to formulate precisely) is that the associative and commutative properties of addition, upon which we rely to add finite lists of numbers, do not hold in this context.

Classifying series by the signs of their terms: If $a_n \geq 0$ for all n, we say the series is *positive.* If $a_n \leq 0$ for all n, we say the series is *negative* (in which case, after factoring out a (-1) we get a positive series). If the signs of a_n alternate, we say the series is *alternating.* There are series which are neither positive, negative nor alternating.

Definitions: Given an infinite series as above, we define the N^{th} *partial sum* of the series to be $S_N = a_1 + a_2 + ... + a_N$. If $\lim_{N\to\infty} S_N = L$, then we say the series *converges* to L and write

$$\sum_{n=1}^{\infty} a_n = L.$$

If $\lim_{N\to\infty} S_N$ does not exist, or is equal to $\pm\infty$, then we say the series *diverges.*

If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say the series $\sum_{n=0}^{\infty} a_n$ *converges absolutely.* By the triangle inequality, if a series converges absolutely then it converges. For a positive or negative series, "convergence" coincides with "absolute convergence". If a series converges absolutely, then the series comprised of its positive terms converges, and the series comprised of its negative terms converges. Also, the terms of an absolutely convergent series can be rearranged arbitrarily without affecting the sum of the series.

If $\sum_{n=0}^{\infty} a_n$ but $\sum_{n=1}^{\infty} |a_n|$ diverges, then we say the series $\sum_{n=0}^{\infty} a_n$ *converges conditionally.* By definition, if a series converges conditionally then it converges. If a series converges conditionally, then the series comprised of its positive terms diverges, and the series comprised of its negative terms diverges. Also, the terms of a conditionally convergent series can be rearranged to obtain a series which converges to any given real number.

Basic concepts:
- The starting index of a series does not affect its convergence (though it changes what number the series converges to).
- A convergent series $\pm$ another convergent series yields a convergent series.
- A convergent series $\pm$ a divergent series yields a divergent series.

- A divergent series $\pm$ a divergent series could yield anything.
- Multiplying a series by a nonzero constant does not affect whether or not a series converges.

Tests for convergence/divergence:

- **Geometric Series Test:** Suppose there is some real number r such that $a_{n+1}/a_n = r$ for all n. Then the series is called *geometric*, the number r is called the *common ratio*, and the series can be rewritten as
$$a_0 \sum_{n=0}^{\infty} r^n.$$
After rewriting the geometric series in this form, the series converges to $\frac{a_0}{1-r}$ if $|r| < 1$ or $a_0 = 0$, and diverges otherwise.

  Remark: We also have a formula for the partial sums of a geometric series; in particular we know
$$a_0 \sum_{n=0}^{N} r^N = a_0(1 + r + r^2 + ... + r^N) = a_0 \left(\frac{1 - r^{N+1}}{1 - r} \right).$$
- **Integral Test:** Let $f(x)$ be a nonnegative, decreasing function such that $f(n) = a_n$ for $n = 1, 2, 3, ...$ Then $\int_1^\infty f(x)\, dx$ converges if and only if $\sum_{n=1}^\infty a_n$ converges.
- **Comparison Test:** Suppose $0 \le a_n \le b_n$ for all n. Then:
  1. If $\sum_{n=1}^\infty a_n$ diverges, so does $\sum_{n=1}^\infty b_n$.
  2. If $\sum_{n=1}^\infty b_n$ converges, so does $\sum_{n=1}^\infty a_n$.
- **p-series Test:** Let p be a constant; the series $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if and only if $p > 1$. In particular the *harmonic* series $\sum_{n=1}^\infty \frac{1}{n}$ diverges.
- n^{th}**−term Test:** If $\lim_{n\to\infty} |a_n| \neq 0$ (equivalently, if $\lim_{N\to\infty} a_n \neq 0$), then the series $\sum_{n=1}^\infty a_n$ diverges.
- **Alternating Series Test:** Suppose $\sum_{n=1}^\infty a_n$ is an alternating series such that $|a_n| \ge |a_{n+1}|$ for all n and $\lim_{n\to\infty} |a_n| = 0$. Then the series converges.
- **Ratio Test:** Given a series $\sum_{n=1}^\infty a_n$, define $\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. If $\rho > 1$, the series diverges. If $\rho < 1$, the series converges absolutely. If $\rho = 1$ or if ρ DNE, then this test is inconclusive.

There are other tests for convergence/divergence of series (the Limit Comparison Test, the Root Test, etc.). Unfortunately, for some complicated series none of these tests work; to determine whether or not these series converge requires techniques beyond the scope of freshman calculus.

Notice also that only the Geometric Series Test provides a technique for finding the numerical value of a convergent infinite series; in all other cases we can show only that a series converges, not the number that it converges to. In general, finding the numerical value of the sum of an arbitrary infinite series is an extremely hard (if not impossible) problem, not just for a calculus student, but for a mathematician!