Important facts to memorize in calculus

Limits worth memorizing:

$$\lim_{x\to\infty} \arctan x = \frac{\pi}{2} \qquad \lim_{x\to\infty} \ln x = \infty \qquad \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e \qquad \lim_{x\to 0^+} \ln x = -\infty$$

Differentiation and integration rules:

	DERIVATIVE RULE(S)	*INTEGRAL RULE(S)*
CONSTANTS	$\frac{d}{dx}(C) = 0$	$\int 0\,dx = C$
POWERS	$\frac{d}{dx}(x^n) = nx^{n-1}$ $(n \neq 0)$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ $\quad (n \neq -1)$
	$\frac{d}{dx}(mx+b) = m$	$\int m\,dx = mx + C$
	$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{2\sqrt{x}}\,dx = \sqrt{x} + C$
	$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$	
	$\frac{d}{dx}(x^2) = 2x$	
TRIG	$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + C$
	$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x\,dx = -\cos x + C$
	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
	$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
	$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
	$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$
EXPONENTIALS	$\frac{d}{dx}(e^x) = e^x$	$\int e^x\,dx = e^x + C$
	$\frac{d}{dx}(e^{rx}) = re^{rx}$	$\int e^{rx}\,dx = \frac{1}{r}e^{rx} + C$
	$\frac{d}{dx}(b^x) = b^x \ln b$	$\int b^x\,dx = \frac{1}{\ln b}b^x + C$
LOGS	$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
INVERSE TRIG	$\frac{d}{dx}(\arctan x) = \frac{1}{x^2+1}$	$\int \frac{1}{x^2+1}\,dx = \arctan x + C$
		$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan \frac{x}{a} + C$
	$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$

Reversal of integration limits: $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

Integration by parts: $\displaystyle\int r\,ds = rs - \int s\,dr$

Geometric Series Test:

Consider a geometric series written in the standard form $\sum\limits_{n=0}^{\infty} ar^n$. Then:

1. The series converges if and only if $|r| < 1$ (or if $a = 0$).
2. The series diverges if and only if $|r| \geq 1$.

   Furthermore, if the series converges, its sum is $\boxed{\sum\limits_{n=0}^{\infty} ar^n = \dfrac{a}{1-r}}$.

Ratio Test:

Suppose $\sum a_n$ is an infinite series and let $\rho = \lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|}$. Then:

1. If $\rho < 1$, then $\sum a_n$ converges absolutely.
2. If $\rho > 1$, then $\sum a_n$ diverges.
3. If $\rho = 1$, or if ρ DNE, then this test tells you nothing.

The "big six" Taylor series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ... \quad \text{(holds for all } x\text{)}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \quad \text{(holds for all } x\text{)}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \quad \text{(holds for all } x\text{)}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ... \quad \text{(holds for } x \in (-1,1)\text{)}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \quad \text{(holds for } x \in (-1,1]\text{)}$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \quad \text{(holds for } x \in [-1,1]\text{)}$$

Fourier series: If f has period T, then

$$f(x) = \frac{1}{T}\int_0^T f(x)\,dx + \sum_{n=1}^{\infty}\left[c_n \cos\left(\frac{2\pi n}{T}x\right) + s_n \sin\left(\frac{2\pi n}{T}x\right)\right]$$

where

$$c_n = \frac{2}{T}\int_0^T f(x)\cos\left(\frac{2\pi n}{T}x\right)\,dx \quad \text{and} \quad s_n = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2\pi n}{T}x\right)\,dx.$$

Calculus 2 Lecture Notes

David M. McClendon
(with Jerome Trouba)

Department of Mathematics
Ferris State University

2023 edition

Contents

Chapter 1

Review of Calculus 1

Big picture issues

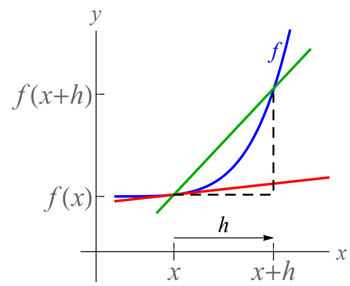
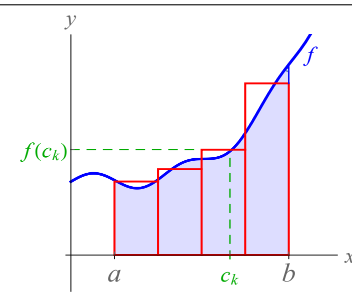
To get our feet wet, let's use what we remember from Calculus 1 to address these questions:

1. What is **calculus**?

   Put another way, what is the difference between a math problem that is a calculus problem and a math problem that is **NOT** a calculus problem?

2. What are some typical kinds of problems you learn how to solve, or mathematical procedures you learn how to do, in Calculus 1? (Brainstorm a list.)

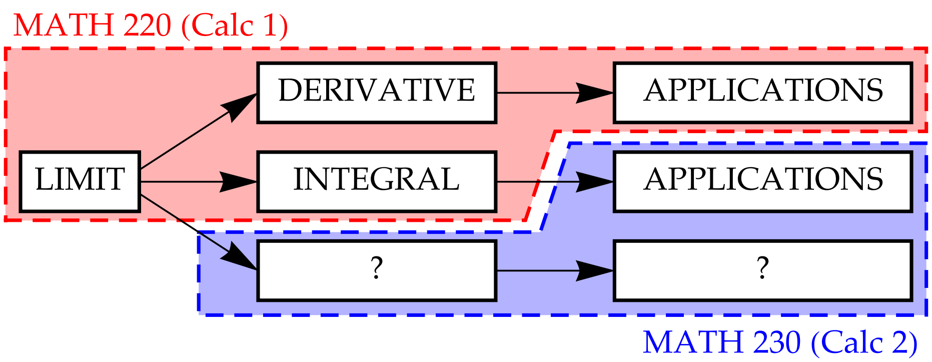
Most of the theory you learn in Calculus 1, in one chart

CALCULUS OBJECT	DERIVATIVE	INTEGRAL
MOTIVATING PROBLEM(S)		
HOW THE SOLUTION IS APPROXIMATED	 slope of secant line	 Riemann sum = area of rectangles
HOW THE APPROXIMATION IMPROVES		
THEORETICAL SOLUTION OF THE MOTIVATING PROBLEM(S)	$f'(x) =$	$\int_a^b f(x)\,dx =$
HOW THIS THEORETICAL SOLUTION IS COMPUTED IN PRACTICE	Memorize some derivatives & **differentiation rules:** - Power Rule - Product Rule - Quotient Rule - Chain Rule etc.	**Fundamental Theorem of Calculus:** $\int_a^b f(x)\,dx = {}'f(x)\big\vert_a^b$ where $'f$ is any antiderivative of f
APPLICATIONS	- slopes of tangent lines - velocity / acceleration - analysis of graphs (increasing/decreasing; concave up/down) - optimization - linear approximation - L'Hôpital's Rule - Newton's method - related rates	- area - displacement - ?

What is calculus? What is the difference between a math problem that is a calculus problem and a math problem that is NOT a calculus problem?

Calculus is the study of **limits**. A limit is a tool to compute exact solutions of problems by way of "better and better" approximating those solutions. A calculus problem is one that contains a limit; a non-calculus problem contains no limit.

A calculus road map

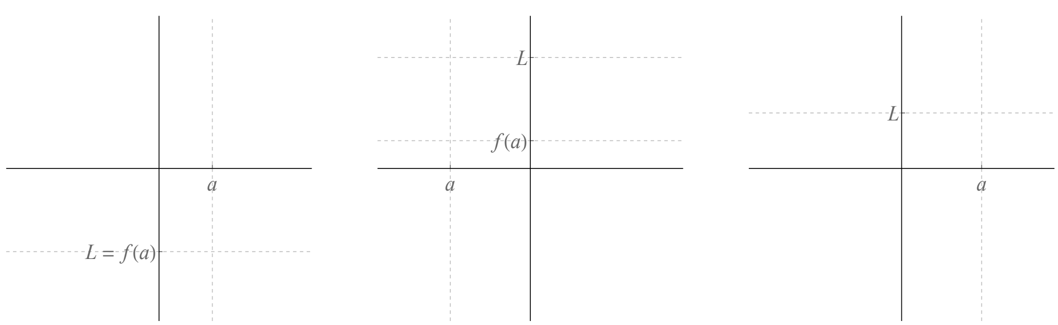


1.1 Limits

Recall: To say

$$\lim_{x \to a} f(x) = L$$

means that as x gets closer and closer to a, then $f(x)$ gets closer and closer to L. This suggests that the graph of f looks like one of the following three pictures:



The graph on the left is "continuous" at a; the other two graphs are not. More precisely,

Definition 1.1 *A function* $f : \mathbb{R} \to \mathbb{R}$ *is* **continuous at** a *if* $\lim\limits_{x \to a} f(x) = f(a)$. f *is called* **continuous** *if it is continuous at every point in its domain.*

Theorem 1.2 *Any function which is the quotient of functions made up of powers of* x*, sines, cosines, arcsines, arctangents, exponentials and/or logarithms is continuous everywhere except where the denominator is zero.*

This theorem suggests that to evaluate most limits, you should start by plugging in a for x. If you get a number, that is usually the answer.

EXAMPLE 1

Evaluate each limit:

a) $\lim\limits_{x \to \pi/3} (3 \sin 2x)$

Solution: $\lim\limits_{x \to \pi/3} (3 \sin 2x) = 3 \sin 2 \cdot \frac{\pi}{3} = 3 \sin \frac{2\pi}{3} = \boxed{\dfrac{3\sqrt{3}}{2}}$.

b) $\lim\limits_{x \to 4} \frac{x+3}{x-2}$

Solution: $\lim\limits_{x \to 4} \frac{x+3}{x-2} = \frac{4+3}{4-2} = \boxed{\dfrac{7}{2}}$.

c) $\lim\limits_{x \to 2^+} \frac{x-5}{x-2}$

d) $\lim\limits_{x \to 5} \frac{x^2-3x-10}{x-5}$

In a limit, when you plug in a for x and you get 0 in a denominator, you have to work a bit harder to determine the answer.

$\frac{\text{nonzero}}{0}$ in limits:

When computing a limit, and you encounter an expression of the form

$$\frac{3}{0} \text{ or } \frac{-5}{0} \text{ or } \frac{1}{0} \text{ or } \frac{\infty}{0} \text{ or anything else of the form } \frac{\text{nonzero}}{0},$$

that expression will evaluate to $\pm\infty$ (you need careful analysis to determine whether it is ∞ or $-\infty$).

In Example 1 (c), $\lim_{x\to 2^+} \frac{x-5}{x-2} = \frac{2-5}{2-2} = \frac{-3}{\pm 0} =$

$\frac{0}{0}$ in limits:

When computing a limit, and you encounter an expression of the form $\frac{0}{0}$, that expression is **indeterminate**, meaning that it might work out to be anything (including 0 , a positive number, a negative number, or $\pm\infty$, and it might not even exist). To evaluate such an expression:

- factor and cancel;
- conjugate square roots;
- clear denominators of "inside" fractions;
- or use L'Hôpital's Rule.

Theorem 1.3 (L'Hôpital's Rule) *Suppose f and g are differentiable functions. Suppose also that either*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad \textit{or} \quad \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty.$$

Then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} \stackrel{L}{=} \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Let's solve Example 1 (d) by factoring and cancelling:

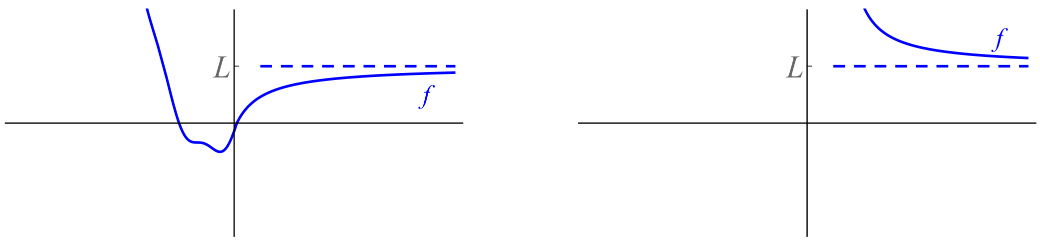
$$\lim_{x\to 5} \frac{x^2-3x-10}{x-5} =$$

Now let's redo Example 1 (d) using L'Hôpital's Rule:

$$\lim_{x\to 5} \frac{x^2-3x-10}{x-5} =$$

Limits at infinity

We can also take a limit of a function as $x \to \infty$. To say $\lim\limits_{x\to\infty} f(x) = L$ means that as x grows larger and larger without bound, then $f(x)$ approaches L. Graphically, this means f has a **horizontal asymptote (HA)** $y = L$:



Although ∞ is not a number, it can be manipulated in some ways as if it is a number, so one can evaluate infinite limits by "plugging in ∞ for x" and applying the following arithmetic rules for ∞:

Adding/subtracting a finite amount to $\pm\infty$ doesn't change it:
For any $c \in \mathbb{R}$, $\infty \pm c = \infty$.

Multiplying/dividing $\pm\infty$ by positive constant doesn't change it:
For any $c > 0$, $c \cdot \infty = \frac{\infty}{c} = \infty$. (This includes $\infty \cdot \infty = \infty$.)

Multiplying/dividing $\pm\infty$ by negative constant reverses it:
For any $c < 0$, $c \cdot \infty = \frac{\infty}{c} = -\infty$. (This includes $-\infty \cdot \infty = -\infty$.)

Dividing a number by infinity gives 0:
For any $c \in \mathbb{R}$, $\frac{c}{\infty} = 0$.

Natural exponentials and logs of ∞ are ∞:
$e^{\infty} = \infty$ and $\ln \infty = \infty$.

Positive powers of ∞ are ∞;
If $c > 0$, then $\infty^c = \infty$. (This includes $\sqrt{\infty} = \infty$ and $\sqrt[n]{\infty} = \infty$.)

Negative powers of ∞ are zero:
If $c < 0$, then $\infty^c = 0$.

WARNING: Here are some expressions that we haven't covered with the rules on the previous page. They are called **indeterminate forms** because they work out to different things depending on the particular limit you are evaluating.

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad \infty - \infty \qquad 0 \cdot \infty \qquad \infty^0 \qquad 0^0 \qquad 1^\infty$$

When you encounter an indeterminate form in a limit, that doesn't mean you are done–you have to do some work to figure out what the limit is.

EXAMPLE 2

Evaluate each limit:

a) $\lim\limits_{x\to\infty} \frac{\ln x}{x}$

b) $\lim\limits_{x\to\infty} \frac{4x^3+2x+1}{2x^3+x^2+2}$

Example 2 (b) above generalizes into the following useful principle:

Theorem 1.4 (Limits at infinity for rational functions) *Suppose f is a rational function, i.e. has form*

$$f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \ldots + a_2x^2 + a_1x + a_0}{b_n x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \ldots + b_2x^2 + b_1x + b_0}.$$

Then:

1. *If $m < n$ (i.e. largest power in numerator < largest power in denominator), then $\lim\limits_{x\to\infty} f(x) = 0$.*

2. *If $m > n$ (i.e. largest power in numerator > largest power in denominator), then $\lim\limits_{x\to\infty} f(x) = \pm\infty$.*

3. *If $m = n$ (i.e. largest powers in numerator and denominator are equal), then $\lim\limits_{x\to\infty} f(x) = \frac{a_m}{b_n}$.*

1.2 Derivatives

Definition 1.5 (Limit definition of the derivative) *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let x be in the domain of f. If the limit*

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists and is finite, say that f is **differentiable at** x*. In this case, we call the value of this limit* **the derivative of** f **at** x *and denote it by $f'(x)$ or $\frac{df}{dx}$ or $\frac{dy}{dx}$ or $Df(x)$.*

Differentiable functions have graphs that are smooth, meaning that they are continuous and do not have sharp corners, vertical tangencies or cusps.

Assuming it exists, the derivative $f'(x)$ computes:

- the slope of the line tangent to f at x;
- the slope of the graph of f at x;
- the instantaneous rate of change of the output y with respect to the input x;
- and the instantaneous velocity at time x
  (under the assumption that f is the object's position at time x).

Definition 1.6 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function.*

- *The* **zeroth derivative** *of f, sometimes denoted $f^{(0)}$, is just the function f itself.*
- *The* **first derivative** *of f, sometimes denoted $f^{(1)}$ or $\frac{dy}{dx}$, is just f'.*
- *The* **second derivative** *of f, denoted f'' or $f^{(2)}$ or $\frac{d^2y}{dx^2}$, is the derivative of f': $f'' = (f')'$.*
- *More generally, the n^{th}* **derivative** *of f, denoted $f^{(n)}$ or $\frac{d^ny}{dx^n}$,is the derivative of $f^{(n-1)}$:*

$$f^{(n)} = ((((f')')\cdots{}')'$$

The first derivative of a function measures its **tone** (i.e. whether it is increasing or decreasing). The second derivative of a function measures its **concavity** (i.e. whether its graph smiles or frowns). In MATH 230, we will learn what the higher-order derivatives of f have to do with the function.

Differentiation rules

We don't compute derivatives using the limit definition. Instead, we use **differentiation rules**, meaning that first, we memorize a bunch of derivatives of common functions:

CLASS OF FUNCTION	*MEMORIZED DERIVATIVE(S)*	
	$\frac{d}{dx}(\quad) =$	
	$\frac{d}{dx}(\quad) =$	
	$\frac{d}{dx}(\quad) =$	$\frac{d}{dx}(\quad) =$
	$\frac{d}{dx}(\quad) =$	$\frac{d}{dx}(\quad) =$
	$\frac{d}{dx}(\quad) =$	
	$\frac{d}{dx}(\quad) =$	
	$\frac{d}{dx}(\quad) =$	
	$\frac{d}{dx}(\quad) =$	
	$\frac{d}{dx}(\quad) =$	
	$\frac{d}{dx}(\quad) =$	
	$\frac{d}{dx}(\quad) =$	
	$\frac{d}{dx}(\quad) =$	
	$\frac{d}{dx}(\quad) =$	
	$\frac{d}{dx}(\quad) =$	
	$\frac{d}{dx}(\quad) =$	
	$\frac{d}{dx}(\quad) =$	

Then, we learn rules telling us how to differentiate more complicated functions in terms of the derivatives we memorize:

DIFFERENTIATION RULE	*FORMULA*
Constant Multiple Rule	$(cf)'(x) = c\, f'(x)$
Sum Rule	$(f+g)'(x) = f'(x) + g'(x)$
Difference Rule	$(f-g)'(x) = f'(x) - g'(x)$

EXAMPLE 3

a) Suppose $f(x) = 3x^6 \sin x$. Compute $f'(x)$.

b) Suppose $y = \frac{4}{x^2} - 2\ln x + 9e^{2x}$. Compute $\frac{dy}{dx}$.

c) Compute the slope of the line tangent to $f(x) = 4\cos x + 2\sin x$ at $x = \frac{\pi}{3}$.

Solution: The slope of the tangent line at $\frac{\pi}{4}$ is given by $f'(\frac{\pi}{3})$. By usual rules,

$$\begin{aligned} f'(x) &= 4 \cdot -\sin x + 2\cos x \\ f'\left(\frac{\pi}{3}\right) &= -4\sin\frac{\pi}{3} + 2\cos\frac{\pi}{3} \\ &= -4 \cdot \frac{\sqrt{3}}{2} + 2 \cdot \frac{1}{2} = \boxed{-2\sqrt{3}+1}. \end{aligned}$$

Tangent line approximation

Derivatives have many applications. The most important (for MATH 230 purposes) is that given a differentiable function f, you can approximate values of f near a using the tangent line to f at a:

Definition 1.7 *Given a differentiable function f and a number a at which f is differentiable, the* **tangent line** *to f at a is the line whose equation is*

$$y = f(a) + f'(a)(x-a) \text{ (a.k.a. } L(x) = f(a) + f'(a)(x-a)).$$

For values of x near a, $f(x) \approx L(x)$; approximating $f(x)$ via this procedure is called **linear approximation**.

EXAMPLE 3(c) (FROM EARLIER)

Compute the equation of the line tangent to $f(x) = 3\cos x + 2\sin x$ at $x = \frac{\pi}{3}$.

Solution: Earlier, we found that $f'(\frac{\pi}{3}) = -2\sqrt{3}+1$.

Now, we compute $f(\frac{\pi}{3}) = 4\cos\frac{\pi}{3} + 2\sin\frac{\pi}{3} = 4(\frac{1}{2}) + 2(\frac{\sqrt{3}}{2}) = 2+\sqrt{3}$.

So by the equation in Definition 1.7, we have

$$y = f(a) + f'(a)(x-a)$$

$$\boxed{y = 2+\sqrt{3} + (-2\sqrt{3}+1)\left(x - \frac{\pi}{3}\right)}.$$

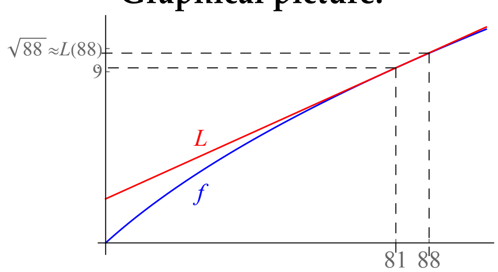
EXAMPLE 4

Approximate $\sqrt{88}$ using tangent line approximation.

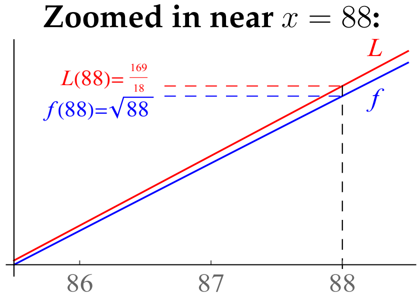
Solution: Let $f(x) = \sqrt{x}$, let $x = 88$ and we choose $a = 81$ (since 81 is close to $x = 88$ and 81 is "easy to work with"). Then $f(a) = \sqrt{81} = 9$ and $f'(a) = \frac{1}{2\sqrt{81}} = \frac{1}{18}$. So by the linear approximation formula,

$$\begin{aligned}\sqrt{88} = f(x) \approx L(x) &= f(a) + f'(a)(x-a) \\ &= 9 + \frac{1}{18}(88-81) = 9 + \frac{7}{18} = \boxed{\frac{169}{18}}.\end{aligned}$$

Graphical picture:



Zoomed in near $x = 88$:



P.S. $\sqrt{88} \approx 9.381$ and $\frac{169}{18} \approx 9.389$, so this estimate is correct to $.08\%$ error.

1.3 The definite integral

Definition 1.8 *Given function $f : [a, b] \to \mathbb{R}$, the* **definite integral** *of f from a to b is*

$$\int_a^b f(x)\,dx = \lim_{||\mathcal{P}|| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k,$$

where the expression inside the limit is a Riemann sum for f.

Note: In MATH 230, the limit above always exists (but it doesn't always exist for crazy functions f... take MATH 430 to learn more about this).

The definite integral of a function is a <u>**number**</u> which is supposed to give the signed area of the region between the graph of f and the x-axis. Area above the x-axis is counted as positive area; area below the x-axis is counted as negative area.

As with derivatives, we don't compute integrals with this limit definition. We use the following important circle of ideas:

Definition 1.9 *Given function f, an* **antiderivative** *of f is a function $'f$ (prononced "f antiprime") such that $('f)' = f$.*

EXAMPLE

$'f(x) = \sin x$ is an antiderivative of $f(x) = \cos x$.

Every continuous function has an antiderivative (although you may not be able to write its formula down); any two antiderivatives of the same function must differ by a constant (so if you know one antiderivative, you know them all by adding a $+C$ to the one you know).

Definition 1.10 *Given function f, the* **indefinite integral** *of f, denoted*

$$\int f(x)\,dx,$$

is the set of all antiderivatives of f.

EXAMPLE

$$\int \cos x\,dx = \sin x + C.$$

Theorem 1.11 (Fundamental Theorem of Calculus Part II) *Let f be continuous on $[a, b]$. Suppose $'f$ is* ***any*** *antiderivative of f. Then*

$$\int_a^b f(x)\,dx = {'f}(x)\Big|_a^b = {'f}(b) - {'f}(a).$$

EXAMPLE

$$\int_0^{\pi/2} \cos x\,dx = \sin x\Big|_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = \boxed{1}.$$

Despite the similar notation, $\int f(x)\,dx$ and $\int_a^b f(x)\,dx$ are very different objects. The first object is a **set** of **functions**; the second object is a **number**.

In MATH 230, we'll start by reviewing the methods of computing integrals you learn in Calculus 1, and then proceeding to more advanced integration techniques. That is the subject matter of Chapter 2.

1.4 Homework exercises

General ground rule in MATH 230: At any time you may be asked to compute a quantity which equals $\pm\infty$ or does not exist. You are responsible for identifying these situations.

Exercises from Section 1.1

In Exercises 1-11, evaluate each given limit.

1. a) $\lim\limits_{x\to 1} \frac{x-1}{x^2+1}$
   b) $\lim\limits_{x\to 4} 7$
2. a) $\lim\limits_{x\to 3} \frac{x^2-9}{x^2+x-12}$
   b) $\lim\limits_{x\to 2^+} \frac{x+2}{x^2-4}$
3. a) $\lim\limits_{x\to\infty} e^{-x}$
   b) $\lim\limits_{x\to -3} \frac{\frac{1}{x}+\frac{1}{3}}{\frac{1}{x+2}+1}$
4. a) $\lim\limits_{x\to\infty} 2^x$
   b) $\lim\limits_{x\to\infty} \frac{3x}{x^2+2}$
5. a) $\lim\limits_{x\to\infty} \frac{-2}{x}$
   b) $\lim\limits_{x\to\infty} \frac{4x^2-3x+1}{3-2x^2}$
6. a) $\lim\limits_{x\to\infty} \frac{4x^2}{e^x-2}$
   b) $\lim\limits_{x\to 0} \frac{e^{2x}-1}{x}$
7. a) $\lim\limits_{x\to 0} \frac{\sin x}{x}$
   b) $\lim\limits_{x\to\infty} \ln x$
8. a) $\lim\limits_{x\to 0^+} \ln x$
   b) $\lim\limits_{x\to 0} e^x$
9. a) $\lim\limits_{x\to 0^+} e^{1/x}$
   b) $\lim\limits_{x\to\infty} \sin x$
10. a) $\lim\limits_{x\to\infty} \arctan x$
    b) $\lim\limits_{x\to 0^+} e^{-1/x}$
11. a) $\lim\limits_{x\to 1^+} f(x)$, where
    $$f(x) = \begin{cases} x^2 & x \leq 1 \\ 5x & x > 1 \end{cases}$$
    b) $\lim\limits_{x\to 0} \frac{|x|}{x}$

Exercises from Section 1.2

In Exercises 12-19, compute the derivative of each indicated function.

12. a) $f(x) = \frac{1}{x}$
    b) $f(x) = 5x + 7$
13. a) $f(x) = \sqrt{\sin x}$
    b) $f(x) = \frac{x^2-1}{x}$
14. a) $g(x) = 3\sqrt{x} - x^{3/7} + \frac{2}{x^8}$
    b) $f(x) = \frac{4}{\sqrt[3]{x^2}}$
15. a) $f(x) = \frac{x^2-3x+4}{2x^3+1}$
    b) $r(x) = x^3 - 3x^4 + 2$
16. a) $g(x) = 2\tan x - 3\cos x$
    b) $h(w) = \frac{w}{\tan w}$
17. a) $f(x) = 4\cot x + 2\sec x - 1$
    b) $y = xe^x$

18. a) $f(t) = 2e^{4t}$

    b) $F(x) = x\ln(3x^2+1)$

19. a) $y = \frac{1}{4}\arctan\frac{t}{2}$

    b) $g(x) = x^3e^{-x\sin x} + \cos\sqrt{\sin x}$

20. a) If $f(x) = 2x^4 - 3x^2 + 5x + 7$, compute $f'(-1)$.

    b) Compute $f'(8)$ if $f(x) = \frac{4}{\sqrt[3]{x}}$.

    c) Compute $f'(0)$ if $f(x) = |x|$.

21. a) If $f(x) = 3\ln x + \frac{e^x}{5}$, compute $f''(x)$.

    b) Let $f(x) = 7e^x - \frac{\sin x}{4}$. Compute the fourth derivative of f.

22. a) Compute the second derivative of $f(x) = \cos(3x^2)$.

    b) Suppose $f(x) = 3\sin 2x$. What is $f^{(199)}(x)$?

    c) Compute the zeroth derivative of $f(x) = 8x^7 + 2\cot 4x$.

23. Compute the slope of the line tangent to $y = 2x^3 - 3x$ when $x = 2$.

24. Compute the equation of the line tangent to $y = 4x(x^2-3)^4$ when $x = 2$.

25. Estimate $\sqrt{150}$ using tangent line approximation.

26. Estimate $\sin\frac{1}{4}$ using tangent line approximation.

27. Suppose that the position of an object at time t (measured in seconds) is given by $f(t) = (3\ln t)^2$ cm. Compute the velocity and acceleration of the object at time $t = e$ sec.

28. a) Let $y = 4x^2$. Compute dy.

    b) Let $y = 2\sin 3x$. Compute dy.

Exercises from Section 1.3

In Exercises 29-33, evaluate each given integral.

29. a) $\displaystyle\int_0^1 2x^3\,dx$

    b) $\displaystyle\int_{-1}^2 \left(9x^2 - 2x^3 + 5\right)\,dx$

30. a) $\displaystyle\int \left(\frac{\sqrt{x}}{3} + \frac{5}{x^4} + \frac{2}{x}\right)\,dx$

    b) $\displaystyle\int_{-3}^5 (4e^x + 1)\,dx$

31. a) $\displaystyle\int 0\,dx$

    b) $\displaystyle\int_0^{\pi/2} 3\cos x\,dx$

32. a) $\displaystyle\int (\sec x\tan x + \csc x\cot x)\,dx$

    b) $\displaystyle\int_1^3 \left(3t^{-1} - t\right)\,dt$

33. a) $\displaystyle\int_{\pi/3}^{\pi/4} 6\sec^2 x\,dx$

    b) $\displaystyle\int \frac{-6}{x^2+1}\,dx$

34. Compute each of these integrals by thinking about the shape of the graph of the integrand:

    a) $\int_0^3 \sqrt{9-x^2}\,dx.$ b) $\int_{-3}^4 |x-2|\,dx.$

35. Compute the derivative of $f(x) = x^2 \sin x$. What does your answer tell you about the value of
$$\int_0^\pi \left(2x\sin x + x^2\cos x\right)\,dx?$$

36. Compute the derivative of the function $F(x) = \int_2^x 3t^2\,dt$.

    Hint: Use the part of the Fundamental Theorem of Calculus not mentioned in these notes.

37. Compute the area under the graph of the function $f(x) = \frac{8}{\sqrt{x}}$ from $x = 1$ to $x = 9$.

38. Suppose that an object's velocity at time t is given by $v(t) = 4\sin t$ in/min. Compute the displacement of the object between times 0 and $\frac{2\pi}{3}$.

39. Suppose that the acceleration of an object is given by $a(t) = 48t$ m/sec^2. If the velocity of the object at time 0 is 12 m/sec and the position of the object at time 0 is 10 m, what is the position of the object at time 3?

Answers

WARNING: I found all answers in these lecture notes by hand, so there may be errors.

1. a) 0 b) 7
2. a) $\frac{6}{7}$ b) ∞
3. a) 0 b) $\frac{1}{9}$
4. a) ∞ b) 0
5. a) 0 b) -2
6. a) 0 b) 2
7. a) 1 b) ∞
8. a) $-\infty$ b) 1
9. a) ∞ b) DNE
10. a) $\frac{\pi}{2}$ b) 0
11. a) DNE b) DNE
12. a) $\frac{-1}{x^2}$ b) 5
13. a) $\frac{1}{2\sqrt{\sin x}}\cos x$ b) $1 + \frac{1}{x^2}$
14. a) $\frac{3}{2\sqrt{x}} - \frac{3}{7}x^{-4/7} - 16x^{-9}$ b) $-\frac{8}{3}x^{-5/3}$
15. a) $\frac{(2x-3)(2x^3+1)-(6x^2)(x^2-3x+4)}{(2x^3+1)^2}$

b) $3x^2 - 12x^3$

16. a) $2\sec^2 x + 3\sin x$
    b) $\frac{\tan w - w\sec^2 w}{\tan^2 w}$

17. a) $-4\csc^2 x + 2\sec x\tan x$
    b) $e^x + xe^x$

18. a) $8e^{4t}$
    b) $\ln(3x^2+1) + \frac{6x^2}{3x^2+1}$

19. a) $\frac{1}{8}\frac{1}{(t/2)^2+1}$
    b) $3x^2e^{-x\sin x} + x^3e^{-x\sin x}(-\sin x - x\cos x) - \frac{\sin\sqrt{\sin x}}{2\sqrt{\sin x}}\cos x$

20. a) 3
    b) $\frac{-1}{12}$
    c) DNE

21. a) $\frac{-3}{x^2} + \frac{e^x}{5}$
    b) $7e^x - \frac{\sin x}{4}$

22. a) $-6\sin(3x^2) - 36x^2\cos(3x^2)$
    b) $-3\cdot 2^{199}\cos 2x$
    c) $8x^7 + 2\cot 4x$

23. 21

24. $y = 8 + 132(x-2)$

25. $\frac{49}{4}$

26. $\frac{1}{4}$

27. $v(e) = \frac{18}{e}$ cm/sec;
    $a(e) = 0$ cm/sec^2

28. a) $dy = 8x\,dx$
    b) $dy = 6\cos 3x\,dx$

29. a) $\frac{1}{2}$
    b) $\frac{69}{2}$

30. a) $\frac{2}{9}x^{3/2} - \frac{5}{3}x^{-3} + 2\ln x + C$
    b) $4e^5 - 4e^{-3} + 8$

31. a) C
    b) 3

32. a) $\sec x - \csc x + C$
    b) $3\ln 3 - 4$

33. a) $6 - 6\sqrt{3}$
    b) $-6\arctan x + C$

34. a) $\frac{9\pi}{4}$
    b) $\frac{29}{2}$

35. $f'(x) = 2x\sin x + x^2\cos x$;
    this tells you that the value of the integral is $[x^2\sin x]_0^\pi = 0$.

36. $3x^2$

37. 32

38. 6 in

39. 262 m

Chapter 2

Integration techniques

2.1 Basic integration rules

REVIEW EXERCISE

Let $f : \mathbb{R} \to \mathbb{R}$. Describe what is meant by each phrase below:

- *antiderivative* (of f)

- *indefinite integral* (of f)

- *definite integral* (of f)

EXAMPLE 1

We know that $\frac{d}{dx}(x^7) = 7x^6$. Therefore:

a) An antiderivative of $f(x) = 7x^6$ is

b) Two other antiderivatives of $f(x) = 7x^6$ are

c) The indefinite integral $\int 7x^6\,dx =$

d) The definite integral $\int_{-1}^{3} 7x^6\,dx =$

EXAMPLE 2

We know that $\frac{d}{dx}(\arctan x) = \frac{1}{x^2+1}$. Therefore:

a) If $f(x) = \frac{1}{x^2+1}$, then $'f(x) =$

b) $\displaystyle\int \frac{1}{x^2+1}\,dx =$

c) $\displaystyle\int_0^2 \frac{1}{x^2+1}\,dx =$

Examples 1 and 2 illustrates the following theoretical concepts:

Theorem 2.1 (Antiderivative Theorem) *Any two antiderivatives of a function can differ by at most a constant. So if $'f$ is an antiderivative of f, then*

$$\int f(x)\,dx = 'f(x) + C.$$

Theorem 2.2 (Fundamental Theorem of Calculus Part II) *Let f be continuous on $[a,b]$. Suppose $'f$ is* ***any*** *antiderivative of f. Then*

$$\int_a^b f(x)\,dx = 'f(x)\big|_a^b = 'f(b) - 'f(a).$$

This theory means that we learn how to compute basic integrals by “reversing” the differentiation rules we learn in Calculus 1.

EXERCISE

Without looking anything up, evaluate each of the following indefinite integrals, if they are "easy" to solve. If the integral isn't easily solved, put a "?". The idea here is that an integral which is "easy" should use facts from algebra regarding exponents, logarithms, etc. and the rules memorized in Calculus 1. "Easy" integrals do not use sophisticated calculus techniques or trig identities.

1. $\int 0\, dx =$
2. $\int 1\, dx =$
3. $\int 5\, dx =$
4. $\int x^4\, dx =$
5. $\int x\, dx =$
6. $\int \frac{1}{x}\, dx =$
7. $\int \frac{1}{x^3}\, dx =$
8. $\int x^{-5}\, dx =$
9. $\int \frac{1}{x^2+1}\, dx =$
10. $\int \frac{1}{x^3+1}\, dx =$
11. $\int \sqrt{x}\, dx =$
12. $\int \sqrt[3]{x}\, dx =$
13. $\int \sqrt[3]{x^2}\, dx =$
14. $\int x^{2/5}\, dx =$
15. $\int x^{-1/2}\, dx =$
16. $\int \frac{1}{\sqrt[5]{x}}\, dx =$
17. $\int \sin x\, dx =$
18. $\int \cos x\, dx =$
19. $\int \tan x\, dx =$
20. $\int \sec x\, dx =$
21. $\int \csc x\, dx =$
22. $\int \cot x\, dx =$
23. $\int \sin^2 x\, dx =$
24. $\int \cos^2 x\, dx =$
25. $\int \sec^2 x\, dx =$
26. $\int \csc^2 x\, dx =$
27. $\int \tan^2 x\, dx =$
28. $\int \cot^2 x\, dx =$
29. $\int \sec x \tan x\, dx =$
30. $\int \sec x \cot x\, dx =$
31. $\int \csc x \tan x\, dx =$
32. $\int \csc x \cot x\, dx =$
33. $\int \csc x \tan x\, dx =$
34. $\int \csc x \sec x\, dx =$
35. $\int e^x\, dx =$
36. $\int e^{x^2}\, dx =$
37. $\int 2^x\, dx =$
38. $\int \ln x\, dx =$
39. $\int \log_{10} x\, dx =$
40. $\int \arctan x\, dx =$
41. $\int \arcsin x\, dx =$

Theorem 2.3 (Integrals that we memorize)

$$\begin{aligned}
\textit{CONSTANTS:}\ & \int 0\,dx = C \\
& \int M\,dx = Mx + C \\
\textit{POWERS:}\ & \int x^n\,dx = \frac{x^{n+1}}{n+1} + C \ \text{ whenever } n \neq -1 \\
& \int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C \\
& \qquad \textit{(I don't care so much about the } |\ | \textit{ here)} \\
\textit{TRIG:}\ & \int \sin x\,dx = -\cos x + C \\
& \int \cos x\,dx = \sin x + C \\
& \int \sec^2 x\,dx = \tan x + C \\
& \int \csc^2 x\,dx = -\cot x + C \\
& \int \sec x \tan x\,dx = \sec x + C \\
& \int \csc x \cot x\,dx = -\csc x + C \\
\textit{EXPONENTIALS:}\ & \int e^x\,dx = e^x + C \\
& \int b^x\,dx = \frac{1}{\ln b} b^x + C \\
\textit{INVERSE TRIG:}\ & \int \frac{1}{x^2+1}\,dx = \arctan x + C \\
& \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C
\end{aligned}$$

One rule that might be new here is

$$\int b^x\,dx = \frac{1}{\ln b} b^x + C.$$

Let's see where that comes from:

Linearity rules

We also learn the following rules which allow us to split integrals into pieces:

Theorem 2.4 (Linearity of Integration) *Suppose f and g are integrable functions. Then:*

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx;$$
$$\int_a^b [k \cdot f(x)]\,dx = k \int_a^b f(x)\,dx \text{ for any constant } k;$$

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx;$$
$$\int [k \cdot f(x)]\,dx = k \int f(x)\,dx \text{ for any constant } k.$$

WARNING: Integration is neither multiplicative nor divisive:

$$\int f(x)g(x)\,dx \neq \left(\int f(x)\,dx\right) \cdot \left(\int g(x)\,dx\right)$$

$$\int \left(\frac{f(x)}{g(x)}\right) dx \neq \frac{\int f(x)\,dx}{\int g(x)\,dx}$$

EXAMPLE 3

Evaluate each integral:

a) $\displaystyle\int \frac{-1}{4} \cos x\,dx$

b) $\displaystyle\int_0^1 \left(2 - \frac{x}{3} + x^9\right) dx$

c) $\displaystyle\int \left(4\sec^2 x - 2x^4 + 7e^x\right) dx = \boxed{4\tan x - \frac{2}{5}x^5 + 7e^x + C}$.

d) $\displaystyle\int \left(\frac{4}{\sqrt[3]{x}} - \frac{8}{x}\right) dx$

2.2 Linear Replacement Principle

Let's start with some integral you "know". Call this integral a *prototype*:

Specific example	General situation
$\int \sec^2 x \, dx =$	$\int f(x)\, dx = {}'f(x) + C$

We want to look at what happens when you replace each x in the prototype integral with a linear expression of the form $mx + b$:

Specific example	General situation
$\int \sec^2(5x - 7)\, dx$	$\int f(mx + b)\, dx$

The big idea here is that if you remember how this general situation works, you can quickly integrate lots of functions of the form $f(mx + b)$. These integrals come up often in applications and in advanced math courses, so it is useful to integrate them without actually writing out a u-substitution.

Theorem 2.5 (Linear Replacement Principle (LRP)) *Suppose you know a "prototype" integral formula*

$$\int f(x)\, dx = {}'f(x) + C.$$

Then for any constants m and b (with $m \neq 0$),

$$\int f(mx + b)\, dx = \frac{1}{m}\,{}'f(mx + b) + C.$$

EXAMPLE 4

In each integral given below, determine if the Linear Replacement Principle can be used to evaluate the integral. If so, give the prototype integral formula, the values of m and b, and evaluate the integral.

	$m = ?$ $b = ?$	PROTOTYPE / SOLUTION
a) $\displaystyle\int_0^2 e^{3x}\,dx$	$m =$ $b =$	
b) $\displaystyle\int \cos\frac{x}{9}\,dx$	$m =$ $b =$	
c) $\displaystyle\int (5x-2)^{12}\,dx$	$m =$ $b =$	
d) $\displaystyle\int (4x^2+1)^8\,dx$	$m =$ $b =$	
e) $\displaystyle\int \sin^2 x\,dx$	$m =$ $b =$	
f) $\displaystyle\int \frac{1}{4-x}\,dx$	$m =$ $b =$	

EXAMPLE 5

Evaluate each integral:

a) $\displaystyle\int_{-5}^{20} e^{-x/5}\,dx$

b) $\int 2e^{5-4x}\,dx$

c) $\int \frac{4}{5+3x}\,dx$

d) $\int \left[(x+7)^8 - 2\sqrt{\frac{x}{3}-2}\right]\,dx$

e) $\int \left(\frac{4}{\sqrt[3]{5-4x}} - 6\cos\frac{x-3}{4}\right)\,dx$

f) $\int \frac{3\csc^2 4x}{5}\,dx$

Solution: $\int \frac{3\csc^2 4x}{5}\,dx = \frac{3}{5}\cdot\frac{1}{4}\left(-\cot 4x\right) + C = \boxed{-\frac{3}{20}\cot 4x + C}.$

g) $\int_{\pi/3}^{\pi/2} \sin 3x\,dx$

Solution: $\int_{\pi/3}^{\pi/2} \sin 3x\,dx = \frac{1}{3}(-\cos 3x)\Big|_{\pi/3}^{\pi/2} = -\frac{1}{3}\cos\frac{3\pi}{2} + \frac{1}{3}\cos\pi = 0 - \frac{1}{3} = \boxed{-\frac{1}{3}}.$

2.3 Rewriting the integrand

It is often useful to rewrite the integrand using algebra, a log rule or a trig identity. Sometimes, after rewriting, the LRP can be helpful.

EXAMPLE 6

Evaluate each integral:

a) $\int \tan^2 x \, dx$

b) $\int_2^3 \frac{(x^2-1)^2}{x} \, dx$

c) $\int \ln(2^x) \, dx$

d) $\int \sin^2 x \, dx$

e) $\int \frac{1}{x^2 + a^2} \, dx$

Here, treat a as a constant.

Comments on the last two examples

Here are two identities which are useful for integrating even powers of sine and/or cosine (for instance, part (d) of Example 6 above):

Theorem 2.6 (Power reducing identities)

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

EXAMPLE 7

Evaluate $\int \cos^4 x\, dx$ by writing the integrand as $(\cos^2 x)(\cos^2 x)$, then applying the power reducing identities, FOILING, and applying the power reducing identity one more time.

The method of Example 6 part (e) led to this general formula, which I think is useful to memorize:

Theorem 2.7 (General arctan integral formula)

$$\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a} \arctan \frac{x}{a} + C.$$

EXAMPLE 8

Evaluate each integral:

a) $\int \frac{2}{x^2+25}\, dx$

b) $\int \frac{1}{3x^2+18}\, dx$

2.4 Elementary u-substitutions

In Calculus 1 we learn how to compute integrals using a $u-$substitution:

Theorem 2.8 (Integration by $u-$substitution - Indefinite Integrals)

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$$

by setting $u = g(x)$.

Theorem 2.9 (Integration by $u-$substitution - Definite Integrals)

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

by setting $u = g(x)$.

For these formulas to work, notice the integrand must have the form

$$\int f\Big(\underset{u\,\nearrow}{\boxed{g(x)}}\Big) \overset{D}{\cdot} \boxed{g'(x)}\;dx$$

The arrow with the "D" represents the idea that the derivative of the green box is, **up to a constant multiple**, the thing in the red box. The green box, where the "D arrow" starts, is what we set equal to u.

Restated, for an elementary $u-$substitution to be a valid integration technique, the integrand **must consist of two terms multiplied together**, where **one term is, up to a constant multiple, the derivative of part of the other term**.

EXAMPLE 9

Evaluate each integral:

a) $\displaystyle\int 5x^3 \cos x^4\,dx$

b) $\displaystyle\int_0^{\pi/3} e^{\cos x} \sin x \, dx$

c) $\displaystyle\int_1^5 \frac{x}{x^2+1} \, dx$

Remark: Suppose we flipped the fraction in (c) upside-down. What then?

$$\int_1^5 \frac{x^2+1}{x} \, dx$$

WARNING! Similar-looking integrals don't necessarily work the same way.

d) $\displaystyle\int \frac{(4\sqrt{x}+5)^{3/4}}{\sqrt{x}}\,dx$

e) $\displaystyle\int \tan x\,dx$

Hint: Rewrite, then perform a u-sub.

2.5 More complicated u-substitutions

We have seen how to perform a $u-$sub when the integral consists of two terms multiplied together.

$$\int f\Big(\underset{u\,\nearrow}{\boxed{g(x)}}\Big)\overset{D}{\curvearrowright}\cdot\ \boxed{g'(x)}\ dx$$

In some cases, we can also perform a $u-$sub when the integral consists of **three** terms multiplied together, so long as one term is the derivative of part of one of the other terms.

$$\int f\Big(\underset{u\,\nearrow}{\boxed{g(x)}}\Big)\overset{D}{\curvearrowright}\cdot\ \boxed{g'(x)}\ \cdot\ \boxed{h(x)}\ dx$$

The key here is to take the third term (the orange one in the diagram above) and rewrite it in terms of u, starting with the formula for u you write down in the u-sub.

Example 10

$$\int x^3\sqrt{x^2+1}\,dx$$

EXAMPLE 11

$$\int x^2\sqrt{x+3}\,dx$$

At this point, the integral is

$$\begin{aligned}\int (u-3)^2\sqrt{u}\,du &= \int (u^2-6u+9)\sqrt{u}\,du \\ &= \int \left(u^{5/2}-6u^{3/2}+9\sqrt{u}\right)\,du \\ \\ &= \end{aligned}$$

General principle (behind Examples 10 and 11)

To evaluate integrals of the form

$$\int x^n(mx\pm b)^q\,dx$$

where n is an integer but q is not a positive integer, use the $u-$sub $u=mx\pm b$.

EXAMPLE 12

$\int_1^4 \frac{x+5}{2x+1}\,dx$

EXAMPLE 13

$\int_0^2 \frac{x^2+4}{x+2}\,dx$

Solution: Think of the integral as

$$\int_0^2 (x^2+4)\cdot\frac{1}{x+2}\cdot 1\,dx$$

So the u-sub is

$$\begin{aligned} u &= x+2 \qquad \Rightarrow x = u-2 \Rightarrow x^2+4 = (u-2)^2+4 \\ du &= 1\,dx \end{aligned}$$

Substituting in (don't forget to change the limits), we get

$$\int_0^2 (x^2+4)\cdot\frac{1}{x+2}\cdot 1\,dx = \int_2^4 \left[(u-2)^2+4\right]\frac{1}{u}\,du.$$

Now FOIL the integrand, rewrite with algebra and integrate:

$$\begin{aligned}
\int_2^4 \left[(u-2)^2+4\right]\frac{1}{u}\,du &= \int_2^4 \left[u^2-4u+4+4\right]\frac{1}{u}\,du \\
&= \int_2^4 \frac{u^2-4u+8}{u}\,du \\
&= \int_2^4 \left(u-4+\frac{8}{u}\right)du \\
&= \left[\frac{1}{2}u^2-4u+8\ln u\right]_2^4 \\
&= \left[\frac{1}{2}(16)-16+8\ln 4\right]-\left[\frac{1}{2}(4)-8+8\ln 2\right] \\
&= \boxed{8\ln 4-8\ln 2-2}.
\end{aligned}$$

Sometimes you can use trig identities to rewrite an integrand before using a complicated $u-$substitution:

EXAMPLE 14

$$\int \sin^4 x\,\cos^3 x\,dx$$

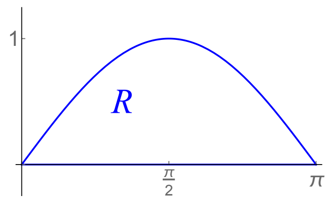
The rest of the integral is straight-forward:

$$\begin{aligned}
\int u^4(1-u^2)\,du &= \int (u^4-u^6)\,du \\
&= \frac{1}{5}u^5-\frac{1}{7}u^7+C \\
\\
&=
\end{aligned}$$

Integrals with powers of tangent and secant, or powers of cosecant and cotangent, are handled similar to Example 14.

2.6 Integration by parts

Consider the region R of points which is above the x-axis, and below the graph of $y = \sin x$ between $x = 0$ and $x = \pi$:



Imagine that R is the shape of some physical object (like a sheet of metal). Many engineering applications require you to know where something called the *center of mass* of this object is (more on what that means in Chapter 4). To compute this center of mass, we will see that you have to evaluate this integral:

$$\int_0^\pi x \sin x \, dx.$$

Integrals like this (where the integrand is x times a trigonometric or exponential function) also arise in actuarial science and mathematical finance, because they can be used to compute "expected" or "average" times until certain events happen (like an insurance policyholder being in an accident or a stock price hitting a certain value).

Question: How would we actually compute $\int_0^\pi x \sin x \, dx$?

- We can't "just do it" (meaning immediately write the answer);
- we can't "split it" (no $+$ or $-$ sign in the integrand);
- the LRP doesn't apply (no $mx \pm b$ present);
- there's no useful way to rewrite the integrand; and
- no u-sub seems to help
  (the integrand is not a product of **related** terms).

To compute this integral, we will need a new method called *integration by parts*.

Background on integration by parts

The Product Rule for differentiation tells us

$$(fg)'(x) =$$

Theorem 2.10 (Integration by Parts (IBP) ("Parts") Formula)

$$\int r\,ds = rs - \int s\,dr.$$

Other textbooks and instructors (and my past exams) use u and v instead of r and s in this formula. We won't do this, because u and v can be hard to tell apart when written by hand.

How to use the parts formula:

1. Choose r and ds so that your integral is $\int r\,ds$.

   (The dx in the integral has to be part of the ds.)

2. Solve for dr (by differentiating r) and s (by integrating ds).

3. Apply the parts formula. If you've done everything right, then the integral $s\,dr$ you are left with should be easier than the integral $\int r\,ds$ you started with.

For reference, the parts formula is $\int r\,ds = rs - \int s\,dr$.

EXAMPLE 15 (REPEATED)

$$\int_0^{\pi} x \sin x \, dx$$

Thought Process:

Solution:

EXAMPLE 16

$$\int x^2 e^x \, dx$$

More generally, to handle integrals like $\int x^n f(x)\,dx$ where $f(x)$ is exponential, sine or cosine, you would (at least theoretically) use integration by parts n times.

Each time you use the part formula, the power on the x drops by 1.

EXAMPLE 17

$$\int \ln x \, dx$$

Remark: $\int \arctan x \, dx$ and $\int \arcsin x \, dx$ are similar to Example 17.

Parts vs. u-substitutions

Question: How do you choose between a u-sub or integration by parts?

$$\int x \sin x \, dx \qquad \text{vs.} \qquad \int \sin x \cos x \, dx$$

How to choose r and ds

Suppose you know you want to use integration by parts. How do you decide which part of the integral should be r and which should be ds?

General principle:

The integral $\int s\,dr$ should be **easier** than $\int r\,ds$. So think ahead!

Specific guidelines for choosing r:

HIGH PRIORITY r (choose these to be r if possible):

MEDIUM PRIORITY r:

LOW PRIORITY r (avoid choosing these as r if possible):

EXAMPLE 18

For each integral, decide if integration by parts is an appropriate method. If so, write what you would choose for r and ds:

a) $\int 16x^3(2x^4+3)^2\,dx$

b) $\int 16x^4(2x^4+3)^2\,dx$

c) $\int 3x^2 \arctan x\,dx$

d) $\int x \ln x\,dx$

e) $\int \frac{\ln x}{x}\,dx$

f) $\int \ln x\,dx$

g) $\displaystyle\int \frac{\ln x}{x^2}\,dx$

h) $\displaystyle\int 2x \sin 4x\,dx$

i) $\displaystyle\int 4x^3 e^{2x}\,dx$

j) $\displaystyle\int 26e^{3x} \cos 2x\,dx$

2.7 Undetermined coefficients

In this section, we introduce a method of solving math problems that may not seem "valid" or "rigorous". But it is actually very important: you'll see it in MATH 330, and mathematicians in industry or academia often approach challenging problems this way.

Informally, this method is "guessing and checking". Formally, it's called the *method of undetermined coefficients*:

Method of undetermined coefficients

1. Guess the general form of the answer. This general form should have some unknown constants in it.

   (These constants are the "undetermined coefficients".)

2. Determine what the constants are by working backwards from your guess to the original problem.

EXAMPLE 19

Without the LRP or a u-sub, compute $\int e^{8x}\,dx$.

EXAMPLE 20 (PART (i) OF EXAMPLE 18)

$$\int 4x^3 e^{2x}\, dx$$

Thought process:

Guess the answer:

Determine the constants:

EXAMPLE 21 (PART (j) OF EXAMPLE 18)

$$\int 26e^{3x}\cos 2x\,dx$$

Thought process:

Guess the answer:

Determine the constants:

$$\frac{d}{dx}(guess) = \left[3Ae^{3x}\cos 2x - 2Ae^{3x}\sin 2x\right] + \left[3Be^{3x}\sin 2x + 2Be^{3x}\cos 2x\right]$$

$$= [3A + 2B]\,e^{3x}\cos 2x + [-2A + 3B]\,e^{3x}\sin 2x.$$

To make this match the integrand $26e^{3x}\cos 2x$, we need

$$\left\{ \begin{array}{l} \\ \\ \end{array} \right.$$

Solve this system to get $A = 6$, $B = 4$. Therefore the answer is

$$\boxed{6e^{3x}\cos 2x + 4e^{3x}\sin 2x + C}.$$

EXAMPLE 22

$$\int x^2 \cos x \, dx$$

Solution: We could do this with integration by parts twice, but let's use undetermined coefficients. First, guess the answer. Similar to Example 22, it probably has the following kind of terms in it:

$$x^2 \cos x \qquad x^2 \sin x \qquad x \cos x \qquad x \sin x \qquad \cos x \qquad \sin x$$

So we will guess the answer is

$$guess = Ax^2 \cos x + Bx^2 \sin x + Cx \cos x + Dx \sin x + E \cos x + F \sin x.$$

Now let's figure out the constants. Differentiate the guess:

$$\begin{aligned}\frac{d}{dx}(guess) &= 2Ax \cos x - Ax^2 \sin x + 2Bx \sin x + Bx^2 \cos x + C \cos x - Cx \sin x \\ &\quad + D \sin x + Dx \cos x - E \sin x + F \cos x \\ &= -Ax^2 \sin x + Bx^2 \cos x + (2B - C)x \sin x + (2A + D)x \cos x \\ &\quad + (C + F) \cos x + (D - E) \sin x.\end{aligned}$$

This must equal the original integrand $x^2 \cos x$, so by equating like terms we have

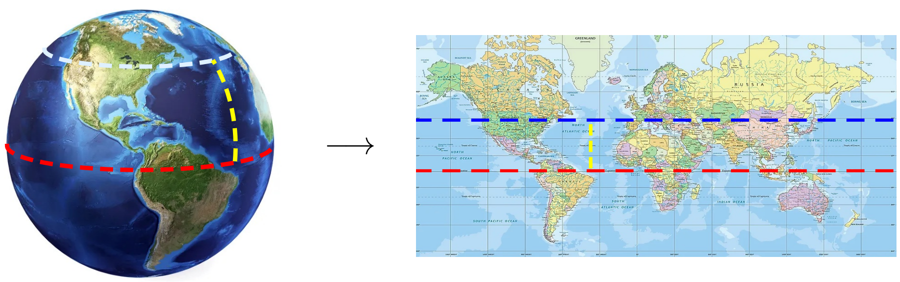
$$\begin{cases} -A &= 0 \\ B &= 1 \\ 2B - C &= 0 \\ 2A + D &= 0 \\ C + F &= 0 \\ D - E &= 0 \end{cases} \implies \begin{cases} A &= 0 \\ B &= 1 \\ C &= 2 \\ D &= 0 \\ F &= -2 \\ E &= 0 \end{cases}$$

So the solution is $\boxed{x^2 \sin x + 2x \cos x - 2 \sin x + C}$.

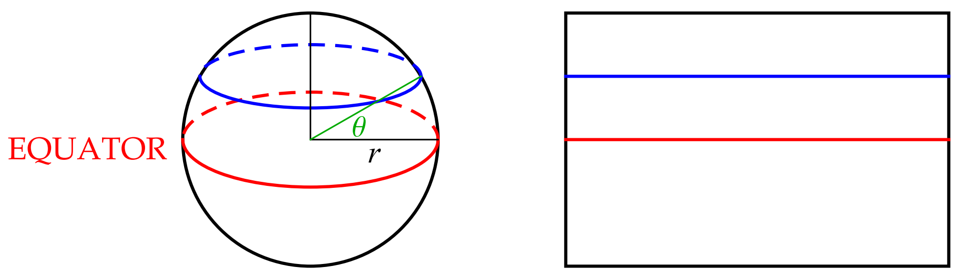
2.8 Partial fractions

MOTIVATION

In large-scale surveying, we produce maps of regions of the Earth. While the Earth is (roughly) spherical, maps are flat. This means we need to somehow "project" the round Earth onto a flat surface. A classical way of doing this is called the *Mercator projection*:



Under this projection, horizontal distances get stretched (and vertical distances may get stretched/shrunk as well to preserve the "shapes" of land masses):



To account for this stretching and derive a formula that will allow us to determine where a point on the Earth's surface should end up on this map, we need to integrate $\sec\theta$. Here's how you might do this:

$$\int \sec\theta \, d\theta = \int \frac{1}{\cos\theta} \, d\theta = \int \frac{\cos\theta}{\cos^2\theta} \, d\theta = \int \frac{\cos\theta}{1-\sin^2\theta} \, d\theta$$

Question: How we we integrate $\displaystyle\int \frac{1}{1-u^2} \, du$?

- we can't just do it;
- the integrand can't be split;
- no u-sub seems to work (no product of related terms);
- maybe parts with $r = \frac{1}{1-u^2}$, $ds = du$? (spoiler alert: this doesn't work)

The new technique we need to do this: Consider the following expression:

$$\frac{2}{x-3} + \frac{5}{x-1}$$

These terms can be combined with algebra to get

$$\frac{2(x-1)}{(x-3)(x-1)} + \frac{5(x-3)}{(x-1)(x-3)} = \frac{[2x-2]+[5x-15]}{(x-1)(x-3)} = \frac{7x-17}{x^2-4x+3}.$$

This means $\int \left(\frac{2}{x-3} + \frac{5}{x-1}\right) dx = \int \frac{7x-17}{x^2-4x+3}\, dx.$

Goal: Rewrite expressions like $\frac{7x-17}{x^2-4x+3}$ as $\frac{2}{x-3} + \frac{5}{x-1}$.

Method: We'll use a version of undetermined coefficients.

Terminology

Before we get to the details, however, we first need to develop some vocabulary that will give us a way to formally state the procedure we are covering in this section:

Definition 2.11 *A* **polynomial** *in x is an expression of the form*

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

where a_0, a_1, ..., a_n are constants. The **degree** *of a polynomial is the highest power of x that appears. A polynomial is called* **irreducible** *if it cannot be factored into two polynomials, both of lower degree.*

EXAMPLES

- $x^4 - 3x + 1$ is a degree 4 polynomial
- $x^2 - 3x^9 - 7x + 5x^6$ is a degree 9 polynomial
- $\frac{x^7+2}{x}$, $\sqrt{x}$, $\frac{1}{x}$, $\sin x$, $4x^\pi$, ... are not polynomials
- $x^2 + A$ is an irreducible polynomial if $A > 0$
- $x^2 - 4$ is not irreducible
- $Ax \pm B$ is irreducible

Theorem 2.12 (Fundamental Theorem of Algebra) *No polynomial of degree ≥ 3 is irreducible.*

Similar to how every whole number factors into a product of primes, we have:

Theorem 2.13 *Every polynomial factors into a product of irreducibles.*

Now we can precisely describe the problem we're interested in solving:

Definition 2.14 *Given a rational function $\frac{p(x)}{q(x)}$ (a* **rational function** *is the quotient of any two polynomials) with degree$(p) <$ degree(q), write $\frac{p(x)}{q(x)}$ as*

$$\frac{p(x)}{q(x)} = \frac{p_1(x)}{q_1(x)} + \frac{p_2(x)}{q_2(x)} + \ldots + \frac{p_n(x)}{q_n(x)}$$

where:

- *degree$(p_j) <$ degree(q_j) for all j; and*
- *every $q_j(x)$ is a power of an irreducible polynomial, i.e.*

$$q_j(x) = ax \pm b \ \textit{ or } \ q_j(x) = (ax \pm b)^r \ \textit{ or}$$

$$q_j(x) = (ax^2 \pm bx \pm c)^r \textit{ where } ax^2 \pm bx \pm c \textit{ doesn't factor}$$

This procedure is called **splitting $\frac{p(x)}{q(x)}$ into partial fractions** *or* **finding the partial fraction decomposition of $\frac{p(x)}{q(x)}$**.

EXAMPLE 23

Find the partial fraction decomposition of $\dfrac{1}{1-u^2}$

(so that we can finish the integral $\int \sec\theta\,d\theta$; recall $u = \sin\theta$).

STEP 1: Factor the denominator completely.

STEP 2: Guess the form of the decomposition.

STEP 3: Multiply through each term in your equation from Step 2 by the original (common) denominator.

STEP 4: Find the unknowns A, B, C, ... (two methods)

METHOD 1 (quick, but sometimes fails): Plug in some carefully selected u (or x)-values to the equation obtained in Step 3, and solve for the constants.

METHOD 2 (slow, but always works): Multiply out the right-hand side; combine like powers of u (or x); equate coefficients on the powers of u (or x); solve the resulting system of equations.

STEP 5: Write the answer.

Remark: There is no calculus in the partial fraction decomposition itself (you do no limit, derivative or integral in Steps 1-6 above). This is really an algebraic technique, whose most common use is to rewrite integrands.

If there's an integral, you integrate after doing the partial fraction procedure. Useful integration rules here include:

$$\int \frac{1}{x+a}\,dx = \ln(x+a) + C \qquad \int \frac{1}{(x+a)^2}\,dx = \frac{-1}{x+a} + C \qquad \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$$

EXAMPLE 24

$$\int \frac{3x^2+10x-24}{x^3+2x^2-8x}\,dx$$

EXAMPLE 25

Find the partial fraction decomposition of $\dfrac{2x^3 - x^2 - x + 1}{x^2(x-1)^2}$.

Attempted solution: Similar to what we've done before, let's try

$$\frac{2x^3 - x^2 - x + 1}{x^2(x-1)^2} = \frac{A}{x^2} + \frac{B}{(x-1)^2} = \frac{A(x-1)^2 + Bx^2}{x^2(x-1)^2}$$

$$\Rightarrow 2x^3 - x^2 - x + 1 = A(x-1)^2 + Bx^2$$

This is a problem! Can you see why?

EXAMPLE 25 (CONTINUED)

On the previous page, we obtained

$$2x^3 - x^2 - x + 1 = Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2$$

and we figured out that $B = 1$ and $D = 1$, so now we have

$$\begin{aligned}
2x^3 - x^2 - x + 1 &= Ax(x-1)^2 + (x-1)^2 + Cx^2(x-1) + x^2 \\
&= Ax(x^2 - 2x + 1) + 1(x^2 - 2x + 1) + C(x^3 - x^2) + 1x^2 \\
&= Ax^3 - 2Ax^2 + Ax + x^2 - 2x + 1 + Cx^3 - Cx^2 + x^2 \\
&= (A + C)x^3 + (-2A + 2 - C)x^2 + (A - 2)x + 1
\end{aligned}$$

which means, by considering the x terms, that

$$A - 2 = -1 \quad \Rightarrow \quad A = 1$$

and finally, by considering the x^3 terms, that

$$2 = A + C = 1 + C \quad \Rightarrow \quad C = 1.$$

All together, we have $A = B = C = D = 1$ so

$$\boxed{\frac{2x^3 - x^2 - x + 1}{x^2(x-1)^2} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x-1} + \frac{1}{(x-1)^2}.}$$

P.S. This means

$$\begin{aligned}
\int \frac{2x^3 - x^2 - x + 1}{x^2(x-1)^2}\,dx &= \int \left[\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x-1} + \frac{1}{(x-1)^2}\right] dx \\
&= \boxed{\ln x - \frac{1}{x} + \ln(x-1) - \frac{1}{x-1} + C}.
\end{aligned}$$

How to guess the partial fraction decomposition

It is hard to write down a "one-size fits all" rule that tells you what your guess needs to be. It's easier to look at some examples to see the pattern:

	$\frac{p(x)}{q(x)}$	GUESSED FORM OF DECOMPOSITION
DISTINCT LINEAR FACTORS	$\frac{6x-1}{(x-5)(x+2)}$	$\frac{A}{x-5} + \frac{B}{x+2}$
	$\frac{-4}{x(x-3)(x+4)}$	$\frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+4}$
REPEATED LINEAR FACTORS	$\frac{2}{(x-4)^2(x+6)}$	
	$\frac{7}{x^4(x-1)^3(x+2)}$	$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x-1} + \frac{F}{(x-1)^2} + \frac{G}{(x-1)^3} + \frac{H}{x+2}$
UNFACTORABLE QUADRATIC FACTORS	$\frac{x-1}{(x^2+5)(x-2)}$	
	$\frac{x^3-3x+4}{(x^2+1)^3x^2(x-7)(x^2+6)}$	$\frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{Ex+F}{(x^2+1)^3} + \frac{G}{x} + \frac{H}{x^2} + \frac{I}{x-7} + \frac{Jx+K}{x^2+6}$

General guidelines for guessing form of partial fraction decomposition:

1. Make sure the denominator is factored completely first.
2. For every linear term $(x \pm a)^r$ in the denominator, the guessed form needs r terms of the form
$$\frac{A}{x \pm a} + \frac{B}{(x \pm a)^2} + \frac{C}{(x \pm a)^3} + \dots + \frac{const}{(x \pm a)^r}$$
3. For every unfactorable quadratic $(x^2 + ax + b)^r$ in the denominator, the guessed form needs r terms of the form
$$\frac{Ax + B}{x^2 + ax + b} + \frac{Cx + D}{(x^2 + ax + b)^2} + \frac{Ex + F}{(x^2 + ax + b)^3} + \dots + \frac{const\, x + const}{(x^2 + ax + b)^r}$$

Partial fractions is a useful for evaluating integrals in these situations:

1. The integrand must be a rational function (i.e. contain only nonnegative powers of x and no trig/exp/log functions).
2. The degree of the numerator must be less than the degree of the denominator.
3. The denominator must be easily factored.
4. The numerator must be "sufficiently complicated" (see Example 26 below).

EXAMPLE 26

$$\int \frac{x}{(x-2)^3}\,dx$$

Solution # 1: Use partial fractions:

$$\begin{aligned}
\frac{x}{(x-2)^3} &= \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} \\
\frac{x}{(x-2)^3} &= \frac{A(x-2)^2}{(x-2)^3} + \frac{B(x-2)}{(x-2)^3} + \frac{C}{(x-2)^3} \\
x &= A(x-2)^2 + B(x-2) + C \\
x &= A(x^2-4x+4) + Bx - 2B + C \\
x &= Ax^2 + (B-4A)x + (4A-2B+C)
\end{aligned}$$

Thus $A = 0$, $B - 4A = 1$ and $4A - 2B + C = 0$. That means $B = 1$ and $C = 2$. So

$$\int \frac{x}{(x-2)^3}\,dx = \int \left(\frac{1}{(x-2)^2} + \frac{2}{(x-2)^3} \right) dx = \boxed{\frac{-1}{x-2} + \frac{-1}{(x-2)^2} + C}.$$

Solution # 2:

EXAMPLE 27

$$\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\, dx$$

2.9 Adventures in integration

In the previous eight sections, we've seen "standard" ways of computing integrals.

In this section, we are going to do some examples that illustrate some more exotic techniques of integration.

One of the things I want you to take from this section is that when you are faced with a hard problem, "try something". If it doesn't work, try something else.

That said, I won't ask you any integrals like this on an exam (except perhaps for extra credit).

Desperado substitutions

EXAMPLE 28

$$\int \frac{1}{x\sqrt{x-1}}\,dx$$

EXAMPLE 29

$$\int \sqrt{9-\sqrt{x}}\,dx$$

At this point the integral can be done by distributing and using the Power Rule:

$$\int \sqrt{u}\cdot 2(u-9)\,du = \int \left[2u^{3/2} - 18u^{1/2}\right]\,du = \frac{4}{5}u^{5/2} - 12u^{3/2} + C.$$

Last, back-substitute to get $\boxed{\frac{4}{5}(9-\sqrt{x})^{5/2} - 12(9-\sqrt{x})^{3/2} + C}$.

EXAMPLE 30

$$\int \frac{1}{e^x+1}\,dx$$

At this point, we can use partial fractions (work omitted):

$$\int \frac{1}{u(u-1)}\,du = \int \left[\frac{-1}{u} + \frac{1}{u-1}\right]\,du = -\ln u + \ln(u-1) + C.$$

Last, back-substitute and simplify:

$$\begin{aligned} -\ln(e^x+1) + \ln(e^x+1-1) + C &= -\ln(e^x+1) + \ln e^x + C \\ &= \boxed{-\ln(e^x+1) + x + C}. \end{aligned}$$

Insane rewrites

EXAMPLE 31

$$\int \frac{1}{x^2 + 8x + 19}\,dx$$

EXAMPLE 32

$$\int \sec x\,dx$$

EXAMPLE 30 (REPEATED FROM BEFORE)

$$\int \frac{1}{e^x + 1}\,dx$$

EXAMPLE 33

$$\int \frac{2}{(x^2+1)^2}\,dx$$

SOHCAHTOA substitutions

EXAMPLE 34

$$\int \sqrt{16 - x^2}\, dx$$

We end up with

$$\begin{aligned}
\int 4\cos u \cdot 4\cos u\, du &= \int 16\cos^2 u\, du \\
&= 16\left(\frac{1+\cos 2u}{2}\right)\, du \\
&= \int (8 + 8\cos 2u)\, du \\
&= 8u + 4\sin 2u + C \\
&= 8u + 4(2\sin u\cos u) + C \quad \text{(by trig identity } \sin 2u = 2\sin u\cos u\text{)} \\
&= 8u + 8\sin u\cos u + C.
\end{aligned}$$

Finally, back-substitute using SOHCAHTOA in our triangle to get

$$\begin{aligned}
8u + 8\sin u\cos u + C &= 8\arcsin\frac{x}{4} + 8\left(\frac{x}{4}\right)\left(\frac{\sqrt{16-x^2}}{4}\right) + C \\
&= \boxed{8\arcsin\frac{x}{4} + \frac{1}{2}x\sqrt{16-x^2} + C}.
\end{aligned}$$

2.10 Summary of integration techniques

1. **Just do it:** check to see if the integral is one you can "just write the answer to".

2. **Split it:** if the integrand contains terms which are added or subtracted, consider using linearity rules to split the integrand into pieces you can "just write the answer to".

3. **Linear Replacement Principle:** if the integrand is a function you can integrate by hand with a $mx \pm b$ instead of an x, use the LRP.

4. **Rewrite it:** see if you can rewrite the integrand using algebra, log rules or a trig identity.

5. u**-sub:** if the integrand contains terms which are multiplied together, where one part is the derivative of something in the other part (up to a constant), try a $u-$sub.

6. **Parts:** if the integrand has terms multiplied together which are unrelated, try parts.

7. **Partial fractions:** If the integrand is a rational function where the denominator factors (and the degree of the denominator is greater than the degree of the numerator), use partial fractions to decompose the integrand.

8. **Use a computer:** *Mathematica* commands are as follows:

TASK	COMMAND
Indefinite integral $\int f(x)\,dx$	`Integrate[`$function$`, x]`
Definite integral $\int_a^b f(x)\,dx$	Exact value: `Integrate[`$function$`, {x,`a`,`b`}]` Numerical approximation: `NIntegrate[`$function$`, {x,`a`,`b`}]`
Partial fraction decomposition (without integrating)	`Apart[`$expression$`]`

9. **Undetermined coefficients:** guess the answer with unknown constants, and then try to figure out what the constants are.

10. **Get creative:** try a complicated u-sub and/or creatively rewrite the integrand.

2.11 Homework exercises

Exercises from Section 2.1

1. Suppose that an antiderivative of function f is $'f(x) = x\ln(\tan x + 1)$.

   a) Write down two other antiderivatives of f.

   b) Evaluate $\int f(x)\,dx$.

   c) Evaluate $\int_0^{\pi/4} f(x)\,dx$.

   d) What is $f(x)$?

2. Suppose that you know $\int f(x)\,dx = F(x) + C$ and $\int g(x)\,dx = G(x) + C$. For each given expression, determine if you can write the answer in terms of F and/or G (together with other stuff you know). If so, write the answer in terms of F and/or G.

   a) $\int 3f(x)\,dx$

   b) $\int [g(x) - 2f(x)]\,dx$

   c) $\int f(x)g(x)\,dx$

   d) $\int \left[g(x) + 6x^2\right]\,dx$

   e) $\int \frac{f(x)}{4}\,dx$

   f) $\int \frac{4}{f(x)}\,dx$

   g) $\int \frac{g(x)}{f(x)}\,dx$

   h) $\int [5f(x) + 3g(x) - 1]\,dx$

   i) $\int xf(x)\,dx$

   j) $\int f(g(x))\,dx$

In Exercises 3-8, evaluate each integral by hand:

3. $\int \left(2e^x - \frac{\sin x}{5} + 1\right)\,dx$

4. $\int 4^x\,dx$

5. $\int \left(\frac{4}{\sqrt{1-x^2}} + 8\sec^2 x + 12\sqrt{x}\right)\,dx$

6. $\int \left(7e^{3t} + \cos t - 5t\right)\,dt$

7. $\int_1^2 \left(12w^2 + \frac{9}{\sqrt{w}} - 3w^{5/2}\right)\,dw$

8. $\int_{\pi/6}^{\pi/4} 3\sec x\tan x\,dx$

Exercises from Section 2.2

In Exercises 9-18, evaluate each integral by hand:

9. $\int e^{7-3x}\,dx$

10. $\int_3^5 \frac{1}{7x+1}\,dx$

11. $\int \sin 4x\,dx$

12. $\int \cos\frac{x}{3}\,dx$

13. $\int \left(\cos\left(\frac{3}{7}x\right)+\frac{4}{x}\right)\,dx$

14. $\int_0^{\pi/9} 2\sec^2 3x\,dx$

15. $\int_{-4}^{10} \left(e^{x/2}+3x\right)\,dx$

16. $\int \csc\pi x\cot\pi x\,dx$

17. $\int \left(\frac{1}{(4x+5)^3}+\frac{2}{\sqrt{2-3x}}\right)\,dx$

18. $\int \left(e^{2s}+e^{-s/2}\right)\,ds$

19. Suppose you know $\int f(x)\,dx = F(x)+C$. For each given expression, determine if you can write the answer in terms of F; if so, write the answer.

    a) $\int 6f(x)\,dx$

    b) $\int f(3x)\,dx$

    c) $\int 2f\left(\frac{x}{2}\right)\,dx$

    d) $\int \frac{f(4x+1)}{8}\,dx$

    e) $\int f\left(1-\frac{2x}{9}\right)\,dx$

    f) $\int f\left(\frac{3}{x}\right)\,dx$

20. Suppose you know $\int_0^{12} f(x)\,dx = 5$.

    a) Do you know the value of $\int_0^{12} 4f(x)\,dx$? If so, what is it?

    b) Do you know the value of $\int_0^{12} f(4x)\,dx$? If so, what is it?

    c) For what b do you know the value of $\int_0^{b} f(4x)\,dx$?

    d) For the value of b that works in part (c), what is $\int_0^{b} f(4x)\,dx$?

Exercises from Section 2.3

In Exercises 21-32, evaluate each integral by hand:

21. $\displaystyle\int t^2\left(t-\frac{2}{t}\right)dt$

22. $\displaystyle\int_0^4 (3-x)\sqrt{x}\,dx$

23. $\displaystyle\int (1+2x^2)^2\,dx$

24. $\displaystyle\int \cos^2 3x\,dx$

25. $\displaystyle\int x\left(1+\frac{1}{x}\right)^2 dx$

26. $\displaystyle\int \frac{(x-7)^2}{x}\,dx$

27. $\displaystyle\int \sin^2\frac{x}{4}\,dx$

28. $\displaystyle\int_{\pi/4}^{\pi/3} 3\cot^2 x\,dx$

29. $\displaystyle\int \sin^4\theta\,d\theta$

30. $\displaystyle\int \frac{8}{x^2+16}\,dx$

31. $\displaystyle\int \left(\frac{1}{6x^2+48}+x^4\right)dx$

32. $\displaystyle\int \cos x\sec x\,dx$

33. Consider the integral $\displaystyle\int 3^{4x}\,dx$.

    a) Compute this integral by rewriting it in the form $\displaystyle\int b^x\,dx$, and then evaluating that integral. In particular, what is the value of b?
    b) Compute this integral by using the Linear Replacement Principle, together with the rule for $\displaystyle\int 3^x\,dx$.
    c) Reconcile the apparently different-looking answers you obtain in parts (a) and (b) of this question.

Exercises from Section 2.4

In Exercises 34-43, evaluate each integral by hand:

34. $\displaystyle\int x^2\sqrt{x^3-1}\,dx$

35. $\displaystyle\int \frac{x^2+1}{x^3+3x-2}\,dx$

36. $\displaystyle\int \frac{e^{2x}}{e^{2x}+1}\,dx$

37. $\displaystyle\int_0^3 \frac{e^{2x}+1}{e^{2x}}\,dx$

38. $\displaystyle\int \cot x\,dx$

39. $\displaystyle\int_1^3 x\,2^{x^2-1}\,dx$

40. $\displaystyle\int \frac{\sin w}{\sqrt{\cos w}}\,dw$

41. $\displaystyle\int \left(\cos^3 x\sin x-\sin 2x\right)dx$

42. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

43. $\int \sin^4 2x \cos 2x\,dx$

Exercises from Section 2.5

In Exercises 44-53, evaluate each integral by hand:

44. $\int \frac{3x+2}{5x-1}\,dx$

45. $\int_{-3}^{-2} \frac{x^2}{x-1}\,dx$

46. $\int \left(\frac{2x}{x-4}+\frac{3}{2x-1}\right)\,dx$

47. $\int_0^1 t\sqrt{t+1}\,dt$

48. $\int x^2(x+3)^8\,dx$

49. $\int \sin^5 x\,dx$

50. $\int_0^{\pi/6} \cos^8 x \sin^3 x\,dx$

51. $\int \tan^7 x \sec^4 x\,dx$

52. $\int 3^{\ln w}\,dw$

53. $\int \frac{\ln x^2}{x}\,dx$

54. a) Evaluate the integral $\int \sin x \cos x\,dx$ by performing the $u-$substitution $u = \sin x$.

    b) Evaluate the integral of part (a) by performing the $u-$substitution $u = \cos x$.

    c) Reconcile the (apparently) different answers you get to parts (a) and (b) of this question.

Exercises from Section 2.6

In Exercises 55-65, evaluate each integral by hand.

55. $\int xe^{7x}\,dx$

56. $\int \frac{4x}{e^x}\,dx$

57. $\int x^3 e^{x^4}\,dx$

58. $\int x\ln(x+5)\,dx$

59. $\int \frac{\ln x}{x^4}\,dx$

60. $\int \left(x\cos x + x\cos x^2\right)\,dx$

61. $\int_0^{\pi/2} (x\sin 2x + \sin 3x)\,dx$

62. $\int_0^1 2xe^{-x/2}\,dx$

63. $\int \arctan x\,dx$

64. $\int \log x\,dx$

65. $\int x\arctan x\,dx$

In Exercises 66-75, determine whether it is better to use parts or a $u-$substitution to evaluate the integral. If parts are better, write "PARTS" and give your choices of r and ds. Otherwise, write "$u-$SUB" and give your choice of u. You do not have to evaluate these integrals.

66. $\displaystyle\int \frac{x}{\sqrt{2+3x}}\,dx$

67. $\displaystyle\int x^2 \cos x\,dx$

68. $\displaystyle\int x \ln^2 x\,dx$

69. $\displaystyle\int \arcsin x\,dx$

70. $\displaystyle\int \sin x \cos x\,dx$

71. $\displaystyle\int x^3 4^x\,dx$

72. $\displaystyle\int \frac{\arctan x}{x^2+1}\,dx$

73. $\displaystyle\int 3x^2 e^{2x-1}\,dx$

74. $\displaystyle\int \frac{1}{x\ln x}\,dx$

75. $\displaystyle\int e^x \cos 3x\,dx$

Exercises from Section 2.7

In Exercises 76-78, evaluate each integral by hand.

76. $\displaystyle\int x^4 e^x\,dx$

77. $\displaystyle\int 3x^2 \sin\frac{x}{2}\,dx$

78. $\displaystyle\int \left(-9e^x\cos 2x - 7e^x \sin 2x\right)\,dx$

79. Use the method of undetermined coefficients to find a function $y = f(x)$ so that $y' - 7y = 2e^{3x}$.

    Hint: Guess that the answer is of the form $y = Ae^{3x}$; plug this guess into the left-hand side of the given equation and figure out what A has to be so that you come out with $5e^{3x}$.

80. Use the method of undetermined coefficients to find a function $y = f(x)$ so that $y' + 3y = -\cos x$.

81. Use the method of undetermined coefficients to find a function $y = f(x)$ so that $y'' + 3y' - 5y = 14e^{-2x}$.

82. Find a nonzero function $y = f(x)$ so that $y'' = y^2$.

    Hint: A creative method of undetermined coefficients may be helpful here.

Exercises from Section 2.8

In Exercises 83-86, write the "guessed" form of the partial fraction decomposition of the given expression. You do not need to solve for the constants.

Example: $\dfrac{1}{x^2-1}$ *Solution:* $\dfrac{1}{x^2-1} = \dfrac{A}{x-1} + \dfrac{B}{x+1}$

83. $\dfrac{4}{x^2-3x}$

84. $\dfrac{7x+2}{x^3-5x^2-24x}$

85. $\dfrac{3x}{x^2+4x-5}$

86. $\dfrac{2}{3x^2+9x-54}$

In Exercises 87-90, evaluate each integral by hand.

87. $\displaystyle\int \frac{1}{x^2-16}\,dx$

88. $\displaystyle\int \frac{3}{x^2+x-2}\,dx$

89. $\displaystyle\int \frac{5-x}{2x^2+x-1}\,dx$

90. $\displaystyle\int \frac{5x^2-12x-12}{x^3-4x}\,dx$

In Exercises 91-96, write the "guessed" form of the partial fraction decomposition of the given expression. You do not need to solve for the constants.

91. $\dfrac{2x+3}{x(x^2+1)}$

92. $\dfrac{4}{(x-3)(x+2)^2}$

93. $\dfrac{5-x^2}{x^3(x-1)^2}$

94. $\dfrac{1}{(x-2)^2(x^2+4)^2}$

95. $\dfrac{x+1}{x^3(x^2+x+1)(x-7)}$

96. $\dfrac{7}{(x+4)^3}$

In Exercises 97-102, evaluate each integral by hand.

97. $\displaystyle\int \frac{4x^2+2x-1}{x^3+x^2}\,dx$

98. $\displaystyle\int \frac{x^2}{x^4-2x^2-8}\,dx$

99. $\displaystyle\int_{-1}^{1} \frac{x}{(x+2)^5}\,dx$

100. $\displaystyle\int \frac{x^2-x+9}{(x^2+9)^2}\,dx$

101. $\displaystyle\int_{1}^{5} \frac{x-1}{x^2(x+1)}\,dx$

102. $\displaystyle\int \frac{3x+4}{x^3+2x^2+10x}\,dx$

103. a) Evaluate the integral $\int \frac{2x-3}{(x-1)^3}\,dx$ by performing a $u-$substitution.
   b) Evaluate the integral from part (a) via partial fractions.
   c) Verify that the answers you get to parts (a) and (b) are equal.
   d) Evaluate the integral using *Mathematica*. Which method do you think *Mathematica* used to compute the integral?

Exercises from Section 2.9

These exercises should be considered examples of questions that would be bonus questions on an exam (so they aren't mandatory). In each exercise, evaluate the integral.

104. $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$

105. $\displaystyle\int \frac{1+\sin x}{\cos x}\,dx$

106. $\displaystyle\int \frac{1}{\cos t - 1}\,dt$

107. $\displaystyle\int \csc x\,dx$

108. $\displaystyle\int \sin v \cos 2v\,dv$

109. $\displaystyle\int \frac{8}{x^2-12x+50}\,dx$

110. $\displaystyle\int \frac{\sqrt{x^2+4}}{x^4}\,dx$

111. $\displaystyle\int \sin(\ln x)\,dx$

112. $\displaystyle\int \left(\frac{\cos x}{x^3} - \frac{3\sin x}{x^4}\right)\,dx$

113. $\displaystyle\int \arcsin\sqrt{x}\,dx$

Answers

Note: with indefinite integrals, answers can sometimes vary a little bit.

1. a) Answers include
   $'f(x) = x\ln(\tan x + 1) + 1$;
   $'f(x) = x\ln(\tan x + 1) + 2\sqrt{6}$;
   $'f(x) = x\ln(\tan x + 1) - 3.7$;
   etc.
   b) $x\ln(\tan x + 1) + C$
   c) $\frac{\pi \ln 4}{2}$
   d) $f(x) = \ln(\tan x + 1) + \frac{x\sec^2 x}{\tan x + 1}$
2. a) $3F(x) + C$
   b) $G(x) - 2F(x) + C$
   c) cannot determine
   d) $G(x) + 2x^3 + C$
   e) $\frac{1}{4}F(x) + C$
   f) cannot determine
   g) cannot determine
   h) $5F(x) + 3G(x) - x + C$
   i) cannot determine
   j) cannot determine
3. $2e^x + \frac{1}{5}\cos x + x + C$
4. $\frac{1}{\ln 4}4^x + C$
5. $4\arcsin x + 8\tan x + 8x^{3/2} + C$
6. $\frac{7}{3}e^{3t} + \sin t - \frac{5}{2}t^2 + C$
7. $\frac{76}{7} + \frac{39}{7}\sqrt{2}$
8. $3\sqrt{2} - 2\sqrt{3}$
9. $\frac{-1}{3}e^{7-3x} + C$
10. $\frac{1}{7}(\ln 36 - \ln 22)$
11. $\frac{-1}{4}\cos 4x + C$
12. $3\sin\frac{x}{3} + C$
13. $\frac{7}{3}\sin\frac{3x}{7} + 4\ln x + C$
14. $\frac{2\sqrt{3}}{3}$

15. $2e^5 - 2e^{-2} + 126$
16. $\frac{-1}{\pi} \csc \pi x + C$
17. $-\frac{4}{3}\sqrt{2-3x} - \frac{1}{8}(5+4x)^{-2} + C$
18. $\frac{1}{2}e^{2s} - 2e^{-s/2} + C$
19.
    a) $6F(x) + C$
    b) $\frac{1}{3}F(3x) + C$
    c) $4F\left(\frac{x}{2}\right) + C$
    d) $\frac{1}{32}F(4x+1) + C$
    e) $-\frac{9}{2}F\left(1 - \frac{2x}{9}\right) + C$
    f) cannot determine
20.
    a) Yes; 20
    b) No
    c) $b = 3$
    d) $\frac{5}{4}$
21. $\frac{1}{4}t^4 - t^2 + C$
22. $\frac{16}{5}$
23. $\frac{4}{5}x^5 + \frac{4}{3}x^3 + x + C$
24. $\frac{1}{2}x + \frac{1}{12}\sin 6x + C$
25. $\frac{1}{2}x^2 + 2x + \ln x + C$
26. $\frac{1}{2}x^2 - 14x + 49\ln x + C$
27. $\frac{1}{2}x - \sin\frac{x}{2} + C$
28. $3 - \sqrt{3} - \frac{\pi}{4}$
29. $\frac{3}{8}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + C$
30. $2\arctan\frac{x}{4} + C$
31. $\frac{1}{6\sqrt{8}}\arctan\frac{x}{\sqrt{8}} + \frac{1}{5}x^5 + C$
32. $x + C$
33.
    a) $b = 81$; $\int 3^{4x}\,dx = \int 81^x\,dx = \frac{1}{\ln 81}81^x + C$
    b) $\int 3^{4x}\,dx = \frac{1}{4}\cdot\frac{1}{\ln 3}3^{4x} + C$
    c) Since $4\ln 3 = \ln 3^4 = \ln 81$, these answers are the same.
34. $\frac{2}{9}(x^3-1)^{3/2} + C$
35. $\frac{1}{3}\ln(x^3+3x-2) + C$
36. $\frac{1}{2}\ln(e^{2x}+1) + C$
37. $\frac{7}{2} - \frac{1}{2}e^{-6}$
38. $\ln(\sin x) + C$
39. $\frac{255}{2\ln 2}$
40. $-2\sqrt{\cos w} + C$
41. $\frac{-1}{4}\cos^4 x + \frac{1}{2}\cos 2x + C$
42. $2\sin\sqrt{x} + C$
43. $\frac{1}{10}\sin^5 2x + C$
44. $\frac{3}{25}(5x-1) + \frac{13}{25}\ln(5x-1) + C$
45. $\ln 3 - \ln 4 - \frac{3}{2}$
46. $2x + 8\ln(x-4) + \frac{3}{2}\ln(2x-1) + C$
47. $\frac{4}{15} + \frac{4}{15}\sqrt{2}$
48. $\frac{1}{11}(x+3)^{11} - \frac{3}{5}(x+3)^{10} + (x+3)^9 + C$
49. $\frac{-1}{80}\cos 5x + \frac{5}{48}\cos 3x - \frac{5}{8}\cos x + C$
50. $\frac{2}{99} - \frac{153\sqrt{3}}{22528}$
51. $\frac{1}{10}\tan^{10} x + \frac{1}{8}\tan^8 x + C$
52. $\frac{1}{\ln 3+1}w^{\ln 3+1} + C$
53. $\ln^2 x + C$

54. a) $\frac{1}{2}\sin^2 x + C$
    b) $\frac{-1}{2}\cos^2 x + D$
    c) Set the two answers equal to one another; rewriting this equation gives $\frac{1}{2} = D - C$ so since C and D are arbitary constants, as long as they are related by $\frac{1}{2} = D - C$, the answers reconcile.

55. $\frac{1}{7}xe^{7x} - \frac{1}{49}e^{7x} + C$

56. $-4xe^{-x} - 4e^{-x} + C$

57. $\frac{1}{4}e^{x^4} + C$

58. $\frac{-1}{4}x^3 + \frac{5}{2}x + \frac{1}{2}x^2\ln(x+5) - \frac{25}{2}\ln(x+5) + C$

59. $\frac{-1}{9}x^{-3} - \frac{1}{3}x^{-3}\ln x + C$

60. $\cos x + x\sin x + \frac{1}{2}\sin x^2 + C$

61. $\frac{1}{3} + \frac{\pi}{4}$

62. $8 - 12e^{-1/2}$

63. $x\arctan x - \frac{1}{2}\ln(1+x^2) + C$

64. $\frac{1}{\ln 10}(x\ln x - x) + C$

65. $\frac{1}{2}x^2\arctan x + \frac{1}{2}\arctan x - \frac{x}{2} + C$

66. u-SUB; $u = 2 + 3x$

67. PARTS; $r = x^2$; $ds = \cos x\,dx$

68. PARTS; $r = \ln^2 x$; $ds = x\,dx$

69. PARTS; $r = \arcsin x$; $ds = dx$

70. u-SUB; $u = \sin x$ or $u = \cos x$

71. PARTS; $r = x^3$; $ds = 4^x\,dx$

72. u-SUB; $u = \arctan x$

73. PARTS; $r = 3x^2$; $ds = e^{2x-1}\,dx$

74. u-SUB; $u = \ln x$

75. PARTS; $r = e^x$; $ds = \cos 3x\,dx$ (or vice versa)

76. $x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C$

77. $-6x^2\cos\frac{x}{2} + 24x\sin\frac{x}{2} + 48\cos\frac{x}{2} + C$

78. $e^x\cos 2x - 5e^x\sin 2x + C$

79. $y = \frac{-1}{2}e^{3x}$

80. $y = \frac{-3}{10}\cos x - \frac{1}{10}\sin x$

81. $y = -2e^{-2x}$

82. $y = 6x^{-2}$

83. $\frac{A}{x} + \frac{B}{x-3}$

84. $\frac{A}{x} + \frac{B}{x-8} + \frac{C}{x+3}$

85. $\frac{A}{x+5} + \frac{B}{x-1}$

86. $\frac{A}{x+6} + \frac{B}{3x-9}$

87. $\frac{1}{8}\ln(x-4) - \frac{1}{8}\ln(x+4) + C$

88. $\ln(x-1) - \ln(x+2) + C$

89. $\frac{3}{2}\ln(2x-1) - 2\ln(x+1) + C$

90. $3\ln x - 2\ln(x-2) + 4\ln(x+2) + C$

91. $\frac{A}{x} + \frac{Bx+C}{x^2+1}$

92. $\frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

93. $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2}$

94. $\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+4} + \frac{Ex+F}{(x^2+4)^2}$

95. $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+x+1} + \frac{F}{x-7}$

96. $\frac{A}{x+4} + \frac{B}{(x+4)^2} + \frac{C}{(x+4)^3}$

97. $\frac{1}{x} + 3\ln x + \ln(x+1) + C$

98. $\frac{\sqrt{2}}{6}\arctan\frac{x}{\sqrt{2}} + \frac{1}{6}\ln(x-2) - \frac{1}{6}\ln(x+2) + C$

99. $-\frac{14}{81}$

100. $\frac{1}{2}(x^2+9)^{-1} + \frac{1}{3}\arctan\frac{x}{3} + C$

101. $\frac{-4}{5} + \ln\frac{25}{9}$

102. $\frac{2}{5}\ln x + \frac{13}{15}\arctan\frac{x+1}{3} - \frac{1}{5}\ln(x^2+2x+10) + C$

103. a) $-2(x-1)^{-1} + \frac{1}{2}(x-1)^{-2} + C$
 b) $\frac{1}{2}(x-1)^{-2} - 2(x-1)^{-1} + C$
 c) They are clearly equal.
 d) It isn't clear, because *Mathematica* simplifies the answer so much.

104. $2\sqrt{x} - 2\ln(1+\sqrt{x}) + C$. *Hint:* use a u-sub $u = 1 + \sqrt{x}$.

105. $-\ln(1-\sin x) + C$. *Hint:* first, rewrite the integrand by multiplying through the numerator and denominator by $(1-\sin x)$.

106. $\csc t + \cot t + C$. *Hint:* first, rewrite the integrand by multiplying through the numerator and denominator by $(\cos t + 1)$.

107. $-\ln\cos\frac{x}{2} + \ln\sin\frac{x}{2} + C$. *Hint:* first, rewrite the integrand by multiplying through the numerator and denominator by $\csc x + \cot x$.

108. $\cos v - \frac{2}{3}\cos^3 v + C$. *Hint:* first, rewrite the integrand using the trig identity $\cos 2v = 2\cos^2 v - 1$.

109. $\frac{8}{\sqrt{14}}\arctan\frac{x-6}{\sqrt{14}} + C$. *Hint:* complete the square in the denominator.

110. $-\frac{(x^2+4)^{3/2}}{12x^3} + C$. *Hint:* use the SOHCAHTOA substitution $u = \arctan\frac{x}{2}$ (a.k.a. $x = 2\tan u$).

111. $\frac{1}{2}[x\sin(\ln x) - x\cos(\ln x)] + C$. *Hint:* use integration by parts twice to recover the original integral.

112. $\frac{\sin x}{x^3} + C$. *Hint:* Combine the two terms in the integrand and write them with a denominator of x^6.

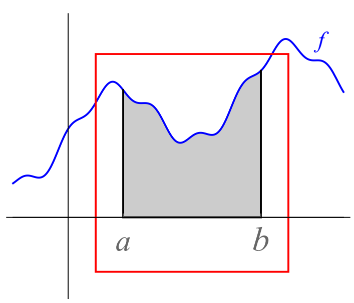
113. $x\arcsin\sqrt{x} + \frac{1}{2}\arcsin\sqrt{1-x} + \frac{1}{2}\sqrt{x-x^2}$. *Hint:* start with the u-sub $u = \sqrt{x}$; then use parts; then a SOHCAHTOA substitution $w = \sin u$. *(This is pretty much the hardest integral I know how to do!)*

Chapter 3

Improper integrals

Recall: Let f be continuous and let $a, b \in \mathbb{R}$. Then $\int_a^b f(x)\,dx$ gives the area under f from $x = a$ to $x = b$.

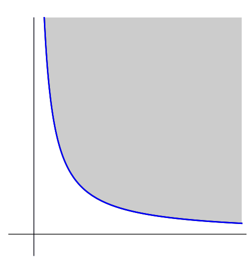
Since f is continuous on $[a, b]$, it must be that the graph of f encloses a **bounded** region (in the sense that one can draw a box around it):



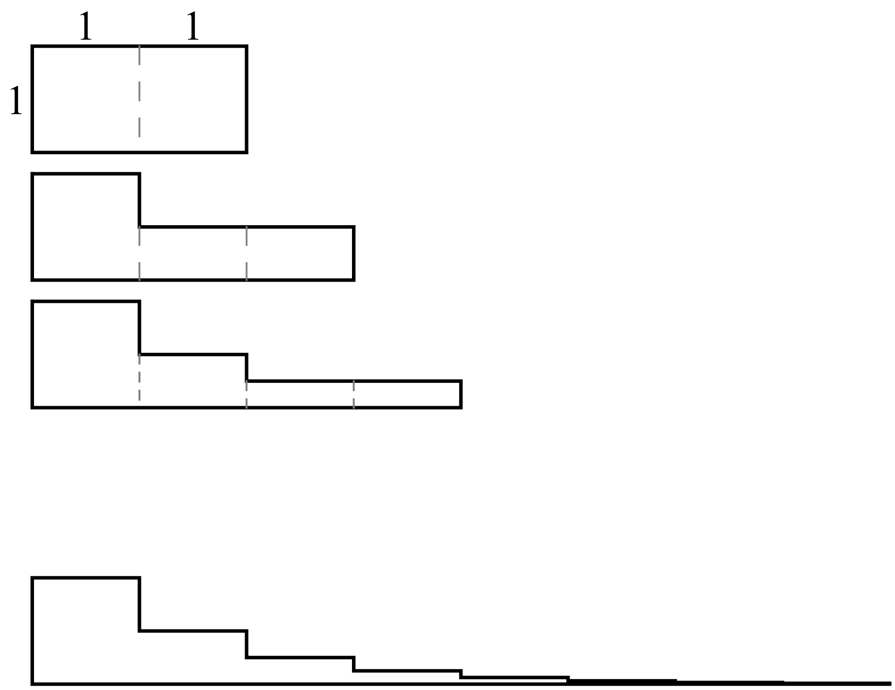
This guarantees that $\int_a^b f(x)\,dx < \infty$.

New question: Can you compute the area of an *unbounded* region?

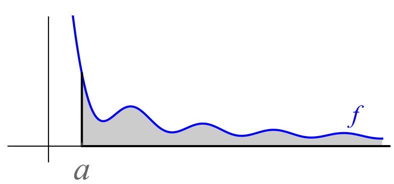
A typical unbounded region has infinite area:



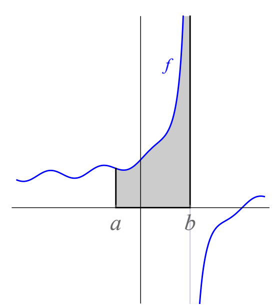
But some unbounded regions have finite area! Here is an example:



Notice: in the previous picture, while the region obtained is unbounded, in the direction it is unbounded, the width/thickness of the region decreases to zero. Another situation where the width/thickness of an unbounded region decreases to zero is when the region is described by a graph with an asymptote:



"Horizontally unbounded region"



"Vertically unbounded region"

Of these two types, **horizontally** unbounded regions are far more important.

Next question: How are such unbounded regions described?

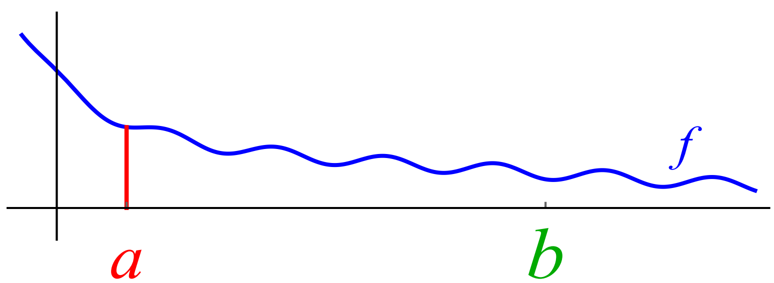
3.1 Horizontally unbounded regions

Definition 3.1 (Improper integrals - horizontally unbounded regions) *Let f be continuous on $[a, \infty)$. Define the* **improper integral** $\int_a^\infty f(x)\,dx$ *to be*

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

If this limit exists and is equal to a finite number L, we write $\int_a^\infty f(x)\,dx = L$ and say $\int_a^\infty f(x)\,dx$ **converges (to L)**. *If this limit does not exist or is equal to $\pm\infty$, we say $\int_a^\infty f(x)\,dx$* **diverges**.

A picture to explain the definition:



<u>Remark</u>: If $\int_a^\infty f(x)\,dx$ converges for a nonnegative function f (i.e. $f(x) \geq 0$ for all x), it must be the case that

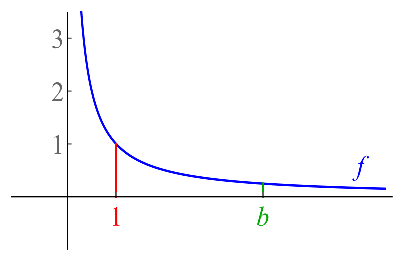
$$\lim_{x\to\infty} f(x) = 0.$$

That way, the region whose area you are considering gets narrower in the unbounded direction.

<u>HOWEVER</u>, the converse of this statement is false (see Example 1 below):

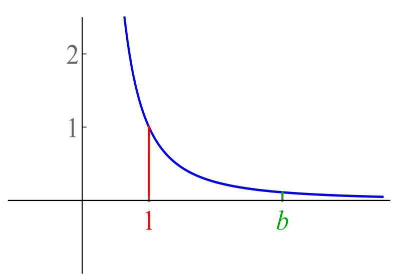
EXAMPLE 1

$$\int_1^\infty \frac{1}{x}\,dx$$



EXAMPLE 2

$$\int_1^\infty \frac{1}{x^2}\,dx.$$



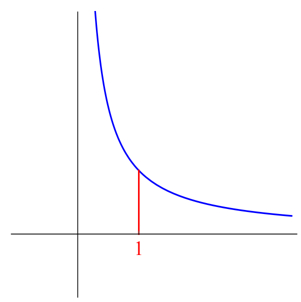
p-integrals

Let's generalize the results of Examples 1 and 2 above:

EXAMPLE 3

Let $p > 0$ be a constant. Compute the following improper integral:

$$\int_1^\infty \frac{1}{x^p}\,dx.$$



We already know:

from Example 1: When $p = 1$, $\int_1^\infty \frac{1}{x^1}\,dx = \int_1^\infty \frac{1}{x}\,dx$ diverges, meaning the area under the curve is infinite.

from Example 2: When $p = 2$, the integral $\int_1^\infty \frac{1}{x^2}\,dx$ converges to 1, meaning the area under the curve is 1.

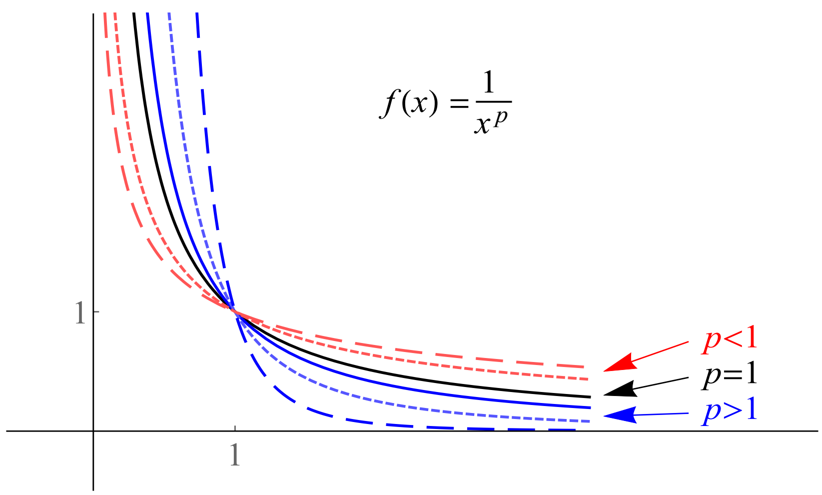
Solution for $p \neq 1$, $p > 0$:

The results of the previous examples are summarized in the following theorem, which should be memorized:

Theorem 3.2 (Convergence/divergence of p-integrals) *The improper integral*

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges to $\dfrac{1}{p-1}$ *if* $p > 1$*, but diverges if* $p \leq 1$.



EXAMPLE 4

Determine whether or not the following improper integral converges or diverges:

$$\int_e^\infty \frac{1}{x\ln^3 x}\,dx$$

In Example 4, we started our computation by rewriting the improper integral as a limit of a definite integral:

$$\int_e^\infty \frac{1}{x\ln^3 x}\,dx = \lim_{b\to\infty}\int_e^b \frac{1}{x\ln^3 x}\,dx.$$

In practice, we don't actually do this. We just leave the "∞" in the integral, and do the limit process in our head. This is technically "wrong", but it's OK because it shouldn't lead you to the wrong answer.

Solution of Example 4 (shorthand):

Let $u = \ln x$ so that $du = \frac{1}{x}\,dx$. Then, after this u-sub, the integral becomes

$$\begin{aligned}
\int_e^\infty \frac{1}{x\ln^3 x}\,dx &= \int_1^\infty \frac{1}{u^3}\,du \\
&= \left.\frac{-1}{2}u^{-2}\right|_1^\infty \\
&= \frac{-1}{2(\infty^2)} - \frac{-1}{2(1^2)} \\
&= \frac{-1}{\infty} + \frac{1}{2} \\
&= 0 + \frac{1}{2} \\
&= \boxed{\frac{1}{2}}.
\end{aligned}$$

One situation where you need to be careful is if you get an indeterminate form like $\frac{\infty}{\infty}$ in a computation like this. In this case, you need to write out the limit and evaluate it using L'Hôpital's Rule (see Example 5 on the next page).

EXAMPLE 5

Determine whether or not the following improper integral converges or diverges:

$$\int_1^\infty \frac{\ln x}{x^2}\,dx$$

Solution: To integrate this function, we use parts:

$$\begin{cases} \text{Let } r = \ln x \\ \text{Let } ds = \frac{1}{x^2}\,dx \end{cases} \implies \begin{cases} dr = \frac{1}{x}\,dx \\ s = \frac{-1}{x} \end{cases}$$

Therefore, by the parts formula, we get

$$\begin{aligned} \int_1^\infty \frac{\ln x}{x^2}\,dx = \int_1^\infty r\,ds = [rs]_1^\infty - \int_1^\infty s\,dr \\ &= \left[\frac{-\ln x}{x}\right]_1^\infty - \int_1^\infty \frac{-1}{x^2}\,dx \\ &= \left[\frac{-\ln \infty}{\infty}\right] - \left[\frac{-\ln 1}{1}\right] + \int_1^\infty \frac{1}{x^2}\,dx \\ &= \left[\frac{-\ln \infty}{\infty}\right] - 0 + \frac{1}{2-1} \qquad (p\text{-integral, } p = 2) \\ &= \left[\frac{-\ln \infty}{\infty}\right] + 1 \end{aligned}$$

So really, the expression above needs to be evaluated as

$$\begin{aligned} \lim_{b\to\infty} \frac{-\ln b}{b} + 1 &= \text{"}\frac{\infty}{\infty}\text{"} + 1 \\ &\overset{L}{=} \lim_{b\to\infty} \frac{-1/b}{1} + 1 \\ &= \frac{0}{1} + 1 = \boxed{1}. \end{aligned}$$

(In other words, the integral converges to $\boxed{1}$.)

3.2 Vertically unbounded regions

When the region under consideration is unbounded in a vertical direction, we need to be more careful in rewriting the improper integral as a limit. There are three situations:

Definition 3.3 (Improper integrals - vertically unbounded regions)

VA on right edge of region: *Let f be cts on $[a, c)$ where $\lim_{x\to c^-} f(x) = \pm\infty$. Then*

$$\int_a^c f(x)\,dx = \lim_{b\to c^-} \int_a^b f(x)\,dx.$$

VA on left edge of region: *Let f be cts on $(a, c]$ where $\lim_{x\to a^+} f(x) = \pm\infty$. Then*

$$\int_a^c f(x)\,dx = \lim_{b\to a^+} \int_b^c f(x)\,dx.$$

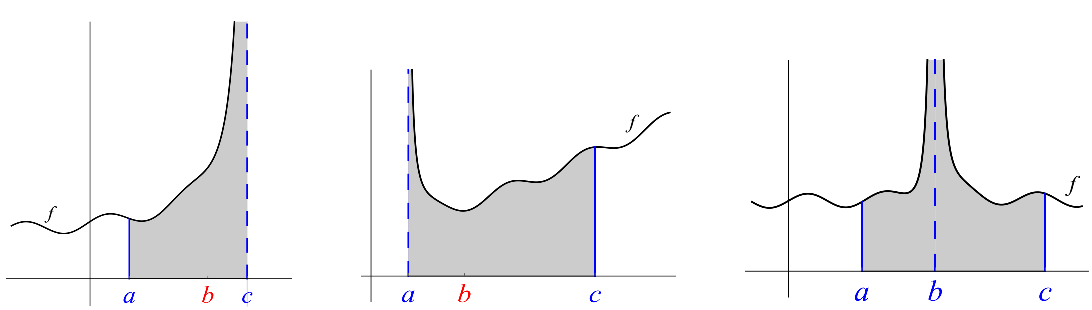
In either of these cases, if the limit exists and is equal to a finite number L, we write $\int_a^c f(x)\,dx = L$ and say the **improper integral** $\int_a^c f(x)\,dx$ **converges (to L)**. *If this limit does not exist or is equal to $\pm\infty$, we say* $\int_a^c f(x)\,dx$ **diverges**.

VA in middle of region: *Let f be cts on $[a, c]$ except at a single point $b \in (a, c)$, where $\lim_{x\to b^+} f(x) = \lim_{x\to b^-} f(x) = \pm\infty$. Then the* **improper integral** $\int_a^c f(x)\,dx$ *is said to* **converge** *only if* $\int_a^b f(x)\,dx$ *and* $\int_b^c f(x)\,dx$ *converge (in the sense of the definitions given above). In this case we set*

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

where the two integrals on the right side are defined as above.

Pictures to explain:



Key concept: if the VA is in the middle of the region of integration, you must consider the two regions above separately.

EXAMPLE 6

Determine whether the following improper integral converges or diverges:

$$\int_0^{\pi/2} \tan x \, dx$$

We've seen how to integrate $\tan x$ before; as a reminder, we rewrite the integrand as

$$\int_0^{\pi/2} \tan x \, dx = \int_0^{\pi/2} \frac{\sin x}{\cos x} \, dx = \int_0^{\pi/2} \sin x \cdot \frac{1}{\cos x} \, dx.$$

Now use the u-sub $u = \cos x \;\Rightarrow\; \begin{aligned} du &= -\sin x \, dx \\ -du &= \sin x \, dx \end{aligned}$ to get

$$\begin{aligned} \int_0^{\pi/2} \tan x \, dx = \int_0^{\pi/2} \frac{\sin x}{\cos x} \, dx &= \int_1^0 -\frac{1}{u} \, du \\ &= \int_0^1 \frac{1}{u} \, du \\ &= \ln u \Big]_0^1 \\ &= \ln 1 - \ln 0 \\ &= 0 - (\qquad\qquad) \end{aligned}$$

EXAMPLE 7

Determine whether the following improper integral converges or diverges:

$$\int_0^\infty \frac{1}{\sqrt{x}(x+1)}\,dx$$

After this u-sub, the first integral becomes

$$\int_0^1 2\cdot\frac{1}{u^2+1}\,du = 2\arctan u\big|_0^1 = 2\left(\frac{\pi}{4}\right) - 2(0) = \frac{\pi}{2}.$$

For the integral $\int_1^\infty \frac{1}{\sqrt{x}(x+1)}\,dx$, use the same u-sub:

$$\begin{aligned}\int_1^\infty \frac{1}{\sqrt{x}(x+1)}\,dx = \int_1^\infty \frac{2}{u^2+1}\,du &= 2\arctan u\big|_1^\infty \\ &= 2\arctan\infty - 2\arctan 1 \\ &= 2\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}.\end{aligned}$$

Last, since $\int_0^1 \frac{1}{\sqrt{x}(x+1)}\,dx$ and $\int_1^\infty \frac{1}{\sqrt{x}(x+1)}\,dx$ both converge,

$$\int_0^\infty \frac{1}{\sqrt{x}(x+1)}\,dx = \int_0^1 \frac{1}{\sqrt{x}(x+1)}\,dx + \int_1^\infty \frac{1}{\sqrt{x}(x+1)}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}.$$

3.3 Theoretical approaches

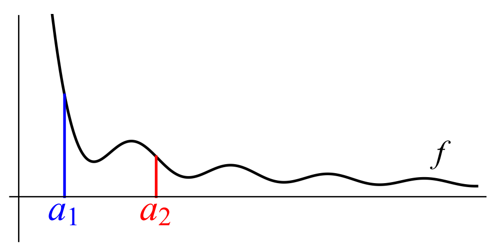
Frequently in mathematics we encounter quantities (such as improper integrals) where it is just as important (if not moreso) to determine **whether or not the quantity converges** than it is to determine **what number the quantity converges to**. It is useful to know some "tricks" based on theoretical concepts which help tell us whether or not an improper integral converges.

The first "trick" is that when integating over a horizontally unbounded region, the starting index of the integral doesn't affect whether or not the integral converges.

Theorem 3.4 (Starting index is irrelevant to convergence/divergence) *Suppose $a_1 < a_2$ and f is continuous on $[a_1, a_2]$. Then:*

1. *if $\int_{a_1}^{\infty} f(x)\,dx$ converges, so does $\int_{a_2}^{\infty} f(x)\,dx$;*
2. *if $\int_{a_1}^{\infty} f(x)\,dx$ diverges, so does $\int_{a_2}^{\infty} f(x)\,dx$.*

A picture to explain:



WARNING: Theorem 3.4 does <u>not</u> say that

$$\int_{a_1}^{\infty} f(x)\,dx = \int_{a_2}^{\infty} f(x)\,dx.$$

If f is positive and $a_1 < a_2$, then (assuming these integrals converge) it is clear that

$$\int_{a_1}^{\infty} f(x)\,dx > \int_{a_2}^{\infty} f(x)\,dx$$

because you are integrating a positive function over a larger region.

The next set of results describe how one can "split up" improper integrals whose integrands are composed of terms added/subtracted together:

Theorem 3.5 (Linearity of improper integrals I) *Suppose* $\int_a^\infty f(x)\,dx = L$ *and* $\int_a^\infty g(x)\,dx = M$. *Then:*

1. $\int_a^\infty [f(x) + g(x)]\,dx = L + M$;
2. $\int_a^\infty [f(x) - g(x)]\,dx = L - M$;
3. $\int_a^\infty kf(x)\,dx = kL$ *for any constant* k.

Theorem 3.6 (Linearity of improper integrals II) *Suppose* $\int_a^\infty f(x)\,dx = L$ *but* $\int_a^\infty g(x)\,dx$ *diverges. Then:*

1. $\int_a^\infty [f(x) + g(x)]\,dx$ *diverges;*
2. $\int_a^\infty [f(x) - g(x)]\,dx$ *diverges;*
3. $\int_a^\infty kg(x)\,dx$ *diverges, for any constant* $k \neq 0$.

Theorem 3.7 (Linearity of improper integrals III) *Suppose that both* $\int_a^\infty f(x)\,dx$ *and* $\int_a^\infty g(x)\,dx$ *diverge. Then* **you know nothing** *about whether or not the improper integrals* $\int_a^\infty [f(x) + g(x)]\,dx$ *and* $\int_a^\infty [f(x) - g(x)]\,dx$ *converge or diverge.*

The preceding three theorems can be summarized in the following three "morals", which we will see again and which hold throughout mathematics:

1.

2.

3.

EXAMPLE 8

Determine whether the following improper integrals converge or diverge:

a) $\int_1^\infty \left(\frac{3}{\sqrt{x}} + \frac{2}{x^3} \right) dx$

b) $\int_6^\infty \left(\frac{1}{5x^2} - \frac{3}{x^{3/2}} \right) dx$

Our last theoretical result is based on a very important and very general mathematical idea of reasoning by way of inequalities:

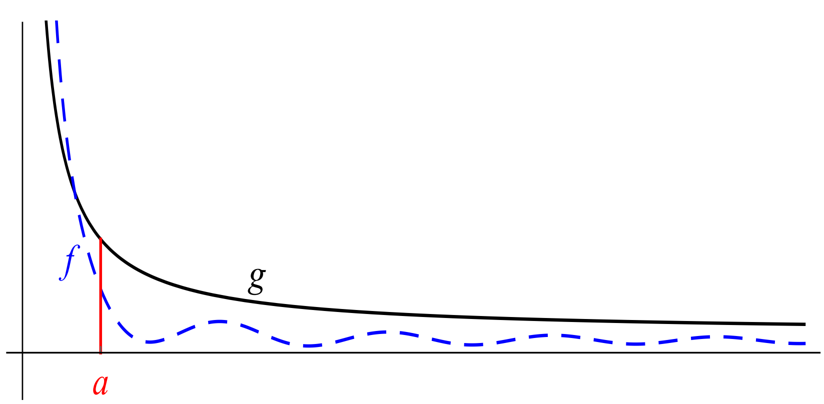
Theorem 3.8 (Comparison Test for Improper Integrals) *Suppose that*

$$0 \leq f(x) \leq g(x)$$

for all $x \geq a$. Then:

1. *if $\int_a^\infty g(x)\,dx$ converges, so does $\int_a^\infty f(x)\,dx$;*
2. *if $\int_a^\infty f(x)\,dx$ diverges, so does $\int_a^\infty g(x)\,dx$.*

A picture to explain:



EXAMPLE 9

Determine whether the following improper integral converges or diverges:

$$\int_1^\infty \frac{3}{x^5 + 2x + 2}\,dx$$

EXAMPLE 10

Determine whether the following improper integral converges or diverges:

$$\int_{12}^\infty \frac{2}{\sqrt{x-7}}\,dx$$

EXAMPLE 11

Determine whether the following improper integral converges or diverges:

$$\int_2^\infty \frac{\sin x^2 + 3}{x^8}\,dx$$

General principles to help construct inequalities

1. **Addition in the denominator:** suppose the integrand is of the form

$$\frac{\Box}{\triangle + \star}$$

where $\Box$, $\triangle$ and $\star$ **are all positive** quantities. Then, we can start with one of these two inequalities

$$\frac{\Box}{\triangle + \star} \leq \frac{\Box}{\triangle} \quad \text{or} \quad \frac{\Box}{\triangle + \star} \leq \frac{\Box}{\star}$$

and try to apply the Comparison Test.

REASON: removing positive terms from the denominator makes the denominator smaller, which makes the entire fraction bigger.

2. **Subtraction in the denominator:** suppose the integrand is of the form

$$\frac{\Box}{\triangle - \star}$$

where $\Box$, $\triangle$ and $\star$ **are all positive** quantities. Then, one can start with this inequality:

$$\frac{\Box}{\triangle - \star} \geq \frac{\Box}{\triangle}$$

and try to apply the Comparison Test.

REASON: removing negative terms from the denominator makes the denominator bigger, which makes the entire fraction smaller.

3. **Terms containing sines or cosines:** suppose the integrand contains some expression of the form $\cos\Box$ or $\sin\Box$ where $\Box$ is some expression. Then, one can start with the inequality

$$-1 \leq \cos\Box \leq 1 \quad (\text{ or } -1 \leq \sin\Box \leq 1)$$

and try to apply the Comparison Test.

Be careful with the logic!

Think of $\displaystyle\int_a^\infty f(x)\,dx$ as the "small integral" and $\displaystyle\int_a^\infty g(x)\,dx$ as the "big integral":

What you can do with the Comparison Test:

- You can conclude that the small integral converges.
- You can conclude that the big integral diverges.

What you cannot do with the Comparison Test:

- The Comparison Test never allows you to conclude that the big integral converges.
- The Comparison Test never allows you to conclude that the small integral diverges.
- The Comparison Test never tells you the value of the small integral, even if it tells you that the small integral converges.

We use the Comparison Test with improper integrals that are similar to a "simpler" integral (often, but not always, a $p-$integral). In general, the reasoning is as follows:

	the given integral $\leq$ the "simpler" integral	the "simpler" integral $\leq$ the given integral
the "simpler" integral converges	**Conclusion:** By the Comparison Test, the given integral **converges**.	No conclusion can be drawn from the Comparison Test
the "simpler" integral diverges	No conclusion can be drawn from the Comparison Test	**Conclusion:** By the Comparison Test, the given integral **diverges**.

3.4 Gamma integrals

Actuarial scientists sometimes encounter integrals of the form

$$\int_0^\infty x^r e^{-\lambda x}\,dx,$$

where $r \geq 0$ and $\lambda > 0$ are constants. These integrals are called **gamma integrals** and they help describe, among other things, the expected amount of time that should pass before a policyholder files a certain number of claims.

Definition 3.9 *For $r \geq 0$ and $\lambda > 0$, let*

$$\gamma(r, \lambda) = \int_0^\infty x^r e^{-\lambda x}\,dx.$$

(γ is the Greek letter "gamma".)

In this section, we'll show that these gamma integrals converge, and find a formula for what they converge to when r is a non-negative integer (i.e. $r \in \{0, 1, 2, 3, ...\}$). Here's the strategy for deriving this formula:

1. We'll compute $\gamma(0, 1)$.
2. We'll develop a formula for $\gamma(r + 1, 1)$ in terms of $\gamma(r, 1)$.
3. We'll use this to find a general formula for all $\gamma(r, 1)$.
4. We'll figure out what $\gamma(r, \lambda)$ is.

Remark: In MATH 414, we discuss what happens if r isn't a non-negative integer ($r = \frac{2}{3}$, $r = \pi$, etc.). But that's beyond the scope of this class.

Step 1: compute $\gamma(0, 1)$

$$\begin{aligned}\gamma(0, 1) = \int_0^\infty x^0 e^{-1x}\,dx = \int_0^\infty e^{-x}\,dx \\ &= -e^{-x}\Big|_0^\infty \\ &= -e^{-\infty} - \left(-e^{-0}\right) \\ &= \end{aligned}$$

Step 2: find formula for $\gamma(r+1,1)$ in terms of $\gamma(r,1)$

$$\gamma(r+1,1) = \int_0^\infty x^{r+1} e^{-1x}\,dx$$

Step 3: use formula from Step 2 to find general formula for $\gamma(r,1)$

From Step 1, we know $\gamma(0,1) = 1$.

From Step 2, we know $\gamma(r+1,1) = (r+1)\,\gamma(r,1)$. In particular, this means

$$\begin{aligned}
\gamma(1,1) &= \\
\gamma(2,1) &= \\
\gamma(3,1) &= \\
\gamma(4,1) &= \\
&\vdots \\
\gamma(r,1) &=
\end{aligned}$$

We have shown:

Theorem 3.10 *For any non-negative integer r,*

$$\gamma(r,1) = \int_0^\infty x^r e^{-x}\,dx = r!.$$

Step 4: find general formula for $\gamma(r, \lambda)$

$$\gamma(r, \lambda) = \int_0^\infty x^r e^{-\lambda x}\, dx$$

We have shown the following formula (very useful in MATH 414):

Theorem 3.11 (Gamma Integral Formula) *Let $r \geq 0$ be an integer and let $\lambda > 0$. Then*

$$\int_0^\infty x^r e^{-\lambda x}\, dx = \frac{r!}{\lambda^{r+1}}.$$

EXAMPLE 12

Compute each improper integral:

a) $\displaystyle\int_0^\infty 5x^4 e^{-3x}\, dx$

b) $\displaystyle\int_0^\infty x^6 e^{-x/2}\, dx$

c) $\displaystyle\int_0^\infty (wx)^3 e^{-2cx}\, dx$

3.5 Homework exercises

Exercises from Section 3.1

In Exercises 1-6, evaluate the improper integral (if the integral diverges, say so).

1. $\displaystyle\int_1^\infty x^{-4}\,dx$
2. $\displaystyle\int_8^\infty \frac{1}{\sqrt[3]{x}}\,dx$
3. $\displaystyle\int_0^\infty xe^{-x}\,dx$
4. $\displaystyle\int_1^\infty \frac{2}{x^2+1}\,dx$
5. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$
6. $\displaystyle\int_3^\infty \frac{2}{(x+3)^{5/3}}\,dx$
7. In class we showed that $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges. Have *Mathematica* evaluate this integral. Does *Mathematica* recognize that this integral diverges?

Exercises from Section 3.2

In each of Exercises 8-15, write the indicated integral as a limit or as a sum of limits. Here are two examples:

Example A: $\displaystyle\int_2^\infty e^{-x}\,dx$ *Solution:* $\displaystyle\lim_{b\to\infty}\int_2^b e^{-x}\,dx$

Example B: $\displaystyle\int_0^4 \frac{2}{x-1}\,dx$ *Solution:* $\displaystyle\lim_{b\to 1^-}\int_0^b \frac{2}{x-1}\,dx + \lim_{B\to 1^+}\int_B^4 \frac{2}{x-1}\,dx$

Then, sketch a picture (similar to those on pages 78 and 84) which reflects how the limit is being used to evaluate the improper integral.

8. $\displaystyle\int_1^2 \frac{1}{x\ln x}\,dx$
9. $\displaystyle\int_0^1 \frac{x}{x^2-1}\,dx$
10. $\displaystyle\int_4^\infty \frac{3}{e^x+x^2}\,dx$
11. $\displaystyle\int_0^\infty \frac{1}{xe^x}\,dx$
12. $\displaystyle\int_3^5 \frac{1}{(x-5)^2}\,dx$
13. $\displaystyle\int_{\pi/2}^\pi \sec x\,dx$
14. $\displaystyle\int_0^\infty \frac{1}{e^x-1}\,dx$
15. $\displaystyle\int_1^5 \frac{1}{(x-3)^2}\,dx$

In Exercises 16-21, evaluate the improper integral (if the integral diverges, say so).

16. $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$
17. $\displaystyle\int_0^1 x\ln x\,dx$

18. $\displaystyle\int_0^e \ln x^2 \, dx$

19. $\displaystyle\int_2^4 \frac{1}{\sqrt{4-x}} \, dx$

20. $\displaystyle\int_1^\infty \frac{1}{x \ln x} \, dx$

21. $\displaystyle\int_1^3 \frac{1}{x^2-1} \, dx$

Exercises from Section 3.3

In Exercises 22-31, determine whether the improper integral converges or diverges (you do not necessarily need to evaluate the integral if it converges).

Note: the intent here is for you to use theory, not to perform lots of computations.

22. $\displaystyle\int_1^\infty \frac{5}{x^8} \, dx$

23. $\displaystyle\int_6^\infty \frac{2}{x^4+1} \, dx$

24. $\displaystyle\int_3^\infty \frac{1}{\sqrt{x-1}} \, dx$

25. $\displaystyle\int_2^\infty 4x^{-3/2} \, dx$

26. $\displaystyle\int_1^\infty \frac{1}{e^x + x} \, dx$

27. $\displaystyle\int_1^\infty \frac{\cos(\pi x) + 3}{x^4} \, dx$

28. $\displaystyle\int_1^\infty \left(\frac{3}{x} - \frac{2}{x^2}\right) \, dx$

29. $\displaystyle\int_2^\infty \left(\frac{2}{x^3} + \frac{5}{x^4}\right) \, dx$

30. $\displaystyle\int_3^\infty \frac{\sin(2x+1) + 4}{\sqrt{x}} \, dx$

31. $\displaystyle\int_8^\infty \frac{1}{2x-1} \, dx$

Exercises from Section 3.4

In Exercises 32-37, compute the improper integral.

32. $\displaystyle\int_0^\infty x^9 e^{-x} \, dx$

33. $\displaystyle\int_0^\infty 4x^5 e^{-x/3} \, dx$

34. $\displaystyle\int_0^\infty (2x)^3 e^{1-4x} \, dx$

35. $\displaystyle\int_0^\infty y(wx)^4 e^{-6x} \, dx$

36. $\displaystyle\int_0^\infty w^5 x^8 e^{-wx-w} \, dx$

37. $\displaystyle\int_4^\infty (x-4)^3 e^{-2x} \, dx$

Answers

1. $\frac{1}{3}$
2. diverges
3. 1
4. $\pi/2$
5. diverges
6. $\frac{3^{1/3}}{2^{2/3}}$
7. Yes, *Mathematica* recognizes that this integral diverges.
8. $\displaystyle\int_1^2 \frac{1}{x\ln x}\,dx = \lim_{b\to 1^+}\int_b^2 \frac{1}{x\ln x}\,dx$
   (Picture looks like the left one on page 84, with the VA at $x = 1$)
9. $\displaystyle\int_0^1 \frac{x}{x^2-1}\,dx = \lim_{b\to 1^-}\int_0^b \frac{x}{x^2-1}\,dx$
   (Picture looks like the left one on page 84, with the VA at $x = 1$)
10. $\displaystyle\int_4^\infty \frac{3}{e^x + x^2}\,dx = \lim_{b\to\infty}\int_4^b \frac{3}{e^x + x^2}\,dx$
    (Picture looks like the one on page 78, with the HA at $y = 0$)
11. $\displaystyle\int_0^\infty \frac{1}{xe^x}\,dx = \lim_{b\to 0^+}\int_b^1 \frac{1}{xe^x}\,dx + \lim_{B\to\infty}\int_1^B \frac{1}{xe^x}\,dx$
    (Picture looks like the one in Example 7, with the VA at $x = 0$ and the HA at $y = 0$)
12. $\displaystyle\int_3^5 \frac{1}{(x-5)^2}\,dx = \lim_{b\to 5^-}\int_3^b \frac{1}{(x-5)^2}\,dx$
    (Picture looks like the left one on page 84, with the VA at $x = 5$)
13. $\displaystyle\int_{\pi/2}^\pi \sec x\,dx = \lim_{b\to \frac{\pi}{2}^+}\int_b^\pi \sec x\,dx$
    (Picture looks like the middle one on page 84, with the VA at $x = \frac{\pi}{2}$)
14. $\displaystyle\int_0^\infty \frac{1}{e^x - 1}\,dx = \lim_{b\to 0^+}\int_b^1 \frac{1}{e^x-1}\,dx + \lim_{B\to\infty}\int_1^B \frac{1}{e^x-1}\,dx$
    (Picture looks like the one in Example 7, with the VA at $x = 0$ and the HA at $y = 0$)
15. $\displaystyle\int_1^5 \frac{1}{(x-3)^2}\,dx = \lim_{b\to 3^-}\int_1^b \frac{1}{(x-3)^2} + \lim_{B\to 3^+}\int_B^5 \frac{1}{(x-3)^2}\,dx$
    (Picture looks like the right one on page 84, with the VA at $x = 3$)

16. diverges
17. $\frac{-1}{4}$
18. 0
19. $2\sqrt{2}$
20. diverges
21. diverges
22. converges
23. converges
24. diverges
25. converges
26. converges
27. converges
28. diverges
29. converges
30. diverges
31. diverges
32. $9!$
33. $4 \cdot 3^6 \cdot 5!$
34. $8e \cdot \frac{4!}{4^4} = \frac{3}{4}e$
35. $yw^4 \cdot \frac{4!}{6^5}$
36. $w^5 e^{-w} \cdot \frac{8!}{w^9}$
37. $e^8 \cdot \frac{3!}{2^4} = \frac{3}{8}e^{-8}$

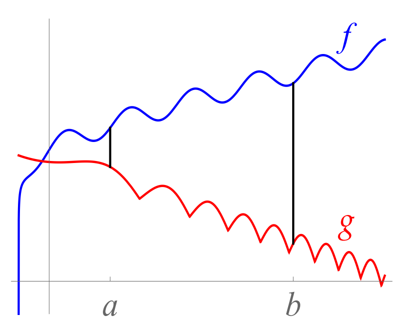
Chapter 4

Applications of integrals

4.1 Area

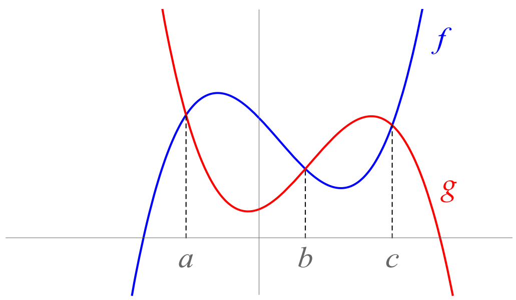
Definite integrals were invented (in part) for the purpose of finding areas of regions. However, there is a slight difference between computing an integral and finding an area, because integrals can be negative (whereas areas are never negative). Formally speaking, integrals compute **signed areas**.

For example, suppose we are asked to compute the area between the graphs of f and g between $x = a$ and $x = b$ where f and g are as in the following picture:



This area is

Here is another example; the problem is to compute the total area enclosed by the graphs of f and g:



EXAMPLE 1

Compute the area between the graphs of $f(x) = x^2 + 2$ and $g(x) = -x - 3$ on $[-2, 3]$.

Mathematica code that will solve Example 1:

In: `f[x_] = x^2 + 2`

In: `g[x_] = -x - 3`

In: `Integrate[Abs[f[x]-g[x]], {x,-2,3}]` $\longleftarrow$ for exact answer

Out: $\frac{235}{6}$

In: `NIntegrate[Abs[f[x]-g[x]], {x,-2,3}]` $\longleftarrow$ for decimal approx.

Out: `39.1667`

EXAMPLE 2

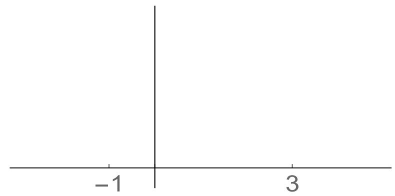
Compute the area of the region enclosed by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.

EXAMPLE 3

Compute the area of the region enclosed by the graphs of $f(x) = x^3 + x^2 - 6x$ and $g(x) = -x^3 + 5x^2$.

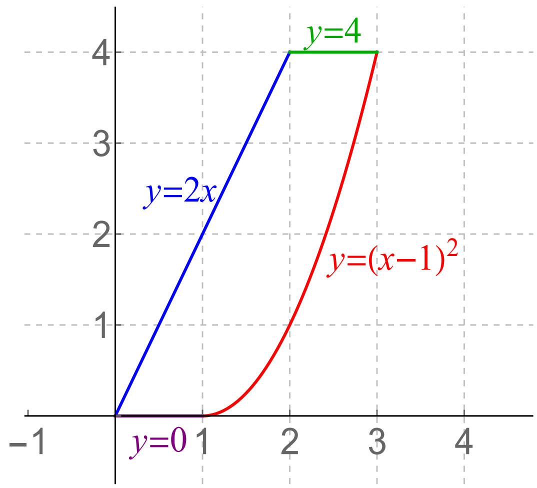
Solution: Start by finding the intersection points of the graphs:

$$\begin{aligned} x^3 + x^2 - 6x &= -x^3 + 5x^2 \\ 2x^3 - 4x^2 - 6x &= 0 \\ 2x(x^2 - 2x - 3) &= 0 \\ 2x(x+1)(x-3) &= 0 \\ x = 0, x = -1, x = 3 & \end{aligned}$$



EXAMPLE 4

Find the area of the region which is above the $x-$axis, below $y = 4$, to the left of $y = (x-1)^2$ and to the right of $y = 2x$. (The region is pictured below.)



$$Area = \int_0^3 (top - bottom)\, dx =$$

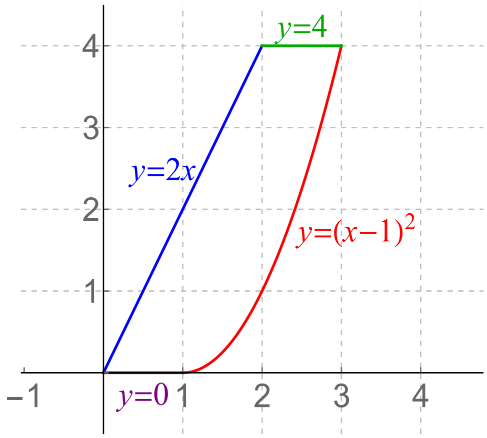
$$=$$

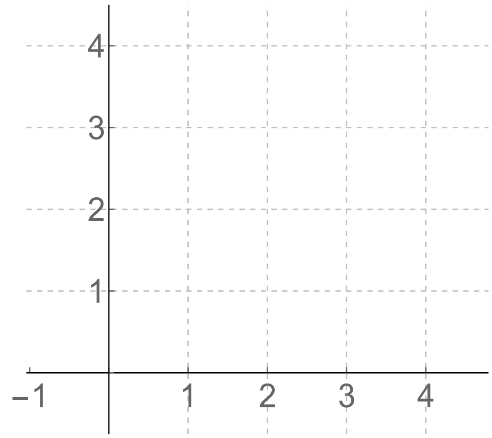
$$= 1 + \left[4 - \frac{1}{3}\right] - [1 - 0] + \left[12 - \frac{8}{3}\right] - \left[8 - \frac{1}{3}\right]$$

$$= 1 + \frac{8}{3} + \frac{5}{3}$$

$$= \boxed{\frac{16}{3}}.$$

EXAMPLE 4, REPEATED





EXAMPLE 5

Write an integral with respect to the variable x which gives the area enclosed by the graphs of $y = \sqrt{x}$ and $y = \frac{1}{6}x$. Write another integral with respect to y that gives the same area.

Solution: Start by finding the intersection points of the graphs:

$$\begin{aligned} \sqrt{x} &= \tfrac{1}{6}x \\ 6\sqrt{x} &= x \\ (6\sqrt{x})^2 &= x^2 \\ 36x &= x^2 \\ 0 &= x^2 - 36x \\ 0 &= x(x-36) \\ &\Rightarrow x = 0, x = 36 \end{aligned}$$

Integration with respect to x

Integration with respect to y

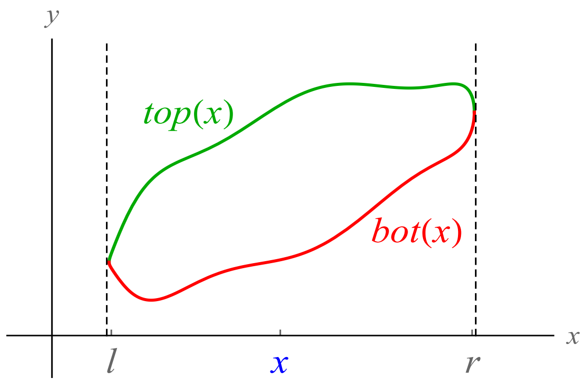
Let's check that these integrals work out to the same thing:

$$\begin{aligned} \int_0^{36} \left(\sqrt{x} - \frac{x}{6}\right) dx &= \left[\tfrac{2}{3}x^{3/2} - \tfrac{x^2}{12}\right]_0^{36} \\ &= \tfrac{2}{3}(6^3) - \tfrac{36^2}{12} \\ &= 144 - 108 \\ &= \boxed{36} \end{aligned}$$

$$\begin{aligned} \int_0^6 (6y - y^2)\, dy &= \left[3y^2 - \tfrac{1}{3}y^3\right]_0^6 \\ &= 3(36) - \tfrac{1}{3}(6^3) \\ &= 108 - 72 \\ &= \boxed{36}. \end{aligned}$$

4.2 Volume

In the previous section we talked about **area**, which is a 2-dimensional problem:



A way to think about the computation of these areas is as follows:

1. Let x run from l to r (from the left edge to the right edge).

   We think of this as choosing the "x-direction" as the "direction of integration".

   (You can also choose the y-direction, but we won't review that on this page.)

2. At each x, think of an infinitely narrow rectangle. These rectangles are called **cross-sections** for the region.

   Note: the cross-sections are perpendicular to the direction of integration.

3. Find the length of this rectangle (call this length $L(x)$).

   - the length depends on x
   - the length is usually $top(x) - bot(x)$, where top (a.k.a. t) is the "top function" and bot (a.k.a. b) is the "bottom function"

4. The rectangles' width is dx, so the area of the narrow rectangle is

$$\text{(length)(width)} =$$

5. "Add up" the areas of these rectangles to get the area of the region:

$$\begin{aligned} A &= \int_l^r L(x)\,dx \\ &= \int_l^r [top(x) - bot(x)]\ dx \\ &= \int_l^r [t(x) - b(x)]\ dx. \end{aligned}$$

We can modify the reasoning on the previous page to compute **volumes** of 3-dimensional solids:

Think of "slicing a potato into potato chips".

1. Let x run from l (the left edge of the solid) to r (the right edge of the solid), i.e. choose a direction of integration.

   (You can also choose the y-direction; more on that later.)

2. At each x, think of an infinitely thin plane figure perpendicular to the $x-$axis. These shapes are called **cross-sections** of the solid.

   Note: the cross-sections are perpendicular to the direction of integration.

3. Find the area of this shape (call this area $A(x)$).

   - the area depends on x
   - use geometry formulas (area of circle, square, etc.)
   - you could have to use an integral to find $A(x)$ (especially in MATH 320)

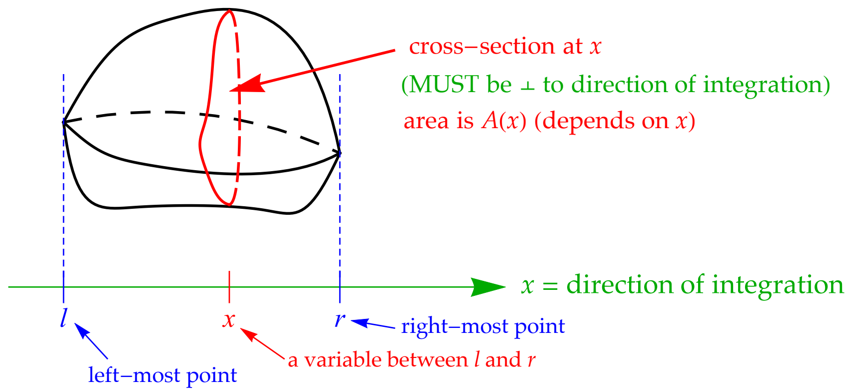
4. The thickness of the cross-sections is dx, so the volume of each thin cross-section is

   $$(\text{area})(\text{thickness}) =$$

5. "Add up" the volumes of these cross-sections to get the volume of the solid:

   $$V = \int_l^r A(x)\,dx.$$

<u>Generic volume formula</u>: If $A(x)$ = area of the cross-section at x, and the solid runs from $x = l$ to $x = r$, then the volume of the solid is



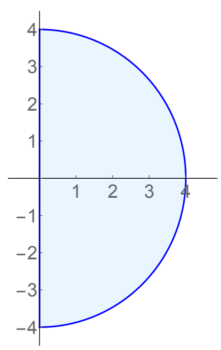
$$V = \int_l^r A(x)\, dx.$$

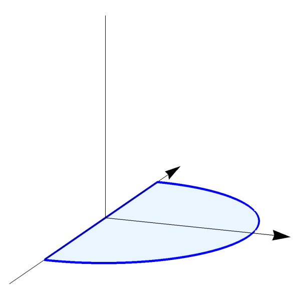
EXAMPLE 6

Consider a solid whose base consists of the region between the y-axis and the right-half of the circle $x^2 + y^2 = 16$. Cross-sections of the solid parallel to the y-axis are squares with bases in the xy-plane. Find the volume of the solid.

Solution: To get started, let's draw some pictures:

<u>BASE</u> (top view) <u>3-D PICTURE</u> <u>CROSS-SECTION</u> (at x)





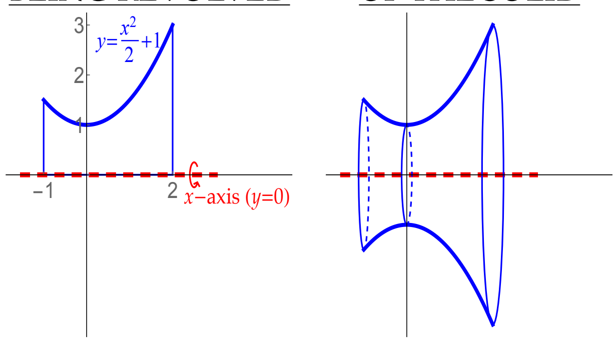
We especially want to compute the volume of solids that are obtained by revolving a two-dimensional region around some axis. We will discuss three methods to do this: the disc method, the washer method, and the shell method.

Volume of solids of revolution: the disc method

EXAMPLE 7

The region R between the graph of $f(x) = \frac{1}{2}x^2 + 1$ and the $x-$axis on $[-1, 2]$ is revolved around the $x-$axis to produce a solid. Find the volume of the solid.

REGION BEING REVOLVED | 3-D PICTURE OF THE SOLID | CROSS-SECTION



$$V = \int_l^r A(x)\,dx =$$

$$= \pi\left[\frac{x^5}{20} + \frac{x^3}{3} - x\right]_{-1}^{2}$$

$$= \pi\left[\frac{32}{20} + \frac{8}{3} - 2\right] - \pi\left[\frac{-1}{20} + \frac{-1}{3} - (-1)\right] = \boxed{\frac{153}{20}\pi}.$$

The disc method

If you take the region between the graph of f and the $x-$axis between $x = l$ and $x = r$ and revolve that region around the $x-$axis to produce a solid, the volume of the solid generated is

$$V = \int_l^r \pi[f(x)]^2\,dx.$$

This formula is called the **disc method** because cross-sections are circles/disks.

Volume of solids of revolution: the washer method

EXAMPLE 8

Write an integral which, when computed, would give the volume of the solid obtained by revolving the region enclosed by the graphs of $f(x) = x^2$ and $g(x) = x$ about the line $y = 3$.

The washer method

If a region is revolved around a line to produce a solid with a hole (like a donut), the cross-sections are washers and the volume is:

$$V = \int_l^r \left(\pi R^2 - \pi r^2\right)\, dx.$$

This formula is called the **washer method** because cross-sections are washer-shaped.

You can do $y-$integration with volume as well:

EXAMPLE 9

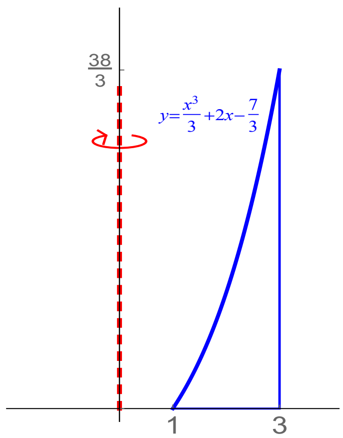
The region between $y = \sqrt{x}$ and $y = \frac{1}{4}x$ is revolved around the $y-$axis to produce a solid. Find its volume.

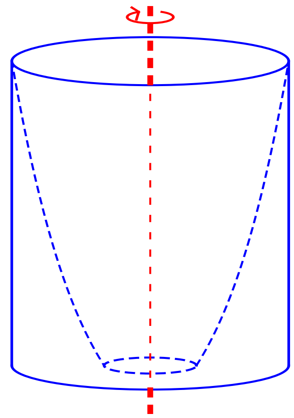
This integral evaluates as

$$V = \int_0^4 \left[16\pi y^2 - \pi y^4\right] \, dy = \left[\frac{16}{3}\pi y^3 - \frac{\pi}{5}y^5\right]_0^4 = \boxed{\frac{2048}{15}\pi}.$$

EXAMPLE 10

A solid is formed by revolving the region enclosed by the graphs of $y = \frac{1}{3}x^3 + 2x - \frac{7}{3}$, $x = 3$ and the $x-$axis about the $y-$axis. Find its volume.



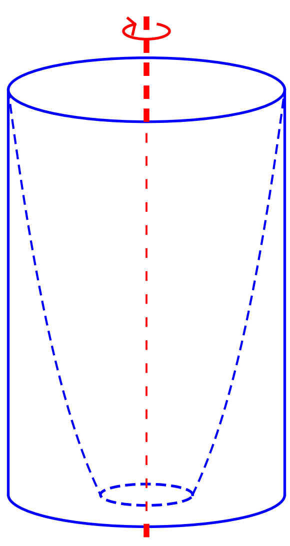


First attempt at at solution: use washer method, integrate with respect to y.

Problem with this method:

We need a new method, which is called the **shell method** or **cookie-cutter method** or **toilet paper roll method**.

We will integrate "inside-out" with respect to x, and take cylindrical cross-sections.



EXAMPLE 11

The region enclosed by $y = \sqrt{x}$, $y = 2$ and $x = 0$ is revolved around the line $y = -1$ to produce a solid. Compute the volume of this solid.

These integrals work out to be the same thing:

$$V = \int_0^4 \left[\pi \cdot 3^2 - \pi(\sqrt{x}+1)^2\right]\,dx = \int_0^2 2\pi(y+1)y^2\,dy = \boxed{\frac{40}{3}\pi}.$$

The various methods of finding volumes of solids formed by revolution of a region about a horizontal or vertical line are summarized in the chart on the next page:

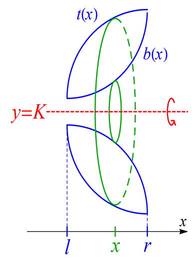
	SOLID COMES FROM REVOLUTION ABOUT A HORIZONTAL LINE $y = K$	SOLID COMES FROM REVOLUTION ABOUT A VERTICAL LINE $x = K$

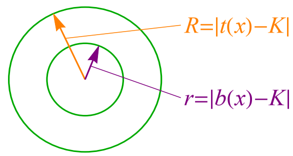
INTEGRAL W/ RESPECT TO x (Solve eqns for y in terms of x)

Revolution about a horizontal line $y = K$:

disk/washer method

cross-section at x:



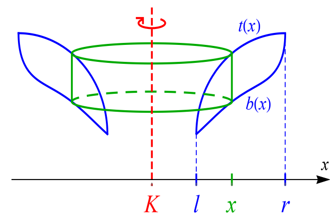


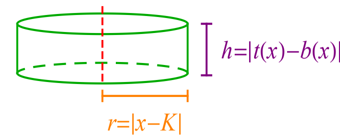
$$V = \int_l^r \pi \left[R^2 - r^2\right]\, dx = \boxed{\int_l^r \pi \left[(t(x)-K)^2 - (b(x)-K)^2\right]\, dx}$$

Revolution about a vertical line $x = K$:

shell method

cross-section at x:





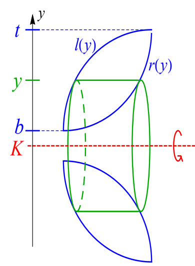
$$V = \int_l^r 2\pi r h\, dx = \boxed{\int_l^r 2\pi\, |x-K|\, |t(x)-b(x)|\, dx}$$

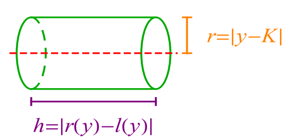
INTEGRAL W/ RESPECT TO y (Solve eqns for x in terms of y)

Revolution about a horizontal line $y = K$:

shell method

cross-section at y:



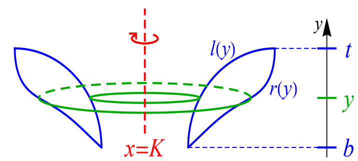


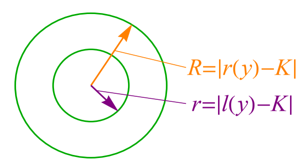
$$V = \int_b^t 2\pi r h\, dy = \boxed{\int_b^t 2\pi\, |y-K|\, |r(y)-l(y)|\, dy}$$

Revolution about a vertical line $x = K$:

disk/washer method

cross-section at y:





$$V = \int_b^t \pi \left[R^2 - r^2\right]\, dy = \boxed{\int_b^t \pi \left[(r(y)-K)^2 - (l(y)-K)^2\right]\, dy}$$

4.3 General principles behind all applications of integration

Before turning to our next application of integration, something more general. All applications of integration are based on the same principle:

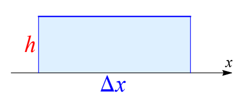
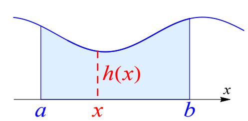
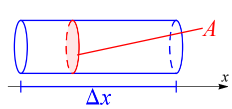
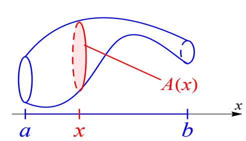
Theorem 4.1 (General principle of application of integration) *Suppose that A and Q are two quantities such that if quantity Q is constant, then A is obtained from Q by multiplying Q by the change in another quantity x, i.e.*

$$A = Q \cdot \Delta x \text{ when } Q \text{ is constant.}$$

Then, when Q becomes a quantity that depends on x, we have

$$A = \int_a^b Q(x)\,dx.$$

Here are some examples, most of which we have seen before, where this general principle is at work:

A	Q	formula when Q is constant	integral formula
displacement d	velocity v	$d = v \cdot \Delta t$ ($t =$ time)	$d = \int_a^b v(t)\,dt$
area A	cross-sectional height h	$A = h \cdot \Delta x$ ($x =$ length) 	$A = \int_a^b h(x)\,dx$ 
volume V	cross-sectional area A	$V = A \cdot \Delta x$ ($x =$ length) 	$V = \int_a^b A(x)\,dx$ 
work W (energy)	force F	$W = F \cdot \Delta x$ ($x =$ distance)	$W = \int_a^b F(x)\,dx$
charge q	current I	$q = I \cdot \Delta t$ ($t =$ time)	$q = \int_a^b I(t)\,dt$

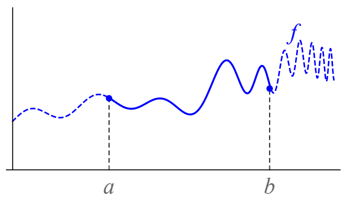
In the next few sections, we will use this general principle to develop formulas which give more applications of integration.

4.4 Arc length

<u>Goal</u>: Find the length of curves which are pieces of the graph of a function $y = f(x)$.

We denote these lengths by the letter s, perhaps because the Latin word for distance is *spatium*.

Question: What exactly does one mean by **length**?



So phrased more precisely, our problem is to determine the length s of the piece of the graph of $y = f(x)$ between $x = a$ and $x = b$.

Solution: Let's suppose first that the slope of $y = f(x)$ is constant (so that the slope is going to be like the Q of the chart on the previous page). That means we are assuming $f(x)$ is...

Thus

$$\begin{aligned} s &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \sqrt{(\Delta x)^2 + (m\Delta x)^2} \\ &= \sqrt{(\Delta x)^2(1 + m^2)} \\ &= \sqrt{1 + m^2}\,\Delta x. \end{aligned}$$

On the previous page, we found that when the slope m is constant, the arc length from $x = a$ to $x = b$ is

$$s = \sqrt{1 + m^2}\,\Delta x.$$

That means, by applying the general principle of integration applications from the previous section, that if the slope of $y = f(x)$ is <u>nonconstant</u> (and depends on x), the arc length from $x = a$ to $x = b$ is

$$s =$$

We have derived the following theorem:

Theorem 4.2 (Arc length formula) *Let f be a differentiable function on $[a, b]$. Then the length of the graph of $y = f(x)$ from $x = a$ to $x = b$ is*

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

EXAMPLE 12

Write an integral which computes the length of the curve $y = \sin x$ from $x = 0$ to $x = \pi/2$.

Mathematica commands to evaluate this integral:

In: `NIntegrate[Sqrt[1 + Cos[x]^2], {x,0,Pi/2}]`

Out: `1.9101`

EXAMPLE 13

Write an integral which computes the length of the curve $y = \ln x$ from $x = 1$ to $x = 5$.

Mathematica commands to evaluate this integral:

In: `NIntegrate[Sqrt[1 + (1/x)^2], {x,1,5}]`

Out: `4.36749`

EXAMPLE 14

Find the length of the curve $y = (4 - x^{2/3})^{3/2}$ from $x = 1$ to $x = 3$.

Note: Observe that in order to get an arc length integral that you can actually work out, you have to start with a weird function.

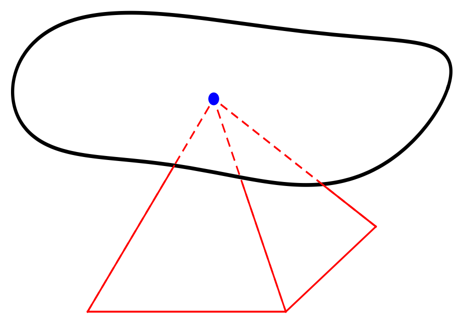
This is your first exposure to the notion that *integrals representing lengths of curves are harder to compute exactly than those representing areas or volumes*. In fact there is an entire branch of math research devoted to studying integrals which represent the lengths of curves; this is the theory of **elliptic integrals**.

4.5 One-dimensional moments and centers of mass

This section deals with the computation of the *center of mass* of a physical system (and related topics).

What do we mean by "center of mass"?

Imagine you had to balance a weirdly-shaped thin sheet of metal on the top of a pyramid:



The **center of mass** of the piece of metal is the place where you would have to put the point of the pyramid to balance it.

If the sheet of metal is circular and made of a uniform material, this is easy. You just put the center of the table on top of the pyramid. But if the tabletop isn't round (or otherwise symmetric) and/or if the tabletop has nonuniform density (i.e. it is made from a porous material with varying amounts of air pockets in it), it isn't so easy to see immediately where the center of mass is. Our first goal in this section is to determine how to find the center of mass in these situations.

Why do we care about this?

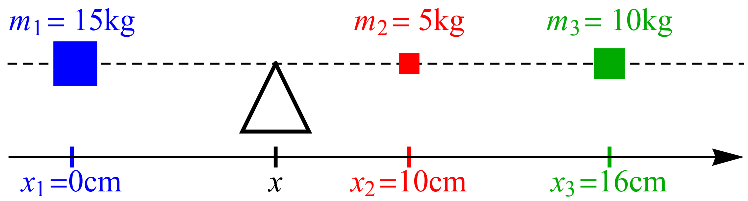
If you are studying the motion of a large object (like a comet or a space station orbiting the earth), you can treat the object as a single point (which is easier to study mathematically than a large blob) **so long as the point you use is the center of mass of the object**.

Centers of mass are also used to compute stresses on beams, and are applied in astronomy and kinesiology.

Discrete masses along a line

We will start by considering the problem of finding the center of mass for a **one-dimensional** physical system (that is, where the mass is distributed along a line, i.e. instead of a tabletop we are thinking of a pencil or metal bar or stick, etc.).

Place some individual masses $m_1, m_2, ..., m_n$ at certain points $x_1, x_2, ..., x_n$ along a number line, and imagine that the number line is supported at some point x called the **fulcrum**:



Physics tells us that each mass exerts **torque** (i.e. rotational force) on the system, causing the system to want to rotate about the fulcrum. Masses on opposite sides of the fulcrum exert torque in opposite directions. In particular, from physics we have

$$\begin{aligned}\text{torque created by } j^{th} \text{ mass} &= (\text{force}) \cdot (\text{distance from fulcrum}) \\ &= (\text{weight}) \cdot (\text{distance from fulcrum}) \\ &= (\text{mass})(\text{acceleration})(\text{distance from fulcrum}) \\ &= m_j\,(9.8\text{m}/\text{sec}^2)\,(x_j - x)\end{aligned}$$

Adding the torque for each of the masses in the system gives the total torque T of the system:

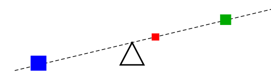
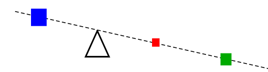
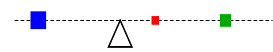
$$T = \sum_{j=1}^{n} m_j (9.8)(x_j - x)$$

In our picture above,

$$T =$$

$$=$$

Remark on units: By the formula above, we see that the standard unit of torque would be $\frac{\text{kg}\cdot\text{m}^2}{\text{sec}^2}$. However, in the SI system, one Newton (1 N) is defined to be $1\frac{\text{kg}\cdot\text{m}}{\text{sec}^2}$, so torque can also be expressed in Newton meters (Nm).

location of fulcrum	total torque T
$x = 8$	
$x = 5$	$T = 2058 - 294(5) = 588$ Nm $T > 0 \Rightarrow$ system will rotate clockwise 
$x = 7$	$T = 2058 - 294(7) = 0$ Nm $T = 0 \Rightarrow$ system is in **equilibrium** (balanced), i.e. $x = 7$ is the center of mass 

When the total torque is zero, we say the system is at **equilibrium** and the corresponding location of the fulcrum is called the **center of mass** of the system and is denoted $\overline{x}$. (In our example, $\overline{x} = 7$.)

Now let's do the same example without using numbers. Suppose individual masses $m_1, m_2, ..., m_n$ at certain points $x_1, x_2, ..., x_n$ along a number line. What is $\overline{x}$?

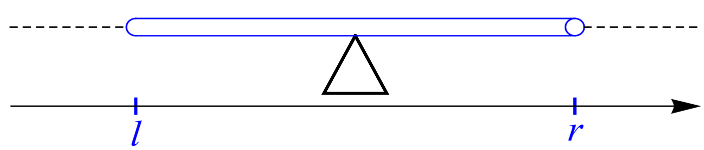
We have derived the following formulas:

Theorem 4.3 (Center of mass for discrete masses along a line) *Suppose discrete masses $m_1, m_2, ..., m_n$ are located at respective points $x_1, x_2, ..., x_n$ along a number line. Then:*

1. *The* **moment about the origin** *of this system is* $M_0 = \sum_{j=1}^{n} m_j x_j$.
2. *The* **total mass** *of this system is* $M = \sum_{j=1}^{n} m_j$.
3. *The* **center of mass** *of the system is* $\overline{x} = \dfrac{M_0}{M}$.

Continuous mass along a line

Consider a wire (or a bar or a stick) situated along the $x-$axis with varying density $\rho(x)$ (ρ is the Greek letter "rho"; sometimes density is denoted by $\delta(x)$).



Question: What is $\overline{x}$?

If the density ρ is constant, then the total mass is $M =$ (density)(length) $= \rho \cdot \Delta x$.

That means, by the general principle of applications of integration, that if the density is a nonconstant function $\rho(x)$, the total mass is

$$M =$$

Note the similarity to the discrete case, where M was $\sum_{j=1}^{n} m_j$. Essentially, the summation is replaced by an integral and the masses m_j are replaced by the density function $\rho(x)$. In the same way, the moment about the origin should be

$$M_0 =$$

Summarizing, we have:

Theorem 4.4 (Center of mass for continuously distributed mass on a line) *If a mass is distributed along a line, running from $x = l$ to $x = r$, such that the density of the mass at position x is $\rho(x)$, then:*

1. *The* **moment about the origin** *of this mass is* $M_0 = \int_l^r x\,\rho(x)\,dx$.
2. *The* **total mass** *of this mass is* $M = \int_l^r \rho(x)\,dx$.
3. *The* **center of mass** *of this system is* $\overline{x} = \dfrac{M_0}{M} = \dfrac{\int_l^r x\,\rho(x)\,dx}{\int_l^r \rho(x)\,dx}$.

EXAMPLE 15

Find the center of mass of a thin rod of length 2 cm, whose density x units from the left edge of the rod is $\rho(x) = x^2 + x$ mg/cm.

EXAMPLE 16

A wire runs from $x = -3$ to $x = 6$ (measured in inches) along the x-axis. If the density of the wire at point x is $\rho(x) = \frac{1}{x^2+1}$ g/in, compute the mass of the wire, and compute the center of mass of the wire.

Solution: The mass of the wire is

$$M = \int_l^r \rho(x)\,dx = \int_{-3}^{6} \frac{1}{x^2+1}\,dx = \arctan x\Big|_{-3}^{6} = \boxed{\arctan 6 - \arctan(-3)\text{ g}}.$$

The moment about the origin of the wire is

$$M_0 = \int_l^r x\rho(x)\,dx = \int_{-3}^{6} \frac{x}{x^2+1}\,dx = \int_{10}^{37} \frac{1}{2u}\,du = \frac{1}{2}\ln u\Big|_{10}^{37} = \boxed{\frac{1}{2}\ln 37 - \frac{1}{2}\ln 10\text{ g}\cdot\text{in}}.$$

Finally, the center of mass of the wire is

$$\overline{x} = \frac{M_0}{M} = \boxed{\frac{\frac{1}{2}\ln 37 - \frac{1}{2}\ln 10}{\arctan 6 - \arctan(-3)}\text{in}} \approx .246419\text{ in}.$$

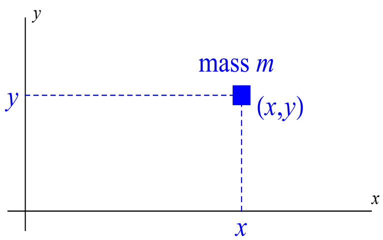
4.6 Two-dimensional moments and centers of mass

Discrete masses in a plane

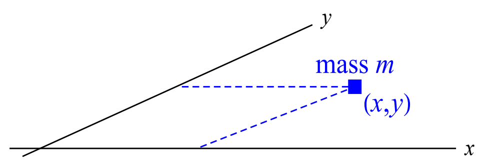
Now, we start considering **two-dimensional** objects (like slabs and tabletops). We start with point masses in a plane; this situation is similar to the situation in the previous section, except that you need to consider $x-$ and $y-$coordinates separately. The center of mass will be a point $(\overline{x}, \overline{y})$.

Consider first a single mass m located at point (x, y).

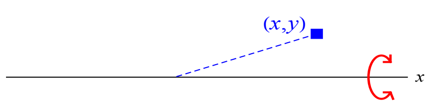
TOP VIEW



3-D VIEW

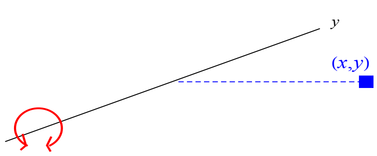


(1) Consider the $x-$axis as an axis of rotation:



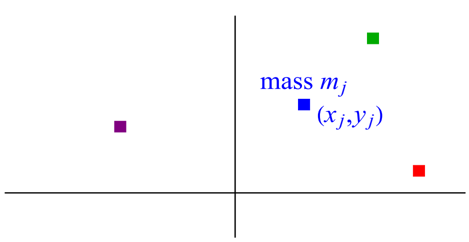
$\Rightarrow$ Moment about $x-$axis of this particle is $M_x = my$.

(2) Now, consider the $y-$axis as an axis of rotation:



$\Rightarrow$ Moment about $y-$axis of this particle is $M_y = mx$.

Now, suppose you have a bunch of masses in a plane:

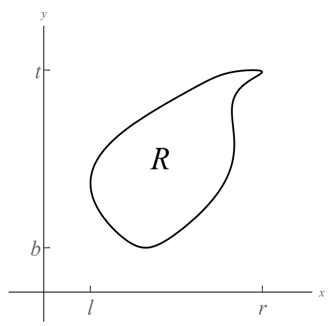


Theorem 4.5 (Center of mass for discrete masses in a plane) *Suppose that discrete masses $m_1, m_2, \ldots, m_n$ are located at respective points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in a plane. Then:*

1. *The* **moment about the $x-$axis** *of this system is* $M_x = \sum_{j=1}^{n} m_j y_j$
2. *The* **moment about the $y-$axis** *of this system is* $M_y = \sum_{j=1}^{n} m_j x_j$.
3. *The* **total mass** *of this system is* $M = \sum_{j=1}^{n} m_j$.
4. *The* **center of mass** *of the system is* $(\overline{x}, \overline{y})$ *where* $\overline{x} = \dfrac{M_y}{M}$ *and* $\overline{y} = \dfrac{M_x}{M}$.

Continuous mass in a plane

Setup:



Question: What is the center of mass $(\overline{x}, \overline{y})$ for a (flat) slab R in the plane, possibly with nonconstant density?

The formulas $\overline{x} = \dfrac{M_y}{M}$, $\overline{y} = \dfrac{M_x}{M}$ still work. But what are M_x, M_y and M?

One important restriction: We assume in MATH 230 that the density is either constant, or depends only on x (i.e. the density is $\rho(x)$, not $\rho(x, y)$). Otherwise one would need machinery you learn in MATH 320.

To compute M_x, M_y and M, we take the formulas from the previous discussion and adapt them in the same way that we adapted the formulas for discrete masses along a line to get to the formulas for continuous mass along a line. This gives:

Theorem 4.6 (Center of mass for continuously distributed mass in a plane) *If a planar region runs from $x = l$ to $x = r$ and has height $L(x)$ at point x, and if the density of this region at point (x, y) is $\rho(x)$, then:*

1. *The* **moment about the $y-$axis** *of this system is $M_y = \int_l^r x\, L(x)\, \rho(x)\, dx$.*
2. *The* **total mass** *of this system is $M = \int_l^r L(x)\, \rho(x)\, dx$.*
3. *The* **center of mass** *of the system is $(\overline{x}, \overline{y})$ where $\overline{x} = \dfrac{M_y}{M}$ and $\overline{y} = \dfrac{M_x}{M}$.*

NOTE: there is no formula for M_x in the above theorem. This is because, when you generalize the formula for M_x obtained earlier in this section, you get

$$M_x = \int_b^t y\, L(y)\, \rho(y)\, dy$$

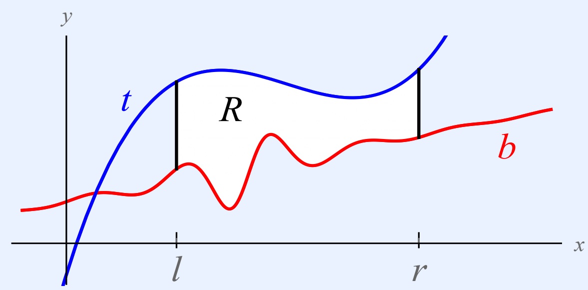
and this is a problem because we are assuming the density depends on x, not on y.

However, this problem is fixable, in the situation that the region lies between the graphs of two functions (see the theorem on the next page).

EXAMPLE 17

Suppose that a slab of wood is described by the region between the graph of $f(x) = \sin x$ and the $x-$axis, from $x = 0$ to $x = \pi$ (x and y are in cm). If the density of the wood at point (x, y) is $\sin x$ g/cm^2, compute the total mass of the wood.

Theorem 4.7 (Center of mass for continuously distributed mass in a plane) *If R is the region lying above function b and below function t for $l \le x \le r$,*



and the density of R at point (x, y) is $\rho(x)$, then:

1. *The* **moment about the** $y-$**axis** *is* $M_y = \displaystyle\int_l^r x\,[t(x) - b(x)]\,\rho(x)\,dx.$

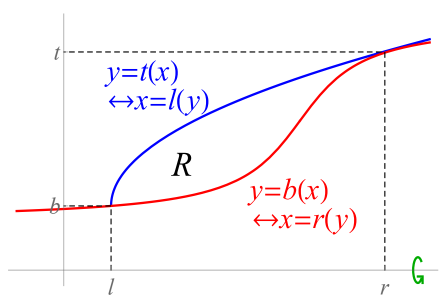
2. *The* **moment about the** $x-$**axis** *is* $M_x = \displaystyle\int_l^r \frac{1}{2}\left([t(x)]^2 - [b(x)]^2\right)\,\rho(x)\,dx.$

3. *The* **total mass** *of R is* $M = \displaystyle\int_l^r [t(x) - b(x)]\,\rho(x)\,dx.$

4. *The* **center of mass** *of R is $(\overline{x}, \overline{y})$ where* $\overline{x} = \dfrac{M_y}{M}$ *and* $\overline{y} = \dfrac{M_x}{M}$.

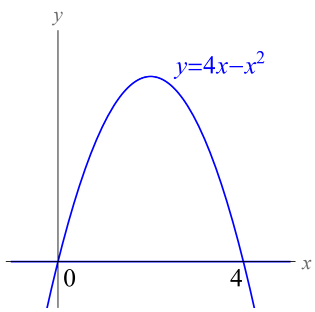
We already have seen statements (1), (3) and (4) in the previous theorem.

To see why (2) is true, let's consider the problem of finding the volume when the region R shown below is revolved around the x-axis. (There are two ways to compute the volume.)



EXAMPLE 18

Find the center of mass of the region enclosed by $y = 4x - x^2$ and the $x-$axis, where the density function is $\rho(x) = x + 1$ kg/ft^2 (and x and y are measured in feet).



Now, apply the formulas in the previous theorem:

$$M = \int_l^r [t(x) - b(x)]\rho(x)\,dx = \int_0^4 (4x - x^2)(x+1)\,dx =$$

$$M_y = \int_l^r x[t(x) - b(x)]\rho(x)\,dx =$$

$$M_x = \int_l^r \frac{1}{2}\left([t(x)]^2 - [b(x)]^2\right)\rho(x)\,dx =$$

Finally,

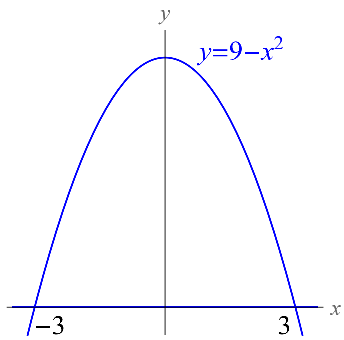
$$\overline{x} = \frac{M_y}{M} = \frac{\frac{1088}{15}}{32} = \frac{34}{15} \text{ ft} \quad \text{and} \quad \overline{y} = \frac{M_x}{M} = \frac{\frac{256}{5}}{32} = \frac{8}{5} \text{ ft}$$

so the center of mass is

$$(\overline{x}, \overline{y}) = \boxed{\left(\frac{34}{15}, \frac{8}{5}\right)}.$$

EXAMPLE 19

Find the center of mass of the region bounded by $y = 9 - x^2$ and the $x-$axis, assuming that the density is a constant ρ kg/cm^2.



First, find intersection points:

$$\begin{aligned} 9 - x^2 &= 0 \\ (3-x)(3+x) &= 0 \end{aligned}$$

$$\Rightarrow x = 3, x = -3$$

Next, apply the formulas:

$$\begin{aligned} M = \int_l^r [t(x) - b(x)]\rho(x)\,dx = \int_{-3}^{3} (9 - x^2)\rho\,dx &= \rho\left[9x - \frac{1}{2}x^2\right]_{-3}^{3} \\ &= \rho\left[27 - \frac{9}{2}\right] - \rho\left[-27 - \frac{9}{2}\right] = \boxed{54\rho \text{ kg}}. \end{aligned}$$

$$\begin{aligned} M_y = \int_l^r x[t(x) - b(x)]\rho(x)\,dx = \int_{-3}^{3} (9x - x^3)\rho\,dx &= \rho\left[\frac{9}{2}x^2 - \frac{1}{4}x^4\right]_{-3}^{3} \\ &= \rho\left[\frac{81}{2} - \frac{81}{4}\right] - \rho\left[\frac{81}{2} - \frac{81}{4}\right] = \boxed{0 \text{ cm} \cdot \text{kg}}. \end{aligned}$$

$$\begin{aligned} M_x = \int_l^r \frac{1}{2}\left([t(x)]^2 - [b(x)]^2\right)\rho(x)\,dx &= \int_{-3}^{3} \frac{1}{2}\left[(9 - x^2)^2 - 0^2\right]\rho\,dx \\ &= \frac{1}{2}\rho \int_{-3}^{3} (81 - 18x^2 + x^4)\,dx \\ &= \frac{1}{2}\rho\left[81x - 6x^3 + \frac{1}{5}x^5\right]_{-3}^{3} \\ &= \frac{1}{2}\rho\left[243 - 54 + \frac{243}{5}\right] - \frac{1}{2}\rho\left[-243 + 54 - \frac{243}{5}\right] \\ &= \boxed{\frac{648}{5}\rho \text{ cm} \cdot \text{kg}}. \end{aligned}$$

Finally,

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{0}{54\rho}, \frac{\frac{648}{5}\rho}{54\rho}\right) = \boxed{\left(0, \frac{12}{5}\right) \text{cm}}.$$

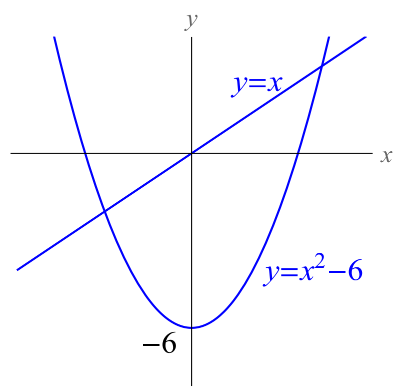
Note: the final answer to Example 19 didn't depend on what ρ was. So we may as well have assumed that ρ was 1. This situation (of constant density) arises fairly often, so we invent the following terminology:

Definition 4.8 *The* **centroid** *of a region is the center of mass of that region, assuming it has constant density.*

(To compute the centroid, just assume $\rho = 1$ and use the formulas already discussed.)

EXAMPLE 20

Write expressions involving one or more integrals that could be evaluated to find the centroid of the region enclosed by $y = x$ and $y = x^2 - 6$.



P.S. These formulas work out to $\overline{x} = \dfrac{1}{2}$ and $\overline{y} = -2$.

4.7 Moments of inertia

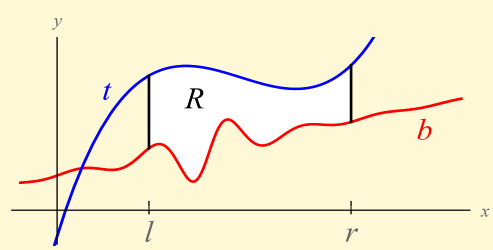
Newton's Second Law tells us that the force required to resist or alter the **linear** motion of an object is proportional to the object's mass:

This means, for example, that it takes more effort to slow down or change the direction of a fully-loaded cart than it does to change the linear motion of an empty cart.

However, the force required to resist or alter a **rotational** motion (here "force" means torque) isn't proportional to the object's mass; rather, it's proportional to something called the *moment of inertia* of the object about the axis around which the object is rotating:

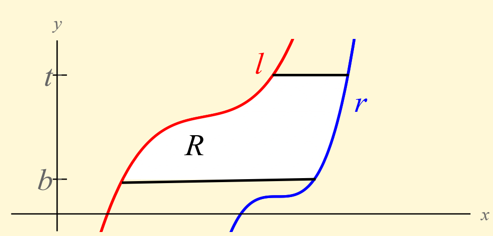
In this section, we'll learn how to compute moments of inertia for thin, flat plates represented as regions in the xy-plane with constant density $\rho = 1$.

Definition 4.9 *Suppose R is the region lying above function $y = b(x)$ and below function $y = t(x)$ for $l \le x \le r$:*



Then, the **moment of inertia of R about the y-axis** *is* $I_y = \displaystyle\int_l^r x^2 \left[t(x) - b(x)\right]\, dx$.

Definition 4.10 *Suppose R is the region lying to the left of function $x = r(y)$ and to the right of function $x = l(y)$ for $b \le y \le t$:*



Then, the **moment of inertia of R about the x-axis** *is* $I_x = \displaystyle\int_t^b y^2 \left[r(y) - l(y))\right]\, dy$.

QUESTION

If x and y are measured in meters, what are the units of I_y (and I_x)?

$$I_y = \int_l^r x^2 \left[t(x) - b(x)\right]\, dx$$

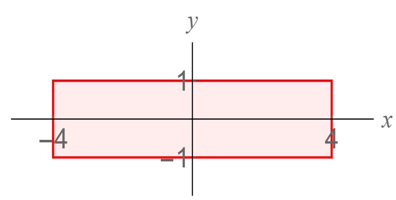
Notice that moments of inertia have nothing to do with the density or mass of the shape–their units depend only on the units in which distances are measured. As such, I_x and I_y are also called **area moments of inertia**.

Moments of inertia are also called **second moments** because of the x^2 or y^2 in their calculation (M_x and M_y are the **first moments** since there is $x = x^1$ and $y = y^1$ in those integrals).

We use the acronym **MOI** for moment(s) of inertia.

EXAMPLE 21

Compute the moments of inertia (a.k.a. MOI) about the x- and y-axes of this rectangle:

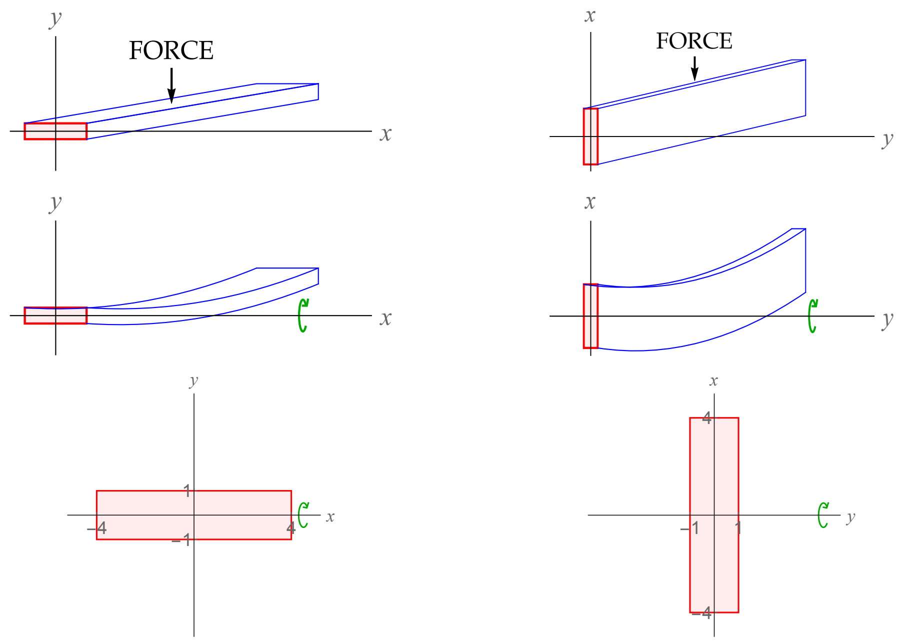


On the previous page, we saw that for a 2×8 rectangle centered at the origin,

$$I_y = \frac{256}{3} \text{ and } I_x = \frac{16}{3}.$$

Question: Why is I_y so much greater than I_x, and what does that mean?

Suppose you have a beam whose cross-section is a rectangle, supported on both ends. If you apply enough force to the center of the beam, it will sag:



Since $I_y > I_x$, it will take a greater force to make the beam sag when it is aligned vertically (in the right-hand column above). In other words, **the beam is stronger when it is turned vertically than it is when it is turned horizontally**.

Enrichment from physics: You may be familiar with the formula for kinetic energy of a moving object:

$$E = \frac{1}{2}mv^2.$$

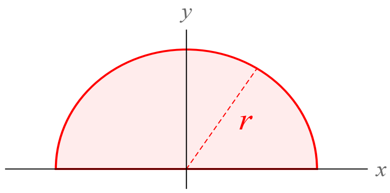
When an object is rotating (rather than moving linearly), the kinetic energy is

$$E = \frac{1}{2}I\omega^2$$

where ω is the object's *angular velocity* and I is the object's moment of inertia about the axis of revolution. This means objects with greater MOI have greater kinetic energy when they twist, bend or rotate (and require more energy to start such movement).

EXAMPLE 22

Textbooks in advanced engineering (technology) courses contain a chart with a bunch of shapes and corresponding formulas for area moments of inertia. One row of such a chart, corresponding to a semicircle, might look like this:



$$I_x = I_y = \frac{1}{8}\pi r^4$$

Where do these formulas on the right-hand side come from?

After this substitution, the integral becomes

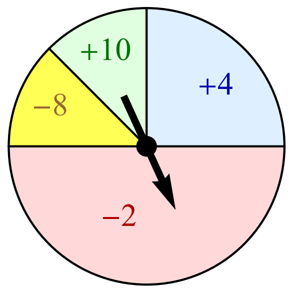
$$\begin{aligned}
I_y &= \int_{\arcsin(-1)}^{\arcsin 1} \left(r^2 \sin^2 u\right)\left(r\cos u\right) r\cos u\,du \\
&= \int_{-\pi/2}^{\pi/2} r^4 \sin^2 u \cos^2 u\,du \\
&= r^4 \int_{-\pi/2}^{\pi/2} \left(\frac{1-\cos 2u}{2}\right)\left(\frac{1+\cos 2u}{2}\right) du \\
&= \frac{1}{4} r^4 \int_{-\pi/2}^{\pi/2} \left(1-\cos^2 2u\right)\, du \\
&= \frac{1}{4} r^4 \int_{-\pi/2}^{\pi/2} \sin^2 2u\,du \\
&= \frac{1}{4} r^4 \int_{-\pi/2}^{\pi/2} \frac{1-\cos 4u}{2}\,du \\
&= \frac{1}{8} r^4 \left[u - \frac{1}{4}\sin 4u\right]_{\pi/2}^{\pi/2} = \frac{1}{8} r^4 \left(\left[\frac{\pi}{2} - 0\right] - \left[\frac{-\pi}{2} - 0\right]\right) = \boxed{\frac{1}{8} r^4 \pi}.
\end{aligned}$$

4.8 Probability

Finite probability

MOTIVATING EXAMPLE

Suppose you spin a spinner and it stops at a random location. You either win or lose money depending on where the spinner lands:



Mathematically, we represent such a situation by an object called a *random variable*:

Definition 4.11 *A* **random variable (r.v.)** *is a quantity X which depends on the result of some random experiment.*

Random variables model all kinds of things, and are studied in detail in MATH 414 and 416. Examples of problems approached with random variables include:

Gambling: outcomes from dice rolls, coin flips, or games of chance;

Actuarial science: the time until an insurance policy holder files a claim (or the size of the claim);

Finance: the price of a stock (or other financial instrument);

Business: the number of customers that enter a store or visit a web site;

Manufacturing: the rate at which parts of a machine will fail;

Sports analytics: the batting average of a baseball player (during the next season); the time it takes a runner to finish a race; etc.

Public health: the life expectancy of a human being; the number of COVID patients in a hospital; etc.

Biology: heights or weights of plants and animals; the population of a bacteria colony; amount of food a wolf eats in a year; etc.

Environmental science: future sea levels; amount of carbon emitted at some future time; etc.

Signal processing: noise (static) in communication systems and cryptology;

Physics: velocities of gas molecules moving around in a chamber.

In the spinner example, X is the amount you win/lose (i.e. X is the number on which the spinner lands). To describe X (i.e. distinguish X from other random variables), we need to know two things about it:

1.

2.

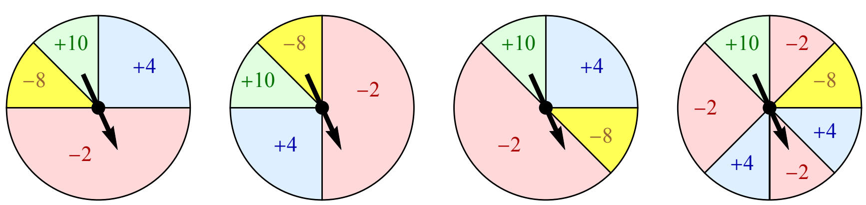
NOTE: There's a difference between X and x. X is a random quantity, and x is a value (i.e. a constant) that might be taken by the random variable X.

In the spinner example, this information can be conveyed by means of a table:

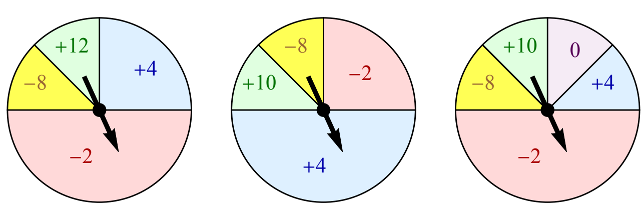
x	$f(x) = P(X = x) =$ probability that r.v. X takes value x
10	
4	
-8	
-2	

Note: The values that X takes (i.e the **range** of X) are the numbers in the left-hand column, and the probability of each value of X is in the right-hand column.

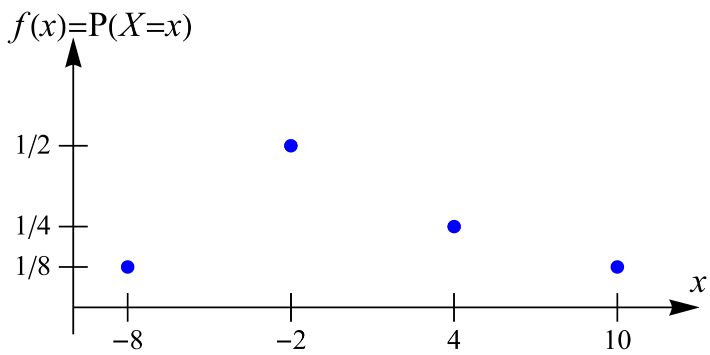
Observe that all these spinners generate the same r.v. as our X, because they lead to the same chart as above:



But these would lead to r.v.s that are all different from our X, and all different from each other:



In fact, this f is a function, so we can visualize f by means of a graph. For our original spinner, this graph is



EXAMPLE 23

In the context of this spinner, let X be the number spun. What is the probability that $X \geq 0$ (in symbols, we are asking for $P(X \geq 0)$)?

Solution:

The preceding example illustrates the following general concept:

Theorem 4.12 *Let X be a random variable taking only the values $x_1, x_2, \ldots, x_n$, with respective probabilities $f(x_1), f(x_2), \ldots, f(x_n)$. Then for any set E, the probability that X is in E is*

$$P(E) = P(X \in E) = \sum_{x_j \in E} f(x_j).$$

In other words, for random variables that only take **finitely many different values**, probabilities are computed **via addition**.

Example 24

Suppose X is a random variable described by the following chart:

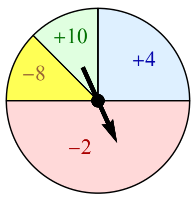
x	5	8	11	12
$f(x) = P(X = x)$	.3	.2	.1	p

1. What is the range of X? (In other words, what values are taken by X?)
2. What is p?
3. What is the probability that X is even?
4. What is the probability that X is less than 3?
5. Compute $P(X < 11)$.

Expected value

NEW QUESTION

Suppose you spin the spinner 800 times. How much would you expect to be ahead or behind after these 800 spins?



This example generalizes in the following formula:

Definition 4.13 *Let X be a random variable taking only the values $x_1, x_2, \ldots, x_n$, with respective probabilities $f(x_1), f(x_2), \ldots, f(x_n)$. The* **expected value** *of X, denoted EX, is*

$$EX = \sum_{j=1}^{n} x_j f(x_j).$$

The expected value of a random variable is the amount you would expect the r.v. to average if you repeat the experiment over and over.

Continuous random variables

The random variable X coming from the spinner in the preceding section is called **finite-valued** because there are only finitely many possible values of X (namely $-8, -2, 4$ and 10).

Many real world r.v.s are not necessarily finite-valued. For example, suppose you want to consider X to be some random amount of time, such as:

- $X =$ amount of time until your cell phone rings next
- $X =$ amount of time until a machine part fails
- $X =$ amount of time until a customer is involved in a traffic accident

(Actuaries working for insurance companies are particularly interested in things like the third X above.)

In all these examples, X is not finite-valued because X could be 1, 5, 100, 3.7, $\frac{19}{3}$, $\sqrt{\pi}$, etc. In this setting X takes values in an *interval* (the interval of values for X in these examples is $[0, \infty)$ but it could be something different in general).

TRIVIAL QUESTION

Suppose X is the amount of time until your cell phone rings. What is the probability that $X = 34$ minutes?

MORE INTERESTING QUESTION

Suppose X is the amount of time until your cell phone rings. What is the probability that $X \geq 34$ minutes?

Definition 4.14 *A random variable X is called* **continuous** *if it takes values in an interval, and if the probability that X takes any one particular value is zero.*

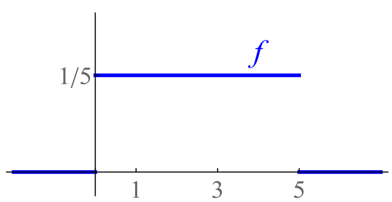
Recall that we described finite-valued random variables by using a chart, or a function f where $f(x) = P(X = x)$, or a graph (consisting only of some dots). How can we describe a continuous random variable?

EXAMPLE 25

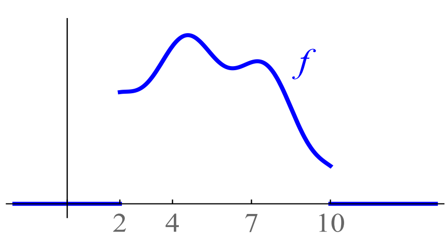
Choose a real number from the interval from the interval $[0,5]$ "uniformly" (this means that all real numbers should be "relatively equally likely" to be chosen). What is the probability that $X \in [1,3]$ (this means the probability that $X \geq 1$ and $X \leq 3$)?

Solution: Think geometrically:

Reinterpretation of this solution: Let $f(x) = \begin{cases} \frac{1}{5} & \text{if } x \in [0,5] \\ 0 & \text{else} \end{cases}$.



This reinterpretation allows us to consider situations where all numbers are **not** relatively equally likely. Suppose $f : \mathbb{R} \to \mathbb{R}$ is some function whose graph is like this:



The function f above is called a *density function* for the random variable X. More generally:

Definition 4.15 *Let X be a continuous random variable. A function $f : \mathbb{R} \to \mathbb{R}$ is called a* **density function** *for X if for every $a \le b$, we have*

$$P(X \in [a, b]) = \int_a^b f(x)\, dx.$$

The definition implies three things that must be true about the density function of any random variable:

1.

2.

3.

Key principles

A continuous random variable X is completely described by specifying its density function f.

Probabilities associated to the **continuous r.v.** X are computed **by integrating** the density function.

In particular, for any number b,

$$P(X = b) = P(X \in [b, b]) = \int_b^b f(x)\, dx = 0$$

and

$$P(X \in (a, b]) = P(X \in [a, b]) - P(X = a) = P(X \in [a, b]) - 0 = \int_a^b f(x)\, dx$$

so for a continuous r.v. X,

$$P(a < X < b),\ P(a \le X < b),\ P(a \le X \le b) \text{ and } P(a < X \le b)$$

are all the same and all equal to

$$\int_a^b f(x)\, dx.$$

Expected value of a continuous random variable

QUESTION

Given continuous r.v. X with density function f, what is EX?

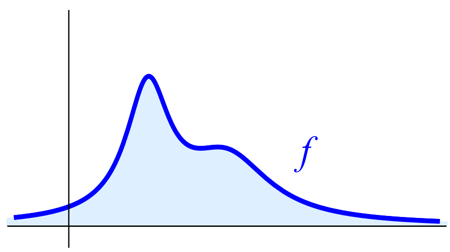
Answer: Recall that if X was finite-valued, $EX = \sum_{j=1}^{n} x_j\, f(x_j)$.

This formula generalizes as an integral:

Definition 4.16 *Let X be a continuous random variable with density function f. Then the* **expected value** *of X, denoted EX, is*

$$EX = \int_{-\infty}^{\infty} x\, f(x)\, dx.$$

This is not entirely different from a concept we have already discussed. Suppose f is a density function of a continuous r.v.



If you wanted to find the centroid of the region between the x-axis and the graph of f, then the x-coordinate of this centroid would be

Even more, if you think of the density function $f(x)$ of a continuous r.v. X as the density of a 1-dimensional rod (i.e. $\rho(x)$), then

$$EX = \int_{-\infty}^{\infty} x\, f(x)\, dx = \frac{\int_{-\infty}^{\infty} x\, f(x)\, dx}{1} = \frac{\int_{-\infty}^{\infty} x\, f(x)\, dx}{\int_{-\infty}^{\infty} f(x)\, dx} = \frac{\int_{-\infty}^{\infty} x\, \rho(x)\, dx}{\int_{-\infty}^{\infty} \rho(x)\, dx} = \frac{M_0}{M} = \overline{x}.$$

This is why we call f the *density function* of the r.v. X. Essentially, **expected value** is a probabilistic interpretation of the concept of **center of mass** (and in probability, the total mass M must always equal 1).

Example 26 (taken from an old actuarial exam)

The amount of time (measured in months) until a driver causes an accident is a continuous random variable whose density function is

$$f(x) = \begin{cases} \frac{25}{12}x^{-3} & \text{if } x \in [1,5] \\ 0 & \text{otherwise} \end{cases}$$

1. Compute the probability that the driver will cause an accident within two months.

2. Compute the expected amount of time until the driver causes an accident.

EXAMPLE 27

Let X be a continuous random variable with density function

$$f(x) = \begin{cases} ce^{-2x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

where c is some constant.

1. Compute c.
2. Compute $P\left(\frac{1}{2} < X < \frac{5}{2}\right)$.
3. Compute $P(X \in [-4, 1))$.
4. Compute $P(X = 7)$.
5. Compute EX.

Solution:

1.

2. $P\left(\frac{1}{2} < X < \frac{5}{2}\right) = \int_{1/2}^{5/2} f(x)\,dx = \int_{1/2}^{5/2} 2e^{-2x}\,dx = -e^{-2x}\Big|_{1/2}^{5/2} = \boxed{-e^{-5} + e^{-1}} \approx .3611$.

3. $P(X \in [-4, 1)) = \int_{-4}^{1} f(x)\,dx = \int_{-4}^{0} 0\,dx + \int_{0}^{1} 2e^{-2x}\,dx =$

4. $P(X = 7) =$

5. $EX = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{0}^{\infty} x\, 2e^{-2x}\,dx =$

4.9 Homework exercises

Exercises from Section 4.1

1. Compute the area of the region enclosed by the graph of $f(x) = x^2 - 6x$ and the $x-$axis.

2. Compute the area of the region enclosed by the graphs of $f(x) = x^2 + 2x + 1$ and $g(x) = 2x + 5$.

3. Compute the area of the region enclosed by the graphs of $f(x) = x^2 - 4x + 3$ and $g(x) = -x^2 + 2x + 3$.

4. Compute the area of the region enclosed by the graphs of $f(x) = (x - 1)^3$ and $g(x) = x - 1$.

5. Compute the area of the region enclosed by the graphs of $f(x) = 1 + \sqrt{3x}$ and $g(x) = x + 1$.

6. Compute the area of the region lying between the graphs of $f(x) = 2\sin x$ and $g(x) = \tan x$ from $x = \dfrac{-\pi}{3}$ to $x = \dfrac{\pi}{3}$.

7. Compute the area of the region lying between the graphs of $f(x) = \cos x$ and $g(x) = 2 - \cos x$ from $x = 0$ to $x = 2\pi$.

8. Let R be the region of the $xy-$plane lying above the $x-$axis, below the line $y = \dfrac{2}{3}x - 4$, and above the curve $y = \dfrac{1}{15}x^2 - \dfrac{2}{3}x + 1$. Compute the area of R.

9. a) Write down the equation of a circle of radius r, centered at the origin.
   b) Write down an expression (in terms of one or more integrals) which will give the area of this circle.
   c) Use *Mathematica* to evaluate the expression you got in part (b).
   d) Evaluate the integral from part (b) by hand. *Hint:* You need to rewrite the integrand and then use the substitution $u = \arcsin \frac{x}{r}$.

10. An **ellipse** (centered at the origin) is the graph of an equation of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Such an ellipse looks like an oval passing through the points $(a, 0)$, $(-a, 0)$, $(0, b)$ and $(0, -b)$.

    a) Write down an expression (in terms of one or more integrals) which will give the area of this ellipse.
    b) Use *Mathematica* to evaluate the expression you got in part (a).

11. Consider the quadrilateral whose vertices are (in order) $(0,2)$, $(4,7)$, $(6,6)$ and $(2,0)$. Write down an expression involving integrals which gives the area of this quadrilateral, and evaluate the expression to find the area of the quadrilateral (you can use *Mathematica* or do the integrals by hand).

12. Let R be the region enclosed by the graphs of $y=x$ and $y=\sqrt{x}$.

    a) Sketch a picture of R, clearly indicating which graph is which.

    b) Write a formula involving one or more integrals with respect to the variable x which gives the area of R.

    c) Write a formula involving one or more integrals with respect to the variable y which gives the area of R.

    d) Use *Mathematica* to evaluate the integrals from parts (b) and (c) (or do them by hand), and verify that you get the same answer.

13. Let R be the region enclosed by the graphs of $x=0$, $3x+y=6$ and $y=x^2-4$.

    a) Sketch a picture of R, clearly indicating which graph is which.

    b) Write a formula involving one or more integrals with respect to the variable x which gives the area of R.

    c) Write a formula involving one or more integrals with respect to the variable y which gives the area of R.

    d) Use *Mathematica* to evaluate the integrals from parts (b) and (c) (or do them by hand), and verify that you get the same answer.

14. Let R be the region enclosed by the graphs of $y=\log_4 x$, $x=16$ and the x-axis.

    a) Write a formula involving one or more integrals with respect to the variable x which gives the area of R.

    b) Write a formula involving one or more integrals with respect to the variable y which gives the area of R.

    c) Use *Mathematica* to evaluate the integrals from parts (a) and (b) (or do them by hand), and verify that you get the same answer.

15. Let R be the region enclosed by the graphs of $y=2\arcsin\frac{x}{3}$, $x=0$ and $y=1$.

    a) Write a formula involving one or more integrals with respect to the variable x which gives the area of R.

    b) Write a formula involving one or more integrals with respect to the variable y which gives the area of R.

c) Use *Mathematica* to evaluate the integrals from parts (a) and (b) (or do them by hand), and verify that you get the same answer.

16. Let R be the region enclosed by the graphs of $y = e^x$ and $y = \frac{x}{3} + 4$. Use *Mathematica* to compute the area of R.

    Hint: you will first need to use *Mathematica* to find decimal approximations of the intersection points of the graphs that describe R.

Exercises from Section 4.2

17. Consider a solid whose base is the triangle in the $xy-$plane whose vertices are $(0,0)$, $(0,2)$ and $(6,0)$. Cross-sections of the solid are squares parallel to the $y-$axis. Compute the volume of the solid.

18. Consider a solid whose base is the region in the $xy-$plane lying above $y = x$ and below $y = 2 - x^2$, such that cross-sections of the solid parallel to the $y-$axis are semicircles whose diameter lies in the $xy-$plane. Compute the volume of the solid.

19. Consider a solid whose base is the interior of the circle $x^2 + y^2 = 16$ in the $xy-$plane. If cross-sections of the solid parallel to the $y-$axis are rectangles which are twice as high as they are wide, compute the volume of the solid.

20. Compute the volume of the solid obtained by revolving the region below the graph of $y = \sec x$ and above the $x-$axis between $x = 0$ and $x = \frac{\pi}{4}$ around the $x-$axis.

21. Compute the volume of the solid obtained by revolving the region below the graph of $y = 2 - x^2$ and above $y = 0$ around the $x-$axis.

In Problems 22-25, let R be the region in the $xy-$plane bounded by the graphs $y = \sqrt{x}$, $y = 0$ and $x = 4$.

22. Let S_1 be the solid obtained by revolving R around the $x-$axis.

    a) Write a formula involving one or more integrals with respect to the variable x which gives the volume of S_1.
    b) Write a formula involving one or more integrals with respect to the variable y which gives the volume of S_1.
    c) Evaluate the integrals from parts (a) and (b) using *Mathematica*, and verify that you get the same answer.

23. Let S_2 be the solid obtained by revolving R around the $y-$axis.

a) Write a formula involving one or more integrals with respect to the variable x which gives the volume of S_2.

b) Write a formula involving one or more integrals with respect to the variable y which gives the volume of S_2.

c) Evaluate the integrals from parts (a) and (b) using *Mathematica*, and verify that you get the same answer.

24. Let S_3 be the solid obtained by revolving R around the line $x = 6$.

a) Write a formula involving one or more integrals with respect to the variable x which gives the volume of S_3.

b) Write a formula involving one or more integrals with respect to the variable y which gives the volume of S_3.

25. Let S_4 be the solid obtained by revolving R around the line $y = -3$.

a) Write a formula involving one or more integrals with respect to the variable x which gives the volume of S_4.

b) Write a formula involving one or more integrals with respect to the variable y which gives the volume of S_4.

In Problems 26-28, let Q be the region in the $xy-$plane lying to the right of the $y-$axis, above the curve $y = 2x^2$ and below the line $y = 8$.

26. Let S_1 be the solid obtained by revolving Q around the $x-$axis.

a) Write a formula involving one or more integrals with respect to the variable x which gives the volume of S_1.

b) Write a formula involving one or more integrals with respect to the variable y which gives the volume of S_1.

27. Let S_2 be the solid obtained by revolving Q around the line $x = -4$.

a) Write a formula involving one or more integrals with respect to the variable x which gives the volume of S_2.

b) Write a formula involving one or more integrals with respect to the variable y which gives the volume of S_2.

28. Let S_3 be the solid obtained by revolving Q around the line $y = 8$.

a) Write a formula involving one or more integrals with respect to the variable x which gives the volume of S_3.

b) Write a formula involving one or more integrals with respect to the variable y which gives the volume of S_3.

29. Use calculus to show that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

    Hint: You can visualize such a sphere by first thinking of a circle of radius r in the plane, and building a solid over the base where the cross-sections of the solid parallel to the $y-$axis are themselves circles. Write down an integral which gives the volume, and evaluate the integral by hand (it's not that hard) to get $\frac{4}{3}\pi r^3$.

30. *(Challenge)* Let R be the region of points in the first quadrant above the graph of $y = x^2$ but below the graph of $y = x$. Revolve R around the **diagonal** line $y = x$ to produce a solid. Compute the volume of this solid.

    Hint: You should integrate in the direction of the diagonal line $y = x$, so your cross-sections have to be perpendicular to this direction of integration. What shape are these cross-sections (this isn't so hard), and what is their area (this is harder)?

Exercises from Section 4.3

31. Suppose a bird flies in a straight line so that its velocity at time t is $50t^2 - 20t$ mi/hr. Find the distance the bird travels between times $t = 0$ and $t = 1$.

32. If the acceleration a of an object is constant, then its change in velocity over elapsed time t is $\Delta v = a \cdot \Delta t$. Suppose an object experiences nonconstant acceleration $a(t) = 4e^{2t}$ m/sec^2 from time $t = -1$ sec to $t = 2$ sec. What is the change in the object's velocity from time $t = -1$ to time $t = 2$?

33. If the density of a metal rod is constant, then the mass of the metal rod is equal to the density times the length of the rod. Suppose a metal rod has a nonconstant density, where the density x units from the left edge of the rod is $2x^2 + x + 1$ grams/unit. Find the mass of the rod, if the wire is 3 units long.

34. If the force applied to an object is constant, then the work done by that force is equal to the force times the distance the object moves. Suppose a force is applied to an object where the force at position x is $2x + 3$ Newtons. Find the work done in moving the object from position 5 meters to position 10 meters.

35. If the current i in an electric circuit is constant, then the charge q that builds up in the circuit is equal to the current times the amount of time that the current is in the circuit. Find the charge in an electric circuit if a current of $\sin t + 2\cos 3t + 3$ amperes is applied to the circuit from time 0 sec to time $\frac{\pi}{6}$ sec.

    Note: An ampere times a second is a Coulomb, the SI unit of charge.

Exercises from Section 4.4

36. Compute (by hand) the length of the curve $y = \frac{2}{3}x^{3/2} + 1$ from $x = 0$ to $x = 1$.

37. Compute (by hand) the length of the curve $y = 2x^{3/2} - 2$ from $x = 0$ to $x = 8$.

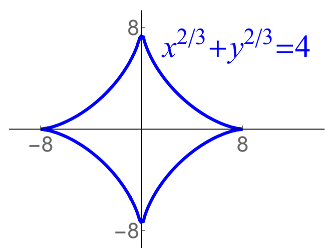
38. Write an integral which gives the length of the curve $y = \ln(\sin x)$ between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$. Then, use *Mathematica* to evaluate the integral.

39. Write an integral which gives the length of the curve $y = \sqrt[3]{x^2}$ between the points $(1, 1)$ and $(8, 4)$. Then, use *Mathematica* to evaluate the integral.

40. Write an integral which gives the length of the curve $y = \frac{1}{x}$ from $x = 1$ to $x = 3$. Then, use *Mathematica* to find a decimal approximation to this integral.

41. Write an integral which gives the length of the curve $y = 2 \sin 4x$ from $x = 0$ to $x = \pi$. Then, use *Mathematica* to find a decimal approximation to this integral.

42. Find the total length (you can use *Mathematica* to do the integral) of the curve $x^{2/3} + y^{2/3} = 4$ (the graph of this curve is found below):



43. Use calculus to show that the circumference of a circle of radius r is $2\pi r$.

    Hint: Write an equation for the top half of the circle; then use the arc length formula to determine the length of half the circle (for the integral, rewrite it so you can recognize $\arcsin$ as part of the answer). Then double that answer to get the circumference.

Exercises from Section 4.5

44. Suppose a system consists of four objects, with respective masses 7 kg, 4 kg, 3 kg and 8 kg, located along an axis with respective positions $-3, -2, 5$ and 6 (measured in meters).

    a) Compute the total mass of the system.
    b) Compute the moment about the origin of the system.
    c) Compute the center of mass of the system.

45. Suppose a system consists of three objects of masses 6 g , 5 g and 3 g, located at positions -5 cm, 1 cm and 3 cm respectively. Compute the center of mass of this system.

46. Suppose two siblings sit on the ends of a see-saw; a brother, who weighs 75 pounds, and a sister, who weighs 50 pounds. If the seesaw is 12 feet long, how far from the brother should the fulcrum be placed so that the see-saw is balanced?

47. A metal rod of length 10 cm has density $\rho(x) = 1 + x^2$ g/cm at a distance x cm from the left end of the rod.

    a) Compute the mass of the rod.
    b) Determine how far from the left end of the rod its center of mass is.

48. A rod with density $\rho(x) = 2 + \sin x$ kg/unit is positioned along the positive $x-$axis, with its left end at $x = 0$ and its right end at $x = \frac{3\pi}{4}$. Compute the $x-$coordinate of the center of mass of the rod.

49. A wire with density $\rho(x) = Cx + e^{-x}$ runs from $x = 0$ to $x = 2$ along the x-axis, where C is an unknown constant. If the center of mass of the wire is $\overline{x} = 1$, what must the value of C be?

50. A rod of length 8 in has density $\rho(x) = 3\sqrt{x}$ mg/cm at a distance x cm from the left end of the rod. If you want to cut the rod into two pieces of equal mass, where should you cut it?

Exercises from Section 4.6

51. Suppose a system consists of three objects in a plane. The first mass is 5g, located at $(2, 2)$; the second mass is 1g, located at $(-3, 1)$, and the last mass is 3g, located at $(1, -4)$. Assume distances are measured in inches.

    a) Compute the total mass of the system.
    b) Compute the moment about the x-axis of the system.
    c) Compute the moment about the y-axis of the system.
    d) Compute the center of mass of the system.

52. Determine the center of mass of a system which consists of the following five masses (distances measured in meters):

mass	3 kg	4 kg	2 kg	1 kg	6 kg
position	$(-2, -3)$	$(5, 5)$	$(7, 1)$	$(0, 0)$	$(-3, 0)$

53. Let R be the region in the $xy-$plane above the $x-$axis, below the graph of $y = \sqrt{x}$ and to the left of the line $x = 9$.

    a) Assuming that R has constant density, compute the center of mass of R.

    b) Assuming that the density of any point (x, y) in R is $\rho(x) = x + 2$, compute the center of mass of R.

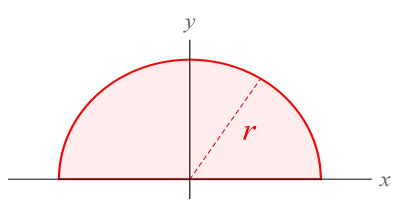
54. Let R be the region in the first quadrant between the graphs of $y = x^2$ and $y = x^3$.

    a) Assuming that R has constant density, compute the center of mass of R.

    b) Assuming that the density of any point (x, y) in R is $\delta(x) = 2 - x^2$, compute the center of mass of R.

55. Compute the centroid of the region bounded by the graphs of $y = xe^{-x}$, $y = 0$ and $x = 5$. (Use *Mathematica* to evaluate the integrals.)

56. Compute the centroid of a semicircle of radius r (a picture is shown below, but you have to figure out what graph is the top of the semicircle, and how far to the left or right it goes).



57. Compute the centroid of a circle of radius r centered at the origin. *Hint:* there is a more clever approach than doing a bunch of integrals.

58. Consider the region in the $xy-$plane bounded by the graphs of $y = x^2$ and $y = b$, where $b > 0$ is a positive constant.

    a) What is the $x-$coordinate of the center of mass of this region?

    b) Is the $y-$coordinate of the center of mass of this region greater than, or less than $\frac{b}{2}$? Explain your reasoning.

Exercises from Section 4.7

59. Compute the moment of inertia about the y-axis for the region bounded by the equations $x = 0$, $x = 2$, $y = x^2$ and $y = 2x^2 + 3$.

60. Compute the moments of inertia about the x- and y-axes for the region consisting of points below the line $y = 4x$ and above the curve $y = x^3$.

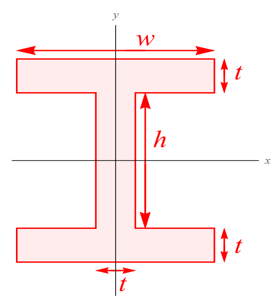
61. Let R be the region consisting of points below the graph of $y = \sin \frac{x}{4}$, to the left of the line $x = \pi$, and above the x-axis.

    a) Write a formula involving one or more integrals that can be used to compute the moment of inertia of R about the x-axis.
    b) Use *Mathematica* to evaluate the formula you wrote down in part (a).
    c) Write a formula involving one or more integrals that can be used to compute the moment of inertia of R about the y-axis.
    d) Use *Mathematica* to evaluate the formula you wrote down in part (c).

62. Finish Example 22 by verifying that $I_x = \frac{1}{8}\pi r^4$ for a semicircle of radius r centered at the origin.

63. Let T be a triangle with vertices $(0,0)$, $(b,0)$ and $(0,h)$. Compute, in terms of b and/or h, the moments of inertia of T about the x- and y-axes.

64. Compute, in terms of the thickness t, the height h and the width w, the moment of inertia about the x-axis for the I-beam pictured below:



Exercises from Section 4.8

65. Let X be a finite-valued random variable whose probabilities are described by the following chart, where p is some unknown constant:

x	-3	-2	-1	0	1	2	3
$f(x) = P(X = x)$	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{1}{10}$	p	$\frac{1}{10}$	$\frac{3}{20}$	$2p$

    a) Compute the value of p.
    b) Compute $P(X = -2)$.
    c) Compute $P(X = 4)$.
    d) Compute $P(X < 0)$.
    e) Compute $P(X \leq 0)$.
    f) Compute $P(X \neq 0)$.

66. Let X be a continuous random variable taking values in the interval $[0,1]$. If the density function of X is $f(x) = cx^2 + x$, find the value of c.

67. Let X be a continuous random variable taking values in the interval $[0, 2]$. If the density function of X is $f(x) = Ce^{-x/2}$, find the value of C.

68. Let X be a continuous random variable taking values in $[0, 4]$ whose density function is $f(x) = \frac{1}{8}x$.

    a) Find the probability that $X > 2$.
    b) Find the probability that $X = 3$.
    c) Find the probability that $X \leq 1$.
    d) Which is more likely, that $X = 1$ or $X = 3$?
    e) Which is more likely, that X is very close to 1 or that X is very close to 3?
    f) Find a number b such that $P(X \geq b) = \frac{2}{3}$.

69. Let X be a continuous random variable whose density function is

$$f(x) = \begin{cases} Ce^{-3x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases},$$

where C is some constant.

    a) What is the value of C?
    b) Find the probability that $X > 10$.
    c) Find the probability that $3 \leq X \leq 4$.

70. Suppose that the lifespan of a certain organism, measured in years, is a continuous random variable X whose density function is

$$\begin{cases} f(x) = 4xe^{-2x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}.$$

    a) Compute the probability that the organism lives for less than one year.
    b) Compute the expected lifespan of the organism.

71. Suppose that the lifetime (in months) of a light bulb is a continuous andom variable X with density function

$$f(x) = \begin{cases} 2x^{-3} & \text{for } x \geq 1 \\ 0 & \text{for } x < 1 \end{cases}.$$

    a) Compute the probability that the light bulb lasts more than four months.
    b) Compute the expected lifetime of the lightbulb.

c) If light bulbs of this type are replaced as soon as they burn out, how many light bulbs would you expect to use in the next 88 months?

72. Find the expected value of a continuous random variable whose density function is
$$f(x) = \begin{cases} \frac{1}{2}\sin x & \text{for } 0 \le x \le \pi \\ 0 & \text{else} \end{cases}.$$

73. Suppose that the time (measured in years) until an insurance policyholder files a claim is a continuous random variable X with density function
$$f(x) = \begin{cases} Cx^3 e^{-x/5} & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$$
where C is some constant.
   a) What is the value of C?
   b) Compute the average amount of time until the policyholder files a claim.
   c) Write an integral which will compute the probability that the insurance policyholder will file a claim within the next 8 years.
   d) Use *Mathematica* to find a decimal approximation to your answer to part (c).

74. Let X be a continuous random variable with density function f. The **second moment** of X, denoted μ_2, is defined by
$$\mu_2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx.$$
   a) We noted in class that expected value is the probabilistic interpretation of the concept of *center of mass*. What is the second moment the probabilistic interpretation of?
   b) Compute the second moment of the random variable described in Exercise 68.
   c) Compute the second moment of the random variable described in Exercise 73.

75. The **variance** of a random variable X, denoted $Var(X)$, is given by the formula
$$Var(X) = \mu_2 - (EX)^2.$$
The variance of a random variable measures how spread out its values are– the larger the variance, the more spread out the values the random variable takes. Compute the variance of the random variable described in Exercise 69.

76. Let X be a continuous random variable taking values in some interval. The **median** of X is a number m such that $P(X \le m) = \frac{1}{2}$. In each of these examples, find the median of X, if X has the given density function:

   a) $f(x) = \begin{cases} 4x^3 & \text{for } 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$

   b) $f(x) = \begin{cases} 2e^{-2x} & \text{for } x \ge 0 \\ 0 & \text{else} \end{cases}$

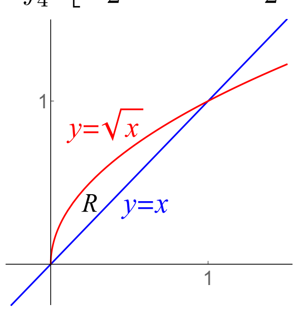
Answers

1. 36
2. $\dfrac{32}{3}$
3. 9
4. $\dfrac{1}{2}$
5. $\dfrac{3}{2}$
6. $2 - \ln 4$
7. 4π
8. $\dfrac{103 - 4\sqrt{10}}{9}$
9. a) $x^2 + y^2 = r^2$

   b) $A = 4\displaystyle\int_0^r \sqrt{r^2 - x^2}\, dx$

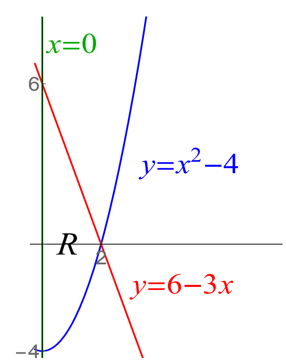
   c) πr^2

   d) πr^2
10. a) $A = 4\displaystyle\int_0^a b\sqrt{1 - \frac{x^2}{a^2}}\, dx$

   b) πab
11. $$A = \int_0^2 \left[(\tfrac{5}{4}x + 2) - (2 - x) \right] dx + \int_2^4 \left[(\tfrac{5}{4}x + 2) - (\tfrac{3}{2}x - 3) \right] dx$$
$$+ \int_4^6 \left[(\tfrac{-1}{2}x + 9) - (\tfrac{3}{2}x - 3) \right] dx = 18$$
12. a) 

   b) $\displaystyle\int_0^1 (\sqrt{x} - x)\, dx$

   c) $\displaystyle\int_0^1 (y - y^2)\, dy$

   d) $\dfrac{1}{6}$
13. a) 

   b) $\displaystyle\int_0^2 \left[(6 - 3x) - (x^2 - 4) \right] dx$

   c) $\displaystyle\int_{-4}^0 \sqrt{y + 4}\, dy + \int_0^6 (2 - \frac{y}{3})\, dy$

d) $\dfrac{34}{3}$

14. a) $\int_1^{16} \log_4 x\,dx$

b) $\int_0^2 [16 - 4^y]\;dy$

c) $\dfrac{-15 + 16\ln 16}{\ln 4} = 32 - \dfrac{15}{\ln 4}$
(these are the same thing)

15. a) $\int_0^1 \left[2\arcsin\frac{1}{3} - 2\arcsin\frac{x}{3}\right]\;dx$

b) $\int_0^{2\arcsin\frac{1}{3}} 3\sin\frac{y}{2}\,dy$

c) $6 - 4\sqrt{2}$

16. 25.8855

17. $V = \int_0^6 [2 - \frac{1}{3}x]^2\,dx = 8$

18. $V = \int_{-2}^1 \frac{1}{2}\pi\left(\frac{2 - x^2 - x}{2}\right)^2\;dx = \frac{81}{80}\pi$

19. $V = \int_{-4}^4 8\left(\sqrt{16 - x^2}\right)^2\;dx = \frac{2048}{3}$

20. $V = \int_0^{\pi/4} \pi\sec^2 x\,dx = \pi$

21. $V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi(2 - x^2)^2\,dx = \frac{64\sqrt{2}}{15}\pi$

22. a) $V = \int_0^4 \pi x\,dx$

b) $V = \int_0^2 2\pi y(4 - y^2)\,dy$

c) 8π

23. a) $V = \int_0^4 2\pi x\sqrt{x}\,dx$

b) $V = \int_0^2 \left[\pi 4^2 - \pi(y^2)^2\right]\;dy$

c) $\dfrac{128}{5}\pi$

24. a) $V = \int_0^4 2\pi(6 - x)\sqrt{x}\,dx$

b) $V = \int_0^2 \left[\pi(6 - y^2)^2 - \pi(6 - 4)^2\right]\;dy$

25. a) $V = \int_0^4 \left[\pi(\sqrt{x} + 3)^2 - \pi(0 + 3)^2\right]\;dx$

b) $V = \int_0^2 2\pi(y + 3)[4 - y^2]\,dy$

26. a) $V = \int_0^2 \left[\pi(8)^2 - \pi(2x^2)^2\right]\;dx$

b) $V = \int_0^8 2\pi y\sqrt{\frac{y}{2}}\,dy$

27. a) $V = \int_0^2 2\pi(x + 4)[8 - 2x^2]\,dx$

b) $V = \int_0^8 \left[\pi(\sqrt{\frac{y}{2}} + 4)^2 - \pi(0 + 4)^2\right]\;dy$

28. a) $V = \int_0^2 \pi(8 - 2x^2)^2\,dx$

b) $V = \int_0^8 2\pi(8 - y)[\sqrt{\frac{y}{2}}]\,dy$

29. $V = \int_{-r}^{r} \pi(\sqrt{r^2 - x^2})^2 \, dx = \frac{4}{3}\pi r^3$

30. $\frac{41}{30}\pi$

31. $\frac{20}{3}$ mi

32. $2e^4 - 2e^{-2}$ m/sec

33. $\frac{51}{2}$ g

34. 90 Nm

35. $2 + 3\pi$

36. $\frac{-2 + 2(2)^{3/2}}{3}$

37. $\frac{-2 + 2(73)^{3/2}}{27}$

38. $s = \int_{\pi/4}^{\pi/2} \sqrt{1 + \cot^2 x} \, dx = -\ln(\tan \frac{\pi}{8})$

39. $s = \int_{1}^{8} \sqrt{1 + \left[\frac{1}{2}x^{-1/2}\right]^2} \, dx = \frac{80\sqrt{10} - 13\sqrt{13}}{27}$

40. $s = \int_{1}^{3} \sqrt{1 + [-x^{-2}]^2} \, dx \approx 2.14662$

41. $s = \int_{0}^{\pi} \sqrt{1 + \cos^2 x} \, dx \approx 3.8202$

42. 48

43. Let $f(x) = \sqrt{r^2 - x^2}$; then $C = 2\int_{-r}^{r} \sqrt{1 + [f'(x)]^2} \, dx$ works out to be $2\pi r$.

44. a) 22 kg
 b) 34 kg·m
 c) $\overline{x} = \frac{17}{11}$ m

45. $\overline{x} = \frac{-8}{7}$ cm

46. $\frac{24}{5}$ ft

47. a) $\frac{1030}{3}$ g
 b) $\frac{765}{103}$ cm

48. $\frac{8\sqrt{2} + 6\pi\sqrt{2} + 9\pi^2}{16 + 8\sqrt{2} + 24\pi}$ units

49. $C = 3e^{-2}$

50. $4\sqrt[3]{2}$ cm from the left end

51. a) 9 g
 b) $M_x = -1$ g·in
 c) $M_y = 10$ g·in
 d) $(\overline{x}, \overline{y}) = \left(\frac{10}{9}, \frac{-1}{9}\right)$ in

52. $(\overline{x}, \overline{y}) = \left(\frac{10}{21}, \frac{13}{21}\right)$ m

53. a) $(\overline{x}, \overline{y}) = \left(\frac{27}{5}, \frac{9}{8}\right)$
 b) $(\overline{x}, \overline{y}) = \left(\frac{1593}{259}, \frac{45}{37}\right)$

54. a) $(\overline{x}, \overline{y}) = \left(\frac{3}{5}, \frac{12}{35}\right)$
 b) $(\overline{x}, \overline{y}) = \left(\frac{4}{7}, \frac{13}{42}\right)$

55. $(\overline{x}, \overline{y}) = \left(\frac{-37 + 2e^5}{-6 + e^5}, \frac{-61 + e^{10})}{8e^5(-6 + e^5)}\right)$

56. $(\overline{x}, \overline{y}) = \left(0, \frac{4}{3}\pi r\right)$

57. $(\overline{x}, \overline{y}) = (0, 0)$

58. a) $\overline{x} = 0$

    b) This should be greater than $\frac{b}{2}$, since the region is wider at the top than it is at the bottom.

59. $M_y = \dfrac{72}{5}$

60. $M_x = \dfrac{256}{5}; M_y = \dfrac{16}{3}$

61. a) $I_x = \displaystyle\int_0^{\sqrt{2}/2} y^2 \left[\pi - 4 \arcsin y\right]\, dy$

    b) $I_x = \dfrac{1}{9}\left(8 - 5\sqrt{2}\right) \approx .103215$

    c) $I_y = \displaystyle\int_0^{\pi} x^2 \sin \frac{x}{4}\, dx$

    d) $I_y = -128 + 16\pi\sqrt{2} - 2\sqrt{2}(-32 + \pi^2) \approx 5.68034$

62. $I_x = \displaystyle\int_0^r y^2 \left[\sqrt{r^2 - y^2} - \left(-\sqrt{r^2 - y^2}\right)\right]\, dy = \frac{1}{8}\pi r^4$

63. $I_x = \dfrac{1}{12} b^3 h; I_y = \dfrac{1}{12} b h^3$

64. $I_x = \displaystyle\int_{-h/2-t}^{-h/2} y^2 w\, dy + \int_{-h/2}^{h/2} y^2 t\, dy + \int_{h/2}^{h/2+t} y^2 w\, dy = \frac{1}{12} t \left(h^3 + 6h^2 w + 12htw + 8t^2 w\right)$

65. a) $p = \dfrac{3}{20}$

    b) $P(X = -2) = \dfrac{3}{20}$

    c) $P(X = 4) = 0$

    d) $P(X < 0) = \dfrac{3}{10}$

    e) $P(X \leq 0) = \dfrac{9}{20}$

    f) $P(X \neq 0) = \dfrac{17}{20}$

66. $c = \dfrac{3}{2}$

67. $C = \dfrac{e}{2e - 2}$

68. a) $P(X > 2) = \dfrac{3}{4}$

    b) $P(X = 3) = 0$

    c) $P(X \leq 1) = \dfrac{1}{16}$

    d) They are equally likely (the probability of both is zero).

    e) It is more likely that X is very close to 3 since $f(3) > f(1)$.

    f) $b = \dfrac{4}{\sqrt{3}}$.

69. a) $C = 3$
    b) $P(X > 10) = e^{-30}$
    c) $P(3 \le X \le 4) = e^{-9} - e^{-12}$.

70. a) $P(X < 1) = 1 - 3e^{-2}$
    b) $EX = 1$ year

71. a) $P(X > 4) = \dfrac{1}{16}$
    b) $EX = 2$ months
    c) 44 light bulbs

72. $EX = \dfrac{\pi}{2}$

73. a) $C = \dfrac{1}{3750}$
    b) 20 years
    c) $P(X < 8) = \displaystyle\int_0^8 \frac{1}{3750} x^3 e^{-x/5}\,dx$
    d) $P(X < 8) = 1 - \dfrac{1711}{375e^{8/5}} \approx .0788135$

74. a) moment of inertia (about the y-axis)
    b) 8
    c) 500

75. $\dfrac{1}{9}$

76. a) $m = 2^{-1/4}$
    b) $m = \dfrac{1}{2}\ln 2$

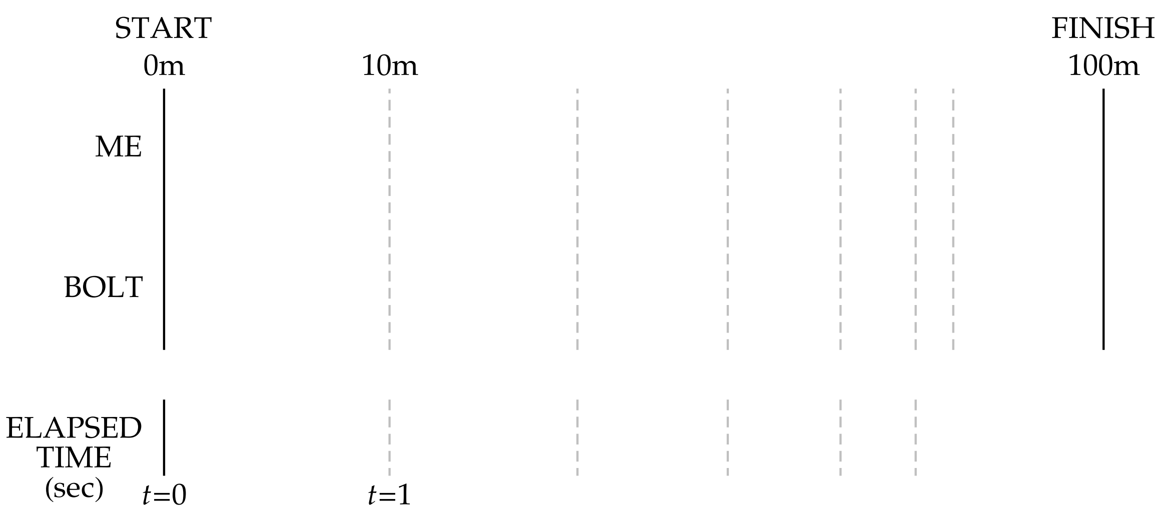
Chapter 5

Introduction to infinite series

5.1 Motivation and big-picture questions

Consider a 100m race between me and Usain Bolt[1]. Let's assume Bolt runs 10m/s and that I run 8m/s. However, I get a head start of 10 m.

Here is an argument that attempts to show why I will win this race:



So the distance I run in the lead is

[1]Usain Bolt is the world record holder and 3-time Olympic champion in the 100m.

Here is an argument which attempts to show why I will not win the race:

QUESTION

How do you reconcile these two arguments?

GENERALIZING THIS RACE

Suppose two runners, a slow runner and a fast runner, are in a race.

The ratio of the runner's speeds (slow runner to fast runner) is r. (In the Bolt example, $r = \frac{8}{10}$.) So if the fast runner's rate is v, the slow runner's rate is ________ .

But the slow runner gets a head start of h.

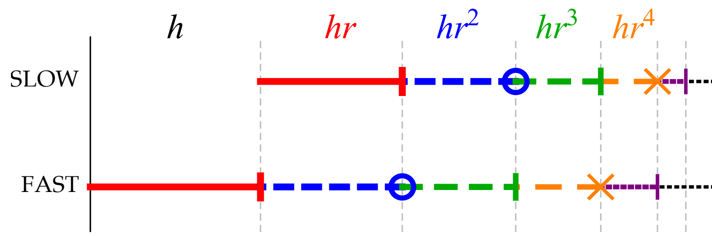
How far does the slow runner run before he is caught?

Solution from physics: the fast runner catches up when

$$\text{position of fast runner} = \text{position of slow runner}$$

So the amount the slow runner runs before being caught is

Solution from mathematics: the amount the slow runner runs in the lead is



The physics and math solutions to this problem should coincide, so

$$h + hr^2 + hr^3 + hr^4 + hr^5 + \ldots = \frac{h}{1-r}$$
$$h\left(1 + r^2 + r^3 + r^4 + r^5 + \ldots\right) = \frac{h}{1-r}$$

This formula works for any $r \in [0, 1)$ (and we will justify this formally once we have the right theory developed).

Question 1: What would happen if $r \geq 1$?

Question 2: What would happen if $r < 0$?
We will return to Question 2 later, but for now, just be aware that

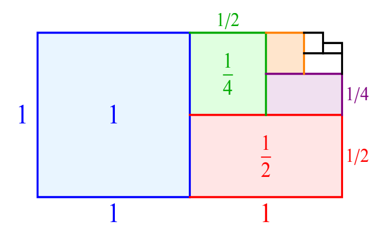
$$\boxed{\text{If } r \in [0,1), \text{ then } 1 + r^2 + r^3 + r^4 + r^5 + \ldots = \frac{1}{1-r}.}$$

EXAMPLE 1

Let $r = \dfrac{1}{2}$ in the formula on the previous page. This yields

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \ldots = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

$$=$$



EXAMPLE 2

Let $r = -1$ in the formula on the previous page. This yields what is called the **Dirichlet series**:

$$\begin{aligned}&1 + (-1) + (-1)^2 + (-1)^3 + (-1)^4 + (-1)^5 + \ldots\\ &= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1\ldots\end{aligned}$$

Solution #1: $\quad 1 - 1 \quad + \quad 1 - 1 \quad + \quad 1 - 1 \quad + \quad 1 - 1 \quad + \ldots$

Solution #2: $\quad 1 \quad - 1 + 1 \quad - 1 + 1 \quad - 1 + 1 \quad - 1 + \ldots$

Solution #3:

Which (if any) of these solutions to the Dirichlet series is correct?

The Dirichlet series illustrates a major problem with trying to study infinite series. Irrespective of which of these solutions is correct, what we know is that **they cannot all be correct**. This means that we definitely **cannot legally rearrange or regroup terms when adding up infinitely many numbers**.

In the grand scheme of things, we might want to add (or try to add) infinite lists of numbers like

$$a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$$

that aren't necessarily of the form

$$1 + r + r^2 + r^3 + r^4 + \cdots$$

General questions in this context

1. **Classification problem:** Can you add $a_1 + a_2 + a_3 + \cdots$ and get a finite number for the answer?
2. **Computation problem:** If so, what is the numerical value of $a_1 + a_2 + a_3 + \cdots$?
3. **Rearrangement problem:** When, if ever, can you legally rearrange or regroup the terms of the sum $a_1 + a_2 + a_3 + \cdots$?

EXAMPLES TO PONDER

a) $1 + 1 + 1 + 1 + 1 + 1 + \cdots$

b) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$

c) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$

d) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

Why these questions are hard

Formally speaking, addition is a *binary operation*, meaning that addition has <u>two inputs</u> and <u>one output</u>:

Suppose you have to add six numbers together, like

$$3 + 8 + 4 + 7 + 2 + 9$$

The reason this procedure works is that you can add any two numbers at a time and get the same answer (this is called the *associative property of addition*). For example,

$$3 + 8 + 4 + 7 + 2 + 9$$

$$(3 + (8 + (4 + 7))) + (2 + 9) \qquad ((3 + 8) + (4 + 7)) + (2 + 9)$$

Secondly, adding two numbers from a list reduces the # of numbers left to add:

But, with an infinite list of numbers there are two problems with this approach:

1.

2.

Therefore **we need a mechanism to add infinite lists of numbers which goes beyond basic arithmetic**. It turns out that calculus can be used to address this problem.

5.2 Convergence and divergence

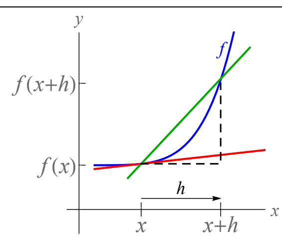
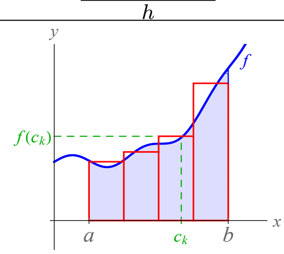
QUESTION

What does calculus have to do with adding infinite lists?

Recall: adding up an infinite list of numbers is hard because

1. when adding numbers two at a time, you never run out of numbers to add; and
2. the associative property fails (i.e. regrouping and rearranging terms of an infinite sum is generally illegal).

In calculus, we have encountered other problems which are difficult to solve:

PROBLEM	APPROXIMATION OF THE SOLUTION	HOW THE APPROX. IMPROVES	THEORETICAL SOLUTION
Find slope of the tangent line to function f at x	 slope of secant line $\frac{f(x+h)-f(x)}{h}$	As $h \to 0$	the *derivative* $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
Find area under the graph of function f from a to b	 Riemann sum $\sum_{k=1}^{n} f(c_k)\Delta x_k$	As $\|\|\mathcal{P}\|\| \to 0$ (i.e. $\frac{b-a}{n} \to 0$, i.e. $n \to \infty$)	the *integral* $\int_a^b f(x)\,dx = \lim_{\|\|\mathcal{P}\|\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$
Find sum of an infinite list of numbers $a_1 + a_2 + \cdots$			

Partial sums

Definition 5.1 *Given an infinite series $a_1 + a_2 + a_3 + a_4 + ...$, and given any index N, the N^{th}* **partial sum** *of the series, denoted S_N, is*

$$S_N = a_1 + a_2 + a_3 + ... + a_N.$$

Note: S_N is always defined, since it is a sum of finitely many numbers.

Remark: The indexing of an infinite series does not always start with the index 1. In general, the N^{th} partial sum of an infinite series is the sum of all terms in the series whose index is $\leq N$. For example, if your series is

$$a_5 + a_6 + a_7 + a_8 + a_9 + ...,$$

then the seventh partial sum of this series is

In particular, to get S_N, you add up terms until you get to index N (you don't necessarily add up N terms).

EXAMPLE 3

Find the second, fourth and fifth partial sums of each of the following infinite series (assume that each series starts with the term a_1):

a) $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$

b) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

Solution:

$$S_2 = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}.$$

$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \boxed{\frac{7}{12}}.$$

$$S_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = \boxed{\frac{47}{60}}.$$

Definitions of convergence and divergence

Definition 5.2 *Let $a_1 + a_2 + a_3 + a_4 + \cdots$ be an infinite series. For each N, let*

$$S_N = a_1 + a_2 + ... + a_N$$

be the N^{th} partial sum of the series. Then:

1. *If L is a real number such that $\lim_{N\to\infty} S_N = L$, then we say that the infinite series $a_1 + a_2 + a_3 + \cdots$* **converges (to L)** *and we write*

$$a_1 + a_2 + a_3 + \cdots = L.$$

   In this setting L is called the **sum** *of the series.*

2. *If $\lim_{N\to\infty} S_N = \pm\infty$ or if $\lim_{N\to\infty} S_N$ DNE, then we say that the infinite series $a_1 + a_2 + a_3 + \cdots$* **diverges**.

General questions related to infinite series

1. **Classification problem:** Does the infinite series $a_1 + a_2 + a_3 + \cdots$ converge or diverge?

2. **Computation problem:** If the infinite series $a_1 + a_2 + a_3 + \cdots$ converges, what is its sum?

3. **Rearrangement problem:** When, if ever, can you legally rearrange or regroup the terms of the infinite series $a_1 + a_2 + a_3 + \cdots$ without affecting whether or not the series converges and without affecting the sum of the series?

Before addressing these questions, we turn to issues with notation.

5.3 Σ-notation

Writing $a_1 + a_2 + a_3 + ...$ over and over again is annoying. We shorthand this expression by writing

$$\sum_{n=1}^{\infty} a_n$$

Using this notation,

- n is called the **index of summation**, a.k.a. the **variable of summation**;
- 1 is called the **initial index**, a.k.a. the **starting index**, a.k.a. the **lower index**;
- ∞ is called the **end index**, a.k.a. **last index**, a.k.a. **upper index**;
- the individual numbers $a_1, a_2, a_3, ...$ are called the **terms** of the series;
- and the number a_n is called the n^{th} **term**.

  (It is also OK to call the "first term" the first one you write down, but we usually won't do this.)

EXAMPLE 4

Write out the following series with + signs, and identify its second term:

$$\sum_{n=1}^{\infty} \frac{n}{n+3}$$

NOTE: We will see that some of the things we want to say about series do not depend on the starting index of the series. In this setting, we will just write $\sum a_n$ to represent the series. If you see $\sum a_n$ (without the upper and lower indices indicated), this means one of two things:

1. The starting index was given earlier in the problem and is being omitted solely for the sake of brevity (while this is "legal", it is not recommended that you do this), or
2. Some property of the series is being described which does not depend on the starting index of the series (so the starting index is irrelevant to the context).

EXAMPLE 5

For each given infinite series:

a) Write the series out with + signs.
b) Identify the second term of the series.
c) Compute the third partial sum of the series.
d) Identify the ninth term of the series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

2. $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n^2+2}$

3. $\displaystyle\sum_{n=0}^{\infty} n$

Solution: (a) $\displaystyle\sum_{n=0}^{\infty} n = \boxed{0+1+2+3+4+5+\cdots}$

(b) $a_2 = \boxed{2}$.

(c) $S_3 = a_0 + a_1 + a_2 + a_3 = 0 + 1 + 2 + 3 = \boxed{6}$.

(d) $a_9 = \boxed{9}$.

EXAMPLE 6

Write each of the following series in Σ-notation:

a) $\frac{5}{7}+\frac{5}{8}+\frac{5}{9}+\frac{5}{10}+\frac{5}{11}+\cdots$

b) $\frac{2}{3}+\frac{2}{7}+\frac{2}{11}+\frac{2}{15}+\frac{2}{19}+\cdots$

c) $\frac{7}{8}+\frac{10}{16}+\frac{13}{2^5}+\frac{16}{2^6}+\frac{19}{2^7}+\cdots$

d) $1-\frac{1}{5}+\frac{1}{9}-\frac{1}{13}+\frac{1}{17}-\frac{1}{21}+\cdots$

5.4 Elementary properties of convergence and divergence

First, we restate the definition of convergence using $\Sigma-$notation:

Definition 5.3 *Let $\sum a_n$ be an infinite series. For each N, let S_N be the N^{th} partial sum of the series; this is defined to be the sum of all the a_n for which $n \leq N$. Then:*

1. *If L is a real number such that $\lim\limits_{N\to\infty} S_N = L$, then we say the infinite series $a_1 + a_2 + a_3 + \ldots$* **converges (to** L**)** *and write $\sum a_n = L$. In this setting L is called the* **sum** *of the series.*

2. *If $\lim\limits_{N\to\infty} S_N = \pm\infty$ or if $\lim\limits_{N\to\infty} S_N$ DNE, then we say the infinite series $\sum a_n$* **diverges**.

IMPORTANT: There is a big difference between saying "$\sum a_n$ converges" and saying "a_n converges".

In particular, **you should never omit the Σ when describing whether or not an infinite series converges**.

Now, we list some elementary results about convergence of series. They should remind you of similar results for improper integrals (which should make some sense, since integration is like "continuous addition").

Theorem 5.4 (Linearity I) *Suppose $\sum a_n$ is an infinite series that converges to L and $\sum b_n$ is an infinite series that converges to M. Then:*

1. *The series $\sum(a_n + b_n)$ converges to $L + M$.*

2. *The series $\sum(a_n - b_n)$ converges to $L - M$.*

3. *For any constant k, the series $\sum(k\, a_n)$ converges to kL.*

Theorem 5.5 (Linearity II) *Suppose $\sum a_n$ is an infinite series that converges to L and $\sum b_n$ is an infinite series that diverges. Then:*

1. *The series $\sum(a_n + b_n)$ diverges.*
2. *The series $\sum(a_n - b_n)$ diverges.*
3. *For any constant $k \neq 0$, the series $\sum(k\, b_n)$ diverges.*

Theorem 5.6 (Linearity III) *Suppose $\sum a_n$ is an infinite series that diverges and $\sum b_n$ is an infinite series that diverges. Then:*

1. *The series $\sum(a_n + b_n)$ might converge or diverge.*
2. *The series $\sum(a_n - b_n)$ might converge or diverge.*

<u>Essential content of these three theorems</u>

Linearity I:	"convergent $\pm$ convergent = convergent" "constant times convergent = convergent"
Linearity II:	"convergent $\pm$ divergent = divergent" "nonzero constant times divergent = divergent"
Linearity III:	"divergent $\pm$ divergent = unknown"

(These are the same maxims that applied to improper integrals.)

Theorem 5.7 (Starting Index is Irrelevant) *Suppose $\sum\limits_{n=M_1}^{\infty} a_n$ is an infinite series. Then, so long as all the terms are defined, for any constant M_2 we have:*

1. *$\sum\limits_{n=M_1}^{\infty} a_n$ converges if and only if $\sum\limits_{n=M_2}^{\infty} a_n$ converges.*
2. *$\sum\limits_{n=M_1}^{\infty} a_n$ diverges if and only if $\sum\limits_{n=M_2}^{\infty} a_n$ diverges.*

NOTE: Suppose that $\sum\limits_{n=M_1}^{\infty} a_n$ converges. Then since the starting index is irrelevant, then $\sum\limits_{n=M_2}^{\infty} a_n$ also converges. But in general, these two series **do not converge to the same sum**, as we see in the next example.

EXAMPLE 7

We may see later why $\sum_{n=2}^{\infty} \frac{n}{3^n} = \frac{5}{12}$ and $\sum_{n=2}^{\infty} \frac{n^2}{3^n} = \frac{7}{6}$. Assuming these two facts, compute the sum of each of these series:

a) $\sum_{n=0}^{\infty} \frac{n}{3^n}$

b) $\sum_{n=4}^{\infty} \frac{12n^2}{3^n}$

c) $\sum_{n=2}^{\infty} \left(\frac{4n - 3n^2}{3^n} \right)$

d) $\sum_{n=2}^{\infty} \frac{18n^2}{3^{n+3}}$

5.5 Changing indices

MOTIVATING EXAMPLE

Write out each of the following two series:

$$\sum_{n=2}^{\infty} \frac{1}{n+3} \qquad\qquad \sum_{n=1}^{\infty} \frac{1}{n+4}$$

It is very useful to master the ability to change the starting index of a series (see Example 12 two pages from now for one reason why).

EXAMPLE 8

Rewrite the infinite series $\displaystyle\sum_{n=3}^{\infty} \frac{(n-1)^4}{3^n}$ so that its starting index is $n = 1$.

Solution # 1: Write the series out, then put it back into Σ-notation.

Solution # 2: Perform a substitution which replaces "old n" with "new n".

EXAMPLE 9

Rewrite the infinite series $\sum_{n=1}^{\infty} \frac{4(n+2)^5}{n^n}$ so that its starting index is $n = 0$.

EXAMPLE 10

Rewrite the infinite series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^4+1}$ so that its starting index is $n = 2$.

EXAMPLE 11

Rewrite the infinite series $\sum_{n=3}^{\infty} \frac{n^{n-1}}{2n^2+n}$ so that its starting index is $n = 1$.

Solution: the series used to start at $1 + 2 = 3$ and now should start at 1.

So we want "new n" $+2 =$ "old n", i.e. we replace n with $n + 2$.
Therefore

$$\sum_{n=3}^{\infty} \frac{n^{n-1}}{2n^2+n} = \sum_{n+2=3}^{\infty} \frac{(n+2)^{n+2-1}}{2(n+2)^2+n+2} = \boxed{\sum_{n=1}^{\infty} \frac{(n+2)^{n+1}}{2(n+2)^2+n+2}}.$$

EXAMPLE 12

Earlier in this chapter, we found that

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty}\frac{1}{2^n} = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = \frac{1}{1-\frac{1}{2}} = 2.$$

Use this fact to compute the following sums:

1. $\displaystyle\sum_{n=3}^{\infty}\frac{1}{2^n}$

   Solution # 1: add/remove terms

   Solution # 2: change indices

2. $\displaystyle\sum_{n=-1}^{\infty}\frac{4}{2^n}$

   $\displaystyle\sum_{n=-1}^{\infty}\frac{4}{2^n}$

3. $\displaystyle\sum_{n=2}^{\infty}\left(\frac{1}{2^n}+\frac{3}{2^{n+1}}\right)$

   $\displaystyle\sum_{n=2}^{\infty}\left(\frac{1}{2^n}+\frac{3}{2^{n+1}}\right)$

5.6 The Comparison Test for series

Using the same logic we used with improper integrals, we get the following theorem:

Theorem 5.8 (Comparison Test) *Suppose $0 \le a_n \le b_n$ for all n. Then:*

1. *If the infinite series $\sum a_n$ diverges, then $\sum b_n$ diverges.*
2. *If the infinite series $\sum b_n$ converges, then $\sum a_n$ converges.*

Remarks (these should remind you of similar remarks about the Comparison Test for improper integrals):

1. The Comparison Test is only useful for series where all the terms are positive. No subtraction allowed!
2. Be very careful with the logic!

   Thinking of $\sum a_n$ as the "small series" and think of $\sum b_n$ as the "big series":

   What you can do with the Comparison Test:

   - You **can** conclude that the small series converges.
   - You **can** conclude that the big series diverges.

   What you cannot do with the Comparison Test:

   - You **cannot** conclude that the big series converges.
   - You **cannot** conclude that the small series diverges.

Idea: Use this test with series that are similar to a "simpler" series (usually the simpler series is a $p-$series or a geometric series (see the next chapter)). Reason as follows:

	the given series $\le$ the "simpler" series	the "simpler" series $\le$ the given series
the "simpler" series converges	**Conclusion:** By the Comparison Test, the given series **converges**.	No conclusion can be drawn from the Comparison Test
the "simpler" series diverges	No conclusion can be drawn from the Comparison Test	**Conclusion:** By the Comparison Test, the given series **diverges**.

Classes of series which suggest the use of the Comparison Test:

1. Series whose terms contain addition or subtraction in the denominator:

$$\frac{\square}{\triangle + \star} \leq \frac{\square}{\triangle}, \quad \frac{\square}{\triangle + \star} \leq \frac{\square}{\star} \qquad\qquad \frac{\square}{\triangle - \star} \geq \frac{\square}{\triangle}$$

2. Series whose terms contain sines and cosines:

$$-1 \leq \cos\square \leq 1 \qquad\qquad -1 \leq \sin\square \leq 1$$

EXAMPLE 13

Determine, with appropriate justification, whether the series

$$\sum_{n=1}^{\infty} \frac{5}{2^n + n^3 + 4}$$

converges or diverges.

5.7 Harmonic and p-series

Terminology

Recall: When studying the improper integrals called p-integrals, we saw that

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \le 1 \end{cases}$$

Question: If we looked at similar-looking series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, would these series have the same behavior as the integrals?

Definition 5.9 *An infinite series is called a $p-$**series** it is of the form $\sum \frac{1}{n^p}$ for some constant $p > 0$.*

The p-series with $p = 1$ has a special name:

Definition 5.10 ***The*** **harmonic series** *is the infinite series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

A **harmonic series** *is any infinite series of the form*

$$\sum \frac{A}{Bn + C}$$

where A, B and C are constants (with $B \neq 0$).

EXAMPLE 14

Determine if each given series is a p-series; if it is, give the value of p. Also determine if the series is harmonic.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6}$

b) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}}$

c) $\displaystyle\sum_{n=3}^{\infty} n^2$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

e) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{3n+1}$

f) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{4^n}$

Divergence of harmonic series

Theorem 5.11 *The harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges.*

PROOF We will use the Comparison Test to argue this. Let $b_n = \frac{1}{n}$ so that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} b_n \\
&= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{16} + \frac{1}{17} + \cdots \frac{1}{32} + \frac{1}{2^5+1} + \cdots + \frac{1}{2^6} + \frac{1}{2^6+1} + \cdots + \frac{1}{2^7} + \cdots \\
&> \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{16} + \frac{1}{32} + \cdots \frac{1}{32} + \frac{1}{2^6} + \cdots + \frac{1}{2^6} + \frac{1}{2^7} + \cdots + \frac{1}{2^7} + \cdots
\end{aligned}$$

Notice that $\sum a_n =$

Since $0 \leq a_n \leq b_n$, by the Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges. □

Theorem 5.12 *Any harmonic series* $\sum \frac{A}{Bn+C}$ *diverges.*

PROOF First,

$$\sum \frac{A}{Bn+C} = \sum \frac{A}{B(n+\frac{C}{B})} = \frac{A}{B} \sum \frac{1}{n+\frac{C}{B}}.$$

Now, change indices by letting "new n" be "old n" $+\frac{C}{B}$.

This turns the series into $\frac{A}{B} \sum \frac{1}{n}$.

We have a constant times the harmonic series, which diverges. □

Convergence of p-series for $p > 1$

Theorem 5.13 *When $p > 1$, the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.*

PROOF We will again use the Comparison Test.

Suppose $p > 1$. Now, let $r = 2^{1-p}$. Since $p > 1$, $1 - p < 0$ so $r = 2^{1-p} < 2^0 = 1$. Therefore

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \ldots \text{ converges to } \frac{1}{1-r}.$$

(The above series will be our "$\sum b_n$" in the Comparison Test.)
Let $a_n = \frac{1}{n^p}$, so that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} a_n \\
&= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{8^p} + \frac{1}{9^p} + \cdots \\
&= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{7^p} + \frac{1}{8^p} + \cdots + \frac{1}{15^p} + \frac{1}{16^p} + \cdots \frac{1}{31^p} + \frac{1}{(2^5)^p} + \cdots + \frac{1}{(2^6-1)^p} + \frac{1}{(2^6)^p} + \cdots \\
&< 1 + \frac{1}{2^p} + \frac{1}{2^p} + \frac{1}{4^p} + \cdots + \frac{1}{4^p} + \frac{1}{8^p} + \cdots + \frac{1}{8^p} + \frac{1}{16^p} + \cdots \frac{1}{16^p} + \frac{1}{(2^5)^p} + \cdots + \frac{1}{(2^5)^p} + \frac{1}{(2^6)^p} + \cdots \\
&= 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \frac{16}{16^p} + \frac{2^5}{(2^5)^p} + \frac{2^6}{(2^6)^p} + \cdots \\
&= 1 + 2^{1-p} + 4^{1-p} + 8^{1-p} + 16^{1-p} + (2^5)^{1-p} + (2^6)^{1-p} + \cdots \\
&= 1 + 2^{1-p} + (2^2)^{1-p} + (2^3)^{1-p} + (2^4)^{1-p} + (2^5)^{1-p} + (2^6)^{1-p} + \cdots \\
&= 1 + 2^{1-p} + (2^{1-p})^2 + (2^{1-p})^3 + (2^{1-p})^4 + (2^{1-p})^5 + (2^{1-p})^6 + \cdots \\
&= 1 + r + r^2 + r^3 + r^4 + r^5 + r^6 + \cdots \\
&= \sum b_n.
\end{aligned}
$$

Therefore, by the Comparison Test, since $\sum b_n$ converges and $0 \le a_n \le b_n$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ also converges as well. □

The general p-series test

Theorem 5.14 (p-Series Test) *If* $\sum a_n = \sum \frac{1}{n^p}$ *is a* $p-$*series, then*

1. $\sum a_n = \sum \frac{1}{n^p}$ *converges if* $p > 1$.
2. $\sum a_n = \sum \frac{1}{n^p}$ *diverges if* $p \leq 1$.

PROOF The only thing we haven't proven yet is the second statement.

We know $\sum \frac{1}{n}$ diverges (since it is harmonic).

If $p \leq 1$, then $n^p \leq n^1 = n$, so $0 \leq \frac{1}{n} \leq \frac{1}{n^p}$.

So by the Comparison Test, $\sum \frac{1}{n^p}$ also diverges. $\square$

Examples (some applying the Comparison Test)

EXAMPLE 15

Determine, with appropriate justification, whether each series converges or diverges:

a) $\sum \frac{-1}{4n}$

b) $\sum \frac{3}{n\sqrt{n}}$

c) $\sum \left(\frac{7}{5n^3} - \frac{2}{n^8} \right)$

d) $\sum \left(\frac{4}{n^8} + \frac{2}{3n+5} \right)$

Solution: $\sum \left(\frac{4}{n^8} + \frac{2}{3n+5} \right) = 4 \sum \frac{1}{n^8} + \sum \frac{2}{3n+5}$.

The first sum converges (it is a p-series with $p = 8 > 1$).

But, the second series diverges (it is harmonic).

So the entire series is the sum of a convergent and divergent series, which **diverges**.

e) $\displaystyle\sum \frac{2\sqrt[3]{n}}{5\sqrt{n}}$

f) $\displaystyle\sum_{n=3}^{\infty} \frac{7}{(n-3)^5}$

g) $\displaystyle\sum_{n=1}^{\infty} \frac{3n+7n^2}{12n^6}$

Solution: $\displaystyle\sum \frac{3n+7n^2}{12n^6} = \sum \left(\frac{3n}{12n^6} + \frac{7n^2}{12n^6}\right) = \frac{1}{4}\sum \frac{1}{n^5} + \frac{7}{12}\sum \frac{1}{n^4}$

This is the sum of two convergent p-series ($p = 5 > 1$ for the first, $p = 4 > 1$ for the second), so this series **converges**.

h) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^3+3n+8}$

i) $\displaystyle\sum_{n=10}^{\infty} \frac{2}{\sqrt[3]{n}-3}$

j) $\sum_{n=1}^{\infty} \frac{3 - \cos n}{n}$

k) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+4}\right)$

l) $\sum_{n=0}^{\infty} \frac{n}{n^4 + 1}$

Solution: Notice $0 \leq \frac{n}{n^4+1} \leq \frac{n}{n^4} = \frac{1}{n^3}$.

$\sum \frac{1}{n^3}$ converges (it is a p-series with $p = 3 > 1$).

$\Rightarrow \sum a_n$ **converges** by the Comparison Test.

5.8 Homework exercises

Exercises from Sections 5.2 to 5.3

1. Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$.

   a) Perform a partial fraction decomposition on the expression $\frac{1}{n^2+n}$.

   b) Substitute your partial fraction decomposition from part (a) into the series. Then, write out the terms you would have to add to compute the N^{th} partial sum S_N.

   c) Simplify the expression for S_N you found in part (b), by cancelling terms and seeing what is left.

   d) Compute the sum of this series.

2. Compute the sum of the series $\sum_{n=1}^{\infty} \frac{4}{n^2+2n}$.

   Hint: Repeat the steps used in Exercise 1.

In Exercises 3-14, you are given an infinite series, written out with $+$ signs. Write each given series in $\Sigma-$notation.
Note: there are multiple correct answers to these questions.

3. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$

4. $\frac{3}{25} + \frac{4}{125} + \frac{5}{625} + \frac{6}{5^5} + \frac{7}{5^6} + \ldots$

5. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{3^5} + \ldots$

6. $-2 + 2 - 2 + 2 - 2 + 2 - 2 + 2 \ldots$

7. $\frac{4}{8} + \frac{7}{15} + \frac{10}{22} + \frac{13}{29} + \frac{16}{36} + \frac{19}{43} + \ldots$

8. $-\frac{2}{9} - \frac{2}{25} - \frac{2}{49} - \frac{2}{81} - \frac{2}{121} - \ldots$

9. $\frac{1}{16} - \frac{1}{64} + \frac{1}{4^4} - \frac{1}{4^5} + \ldots$

10. $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \ldots$

11. $\frac{1}{14} + \frac{1}{17} + \frac{1}{20} + \frac{1}{23} + \ldots$

12. $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+...$

13. $\frac{1}{3}-\frac{1}{2\cdot 3^2}+\frac{1}{3\cdot 3^3}-\frac{1}{4\cdot 3^4}+...$

14. $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+...$

In Exercises 15-22, you are given an infinite series in $\Sigma-$notation. For each series, find the third term of the series (if it exists), find the ninth term of the series, and find the fourth partial sum of the series.

15. $\sum_{n=1}^{\infty} \frac{2n-1}{n}$

16. $\sum_{n=0}^{\infty} [1+(-1)^n]$

17. $\sum_{n=4}^{\infty} \frac{1}{n}$

18. $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n+1}$

19. $\sum_{n=0}^{\infty} n!$

20. $\sum_{n=2}^{\infty} (-1)^n \frac{3}{2n-1}$

21. $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

22. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Exercises from Sections 5.4 to 5.5

In Exercises 23-30, you are given an infinite series. Rewrite each series in $\Sigma-$notation such that the initial term of each rewritten series corresponds to the given index.

23. $\sum_{n=3}^{\infty} \frac{4}{(n+1)(n+2)}$; starting index 0

24. $\sum_{n=2}^{\infty} \frac{3^{2n-1}}{n!}$; starting index 4

25. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}$; starting index 2

26. $\sum_{n=5}^{\infty} \frac{2n-1}{(n-2)^3-n}$; starting index 1

27. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{e^{-3n}}$; starting index 0

28. $18-6+2-\frac{2}{3}+\frac{2}{9}-\frac{2}{27}+...$; starting index 3

29. $2+\frac{2}{3}+\frac{2}{5}+\frac{2}{7}+\frac{2}{9}+....$; starting index 4

30. $\frac{3}{8}-\frac{4}{11}+\frac{5}{14}-\frac{6}{17}+\frac{7}{20}-\frac{8}{23}+...$; starting index 0

In Exercises 31-38, you are given an infinite series that converges to some number. Then you are given a second infinite series which relates to the first series in some way. Compute the sum of the second series.

31. Given $\sum_{n=0}^{\infty}\frac{1}{n!}=e$, compute $\sum_{n=2}^{\infty}\frac{1}{n!}$.

32. Given $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, compute $\sum_{n=1}^{\infty}\frac{3}{n^2}$.

33. Given $\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{(2n)!}=-1$, compute $\sum_{n=2}^{\infty}\frac{(-1)^n3\pi^{2n}}{(2n)!}$.

34. Given $\sum_{n=0}^{\infty}\frac{e^{-2}2^n}{n!}=1$, compute $\sum_{n=3}^{\infty}\frac{2^{n+3}}{n!}$.

35. Given $\sum_{n=0}^{\infty}\frac{5^n}{n!}=e^5$, compute $\sum_{n=2}^{\infty}\frac{5^n}{(n+2)!}$.

36. Given $\sum_{n=0}^{\infty}\frac{1}{2^n}=2$, compute $\sum_{n=-3}^{\infty}\frac{3}{2^n}$.

37. Given $\sum_{n=0}^{\infty}n\left(\frac{1}{2}\right)^n=2$, compute $\sum_{n=2}^{\infty}n\left(\frac{1}{2}\right)^{n+2}$.

38. Given $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-...=\frac{\pi}{4}$, compute $\frac{2}{7}-\frac{2}{9}+\frac{2}{11}-\frac{2}{13}+\frac{2}{15}-....$

In Exercises 39-46, you are to assume that

$$\sum_{n=0}^{\infty}\frac{n^2}{4^n}=\frac{7}{27}\quad\text{and}\quad\sum_{n=0}^{\infty}\frac{n}{4^n}=\frac{7}{36}\quad\text{and}\quad\sum_{n=0}^{\infty}\frac{1}{7^n}=\frac{7}{6}.$$

Using these facts, compute each expression:

39. $\sum_{n=0}^{\infty}\frac{3n^2+8}{7^n}$

40. $\sum_{n=2}^{\infty}\frac{3n}{5\cdot 7^n}$

41. $\sum_{n=-1}^{\infty}\frac{1}{20\cdot 7^n}$

42. $\sum_{n=0}^{\infty}\frac{(n+3)^2}{7^n}$

43. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{7^{n-2}}$

44. $\displaystyle\sum_{n=0}^{\infty} \frac{n+4}{7^{n+1}}$

45. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{7^{n+3}}$

46. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{7^n} - \sum_{n=2}^{\infty} \frac{n^3}{7^n}$

Exercises from Sections 5.6 to 5.8

In Exercises 47-60, determine whether each given series converges or diverges. Be sure to adequately justify your reasoning, giving arguments like those in the examples of this text.

47. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^4}$

48. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{5n-3}$

49. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\sqrt[3]{n}}$

50. $\displaystyle\sum_{n=1}^{\infty} \frac{-3}{2\sqrt{n}\sqrt[3]{n}\sqrt[4]{n}}$

51. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4}{n^5} + \frac{2}{n}\right)$

52. $\displaystyle\sum_{n=2}^{\infty} (n-1)^{-1/2}$

53. $\displaystyle\sum_{n=4}^{\infty} \frac{1}{\ln(2^n)}$

54. $\displaystyle\sum_{n=1}^{\infty} \frac{2+\cos(3n)}{n}$

55. $\displaystyle\sum_{n=0}^{\infty} \frac{3+\cos(2n)}{4n^2}$

56. $\displaystyle\sum_{n=0}^{\infty} \frac{3+\cos(2n)}{4\sqrt[3]{n}}$

57. $\displaystyle\sum_{n=2}^{\infty} \frac{3+n^3}{n^5+4}$

58. $\displaystyle\sum_{n=1}^{\infty} \frac{4+\sin(n^2+2n)}{\sqrt[3]{n^5+1}}$

59. $\displaystyle\sum_{n=3}^{\infty} \left(\frac{3}{n\sqrt{n}} + \frac{\sqrt{n}}{5n^3}\right)$

60. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

Answers

Note: there are multiple correct answers to numbers 3-12.

1. a) $\displaystyle\frac{1}{n^2+n} = \frac{1}{n} + \frac{-1}{n+1}$

   b) $S_N = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{N}-\frac{1}{N+1}\right)$

   c) $S_N = 1 - \frac{1}{N+1}$

   d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \lim_{N\to\infty} S_N = 1$

2. 3

3. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

4. $\sum_{n=3}^{\infty} \frac{n}{5^{n-1}}$

5. $\sum_{n=0}^{\infty} \frac{1}{3^n}$

6. $\sum_{n=1}^{\infty} 2 \cdot (-1)^n$

7. $\sum_{n=1}^{\infty} \frac{3n+1}{7n+1}$

8. $\sum_{n=1}^{\infty} \frac{-2}{(2n+1)^2}$

9. $\sum_{n=2}^{\infty} \left(\frac{-1}{4}\right)^n$

10. $\sum_{n=0}^{\infty} \frac{1}{n!}$

11. $\sum_{n=1}^{\infty} \frac{1}{3n+11}$

12. $\sum_{n=1}^{\infty} \frac{n}{2^n}$

13. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n3^n}$

14. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

15. third term is $\frac{5}{3}$; ninth term is $\frac{17}{9}$; fourth partial sum is $\frac{71}{12}$.

16. third term is 0; ninth term is 0; fourth partial sum is 6.

17. third term is 0; ninth term is $\frac{1}{9}$; fourth partial sum is $\frac{1}{4}$.

18. third term is $\frac{-1}{4}$; ninth term is $\frac{-1}{10}$; fourth partial sum is $\frac{-13}{60}$.

19. third term is 6; ninth term is 362880; fourth partial sum is 34.

20. third term is $\frac{-3}{5}$; ninth term is $\frac{-3}{17}$; fourth partial sum is $\frac{29}{35}$

21. third term is $\frac{1}{12}$; ninth term is $\frac{1}{90}$; fourth partial sum is $\frac{4}{5}$

22. third term is $\frac{-1}{3}$; ninth term is $\frac{-1}{9}$; fourth partial sum is $\frac{-7}{12}$

23. $\sum_{n=0}^{\infty} \frac{4}{(n+4)(n+5)}$

24. $\sum_{n=4}^{\infty} \frac{3^{2n-5}}{(n-2)!}$

25. $\sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)2^{n-1}}$

26. $\sum_{n=1}^{\infty} \frac{2n+7}{(n+2)^3-n-4}$

27. $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{e^{-3n-3}}$

28. $\sum_{n=3}^{\infty} (-1)^{n+1} 486 \left(\frac{1}{3}\right)^n$

29. $\sum_{n=4}^{\infty} \frac{2}{2n-7}$

30. $\sum_{n=0}^{\infty} (-1)^n \frac{n+3}{3n+8}$

31. $e-2$

32. $\dfrac{\pi^2}{2}$
33. $-6 + \dfrac{3}{2}\pi^2$
34. $8e^2 - 40$
35. $\dfrac{1}{25}e^2 - \dfrac{118}{75}$
36. 48
37. $\dfrac{3}{8}$
38. $\dfrac{\pi}{2} - \dfrac{26}{15}$
39. $\dfrac{91}{9}$
40. $\dfrac{13}{420}$
41. $\dfrac{49}{120}$
42. $\dfrac{322}{27}$
43. $\dfrac{1}{6}$
44. $\dfrac{25}{36}$
45. $\dfrac{1}{1323}$
46. $\dfrac{1}{7}$
47. converges
48. diverges
49. diverges
50. converges
51. diverges
52. diverges
53. diverges
54. diverges
55. converges
56. diverges
57. converges
58. converges
59. converges
60. converges

Chapter 6

Geometric series and the Ratio Test

6.1 Definitions

By far, the most important class of infinite series are *geometric series*. We first encountered this class of series when we discussed the race between Usain Bolt and me at the beginning of Chapter 5.

Definition 6.1 *A series $\sum a_n$ is called* **geometric** *if there exists a real number r such that $a_{n+1} = ra_n$ for all n. The number r is called the* **common ratio** *of the series.*

Why is r called the "common ratio"?

Consequence: suppose the initial index of the geometric series is $n = 0$. Then, by repeated application of the formula $a_{n+1} = ra_n$, we see:

$$\begin{aligned} &a_0 \\ &a_1 = \\ &a_2 = \\ &a_3 = \\ &\vdots \end{aligned}$$

On the previous page, we saw that for any geometric series with initial term a_0 and common ratio r, the n^{th} term a_n is

$$a_n = a_0 r^n.$$

If we change notation and call the number a_0 just "a", we get the following important characterization of geometric series:

Theorem 6.2 (Characterization of geometric series) *Every infinite geometric series can be written in the standard form*

$$\sum_{n=0}^{\infty} ar^n$$

where a is the initial term of the series and r is the common ratio of the series.

In other words, **every geometric series is the sum of a constant times all the non-negative powers of the common ratio**.

If you are given a geometric series, you can always rewrite the series in the standard form given in the theorem above, where the starting index is $n = 0$.

To study a geometric series, **your first step should almost always be to rewrite the series in this standard form**. To do this, simply write the terms out and factor out the initial term of the series.

EXAMPLE 1

For each given series, determine if the series is geometric. If it is, write the series in standard form and identify the common ratio of the series.

a) $\displaystyle\sum_{n=1}^{\infty} 3 \cdot \left(\frac{7}{8}\right)^{n+1}$

b) $2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \frac{32}{81} - \ldots$

b) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

6.2 The Geometric Series Test

TWO OF OUR BIG PICTURE QUESTIONS

Classification problem: When does a geometric series converge (or diverge)?

Computation problem: If a geometric series converges, what is its sum?

These questions are answered by a theorem called the *Geometric Series Test*:

Theorem 6.3 (Geometric Series Test (GST)) *Consider a geometric series written in standard form* $\sum_{n=0}^{\infty} ar^n$*. Then:*

1. *The series converges if and only if* $|r| < 1$ *(or if* $a = 0$*).*
2. *The series diverges if and only if* $|r| \geq 1$*.*

Furthermore, if the series converges, its sum is $\dfrac{a}{1-r}$*.*

PROOF OF THE GEOMETRIC SERIES TEST

There are several cases to consider:

Case 1: $a = 0$

In this case, every term of the series is $\boxed{}$.

Therefore the N^{th} partial sum of the series is

$S_N = \sum_{n=0}^{N} ar^n = \sum_{n=0}^{N} 0 = 0$

and since $\lim_{N\to\infty} S_N = \lim_{N\to\infty} 0 = \boxed{}$, the series ______________ to $\boxed{}$.

Case 2: $a \neq 0, r = 1$

In this case, $S_N = \sum_{n=0}^{N} ar^n = \sum_{n=0}^{N} a1^n = \sum_{n=0}^{N} a = a + a + ... + a = \boxed{}$.

Therefore $\lim_{N\to\infty} S_N = \lim_{N\to\infty} a(N+1) = \boxed{}$ so the series $\sum a_n$ **diverges**.

Case 3: $a \neq 0, r = -1$

In this case, $S_N = \sum_{n=0}^{N} ar^n = \sum_{n=0}^{N} a(-1)^n = a - a - a + a - ... \pm a$

$$= \boxed{\left\{ \right.}.$$

Therefore $\lim_{N\to\infty} S_N \boxed{}$ (S_N alternates between 0 and a),

so the series $\sum a_n$ **diverges**.

(proof continues on next page)

Case 4: $a \neq 0, |r| \neq 1$

Here,

$$S_N \;=\; \sum_{n=0}^{N} ar^n \;=\; a + ar + ar^2 + ar^3 + \cdots + ar^N$$

Therefore $\lim\limits_{N\to\infty} S_N = \lim\limits_{N\to\infty} a\dfrac{1-r^{N+1}}{1-r} = \dfrac{a}{1-r} \lim\limits_{N\to\infty}(1-r^{N+1})$

$$= \begin{cases} & \text{if } |r| < 1 \\ & \\ & \text{if } |r| > 1 \end{cases}$$

$$= \begin{cases} & \text{if } |r| < 1 \\ & \\ & \text{if } |r| > 1 \end{cases}$$

Therefore $\sum\limits_{n=0}^{\infty} ar^n = \begin{cases} \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$

This takes care of all possible cases, so the proof of the GST is complete. □

Remarks on the proof of the Geometric Series Test

Remark 1: Since you can always factor a constant a out of a series, the simplest way to state the content of the Geometric Series Test is as follows:

$$\sum_{n=0}^{\infty} r^n \begin{cases} = \dfrac{1}{1-r} & \text{if } |r|<1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases} \qquad \left(\text{i.e. } \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ for } |r|<1\right)$$

If you learn nothing else from me this semester, learn this fact above.

Remark 2: The geometric series with $a=1$ and $r=-1$ is the Dirichlet series

$$\sum_{n=0}^{\infty} 1(-1)^n = 1-1+1-1+1-1+1-1+\ldots$$

which we encountered in Chapter 5. According to the Geometric Series Test, the Dirichlet series **diverges** (since $|r| = |-1| = 1$), so none of the values for this sum obtained earlier by various regrouping procedures are correct. This series **diverges** and cannot be legally added.

Remark 3: In the context of proving the Geometric Series Test, we proved a formula for the partial sums of a geometric series.

We restate this formula as a theorem, as this result will be used in examples that follow.

In particular, this formula holds in any situation where $r \neq 1$ (**even if** $|r|>1$):

Theorem 6.4 (Finite Sum Formula for a Geometric Series) *Consider a geometric series written in standard form* $\sum\limits_{n=0}^{\infty} ar^n$ *where* $r \neq 1$*. Then the* N^{th} *partial sum satisfies*

$$\sum_{n=0}^{N} ar^n = a[1+r+r^2+r^3+\ldots+r^N] = \frac{a(1-r^{N+1})}{1-r}.$$

If $a=1$, the above formula reduces to

$$\sum_{n=0}^{N} r^n = 1+r+r^2+r^3+\ldots+r^N = \frac{1-r^{N+1}}{1-r}.$$

EXAMPLE 2

For each given (finite or infinite) series:

i. Determine if the series is geometric.
ii. If the series is geometric, determine if it series converges or diverges.
iii. If the geometric series converges, compute its sum.

a) $\dfrac{2}{3}+\dfrac{2}{27}+\dfrac{2}{3^5}+\dfrac{2}{3^7}+\ldots$

b) $4-2+1-\dfrac{1}{2}+\dfrac{1}{4}-\dfrac{1}{8}+\ldots$

c) $2 + 6 + 18 + 54 + \ldots$

d) $2 + 6 + 18 + 54 + \ldots + 2 \cdot 3^{35}$

e) $\displaystyle\sum_{n=2}^{15} 4\left(\frac{2}{3}\right)^n$

f) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$

g) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{-2}{5}\right)^n$

Solution: This geometric series is already in standard form.

We have $a = 1$, $r = \frac{-2}{5}$ so this series converges to

$$\frac{a}{1-r} = \frac{1}{1-(-2/5)} = \frac{1}{7/5} = \frac{5}{7}.$$

h) $\displaystyle\sum_{n=1}^{\infty}\frac{2\cdot 15^{n+3}}{7^{2n-1}}$

i) $\displaystyle\sum_{n=2}^{\infty}\frac{7(-1)^n 3^{2n-1}}{5\cdot 2^{4n+3}}$

At this point, we have $\displaystyle\sum_{n=0}^{\infty}\frac{7\cdot 3^3}{5\cdot 2^{11}}\left(\frac{-9}{16}\right)^n = \frac{7\cdot 3^3}{5\cdot 2^{11}}\cdot\frac{1}{1-\left(\frac{-9}{16}\right)} = \boxed{\frac{7\cdot 3^3}{5\cdot 2^{11}}\cdot\frac{16}{25}}.$

6.3 Applications of geometric series

Investments

EXAMPLE 3

Suppose you invest $\$1000$ each year in an account which earns 5% interest each year. How much will you have in your account after 30 years (assume that you make your annual deposit on January 1, and that "after 30 years" means "on December 31 of the 30th year")?

To answer this question, we first have to start with a side question.

Side question: Suppose you invest $\$P$ **once** into an account which pays interest rate R, compounded once per time period. How much do you have after n time periods?

Answer to side question:

After 0 time periods:

After 1 time period:

After 2 time periods:

$\vdots$

After n time periods:

Back to the original problem:

Initial deposit:

Second deposit:

$\vdots$

Last deposit:

Repeating decimals

EXAMPLE 4

Write the repeating decimal .243737373737... as a fraction in lowest terms.

$$\begin{aligned}
&= \frac{24}{100} + \frac{37}{10000} \cdot \frac{1}{1 - \frac{1}{100}} \\
&= \frac{24}{100} + \frac{37}{10000} \cdot \frac{100}{99} \\
&= \frac{24}{100} + \frac{37}{9900} \\
&= \frac{24 \cdot 99 + 37}{9900} = \boxed{\frac{2413}{9900}}.
\end{aligned}$$

Pharmacokinetics

EXAMPLE 5

Suppose you give a patient a 10 mg dose of a drug daily. If the patient's bodily functions remove 90% of whatever amount of that drug is in the patient's body,

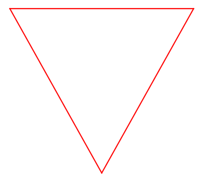
a) How much will be in the patient's body after 14 days (just before he takes the 15th dose)?

b) How much will end up in the patient's body (just before he takes each dose) if he takes the dose indefinitely?

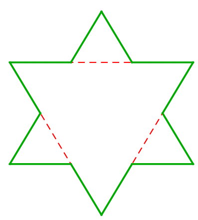
Fractal geometry

EXAMPLE 6

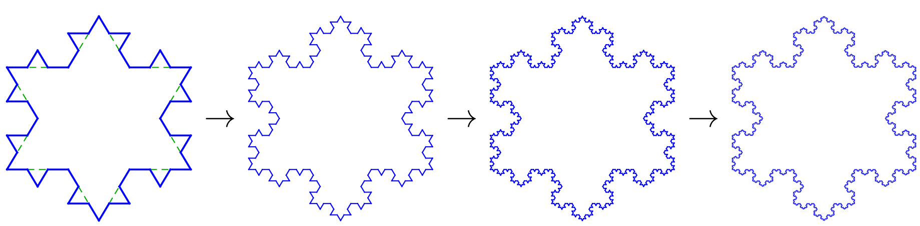
The **Koch snowflake** is a figure constructed by the following procedure: first, start with an equilateral triangle of side length 1 (the area of such a triangle is $\frac{\sqrt{3}}{4}$).



Second, divide each side of the trangle into three segments of equal length and attach an equilateral triangle to the middle of each segment. Then erase the middle of each segment. After doing this, you get the following figure:



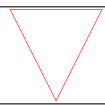
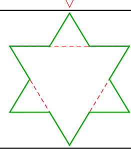
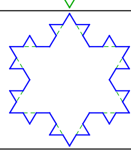
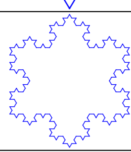
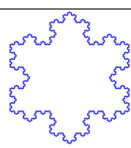
Third, repeat this procedure indefinitely. This means that at each stage, you take each side of the figure, divide it into thirds, attach an equilateral triangle to the middle third of each segment, and the erase the middle of each previous segment. If you carry out this procedure, you get the following sequence of figures in the next three steps:



Repeating this procedure infinitely many times produces a figure called the **Koch snowflake**.

a) What is the perimeter of the Koch snowflake?

b) What is the area enclosed by the Koch snowflake?

STAGE	PICTURE	#△S ADDED	# OF SIDES IN FIGURE	SIDE LENGTH OF EACH △ ADDED	AREA OF EACH △ ADDED	AREA ADDED AT THIS STAGE	ADDITIONAL PERIMETER ADDED
0		1	3	1	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{4}$	3
1							
2							
3							
$\vdots$							
n							

a) perimeter of Koch snowflake $=$

b) total area of Koch snowflake $=$

6.4 The Ratio Test

Recall our three "big picture" questions with infinite series:

General questions related to infinite series

Given an infinite series $\sum a_n$:

1. **Classification problem:** Does $\sum a_n$ converge or diverge?
2. **Computation problem:** If $\sum a_n$ converges, what is its sum?
3. **Rearrangement problem:** When, if ever, can you legally rearrange or regroup the terms of $\sum a_n$ without affecting its convergence?

In this section we address (in part) the first of these questions - determining whether or not a series converges.

Our overall approach to the classification problem is to develop a bunch of "tests" which tell us whether or not certain series converge or diverge. The trick is to figure out which test(s) to use on which series.

We have seen a few tests already: the GST, the p-series test, and the Comparison Test. In this section we develop what is probably the most useful test, based on reasoning coming from geometric series.

Developing the Ratio Test

Recall: a series $\sum a_n$ is called **geometric** if there exists a common ratio r, i.e.

$$\frac{a_1}{a_0} = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \cdots = r.$$

All geometric series can be rewritten as $\sum_{n=0}^{\infty} ar^n$; by the GST we know

$$\sum_{n=0}^{\infty} ar^n \begin{cases} = \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}.$$

Now let $\sum a_n$ be any (not necessarily geometric) infinite series. In this setting

$$\frac{a_1}{a_0}, \ \frac{a_2}{a_1}, \ \frac{a_3}{a_2}, \ \frac{a_4}{a_3}, \ \text{etc.}$$

are not likely to be the same number. However, it may be the case that these ratios get closer and closer to some number ρ (this is the Greek letter "rho"), i.e.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

This means that for large n, $\frac{a_{n+1}}{a_n} \approx \rho$ so $\sum a_n$ should behave like a geometric series with common ratio very, very close to ρ.

Consequences of this reasoning:

1. If $\rho > 1$, then $\sum a_n$ behaves like a geometric series with $|r| > 1$, i.e.

2. If $\rho < 1$, then $\sum a_n$ behaves like a geometric series with $|r| < 1$, i.e.

3. If $\rho = 1$, then $\sum a_n$ behaves like a geometric series with $r \approx 1$.

The conclusion of all this logic is what is called the *Ratio Test*:

Theorem 6.5 (Ratio Test) *Suppose* $\sum a_n$ *is an infinite series; let* $\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. *Then:*

1. *If* $\rho < 1$, *then* $\sum a_n$ *converges.*
2. *If* $\rho > 1$, *then* $\sum a_n$ *diverges.*
3. *If* $\rho = 1$, *or if* ρ *DNE, then this test tells you nothing.*

Remarks:

1. Notice in this theorem we inserted absolute value signs around the terms; this is a computational convenience that will help simplify some examples.

2. Since we are taking the limit of ratios which are positive, the value of ρ must be nonnegative. If you get $\rho < 0$, you have done something wrong (you probably forgot the absolute values inside the limit).

EXAMPLE 7

Determine whether or not the series $\sum_{n=1}^{\infty} \frac{5^n}{n7^n}$ converges or diverges.

Types of series for which the Ratio Test works well

Types of series for which the Ratio Test will NOT work well

Remarks on simplifying expressions in the Ratio Test

The computation of ρ in the Ratio Test often contains either a simplification of exponents of the form

$$\frac{c^{n+1}}{c^n} = c \quad \text{or} \quad \frac{c^n}{c^{n+1}} = \frac{1}{c} \text{ etc.}$$

or a simplification of factorials of the form

$$\frac{n!}{(n+1)!} = \frac{1}{n+1} \quad \text{or} \quad \frac{(n+1)!}{n!} = n \text{ or} \quad \frac{(2n)!}{(2n+2)!} = \frac{1}{(2n+2)(2n+1)} \text{ etc.}$$

When simplifying factorial expressions, it is often useful to write out the terms being multiplied to see how they will be cancelled. For example, the last equality above comes from

$$\begin{aligned}\frac{(2n)!}{(2n+2)!} &= \frac{2n(2n-1)(2n-2)\cdots 3\cdot 2\cdot 1}{(2n+2)(2n+1)2n(2n-1)(2n-2)\cdots 3\cdot 2\cdot 1} \\ &= \frac{1}{(2n+2)(2n+1)}.\end{aligned}$$

In general,

EXAMPLE 8

Determine whether or not each series converges or diverges.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n^2 2^{4n}}{3^{2n}}$

c) $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}$

Solution: Since the series has only multiplication/division and contains factorials, we use the Ratio Test:

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\dfrac{(2(n+1))!}{((n+1)!)^2}\right|}{\left|\dfrac{(2n)!}{(n!)^2}\right|}$$

$$= \lim_{n\to\infty} \frac{(2(n+1))!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)!}{(n+1)!(n+1)!} \cdot \frac{n!\,n!}{(2n)!}$$

$$= \lim_{n\to\infty}$$

d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!}{n^n}$

e) $\sum_{n=2}^{\infty} \frac{2n}{n^2 - 1}$

6.5 Homework exercises

Exercises from Section 6.2

In Exercises 1-20, find the sum of each finite or infinite series (assuming the series converges). If the series diverges, say so.

1. $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$
2. $\sum_{n=0}^{\infty} \frac{-3}{6^n}$
3. $9 - \frac{9}{2} + \frac{9}{4} - \frac{9}{8} + \frac{9}{16} - \frac{9}{32} + \ldots$
4. $\sum_{n=3}^{17} \frac{2}{5^n}$
5. $\sum_{s=2}^{11} \left(\frac{3^s}{4^{2s}}\right)$
6. $2 + 4 + 8 + 16 + \ldots + 2^{100}$
7. $\sum_{n=3}^{\infty} \left(\frac{-4}{3}\right)^n$
8. $\sum_{n=2}^{\infty} \frac{6^{n-1}}{7^{n+1}}$
9. $\sum_{n=0}^{\infty} \left[\frac{3}{2^n} + \left(\frac{2}{3}\right)^n\right]$
10. $\sum_{n=1}^{\infty} \left[\frac{5}{(-1)^n 3^n} - \left(\frac{3}{5}\right)^{n+2}\right]$
11. $80 + 40 + 20 + 10 + 5 + \frac{5}{2} + \frac{5}{4} + \ldots$
12. $\sum_{t=0}^{\infty} \frac{3 \cdot 8^t}{5^{2t-3}}$
13. $\sum_{n=2}^{\infty} \frac{2 \cdot 3^{2n-1}}{5 \cdot 2^{4n+3}}$
14. $\sum_{y=0}^{\infty} 3^{-y}$
15. $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^{-n}$
16. $\sum_{n=1}^{\infty} 4^{1-2n}$
17. $\sum_{n=1}^{\infty} \frac{3^n - 5}{6^n}$
18. $\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n \left[2 + \left(\frac{1}{4}\right)^n\right]$
19. $\sum_{x=0}^{\infty} \left(\frac{1}{8}\right)^{x+1} \left[2^{2x-3} - 4^{x+1}\right]$
20. $\sum_{n=0}^{\infty} 4^n 5^{-n}$
21. Suppose the initial term of a geometric series is 3 and that the series converges to 2. What is the common ratio of the series?
22. A geometric series with common ratio $\frac{2}{3}$ converges to 8. What is the initial term of the series?

Exercises from Section 6.3

23. You invest $\$250$ at the beginning of each month into an account paying 3% interest, compounded monthly. How much will you have in the account after 4 years (meaning at the end of 48 months, but before you make your 49^{th} deposit)?

24. You want to invest a fixed amount annually into an account that earns 6% interest, compounded annually. How much do you need to invest annually so that you will have $\$250,000$ after 40 years (before you make your 41^{st} deposit)?

25. Write the repeating decimal $.71717171717...$ as a fraction in lowest terms.

26. Write the repeating decimal $1.314314314314...$ as a fraction in lowest terms.

27. Write the repeating decimal $.256161616161...$ as a fraction in lowest terms.

28. Write the repeating decimal $.132032032032032...$ as a fraction in lowest terms.

29. A ball is dropped from a height of 15 ft onto a concrete slab. Each time the ball bounces, it rebounds directly to $\frac{2}{3}$ of its previous height. Find the total distance the ball travels before it comes to rest.

30. A worker puts in 16 units of effort on his first day at work. But he gets a little lazier each day, and he only puts in $\frac{7}{8}$ as much effort on each day as he did the previous day.

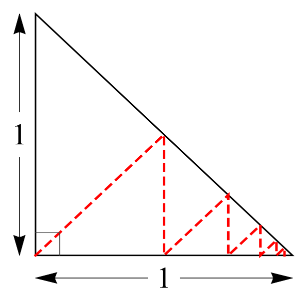
    a) What is the amount of effort the worker puts in on his 20^{th} day at work?
    b) What is the total amount of effort the worker puts in during his first 12 days at work?
    c) If the worker continues to work every day forever, what is the total amount of effort he puts in?

31. Suppose a patient takes 25 mg of a certain drug each day. If 80% of the drug is excreted by bodily functions each day, how much of the drug will be in the patient's body immediately before she takes her 22nd dose of the medicine?

32. Suppose your drinking water contains poison, and as such you ingest 0.25 mg of the poison each day. Although your body gets rid of 10% of the poison in your body each day, when you accumulate 2 mg of the poison in your system you will be dead. How long do you have before you need to stop drinking your water?

33. Suppose your drinking water contains poison, and as such you ingest 2 mg of the poison each day. Suppose further that when you accumulate 250 mg of the poison in your system you will be dead. What percent of the poison in your system does your body need to excrete daily in order to never die from the poison? (Assume that you will live forever if the poison doesn't kill you.)

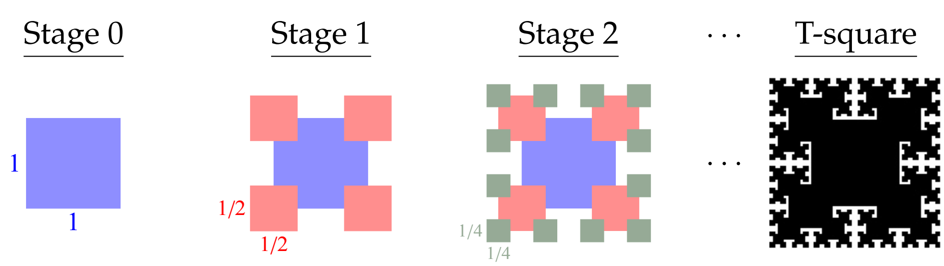
34. In the figure below, the triangle indicated by the solid lines is an isoceles right triangle whose height is 1 unit. All the red dashed line segments drawn are perpendicular to either the base of the triangle or the hypotenuse. If the red dashed line segments continue indefinitely, find the total length of the red dashed line segments.



35. The **T-square** is a shape similar to the Koch snowflake that is constructed as follows:

- **Stage 0:** start with a 1×1 square (which is colored or filled in). See the left-most picture below.
- **Stage 1:** at each corner of the square from Stage 0, draw a square of side length $\frac{1}{2}$ and color those squares in (see the second picture below).
- **Stage 2:** at each corner of the figure from Stage 2, draw a square of side length $\frac{1}{4}$ and color those squares in.
- Continue like this indefinitely: at Stage n, take the figure from Stage $n-1$ and draw a square of side length $\frac{1}{2^n}$; color those squares in.

The colored region so obtained is called the **T-square**.



a) Let $n \geq 2$. How many corners does the shape have after Stage $n-1$?

b) How many squares will be drawn in Stage n?

c) How much **new** area is added to the shape from each new square drawn in Stage n?

d) What is the total area added to the shape in Stage n?

e) Compute the area enclosed by the T-square.

f) How much net perimeter is added to the shape from each new square drawn in Stage n?

g) What is the total net perimeter is added to the shape in Stage n?

h) Compute the perimeter of the T-square.

36. In this problem we construct an example of a set called a **Cantor set**. To do this:

    - **Stage 0:** start with the interval $[0,1]$.
    - **Stage 1:** divide the interval $[0,1]$ into five equal-length pieces, and remove the middle piece (i.e. remove the middle fifth of the interval). This leaves you with two intervals.
    - **Stage 2:** take each interval left after Stage 1, divide that interval into five equal-length pieces, and remove the middle fifth.
    - Continue like this indefinitely: at Stage n, take each interval left from Stage $n-1$, divide that interval into five equal-length pieces, and remove the middle fifth.

    The points that are never removed comprise a set called a **Cantor set** (not "the" Cantor set).

    a) Let $n \geq 1$. After Stage $n-1$, how many intervals are there?

    b) Let $n \geq 1$. After Stage $n-1$, what is the length of each interval that is left?

    c) During Stage n, what is the length removed from each interval?

    d) During Stage n, what is the total length removed?

    e) What is the total length of the points in this Cantor set (meaning the total length of the points that are not removed)?

37. Suppose you roll a fair six-sided die repeatedly.

    a) On any one roll, what is the probability that you roll a 6?

    b) On any one roll, what the probability that you do *not* roll a 6?

    c) What is the probability that the first time you roll a 6 is on the fifth roll?
    Hint: Multiply together the probabilities of what has to happen on each of the first five rolls.

    d) What is the probability that the first time you roll a 6 is on the n^{th} roll?

e) What is the probability that you won't roll a 6 in the first seven rolls, but that you will roll a 6 within the first 18^{th} roll?

38. A machine part will fail with probability $\frac{1}{2000}$ on the first day it operates. However, the part starts to wear, so that its probability that the part fails on each subsequent day is 5% greater than its probability of failure on the preceding day.

    a) What is the probability that the part will fail within 60 days?

    b) What is the longest the part will last before it fails for certain?

Exercises from Section 6.4

In Exercises 39-52, determine, with justification, whether or not each series converges or diverges.

39. $\displaystyle\sum_{n=1}^{\infty} (-2)^n 2^{-n}$

40. $\displaystyle\sum_{n=4}^{\infty} \frac{7^n}{n^8 5^n}$

41. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!}$

42. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n}}{7^n}$

43. $\displaystyle\sum_{n=0}^{\infty} \frac{n^n}{n^3 n!}$

44. $\displaystyle\sum_{n=0}^{\infty} \frac{n^{2014}}{1.01^n}$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n n!}{n^n}$

46. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$

47. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n! + n^2}$

48. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n! 2n!}{(3n)!}$

49. $\displaystyle\sum_{n=0}^{\infty} \frac{n!(3n)!}{[(2n)!]^2}$

50. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^{(n^2)}}$

51. $\displaystyle\sum_{n=4}^{\infty} \frac{(2n)^{2n}}{(n^n)^2}$

52. $\displaystyle\sum_{n=1}^{\infty} \left[\frac{2^n}{n!} + \frac{n^4 3^n}{5^n}\right]$

Answers

1. $\dfrac{1}{4}$

2. $\dfrac{-18}{5}$

3. 6

4. $\dfrac{1}{50}\left[1-\left(\dfrac{1}{5}\right)^{15}\right]$

5. $\dfrac{9}{208}\left[1-\left(\dfrac{3}{16}\right)^{10}\right]$

6. $2(2^{100}-1)$

7. diverges

8. $\frac{6}{49}$

9. 9

10. $\frac{-179}{100}$

11. 160

12. $\frac{9375}{17}$

13. $\frac{27}{2240}$

14. $\frac{3}{2}$

15. diverges

16. $\frac{4}{15}$

17. 0

18. $\frac{45}{11}$

19. $\frac{-31}{32}$

20. 5

21. $r = \frac{-1}{2}$

22. $a_0 = \frac{8}{3}$

23. $\$26,885.16$

24. $\$1523.95$

25. $\frac{71}{99}$

26. $\frac{1313}{999}$

27. $\frac{634}{2475}$

28. $\frac{1319}{9990}$

29. 75 ft

30. a) $16\left(\frac{7}{8}\right)^{19}$ units b) $16(8)\left[1-\left(\frac{7}{8}\right)^{12}\right]$ units c) 128 units

31. 6.25 mg

32. 14 days (on day 15, you'll die).

33. $.92\%$

34. $1+\sqrt{2}$

35. a) $4 \cdot 3^n$
   b) $4 \cdot 3^n$
   c) $\frac{3}{4}\left(\frac{1}{2^n}\right)^2$
   d) $\frac{3^{n+1}}{4^n}$
   e) 10
   f) $\frac{2}{2^n}$
   g) $8\left(\frac{3}{2}\right)^n$
   h) ∞

36. a) 2^{n-1}
   b) $\left(\frac{2}{5}\right)^n$
   c) $\frac{1}{5}\left(\frac{2}{5}\right)^n$
   d) $2^{n-1}\left(\frac{1}{5}\right)\left(\frac{2}{5}\right)^n$
   e) $\frac{1}{2}$

37. a) $\frac{1}{6}$
   b) $\frac{5}{6}$
   c) $\left(\frac{5}{6}\right)^4 \frac{1}{6}$
   d) $\left(\frac{5}{6}\right)^{n-1} \frac{1}{6}$
   e) $\left(\frac{5}{6}\right)^7 \left[1-\left(\frac{5}{6}\right)^{11}\right]$

38. a) $\left(\frac{1}{2000}\right)\frac{1-1.05^{60}}{-.05}$
   b) The part will fail by the 95^{th} day.

39. diverges

40. diverges

41. converges

42. diverges

43. diverges

44. converges

45. diverges

46. converges

47. converges

48. converges

49. diverges

50. converges

51. diverges

52. converges

Chapter 7

More on infinite series

Recall our general questions with infinite series:

General questions related to infinite series

Given an infinite series $\sum a_n$:

1. **Classification problem:** Does $\sum a_n$ converge or diverge?
2. **Computation problem:** If $\sum a_n$ converges, what is its sum?
3. **Rearrangement problem:** When, if ever, can you legally rearrange or regroup the terms of $\sum a_n$ without affecting its convergence?

This chapter is motivated by Question 1 above: determining whether or not a series converges.

It turns out that we will discover the answer to Question 3 as we go along.

7.1 Returning to the classification problem

The n^{th}-term Test

Idea:

> **Theorem 7.1 (n^{th} Term Test)** *Suppose $\sum a_n$ is an infinite series. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.*

IMPORTANT:

Note: $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} |a_n| = 0$. Therefore we can generalize the n^{th} Term Test as follows:

> **Theorem 7.2 (n^{th} Term Test, restated)** *Suppose $\sum a_n$ is an infinite series. If*
>
> $$\lim_{n\to\infty} |a_n| \neq 0$$
>
> *(this includes when this limit is ∞ or DNE), then $\sum a_n$ diverges.*

EXAMPLE 1

Determine whether or not the n^{th} Term Test can be used to determine the convergence or divergence of each series.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{\sqrt[6]{n^5}}$

c) $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{2n+5}$

d) $\displaystyle\sum_{n=0}^{\infty} \sin n$

Classifying series according to sign

At this point, we know five basic tests which tell us whether some infinite series converges or diverges:

1.

2.

3.

4.

5. n^{th} **Term Test:** if $\lim\limits_{n\to\infty} |a_n| \neq 0$, then $\sum a_n$ diverges.

We also know:

- starting index is irrelevant to convergence/divergence;
- convergent $\pm$ convergent $=$ convergent;
- convergent $\pm$ divergent $=$ divergent;
- etc.

QUESTION

What do you do next, if this stuff doesn't help?

Next, look at the signs of the individual terms of the series. This is because the validity of other test depends in part on what the signs of the series are.

Definition 7.3 *An infinite series is called* **positive** *if all its terms are positive; more precisely* $\sum a_n$ *is* **positive** *if* $a_n \geq 0$ *for all* n.

An infinite series is called **negative** *if all its terms are negative; more precisely* $\sum a_n$ *is* **negative** *if* $a_n \leq 0$ *for all* n.

If your series is negative, you can factor out -1 from it, and then what's left will be a positive series (so maybe the Comparison Test will apply to what's left.)

7.2 Alternating series

Recall: we are trying to determine whether a given series $\sum a_n$ converges or diverges.

EXAMPLE 2

Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

This series is called the **alternating harmonic series**. Does it converge or diverge?

- This series is not geometric $\Rightarrow$
- This series does not contain exponentials or factorials $\Rightarrow$
- This series is not a p-series $\Rightarrow$
- This series is neither positive nor negative $\Rightarrow$
- $\lim_{n\to\infty} |a_n| =$

Now what?

Back to the definition of convergence: to say $\sum a_n = L$ means

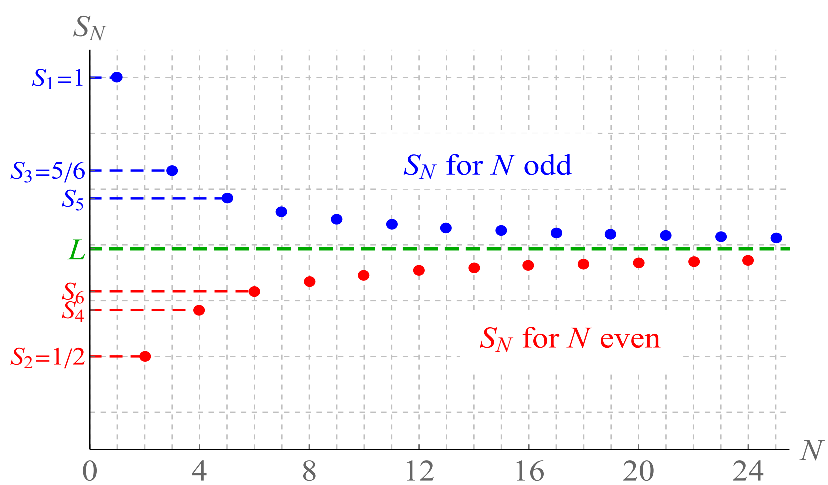
Let's compute the partial sums of the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$:

$$S_1 = a_1 = 1$$

$$S_2 = a_1 + a_2 = 1 - \frac{1}{2} =$$

$$S_3 = a_1 + a_2 + a_3 = 1 - \frac{1}{2} + \frac{1}{3} =$$

$$S_4 =$$



Observations:

This theorem generalizes the argument on the previous pages:

Theorem 7.4 (Alternating Series Test (AST)) *Suppose $\sum a_n$ is an infinite series such that:*

1. *$\sum a_n$ is* **alternating**
   (meaning the terms being added alternate between positive and negative);
2. *$\lim_{n\to\infty} |a_n| = 0$; and*
3. *The values of $|a_n|$ decrease, i.e. $|a_n| \geq |a_{n+1}|$ for all n.*

Then $\sum a_n$ converges.

Important:

How to classify an alternating series

1. Verify that the series is alternating.
2. If the Ratio Test is appropriate, use it.
3. Otherwise, compute $\lim_{n\to\infty} |a_n|$.
   - If this limit is nonzero, the series diverges by the n^{th}-term Test.
   - If this limit is zero, verify that $|a_n| \geq |a_{n+1}|$ (it will in MATH 230). Then the series converges by the AST.

EXAMPLE 3

Determine whether or not each series converges or diverges:

a) $\displaystyle\sum \frac{\cos(\pi n)}{n^2+1}$

b) $\displaystyle\sum_{n=2}^{\infty} (-1)^{n+1} \frac{n}{n+4}$

7.3 Absolute and conditional convergence

The triangle inequality

MOTIVATING EXAMPLE

Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{\sin(e^n)}{|\sin(e^n)|} e^{-n}$$

Problem: This series is not geometric, not a $p-$series, not positive (so the Integral and Comparison Tests are no good) and not alternating (so the Alternating Series Test is no good). The only test we know so far that we could use to study this is the $n^{th}-$term Test:

A new idea in the study of series involves the following important concept about numbers:

Theorem 7.5 (Triangle Inequality for $\mathbb{R}$) *For all real numbers a and b, $|a+b| \leq |a| + |b|$.*

PROOF OF THE TRIANGLE INEQUALITY FOR $\mathbb{R}$:

$$\begin{aligned}
ab &\leq |ab| \\
2ab &\leq 2|ab| \\
a^2 + 2ab + b^2 &\leq |a|^2 + 2|ab| + |b|^2 \\
(a+b)^2 &\leq (|a| + |b|)^2 \\
& \qquad \text{(take } \sqrt{\ } \text{ of both sides; note } \sqrt{z^2} = |z|) \\
|a+b| &\leq ||a| + |b|| = |a| + |b|.
\end{aligned}$$

Reason this is called the Triangle Inequality:

Theorem 7.6 (Generalized Triangle Inequality for $\mathbb{R}$) *For any finite list of real numbers $a_1, a_2, ..., a_n$,*

$$\left|\sum_{j=1}^{n} a_j\right| \leq \sum_{j=1}^{n} |a_j|.$$

Theorem 7.7 (Triangle Inequality for Infinite Series) *Let $\sum a_n$ be an infinite series. If $\sum |a_n|$ converges, then $\sum a_n$ also converges.*

(In this case, we have $\left|\sum a_n\right| \leq \sum |a_n|$.)

MOTIVATING EXAMPLE (FROM EARLIER)

$$\sum_{n=1}^{\infty} \frac{\sin(e^n)}{|\sin(e^n)|} e^{-n}$$

If we denote this series by $\sum a_n$, then

$$\sum |a_n| = \sum \left| \frac{\sin(e^n)}{|\sin(e^n)|} e^{-n} \right| = \sum |\pm 1|\, e^{-n} = \sum e^{-n} = \sum \left(\frac{1}{e}\right)^n$$

Since $\sum |a_n|$ converges, so does $\sum a_n$ by the Triangle Inequality.

PROOF OF THE TRIANGLE INEQUALITY FOR INFINITE SERIES:

Let $b_n = a_n + |a_n|$.

Notice that $a_n + |a_n| = \begin{cases} a_n - a_n = 0 & \text{if } a_n < 0 \\ |a_n| + |a_n| = 2|a_n| & \text{if } a_n \geq 0 \end{cases}$.

Therefore b_n is always either 0 or $2|a_n|$. That means $0 \leq b_n \leq 2|a_n|$.

By hypothesis, $\sum |a_n|$ converges, so by linearity $\sum 2|a_n|$ converges.

Also, by the Comparison Test, $\sum b_n$ converges.

Last,

$$\sum a_n = \sum (b_n - |a_n|) = \sum b_n - \sum |a_n|$$

is the difference of two convergent series, hence $\sum a_n$ converges. $\square$

Absolute and conditional convergence

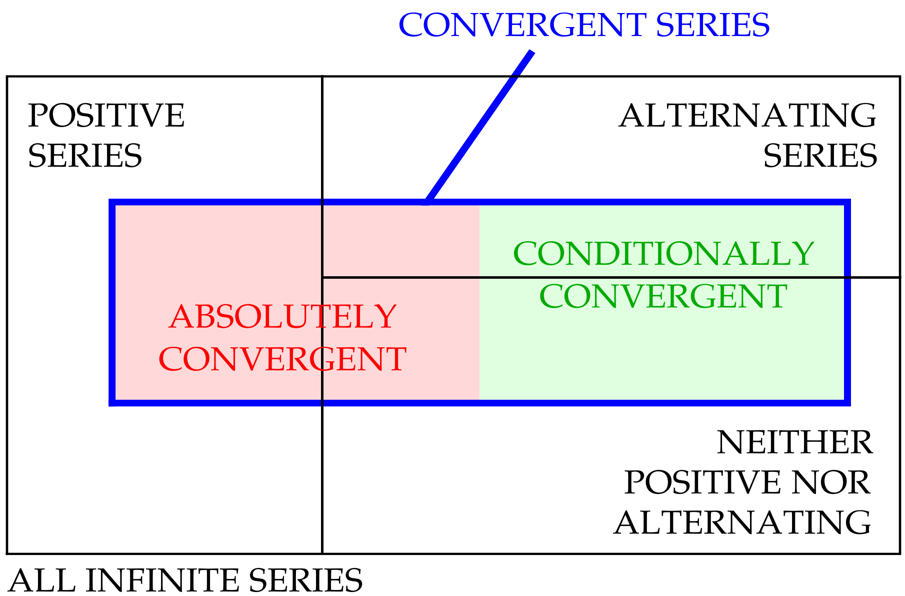
Definition 7.8 *Let $\sum a_n$ be an infinite series. We say the series is* **absolutely convergent** *(or that the series* **converges absolutely***) if $\sum |a_n|$ converges.*

Remarks:

- By the Triangle Inequality for Infinite Series, we know that if a series is absolutely convergent, then it converges.
- If a series diverges, then it cannot converge absolutely (this is the contrapositive of the immediate preceding statement).
- If a series $\sum a_n$ is positive, then there is no difference between $\sum |a_n|$ and $\sum a_n$, so saying that a positive series converges is the same as saying that it absolutely converges.
- If a series is negative, then $\sum |a_n| = -\sum a_n$ so to say a negative series converges is the same as saying that it absolutely converges.
- Based on these observations, there are three possibilities for an infinite series $\sum a_n$:
    1. The series $\sum a_n$ converges absolutely (i.e. $\sum |a_n|$ converges).
    2. The series $\sum a_n$ diverges.
    3. Something else (which given the remarks above must be that $\sum a_n$ converges but $\sum |a_n|$ diverges).

Definition 7.9 *Let $\sum a_n$ be an infinite series. If $\sum a_n$ converges but $\sum |a_n|$ diverges, then we say $\sum a_n$ is* **conditionally convergent** *(or that the series* **converges conditionally***).*

Based on the ideas developed thus far, we can now create a Venn diagram which incorporates the sign of a series together with whether the series converges absolutely, converges conditionally, or diverges. This Venn diagram is on the next page:



We can now refine one of our major questions regarding infinite series:

General questions related to infinite series

Given an infinite series $\sum a_n$:

1. **Classification problem:** Does $\sum a_n$ converge absolutely, converge conditionally, or diverge? (In other words, where on the Venn diagram above does the series belong?)

2. **Computation problem:** If $\sum a_n$ converges, what is its sum?

3. **Rearrangement problem:** When, if ever, can you legally rearrange or regroup the terms of $\sum a_n$ without affecting its convergence?

For now, we turn our attention to the third question: the rearrangement problem.

Rearrangement of infinite series

Goal: Determine the circumstances under which the terms of an infinite series can be legally rearranged without affecting the sum.

MOTIVATING EXAMPLE (FROM EARLIER)

Consider the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$. Does this series converge or diverge?

Does this series converge absolutely or converge conditionally?

Note: This example is a prototype example of a conditionally convergent series.

Now, let $L = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$.

Since the series alternates and has positive initial term, $L \in [S_2, S_1] = [\frac{1}{2}, 1]$ (see the picture in Section 7.2).

Suppose it was legal to rearrange this series. Then:

$$\begin{aligned} L &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots \\ &= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \ldots \\ &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \ldots \\ &= \\ &= \\ &= \end{aligned}$$

Further investigation: Look only at the terms of the alternating harmonic series which are positive:

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ... = \sum_{n=1}^{\infty} \frac{1}{2n+1}$$

Now look only at the negative terms of the alternating harmonic series:

$$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - ... = \sum_{n=1}^{\infty} \frac{-1}{2n} = \frac{-1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$$

Consequence:

What happened when the series was rearranged?

Loosely speaking, the problem is that this series converges conditionally (as opposed to absolutely). Any conditionally convergent series is comprised of an infinite amount of "positive stuff" and an infinite amount of "negative stuff", and the series only converges because the terms cancel each other out in a very delicate way.

On the other hand, an absolutely convergent series (almost by definition) has only a finite amount of "positive stuff" and a finite amount of "negative stuff", so no matter how you rearrange the series you always get the same thing.

To summarize:

Theorem 7.10 (Rearrangement Theorem) *Suppose $\sum a_n$ is an infinite series.*

1. *If $\sum a_n$ converges conditionally, then the terms of that series can be rearranged so that the rearranged series converges to any number you like!*

   The series can also be rearranged so that the rearranged series diverges!

2. *If $\sum a_n$ converges absolutely to L, then no matter how the terms of the series are regrouped or rearranged, the rearranged series still converges absolutely to L.*

Additional remarks:

1. Geometric series always converge absolutely or diverge (they never converge conditionally), since by the GST their convergence depends on $|r|$ (rather than just on r).

2. If a series is shown to converge by the Ratio Test, then the series converges absolutely (since when evaluating ρ one takes absolute values of the terms in the series).

   In particular, for any conditionally convergent series, if you tried the Ratio Test you would get $\rho = 1$. (However, if $\rho = 1$ the series could converge absolutely, converge conditionally or diverge.)

3. p-series converge absolutely when they converge (since they are positive).

4. If a series is positive or negative, it must converge absolutely if it converges.

5. Since the Comparison Test applies only to positive series, if you use the Comparison Test to show that a series converges, then the series must converge absolutely.

6. The Alternating Series Test tells you only that a series converges (not whether the series converges absolutely or conditionally). To distinguish between these situations, you need further analysis.

Putting together everything we know, we have the outline on the next page which describes how to determine whether a given series is absolutely convergent, conditionally convergent or divergent.

7.4 Summary of classification techniques

Directions to classify most infinite series

1. If the series is geometric, then write in the standard form $\sum_{n=0}^{\infty} ar^n$.

   If $|r| \geq 1$, the series diverges by the GST.

   If $|r| < 1$, then the series converges absolutely to $\dfrac{a}{1-r}$ by the GST.

2. If the series contains only multiplication/division and has at least one exponential or factorial term, try the Ratio Test: compute $\rho = \lim_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|}$:

   If $\rho > 1$, the series diverges by the Ratio Test.
   If $\rho < 1$, the series converges absolutely by the Ratio Test.
   If $\rho = 1$, the Ratio Test is inconclusive.

3. If the series is alternating, compute $\lim_{n\to\infty} |a_n|$.

   If this limit is nonzero, the series diverges by the n^{th}-term Test.
   If this limit is zero, verify $|a_n| \geq |a_{n+1}|$ and conclude that the series converges by the AST. Then, consider the series $\sum |a_n|$:
   - If $\sum |a_n|$ converges, then $\sum a_n$ converges absolutely.
   - If $\sum |a_n|$ diverges, then $\sum a_n$ converges conditionally.

4. If the series is a p-series, then by the p-series Test it converges absolutely if $p > 1$ and diverges if $p \leq 1$.

5. If $\lim_{n\to\infty} |a_n| \neq 0$, then the series diverges by the n^{th} Term Test.

6. If the series can be split, analyze the two pieces separately and use linearity rules.

7. If the series is positive (or if it is negative, in which case you first factor out (-1) to leave a positive series), try the Comparison Test.

8. If the series $\sum a_n$ is neither positive, negative nor alternating, try to show $\sum |a_n|$ converges. Then $\sum a_n$ converges absolutely by the Triangle Inequality.

Examples

In Examples 4-8, you are to classify the given series as absolutely convergent, conditionally convergent, or divergent, providing appropriate justification.

EXAMPLE 4

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{4^n}$$

Example 5

$$\sum_{n=1}^{\infty}(-1)^n n^{-1/3}$$

Example 6

$$\sum_{n=1}^{\infty}\frac{2n^4}{n^8+3n^{11}+12}$$

EXAMPLE 7

$$\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2}+\frac{3}{4^n}\right)$$

EXAMPLE 8

$$1-\frac{1}{5}+\frac{1}{25}-\frac{1}{125}+\frac{1}{625}-\frac{1}{5^5}+\dots$$

7.5 Homework exercises

In Exercises 1-6, determine whether the following series converge or diverge. Completely justify your reasoning:

1. $\displaystyle\sum_{n=1}^{\infty} (-1)^n 8n^{-3/4}$
2. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n^2}{3^n}$
3. $\displaystyle\sum_{n=4}^{\infty} (-1)^n \frac{5n^4+3}{20n^4+2n^2+n+1}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{3n+8}}$
5. $\displaystyle\sum_{n=3}^{\infty} \frac{4\cos(\pi n)}{2^n+3}$
6. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n^2}{\ln(n+4)}$

In Exercises 7-22, classify the following statements as true or false:

7. If a negative series converges, then it must converge absolutely.
8. If a series converges conditionally, then its terms can be legally rearranged without affecting the sum.
9. If a series $\sum |a_n|$ diverges, then $\sum a_n$ must also diverge.
10. If a series $\sum |a_n|$ diverges, then $\sum a_n$ cannot converge absolutely.
11. If a series $\sum a_n$ diverges, then $\sum |a_n|$ must also diverge.
12. If a series $\sum a_n$ converges, then $\sum |a_n|$ must also converge.
13. If a series $\sum |a_n|$ converges, then $\sum a_n$ must also converge.
14. It is possible for an alternating series to diverge.
15. It is possible for an alternating series to converge absolutely.
16. It is possible for an alternating series to converge conditionally.
17. It is possible for a positive series to diverge.
18. It is possible for a positive series to converge absolutely.
19. It is possible for a positive series to converge conditionally.
20. If $\lim\limits_{n\to\infty} |a_n| = 1$, then $\sum a_n$ diverges.
21. If $\lim\limits_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.

22. If $\lim_{n\to\infty} |a_n| = 0$, then $\sum a_n$ converges.

In Exercises 23-43, determine whether the following series converge absolutely, converge conditionally, or diverge. You should state the name of the test(s) you use and completely justify your reasoning, giving arguments like those in the examples in this text.

23. $\sum_{n=2}^{\infty} (-1)^n \sin n$

24. $\sum_{n=2}^{\infty} e^{-1/n}$

25. $\sum_{n=4}^{\infty} \frac{n+1}{\ln(2n-5)}$

26. $\sum_{n=2}^{\infty} 5^{-n^2-3n}$

27. $\sum_{k=1}^{\infty} \frac{3}{4+\sin^4(2k)}$

28. $\sum_{n=1}^{\infty} \frac{6^n}{6^{2n}+3}$

29. $\sum_{k=1}^{\infty} \frac{2\cos(\pi k)}{k^2}$

30. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2\sqrt{n+2}}$

31. $\sum_{n=0}^{\infty} \frac{1}{e^n + e^{-n}}$

32. $\sum_{n=2}^{\infty} \frac{2n^2+3}{5n^2-4}$

33. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$

34. $\sum_{n=0}^{\infty} \frac{(-1)^n 2n}{n^3+5}$

35. $\sum_{n=1}^{\infty} \ln n$

36. $\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n}}{7^n n!}$

37. $\sum_{n=1}^{\infty} 4(-1)^{n+1} n^{-3/5}$

38. $\sum_{n=2}^{\infty} \frac{3+2^n}{3^n+4}$

39. $\sum_{n=0}^{\infty} \left[\frac{3}{n^5} + \frac{(-1)^n}{n}\right]$

40. $\sum_{n=1}^{\infty} \frac{(-1)^{[\frac{1}{2}n(n+1)]}}{4^n+n^2}$

41. $\sum_{n=1}^{\infty} \frac{2n^2+3n-2}{n^2+4n+1}$

42. *(Challenge)*

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$$

43. *(Challenge)*

$$1 + \frac{1}{1.1} + \frac{1}{1.11} + \frac{1}{1.111} + \frac{1}{1.1111} + \ldots$$

Answers

1. converges
2. converges
3. diverges
4. converges
5. converges
6. diverges
7. True
8. False
9. False
10. True
11. True
12. False
13. True
14. True
15. True
16. True
17. True
18. True
19. False
20. True
21. False
22. False
23. diverges
24. diverges
25. diverges
26. converges absolutely
27. diverges
28. converges absolutely
29. converges absolutely
30. converges conditionally
31. converges absolutely
32. diverges
33. converges conditionally
34. converges absolutely
35. diverges
36. converges absolutely
37. converges conditionally
38. converges absolutely
39. converges conditionally
40. converges absolutely
41. diverges
42. diverges
43. diverges

Chapter 8

Taylor and Fourier series

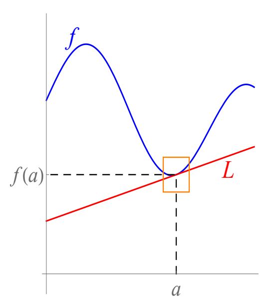
GOAL

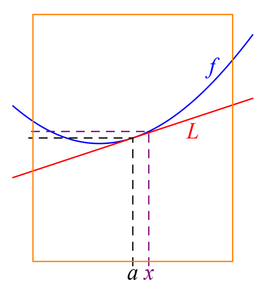
Use the ideas of infinite series to study functions.

Why might we want to do this?

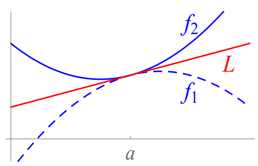
Recall from Calculus 1 that for values of x near $x = a$, we can approximate a differentiable function f by its tangent line at a:

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$





Question: In general, given an arbitrary function f, when will this approximation $L(x)$ overestimate the actual value of $f(x)$, and when will $L(x)$ underestimate $f(x)$?



L underestimates f_2 $\Leftrightarrow$

L overestimates f_1 $\Leftrightarrow$

With this in mind, you may be aware that we can obtain a better approximation to f, which accounts (in part) for how the function f curves, by using *quadratic approximation* at a:

$$f(x) \approx Q(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

Quadratic approximation is better than linear approximation in two ways:

1. For x near a, $Q(x)$ will be closer to $f(x)$ than $L(x)$ was.

2. There is a larger interval of x-values for which the approximation $Q(x) \approx f(x)$ is good, so (loosely speaking) there are more xs that count as being "near a".

Questions:

1. Under what circumstances will $Q(x)$ over/underestimate $f(x)$?

2. How might we get a better approximation to f than L or Q ("better" meaning closer to f and working for a larger interval of x-values)?

8.1 Taylor series

Uniqueness of Taylor coefficients

MAIN PROBLEM

Given a function f, we can approximate f by linear and quadratic functions near $a = 0$ as follows:

$$f(x) \approx L(x) = f(0) + f'(0)(x-0) = f(0) + f'(0)x$$
$$f(x) \approx Q(x) = f(0) + f'(0)(x-0) + \frac{1}{2}f''(0)(x-0)^2 = f(0) + f'(0)x + \frac{f''(0)}{2}x^2.$$

For each N, we want to find numbers a_0, a_1, a_2, ... so that we can approximate f by a ____________________ of the form

$$P_N(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_Nx^N.$$

We will do this by **writing f as an infinite series** of the form

$$f(x) = \sum_{n=0}^{\infty} a_nx^n = a_0 + a_1x + a_2x^2 + a_3x^3 + ... \tag{8.1}$$

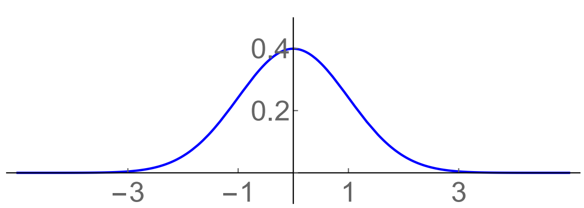
Then $P_N(x)$ will the the N^{th} partial sum of the series representing f, so it will approximate f (and as N gets bigger and bigger, the approximation will get better and better).

An application from statistics

A lot of naturally occurring data (student exam grades, heights/weights/lifespans of species, errors in experiments) is modeled by a class of random variables (remember those?) called *normal random variables*. The simplest of these is called the *standard normal r.v.* and has density function

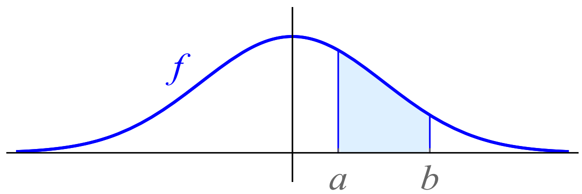
$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

The graph of this density function is called a **bell curve**:



If X is a (standard) normal r.v., then to compute the probability that X is between a and b, we would need to compute

$$P(a \le X \le b) = \int_a^b f(x)\,dx = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx.$$



Problem: We know no method of computing $\int e^{-x^2/2}\,dx$.
In fact, there is no method to compute exact values of this integral.

Potential solution: If we approximate this f by a polynomial, then we can approximate the integral as well.

RECALL

Our goal is to **write f as an infinite series** of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \qquad (\bigstar)$$

THEORETICAL SOLUTION

Suppose we can write f as an infinite series, as in equation ($\bigstar$) above. Then by repeatedly differentiating f, we get

$$\begin{array}{rllllll}
f(x) & = a_0 & +a_1x & +a_2x^2 & +a_3x^3 & +a_4x^4 & +a_5x^5 + \ldots \\
f'(x) & = & a_1 & +2a_2x & +3a_3x^2 & +4a_4x^3 & +5a_5x^4 + \ldots \\
f''(x) & = & & 2a_2 & +3\cdot 2a_3x & +4\cdot 3a_4x^2 & +5\cdot 4a_5x^3 + \ldots \\
f'''(x) & = & & & 3\cdot 2a_3 & +4\cdot 3\cdot 2a_4x & +5\cdot 4\cdot 3a_5x^2 + \ldots \\
f^{(4)}(x) & = & & & & 4\cdot 3\cdot 2a_4 & +5\cdot 4\cdot 3\cdot 2a_5x + \ldots \\
\vdots & & & & & \vdots &
\end{array}$$

Continuing in this fashion, we see

$$f^{(n)}(x) = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 a_n + (const)x + (const)x^2 + \ldots$$

Plugging in $x = 0$ to all the formulas on the previous page, we get

$$\begin{aligned} f(0) &= \\ f'(0) &= \\ f''(0) &= \\ f'''(0) &= \\ \vdots &= \quad \vdots \\ f^{(n)}(0) &= \end{aligned}$$

Importantly, this gives us a formula for all the a_n:

We have proven the following fundamental theorem, called the *uniqueness of power series*:

Theorem 8.1 (Uniqueness of power series) *Suppose f is a function which can be differentiated over and over again at $x = 0$. Then if we write f as a power series of the form*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots,$$

then for all n, the coefficients a_n must satisfy

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

In other words, the ***only*** *power series of the form $\sum a_n x^n$ which can represent f is the series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

This series is called the **Taylor series (centered at 0) of** *f or the* **Maclaurin series of** *f.*

Corollary 8.2 *Suppose you have two power series which represent the same function of x, i.e.*

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all x in some open interval containing 0. Then, for every n, $a_n = b_n$.

Note: There are functions which cannot be represented as a power series. However, we will not encounter those in MATH 230 (so we will assume in our course that every function f is equal to its Taylor series at any x for which the Taylor series converges).

The "big six" Taylor series

It turns out that representing functions by their Taylor series is very doable and very useful. Our next task is to determine the Taylor series of some common functions (which should be memorized); these memorized Taylor series can then be used to determine the Taylor series of lots of other functions.

(This should remind you of how we learn to differentiate functions: you memorize derivatives of elementary functions, and then you learn rules to differentiate more complicated functions.)

EXAMPLE 1

Determine the Taylor series (centered at 0) of $f(x) = e^x$.

Solution: By definition, the Taylor series of any function f is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\left(\text{i.e. } f(x) = \sum_{n=0}^{\infty} a_n x^n, \text{ where } a_n = \frac{f^{(n)}(0)}{n!} \right).$$

Let's compute this series directly. Since the exponential function is its own derivative, we have, for all n:

$$f^{(n)}(x) = e^x \quad \Rightarrow \quad f^{(n)}(0) = e^0 = 1.$$

So for all n, we have $a_n =$

and therefore the Taylor series of f is

To summarize, on the previous page we found the following formula:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

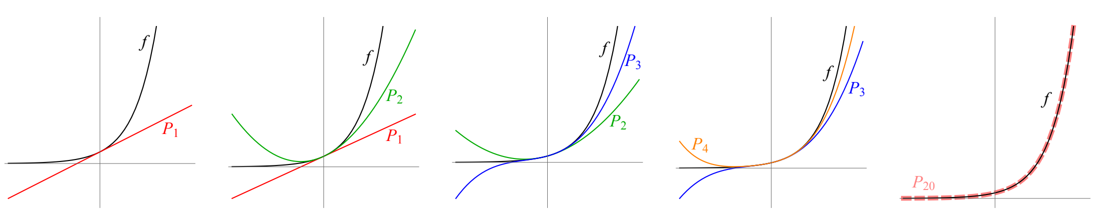
Memorize this (and the other Taylor series we derive in this section)!

P.S. You can show that this series converges for all x using the Ratio Test.

What does this formula mean?

Since $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$, we can approximate e^x by computing a partial sum for this series. In this context, the N^{th} partial sum is denoted $P_N(x)$.

$P_N(x)$ should be a good approximation to e^x, and as N gets bigger, $P_N(x)$ approximates e^x more and more closely, and for larger and larger intervals of x-values:



If you let $N \to \infty$, then $P_N(x)$ becomes the entire series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, which is exactly the function e^x for all x.

EXAMPLE 2

Determine the Taylor series (centered at 0) of $f(x) = \sin x$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$a_n = \dfrac{f^{(n)}(0)}{n!}$
0	$\sin x$		
1	$\cos x$		
2			
3			
4			
5			
6			
7			
8			

Therefore the Taylor series of $\sin x$ is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

This series can also be shown to converge for all x using the Ratio Test.

EXAMPLE 3

Determine the Taylor series (centered at 0) of $f(x) = \cos x$.

We could go through the procedure of Example 2 ($f(x) = \sin x$) again and find a pattern with the a_n. But there is a better way:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

This series can also be shown to converge for all x using the Ratio Test.

EXAMPLE 4

There is one other function whose Taylor series we "know" (even though we didn't call it a "Taylor series" at the time). Suppose we set $a_n = 1$ for all n. Then we get

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 1x^n = \sum_{n=0}^{\infty} x^n$$

This series converges to $\frac{1}{1-x}$, but only when $|x| < 1$. Thus we have:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \text{ for } -1 < x < 1$$

Any known Taylor series can be manipulated to find Taylor series of other functions. For instance, let's start with the formula we discovered in Example 4:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \text{ for } -1 < x < 1$$

If you replace the x on both sides of the above equation with $-x$, we get

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \ldots \\ \frac{1}{1+x} &= \end{aligned}$$

Now if we integrate both sides of the equation above, we get

$$\begin{aligned} \frac{1}{1+x} &= 1 - x + x^2 - x^3 + \ldots \\ \ln(1+x) &= \end{aligned}$$

We have derived:

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots \quad \text{for } -1 < x \leq 1$$

Next, go back to the series for $\frac{1}{1-x}$, and let's do some different manipulations. First, replace x with $-x^2$, and then integrate:

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \ldots \\ \frac{1}{1+x^2} &= \end{aligned}$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots \quad \text{for } -1 \leq x \leq 1$$

To summarize, we have derived Taylor series representations for the following six common functions. **YOU MUST KNOW THESE COLD.**

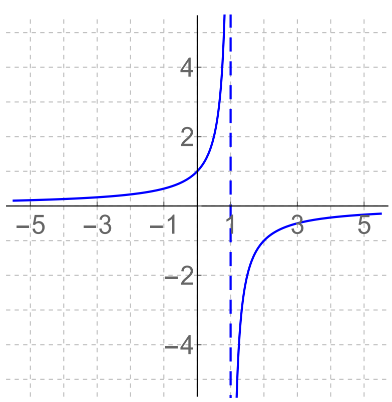
The "big six" Taylor series

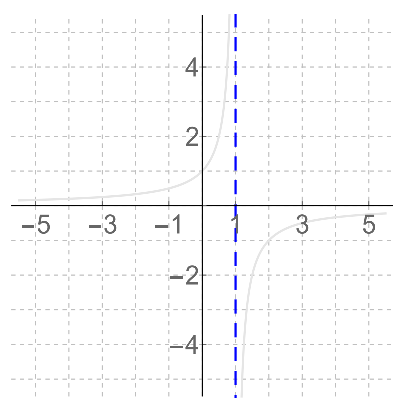
$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} &&= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots && \text{(holds for all } x\text{)}\\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots && \text{(holds for all } x\text{)}\\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} &&= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots && \text{(holds for all } x\text{)}\\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n &&= 1 + x + x^2 + x^3 + \ldots && \text{(holds for } x \in (-1,1)\text{)}\\
\ln(1+x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n &&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots && \text{(holds for } x \in (-1,1]\text{)}\\
\arctan x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} &&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots && \text{(holds for } x \in [-1,1]\text{)}
\end{aligned}$$

From these six Taylor series, we can obtain the Taylor series for lots of other functions via manipulations (see Example 5 below).

Why the values of x for which the series converge are important

$$f(x) = \frac{1}{1-x} \qquad \text{vs.} \qquad f(x) = \sum_{n=0}^{\infty} x^n$$





EXAMPLE 5

Find a power series representation (i.e. find the Taylor series centered at 0) of each of the following functions (give the answer both in Σ notation and in written-out form):

a) $f(x) = \arctan x^3$

b) $f(x) = x^2 e^{-3x^5}$

c) $f(x) = \dfrac{x}{(1-x)^2}$

EXAMPLE 6

Find a power series representation (i.e. find the Taylor series centered at 0) of each of the following functions (give the answer in Σ notation).

a) $f(x) = x^3 \cos(2x)$

Solution: Start with $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

Replace each x with $2x$ to get $\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n}}{(2n)!}$.

Now multiply by x^3 in front to get

$$x^3 \cos 2x = \sum_{n=0}^{\infty} x^3 \frac{(-1)^n 4^n x^{2n}}{(2n)!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+3}}{(2n)!}}.$$

b) $f(x) = 2 \sin x^4$

Solution: Start with $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

Replace each x with x^4 to get $\sin x^4 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^4)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+4}}{(2n+1)!}$.

Now multiply by 2 in front to get

$$2 \sin x^4 = \boxed{\sum_{n=0}^{\infty} 2 \frac{(-1)^n x^{8n+4}}{(2n+1)!}}.$$

c) $f(x) = \dfrac{4}{3+5x}$

8.2 Applications of Taylor series

Evaluation of hard limits without using L'Hôpital's rule

EXAMPLE 7

Evaluate the limit $\lim_{x\to 0} \dfrac{6\sin x - 6x + x^3}{2x^5}$.

Old solution using L'Hôpital's Rule:

$$
\begin{aligned}
&\lim_{x\to 0} \frac{6\sin x - 6x + x^3}{2x^5} = \frac{0}{0}\\
&\stackrel{L}{=} \lim_{x\to 0} \frac{6\cos x - 6 + 3x^2}{10x^4} = \frac{0}{0}\\
&\stackrel{L}{=} \lim_{x\to 0} \frac{-6\sin x + 6x}{40x^3} = \frac{0}{0}\\
&\stackrel{L}{=} \lim_{x\to 0} \frac{-6\cos x + 6}{120x^2} = \frac{0}{0}\\
&\stackrel{L}{=} \lim_{x\to 0} \frac{6\sin x}{240x} = \frac{0}{0}\\
&\stackrel{L}{=} \lim_{x\to 0} \frac{6\cos x}{240} = \frac{6}{240} = \boxed{\frac{1}{40}}.
\end{aligned}
$$

New solution using Taylor series:

EXAMPLE 8

Evaluate the limit $\lim_{x \to 0} \frac{\arctan 3x - 3x}{x^2}$.

Approximation of function values and definite integrals

First, we introduce some vocabulary to describe the partial sums of a Taylor series of a function.

Definition 8.3 *Suppose f is a function which can be differentiated over and over again at $x = 0$; the Taylor series of f is*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

If we truncate this series at the N^{th} power term, we obtain a partial sum of the Taylor series called the N^{th} **Taylor polynomial (centered at 0)** *(a.k.a.* **Taylor polynomial of order** *N) of f. This polynomial is denoted $P_N(x)$.*

EXAMPLE 9

Let $f(x) = e^x$. Then the Taylor series of f is

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

and the Taylor polynomials of $f(x) = e^x$ are:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= 1 + x \\ P_2(x) &= 1 + x + \frac{x^2}{2} \\ P_3(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \\ &\vdots \\ P_N(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ... + \frac{x^N}{N!} \end{aligned}$$

We saw a few pages ago how the graphs of these P_N more closely approximate e^x as N gets larger.

EXAMPLE 10

Let $f(x) = \sin x$. Then the Taylor series of f is

$$\begin{aligned} f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \\ &= 0 + x + 0x^2 - \frac{x^3}{3!} + 0x^4 + \frac{x^5}{5!} + 0x^6 - \frac{x^7}{7!} + ... \end{aligned}$$

and the Taylor polynomials of $f(x) = \sin x$ are:

$P_0(x) =$

$P_1(x) =$

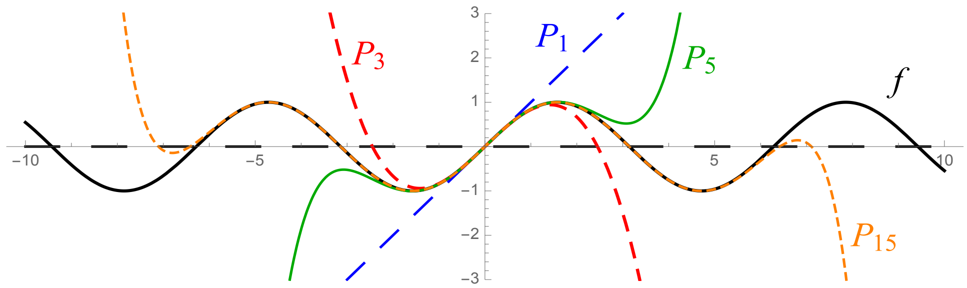
$P_2(x) =$

$P_3(x) =$

$P_4(x) =$

$P_5(x) =$

Some Taylor polynomials of $\sin x$ are graphed on the next page.



General properties of Taylor polynomials

1. $P_N(x)$ is a polynomial of degree $\leq N$;
2. $P_0(x)$ is the constant function of height $f(0)$;
3. $P_1(x)$ is the tangent line to f when $x = 0$;
4. $P_N(x)$ is the best N^{th} degree polynomial approximation to f near 0.

To approximate the value of a function or a definite integral, we can replace the function with its N^{th} Taylor polynomial to get a good approximation.

EXAMPLE 11

Approximate $\ln(1.2)$ using the fourth Taylor polynomial (a.k.a. Taylor polynomial of order 4) for an appropriately chosen function.

This sum evaluates to $\dfrac{1367}{7500} \approx .182267$.

Remark: The actual value of $\ln(1.2)$ is $.182322...$, so our approximation has only $.03\%$ error.

EXAMPLE 12

Approximate $e^{.1}$ using a Taylor polynomial of order 3 for an appropriately chosen function.

This sum evaluates to $\dfrac{6631}{6000} \approx 1.1051666$.

$e^{.1} \approx 1.1051709$, so the error here is $.0003\%$.

EXAMPLE 13

Approximate $\displaystyle\int_0^1 \sin x^4\,dx$ by replacing the integrand with its tenth Taylor polynomial.

The integral $\displaystyle\int_0^1 \sin x^4\,dx$ has an actual value of about $.18757$, so our approximation is good to an error of 6.6%.

If we had used one more nonzero term and approximated $\sin x^4$ by $P_{12}(x) = x^4 - \frac{x^{12}}{6}$, we would obtain an approximation of $\dfrac{73}{390} \approx .187179$, which has an error of $.2\%$.

EXAMPLE 14

The height of a randomly chosen adult man in the United States is represented by adding 70 inches to a continuous random variable with density function

$$f(x) = \frac{1}{3\sqrt{2\pi}} e^{-x^2/18}.$$

Use the second-order Taylor polynomial of the integrand to estimate the probability that a randomly chosen American male is between $5'9$" ($= 69$ inches) and $6'0$" ($= 72$ inches).

Solution: We know

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} &&= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \\ \Rightarrow e^{-x^2/18} & &&= 1 - \frac{x^2}{18} + \frac{x^4}{18^2 \cdot 2} - \frac{x^6}{18^3 \cdot 3!} + \cdots \end{aligned}$$

so by approximating f by its second-order Taylor polynomial, we have

$$\begin{aligned} P(\text{height } \in [69,72]) = P(X \in [-1,2]) &= \int_{-1}^{2} f(x)\,dx \\ &\approx \int_{-1}^{2} P_2(x)\,dx \\ &= \frac{1}{3\sqrt{2\pi}} \int_{-1}^{2} \left(1 - \frac{x^2}{18}\right) dx \\ &= \frac{1}{3\sqrt{2\pi}} \left[x - \frac{x^3}{54}\right]_{-1}^{2} \\ &= \frac{1}{3\sqrt{2\pi}} \left[\frac{17}{6}\right] \\ &= \boxed{\frac{17}{18\sqrt{2\pi}}} \approx \boxed{.3767}. \end{aligned}$$

P.S. In MATH 251 and/or MATH 414, you will learn how to solve this problem using some tables associated to things called *z-scores*; those tables produce an answer of $.3780\ldots$.

Computation of high-order derivatives

EXAMPLE 15

Let $f(x) = x^{20}\cos(3x^8)$. Compute $f^{(100)}(0)$, the 100^{th} derivative of f at zero.

Analysis of numerical series

EXAMPLE 16

Determine whether or not the series $\sum_{n=1}^{\infty}\left(e^{1/n} - 1\right)$ converges or diverges.

EXAMPLE 17

Compute the sum of each series:

a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

b) $1 + \frac{1}{2} + \frac{1}{4 \cdot 2!} + \frac{1}{8 \cdot 3!} + \frac{1}{16 \cdot 4!} + \frac{1}{2^5 \cdot 5!} \cdots$

c) $\pi - \frac{2\pi^3}{2^3 \cdot 3!} + \frac{2\pi^5}{2^5 \cdot 5!} - \frac{2\pi^7}{2^7 \cdot 7!} + \dots$

Series solutions of differential equations

EXAMPLE 18

Suppose f is an unknown function such that $f'(x) = xf(x)$ and $f(0) = 2$. Compute the fourth Taylor polynomial of f, and use that polynomial to estimate $f(1)$.

EXAMPLE 19

Suppose y is an unknown function of x where $y'' + y = 0$, where $y(0) = 2$ and $y'(0) = -1$. Compute the Taylor series of y, and then, if possible, identify y as a common function.

8.3 General theory of power series

In the last section, we saw how to use certain series to study functions. In particular, given a function f, we can define the Taylor series of f:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

The partial sums of this series are denoted $P_N(x)$ and called the Taylor polynomials of f. Ideally, $f(x) \approx P_N(x)$, so f can be replaced by P_N in approximation problems.

Now, we want to study the general behavior of series of the form

$$\sum_{n=0}^{\infty} a_n x^n,$$

where the a_n are constants (which may or may not have anything to do with some function f). Such functions are called *power series (centered at 0)*. In fact, we can study more general objects:

Definition 8.4 *A* **power series (in x) centered at** $x = a$ *is a function of the form*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x-a)^n \\ &= a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + \ldots \end{aligned}$$

where the $a_0, a_1, \ldots$ are real numbers. A **power series (in x)** *is a power series centered at $x = 0$, i.e. a power series is a function of the form*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots \end{aligned}$$

The numbers $a_0, a_1, a_2, \ldots$ are called the **coefficients** *of the power series; the* **terms** *of the series are a_0, $a_1 x$, $a_2 x^2$, ...*

Remark: When evaluating a power series, 0^0 is always taken to be 1 (even though technically 0^0 is an indeterminate form).

Most important question regarding power series: What is the domain of a power series?

EXAMPLE 20

Determine the values x for which the following series converges:

$$\sum_{n=1}^{\infty} \frac{(x-6)^n}{n4^n}$$

Solution: Apply the Ratio Test (we expect ρ to depend on x):

EXAMPLE 21

Determine the values x for which the following series converges:

$$\sum_{n=0}^{\infty} n!(x+1)^n$$

Solution: Again apply the Ratio Test:

EXAMPLE 22

Determine the values x for which the following series converges:

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{3n!}$$

Solution: Again apply the Ratio Test:

$$\rho = \lim_{n\to\infty} \frac{\left|\frac{(x-2)^{n+1}}{3(n+1)!}\right|}{\left|\frac{(x-2)^n}{3n!}\right|} = \lim_{n\to\infty} \frac{|x-2|^{n+1}}{3(n+1)!} \cdot \frac{3n!}{|x-2|^n} = \lim_{n\to\infty} \frac{|x-2|}{n+1} =$$

Notice the pattern in the Examples 20 to 22 above:

Theorem 8.5 (Cauchy-Hadamard (C-H) Theorem) *Let* $\sum_{n=0}^{\infty} a_n(x-a)^n$ *be a power series centered at* a. *As in the Ratio Test, set*

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}.$$

Then, the quantity $R = \dfrac{1}{\rho}$ *(which is either a nonnegative real number or* ∞*) is called the* **radius of convergence** *of the series. One of the following holds:*

1. *If* $0 < R < \infty$ *(i.e.* $0 < \rho < \infty$*), then:*
   a) *the power series converges absolutely for* $x \in (a-R, a+R)$*;*
   b) *the power series diverges for* $x \in (-\infty, a-R)$ *or* $x \in (a+R, \infty)$*;*
   c) *anything can happen when* $x = a - R$ *or* $x = a + R$*.*
2. *If* $R = 0$ *(i.e.* $\rho = \infty$*), then the power series converges absolutely when* $x = a$ *but diverges for all other* x*.*
3. *If* $R = \infty$ *(i.e.* $\rho = 0$*) , the power series converges absolutely for all* x*.*

Remarks:

1. the formula $R = \frac{1}{\rho}$ above is called **Abel's Formula**. To compute ρ, and therefore R, we use only the *coefficients* of the power series and *not the powers of* $(x-a)$.
2. This theorem has nothing to do with numerical series (those with no x^n or $(x-a)^n$); it is only useful for power series.

According to the C-H Theorem, the set of x for which a power series converges is an interval; that interval is called the *interval of convergence* of that series and that interval is the domain of that power series.

General procedure to find the interval of convergence of a power series $\sum\limits_{n=0}^{\infty} a_n(x-a)^n$

1. Read off the value of a (where the power series is centered).
2. Compute ρ using the Ratio Test

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}.$$

   Then the radius of convergence is $R = \dfrac{1}{\rho}$.

3. If $R = 0$, then the series converges absolutely for $x = a$ and diverges for all other x. The interval of convergence is $[a, a]$ or just $\{a\}$.
4. If $R = \infty$, then the series converges absolutely for all x. The interval of convergence is $(-\infty, \infty)$.
5. If $0 < R < \infty$, then by the C-H Theorem:
   - the series converges absolutely on $(a - R, a + R)$;
   - the series diverges on $(-\infty, a - R)$ and $(a + R, \infty)$.

   The interval of convergence in this case always runs from $a - R$ to $a + R$; what you have to determine is whether the endpoints $a - R$ and $a + R$ should be included in the interval. To figure this out, plug each of these endpoints in for x (do them one-by-one) and check whether the numerical series you obtain converge or diverge.

EXAMPLE 23

For each given power series:

i. Determine the radius of convergence of the series.
ii. Give the set of x for which the series converges.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^n}{(2n)!}$

b) $\displaystyle\sum_{n=2}^{\infty} n^n (x-4)^n$

Solution: This power series is centered at $x = \boxed{}$.

Next, use the Ratio Test and Abel's Formula:

$$\begin{aligned}\rho = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n (n+1) &= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n (n+1) \\ &= e \lim_{n\to\infty} (n+1) = \infty,\end{aligned}$$

so $R = \frac{1}{\rho} = \frac{1}{\infty} = \boxed{}$.

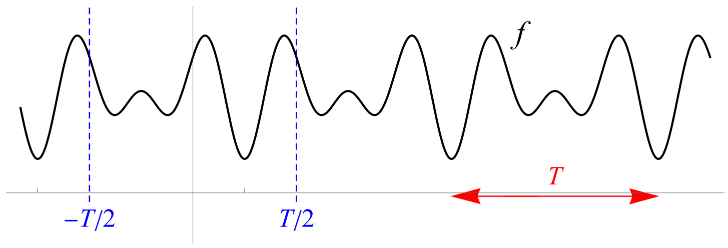
Therefore the series $\displaystyle\sum_{n=2}^{\infty} n^n (x-4)^n$ $\boxed{\begin{cases} \text{converges absolutely} & \text{when } x = 4 \\ \text{diverges} & \text{otherwise} \end{cases}}$.

c) $\sum_{n=0}^{\infty} 3^n x^n$

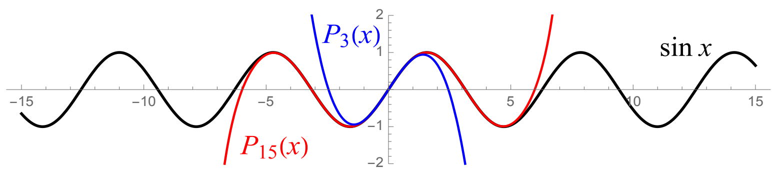
8.4 Fourier series

Periodic functions

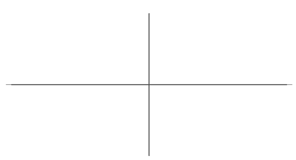
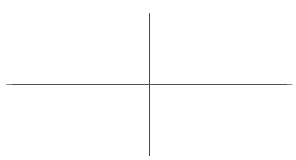
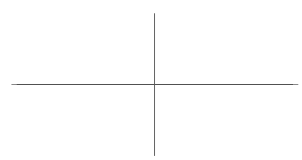
Definition 8.6 *Let $f : \mathbb{R} \to \mathbb{R}$. f is called* **periodic** *if there is a number T such that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$. In this case, the number T is called a* **period** *of f.*



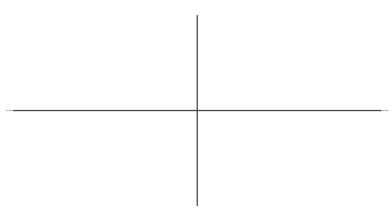
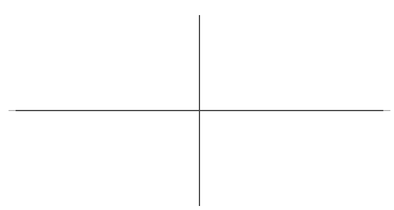
Periodic functions cannot be globally approximated well by a Taylor polynomial, because polynomial graphs have "tails" that point upward or downward, whereas periodic functions keep repeating themselves forever:



Our goal in this section is to come up with a way to globally approximate a periodic function. To do this, we will approximate f by a sum of "basic" periodic functions. We know three basic periodic functions, two with the same period and one that is "easy" but "hard to come up with":

To obtain a basic periodic function with period T, stretch/shrink the first two functions above horizontally:

Remark: If a function has period T, then technically it also has period $2T$, $3T$, etc.

So generally speaking, these functions have period T:

$$\text{constants} \qquad \cos\left(\frac{2\pi n}{T}x\right) \qquad \sin\left(\frac{2\pi n}{T}x\right).$$

Definition of Fourier series

Suppose f is periodic with period T. Our goal is to write f as an infinite series made up of the basic functions we found above that have period T:

To do this, we would need to determine the constant term a_0 and the coefficients c_n and s_n on the cosine and sine terms. Turns out, we can figure out formulas for these coefficients (time permitting, we'll discuss this at the end of this section).

Definition 8.7 *Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with period T. For each $n \in \{1, 2, 3, ...\}$, define the numbers*

$$c_n = \frac{2}{T}\int_0^T f(x)\cos\left(\frac{2\pi n}{T}x\right)\,dx \quad \textit{and} \quad s_n = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2\pi n}{T}x\right)\,dx.$$

Then, the series

$$f(x) = \frac{1}{T}\int_0^T f(x)\,dx + \sum_{n=1}^{\infty}\left[c_n\cos\left(\frac{2\pi n}{T}x\right) + s_n\sin\left(\frac{2\pi n}{T}x\right)\right]$$

is called the **Fourier series of** f.

The n^{th} term of this Fourier series, namely

$$c_n\cos\left(\frac{2\pi n}{T}x\right) + s_n\sin\left(\frac{2\pi n}{T}x\right),$$

is called the n^{th} **harmonic of** f.

The N^{th} partial sum of the Fourier series of f is called the N^{th} **Fourier polynomial of** f *(even though it isn't a polynomial) and is denoted by F_N. In particular,*

$$F_N(x) = \frac{1}{T}\int_0^T f(x)\,dx + \sum_{n=1}^{N}\left[c_n\cos\left(\frac{2\pi n}{T}x\right) + s_n\sin\left(\frac{2\pi n}{T}x\right)\right].$$

Advantages of Fourier series over Taylor series:

1. Fourier series give good global approximations to the behavior of periodic functions.
2. Fourier series approximate integrals of f well.
3. The function f being approximated doesn't have to be differentiable (or even cts).

Disadvantages of Fourier series, relative to Taylor series:

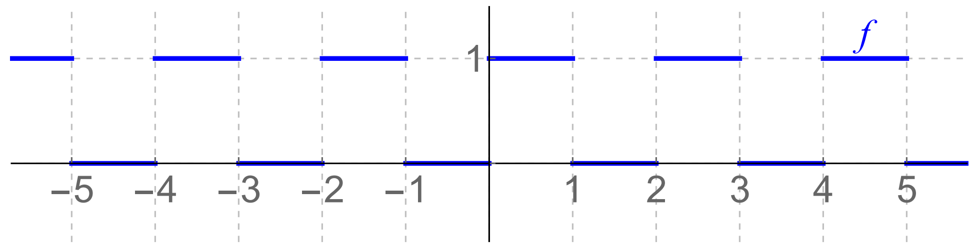
1. Fourier series are no good locally (there's no reason why $F_n(x) \approx f(x)$ for a particular x).
2. The set of x for which the Fourier series converges can be complicated (not necessarily an interval).
3. No hard and fast theoretical rules exist telling us when a Fourier series converges to the original function f.
4. Fourier series coefficients are harder to compute than Taylor series coefficients.

Examples

EXAMPLE 24

Compute the Fourier series of the *square wave* function

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1), [2,3), [4,5), ..., [2n, 2n+1), ... \\ 0 & \text{if } x \in [1,2), [3,4), [5,6), ..., [2n+1, 2n+2), ... \end{cases} .$$



$$a_0 = \frac{1}{T}\int_0^T f(x)\,dx = \frac{1}{2}\int_0^2 f(x)\,dx =$$

$$\begin{aligned} c_n &= \frac{2}{T}\int_0^T f(x)\cos\left(\frac{2\pi n}{T}x\right)\,dx \\ &= \frac{2}{2}\int_0^2 f(x)\cos(\pi n x)\,dx \\ &= \int_0^1 \cos(\pi n x)\,dx \\ &= \left[\frac{1}{\pi n}\sin(\pi n x)\right]_0^1 = \frac{1}{\pi n}\left[\sin \pi n - \sin(0)\right] = \end{aligned}$$

$$\begin{aligned} s_n &= \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2\pi n}{T}x\right)\,dx \\ &= \frac{2}{2}\int_0^2 f(x)\sin(\pi n x)\,dx \\ &= \frac{2}{2}\int_0^1 \sin(\pi n x)\,dx \\ &= \left[-\frac{1}{\pi n}\cos(\pi n x)\right]_0^1 = -\frac{1}{\pi n}\left[\cos \pi n - \cos(0)\right] = \end{aligned}$$

EXAMPLE 24 (CONTINUED)

On the previous page, we found:

$$a_0 = \frac{1}{2} \qquad c_n = 0 \qquad s_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2}{\pi n} & \text{if } n \text{ is odd} \end{cases}$$

Therefore the Fourier series of f is

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[c_n \cos\left(\frac{2\pi n}{T} x\right) + s_n \sin\left(\frac{2\pi n}{T} x\right) \right]$$

$$= a_0 + \sum_{n=1}^{\infty} \left[c_n \cos(\pi n x) + s_n \sin(\pi n x) \right]$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \left[0 \cos(\pi n x) + \left(\begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{\pi n} & \text{if } n \text{ is odd} \end{cases} \right) \sin(\pi n x) \right]$$

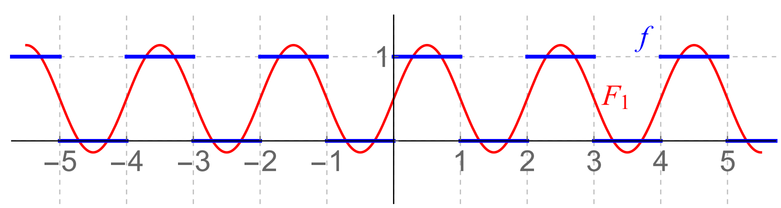
$$= \boxed{\frac{1}{2} + \frac{2}{\pi}\sin(\pi x) + \frac{2}{3\pi}\sin(3\pi x) + \frac{2}{5\pi}\sin(5\pi x) + \ldots}$$

$$= \boxed{\frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n+1)\pi} \sin\left[(2n+1)\pi x\right]}.$$

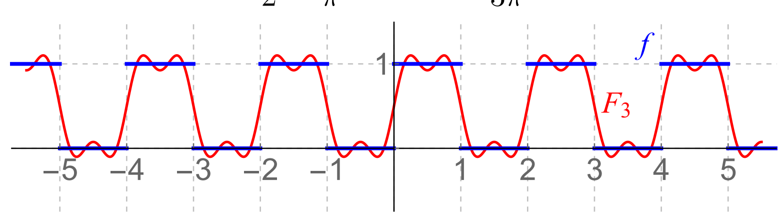
What's the point of this?

Let's take a look at the Fourier polynomials (i.e. the partial sums of this Fourier series):

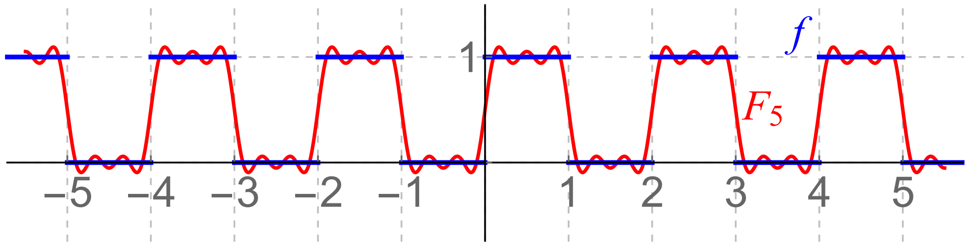
$$F_1(x) = \frac{1}{2} + \frac{2}{\pi}\sin(\pi x)$$



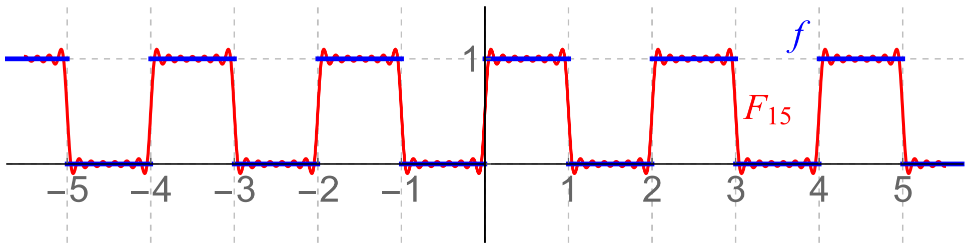
$$F_3(x) = \frac{1}{2} + \frac{2}{\pi}\sin(\pi x) + \frac{2}{3\pi}\sin(3\pi x)$$



$$F_5(x) = \frac{1}{2} + \frac{2}{\pi}\sin(\pi x) + \frac{2}{3\pi}\sin(3\pi x) + \frac{2}{5\pi}\sin(5\pi x)$$

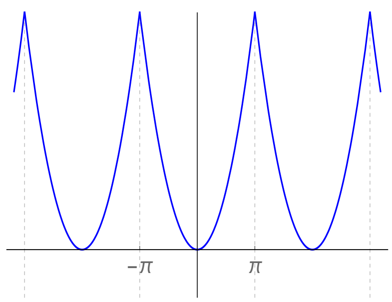


$$F_{15}(x) = \frac{1}{2} + \frac{2}{\pi}\sin(\pi x) + \ldots + \frac{2}{15\pi}\sin(15\pi x)$$



EXAMPLE 25

Use *Mathematica* to compute the fourth Fourier polynomial for the function f which repeats the portion of the graph of $y = x^2$ between $x = -\pi$ and $x = \pi$ periodically, with period 2π:



Solution: We have $T = 2\pi$, so all integrals will run from 0 to 2π. Therefore

$$\begin{aligned} a_0 &= \frac{1}{T}\int_0^T f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \\ &= \frac{1}{2\pi}\int_0^{\pi} x^2\,dx + \frac{1}{2\pi}\int_{\pi}^{2\pi}(x-2\pi)^2\,dx \end{aligned}$$

```
(1/(2 Pi)) Integrate[x^2, {x,0,Pi}] +
 (1/(2 Pi)) Integrate[(x-2 Pi)^2, {x,Pi, 2Pi}
```

$$= \frac{\pi^2}{3}$$

EXAMPLE 25 (CONTINUED)

Also,

$$\begin{aligned}
c_n &= \frac{2}{T}\int_0^T f(x)\cos\left(\frac{2\pi n}{T}x\right)\,dx = \frac{2}{2\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} x^2\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(x-2\pi)^2\cos nx\,dx
\end{aligned}$$

```
(1 / Pi) Integrate[x^2 Cos[n x], {x,0,Pi}] +
(1 / Pi)Integrate[(x-2 Pi)^2 Cos[n x], {x, Pi, 2 Pi}
```

$$\begin{aligned}
&= \frac{4n\pi\cos(n\pi) + 2(-2+n^2\pi^2)\sin(n\pi)}{n^3\pi} \\
&= \frac{4\pi n\cos(n\pi)+0}{n^3\pi} = \frac{4}{n^2}\cos n\pi = \begin{cases} \frac{4}{n^2} & \text{if } n \text{ is even} \\ \frac{-4}{n^2} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

$$\begin{aligned}
s_n &= \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2\pi n}{T}x\right)\,dx = \frac{2}{2\pi}\int_0^{2\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} x^2\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(x-2\pi)^2\sin nx\,dx
\end{aligned}$$

```
(1 / Pi) Integrate[x^2 Sin[n x], {x,0,Pi}] +
(1 / Pi)Integrate[(x-2 Pi)^2 Sin[n x], {x, Pi, 2 Pi}
```

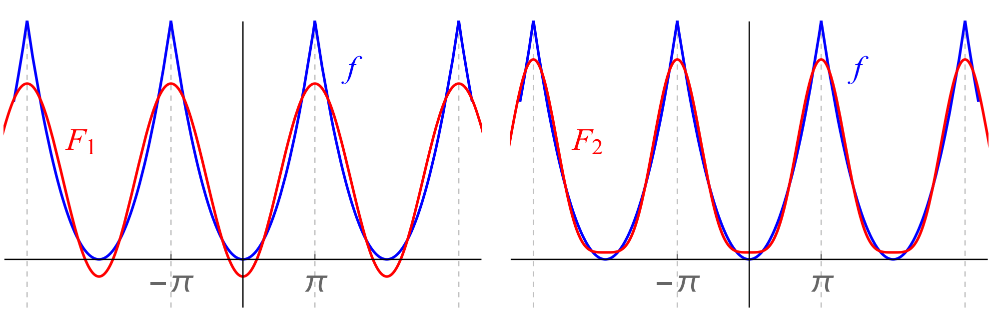
$$= 0.$$

Therefore the Fourier series is

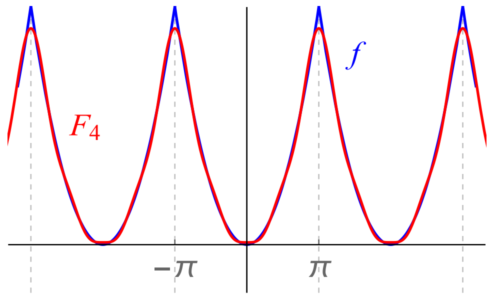
$$\boxed{f(x) = \frac{\pi^2}{3} - \frac{4}{1^2}\cos x + \frac{4}{2^2}\cos 2x - \frac{4}{3^2}\cos 3x + \frac{4}{4^2}\cos 4x - \dots}$$

and some Fourier polynomials are

$$F_1(x) = \frac{\pi^2}{3} - 4\cos x \qquad F_2(x) = \frac{\pi^2}{3} - 4\cos x + \cos 2x$$



$$F_4(x) = \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x + \frac{1}{4}\cos 4x$$

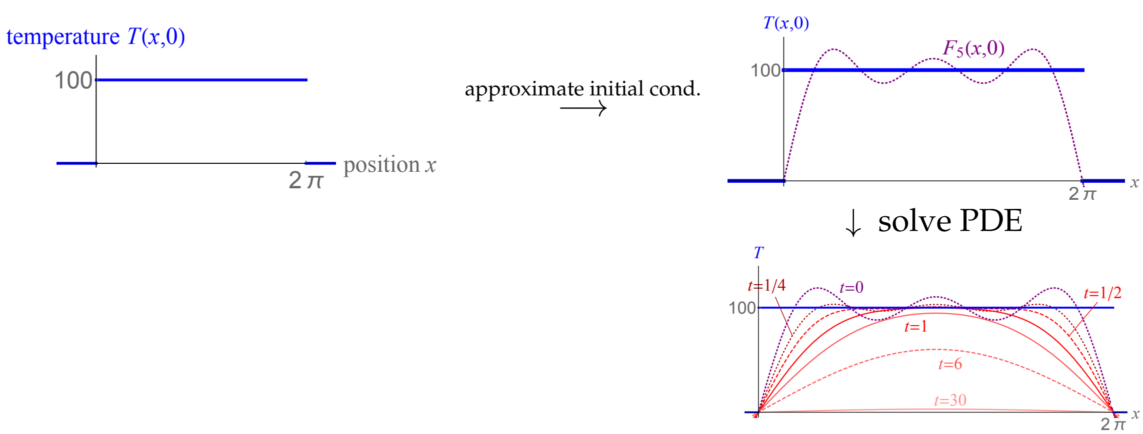


An application in heat transfer

EXAMPLE 26

A pipe of length 2π ft is insulated except at its two ends. The pipe is initially heated to 100° F. Let $T(x,t)$ represent the temperature at position x of the pipe (measured from the left end) t seconds after the ends of the pipe are initally exposed to an outside temperature of 0° F.

In MATH 330, you may learn how to model this situation using a *partial differential equation (PDE)*, which is much easier to approach if you approximate the initial temperature $T(x,0)$ by a Fourier polynomial. If you do that, you get these simulations:



Notice that after about time $t = 1$, the waves from the initial approximation have disappeared, and you see that the pipe cools quickly on the ends and more slowly in the middle (eventually the whole pipe has temperature 0° F).

This enables you to approximate the temperature at any point on the pipe at any time in the future, as the pipe cools.

Where do the Fourier coefficients come from?

In this section, we derive the formulas for a_0, c_n and s_n in the Fourier series

$$f(x) = \frac{1}{T}\int_0^T f(x)\,dx + \sum_{n=1}^{\infty}\left[c_n \cos\left(\frac{2\pi n}{T}x\right) + s_n \sin\left(\frac{2\pi n}{T}x\right)\right].$$

First, we need some basic results about integrals of our basic periodic functions:

Fact 1: $\displaystyle\int_0^T \cos\left(\frac{2\pi n}{T}x\right)\,dx = 0.$

PROOF Use the Linear Replacement Principle to get

$$\frac{T}{2\pi n}\sin\left(\frac{2\pi n}{T}x\right)\Big|_0^T = \frac{T}{2\pi n}\left[\sin(2\pi n) - \sin 0\right] = 0.$$

Fact 2: $\displaystyle\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\,dx = 0.$

PROOF HW (similar to Fact 1).

Fact 3: $\displaystyle\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi k}{T}x\right)\,dx = 0.$

PROOF HW (you have to rewrite the integrand with a weird trig identity).

Fact 4: $\displaystyle\int_0^T \cos\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi k}{T}x\right)\,dx = \begin{cases} \frac{T}{2} & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}.$

PROOF HW you have to rewrite the integrand with a weird trig identity).

Fact 5: $\displaystyle\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\sin\left(\frac{2\pi k}{T}x\right)\,dx = \begin{cases} \frac{T}{2} & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}.$

PROOF HW you have to rewrite the integrand with a weird trig identity).

Now suppose

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[c_n \cos\left(\frac{2\pi n}{T}x\right) + s_n \sin\left(\frac{2\pi n}{T}x\right)\right] \qquad (\bigstar)$$

First, let's find the constant term a_0.

To do this, integrate both sides of (★) over one period of the function f, from 0 to T:

$$\int_0^T f(x)\,dx = \int_0^T a_0\,dx + \sum_{n=1}^{\infty}\left[c_n \int_0^T \cos\left(\frac{2\pi n}{T}x\right)\,dx + s_n \int_0^T \sin\left(\frac{2\pi n}{T}x\right)\,dx\right]$$

$$= a_0\left[\frac{T}{2} - \frac{-T}{2}\right] + \sum_{n=1}^{\infty}\left[c_n(0) + s_n(0)\right] \quad \text{(by Facts 1 and 2 above)}$$

$= a_0 T$. Therefore the constant term is $\boxed{a_0 = \frac{1}{T}\int_0^T f(x)\,dx}$.

Next, we find the values of the cosine coefficients c_k.

To do this, first multiply f by $\cos\left(\frac{2\pi k}{T}x\right)$ and then integrate over one period of the function f, from 0 to T:

$$\int_0^T f(x)\cos\left(\frac{2\pi k}{T}x\right)\,dx =$$

$$a_0\int_0^T \cos\left(\frac{2\pi k}{T}x\right)\,dx + \sum_{n=1}^{\infty}\left[c_n\int_0^T \cos\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi k}{T}x\right)\,dx \right.$$

$$\left. + s_n\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi k}{T}x\right)\,dx\right]$$

$$= 0 + \sum_{n=1}^{\infty}\left[c_n\left(\begin{cases}\frac{T}{2} & \text{if } n = k\\ 0 & \text{otherwise}\end{cases}\right) + s_n(0)\right] \quad \text{(by Facts 1, 4 and 3 above)}$$

$= c_k\frac{T}{2}$. Therefore $\boxed{c_k = \frac{2}{T}\int_0^T f(x)\cos\left(\frac{2\pi k}{T}x\right)\,dx}$.

Last, we find the values of the sine coefficients s_k:

To do this, first multiply f by $\sin\left(\frac{2\pi k}{T}x\right)$ and then integrate over one period of the function f, from 0 to T:

$$\int_0^T f(x)\sin\left(\frac{2\pi k}{T}x\right)\,dx =$$

$$a_0\int_0^T \sin\left(\frac{2\pi k}{T}x\right)\,dx + \sum_{n=1}^{\infty}\left[c_n\int_0^T \cos\left(\frac{2\pi n}{T}x\right)\sin\left(\frac{2\pi k}{T}x\right)\,dx \right.$$

$$\left. + s_n\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\sin\left(\frac{2\pi k}{T}x\right)\,dx\right]$$

$$= 0 + \sum_{n=1}^{\infty}\left[c_n(0) + s_n\left(\begin{cases}\frac{T}{2} & \text{if } n = k\\ 0 & \text{otherwise}\end{cases}\right)\right] \quad \text{(by Facts 2, 3 and 5 above)}$$

$= s_k\frac{T}{2}$. Therefore $\boxed{s_k = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2\pi k}{T}x\right)\,dx}$.

8.5 Homework exercises

Exercises from Section 8.1

1. a) Verify that the Taylor series of e^x converges for all x, by using the Ratio Test and showing that $\rho = 0$ no matter what x is.
   b) Based on the fact that you used the Ratio Test in part (a), does the Taylor series of e^x converge absolutely, or converge conditionally, for all x?

2. a) Verify that the Taylor series of $\sin x$ converges for all x, by using the Ratio Test and showing that $\rho = 0$ no matter what x is.
   b) Based on the fact that you used the Ratio Test in part (a), does the Taylor series of $\sin x$ converge absolutely, or converge conditionally, for all x?

In Exercises 3-23, write the Taylor series of the given function.

3. $f(x) = \frac{3}{1-x}$
4. $f(x) = \frac{1}{1-x^2}$
5. $f(x) = \ln(1+x)$
6. $f(x) = \frac{1}{(1-x)^3}$
   Hint: Differentiate $\frac{1}{1-x}$ twice.
7. $f(x) = \frac{1}{1-x^3}$
8. $f(x) = \frac{2}{2+5x}$
9. $f(x) = \frac{-3}{-2-x}$
10. $f(x) = e^{-x}$
11. $f(x) = \cos 4x$
12. $f(x) = \ln(1-2x)$
13. $f(x) = \frac{1}{1+2x}$
14. $f(x) = \cos x^2$
15. $f(x) = \ln(1-x^2)$
16. $f(x) = x \sin 3x$
17. $f(x) = \frac{x}{e^{x^2}}$
18. $f(x) = x \arctan x^2$
19. $f(x) = \frac{2}{x^2+1}$
20. $f(x) = \frac{1}{(1-x)^2}$
21. $f(x) = x \sin x^2 - x^3$
22. $f(x) = e^x + e^{-x}$
    Hint: Find the Taylor series of e^x and e^{-x} independently, and then add them term-by-term.
23. $f(x) = (x^2+2)\cos x$
    Hint: Find the Taylor series of $x^2 \cos x$ and $2\cos x$ independently, and then add them term-by-term.

Exercises from Section 8.2

For Exercises 24-33, evaluate the following limits without using L'Hopital's Rule:

24. $\lim_{x\to 0} \frac{\sin x}{x}$
25. $\lim_{x\to 0} \frac{\cos x - 1}{x}$
26. $\lim_{x\to 0} \frac{\arctan x - x}{x^3}$
27. $\lim_{x\to 0} \frac{e^x - 1 - x}{2x^2}$
28. $\lim_{x\to 0} \frac{2e^x - 2 - 2x - x^2}{x^3}$
29. $\lim_{x\to 0} \frac{\arctan 6x^2 - 6x^2}{x^6}$
30. $\lim_{x\to 0} \frac{\sin x^8 - x^8}{x^{20}}$
31. $\lim_{x\to 0} \frac{\sin x^8 - x^8}{x^{24}}$
32. $\lim_{x\to 0} \frac{\sin x^8 - x^8}{x^{30}}$
33. $\lim_{x\to 0} \frac{\arctan x^9 - \ln(x^9 + 1)}{x^{18}}$

In Exercises 34-38, approximate each of the following numbers using the second Taylor polynomial of an appropriately chosen function (write your answer as a fraction in lowest terms).

34. $\ln .8$
35. $\sin .3$
36. $e^{3/5}$
37. $\arctan \frac{1}{6}$
38. $\sqrt{2}$

    Hint: Here, the appropriate function is $f(x) = \sqrt{x+1}$. You will have to figure out the second Taylor polynomial of $f(x)$ by computing derivatives of f at zero and using the definition of Taylor series.

In Exercises 39-43, approximate each of the following numbers using the fourth Taylor polynomial of an appropriately chosen function (write your answer as a fraction in lowest terms).

39. $\sin \frac{1}{4}$
40. $\cos .2$
41. $\sqrt{e}$
42. $\arctan \frac{2}{3}$
43. $\sqrt{\frac{3}{2}}$

    Hint: As in Exercise 38, the appropriate function is $f(x) = \sqrt{x+1}$.

44. Approximate $\int_0^{1/2} \cos(4x^2)\, dx$ by replacing the integrand with its fourth Taylor polynomial.

45. Approximate $\int_0^{1/2} \arctan x^2 \, dx$ by replacing the integrand with its fourth Taylor polynomial.

46. Approximate $\int_{-1}^{1} e^{-x^3} \, dx$ by replacing the integrand with its fourth Taylor polynomial.

47. Approximate $\int_0^{1} \ln(2x^2+1) \, dx$ by replacing the integrand with its fourth Taylor polynomial.

48. Approximate $\int_0^{1} x^2 \sin(x^2) \, dx$ by replacing the integrand with its sixth Taylor polynomial.

49. a) Approximate $\int_0^{1} x^8 \sin x \, dx$ by replacing the integrand with its twelfth Taylor polynomial.
    b) Describe the integration technique that one would use to find the exact value of this integral. (Isn't using a Taylor polynomial better?)

50. Suppose f is an unknown function with $f(0) = 2$, $f'(0) = 4$, $f''(0) = -6$, $f'''(0) = 0$ and $f^{(4)}(0) = 8$.
    a) Compute $P_3(x)$.
    b) Approximate $f(2)$ using the fourth Taylor polynomial of f.
    c) Approximate $\int_{-1}^{1} f(x) \, dx$ using the fourth Taylor polynomial of f.
    d) Compute $\lim_{x\to 0} \dfrac{f(x) - 4x - 2}{x^2}$.

51. Compute $f^{(6)}(0)$, if $f(x) = \sin x^2$.

52. Compute $f^{(36)}(0)$, if $f(x) = \cos x^2$.

53. Compute $f^{(100)}(0)$, if $f(x) = 4\ln(2x^2 + 1)$.

54. Compute $f^{(40)}(0)$, if $f(x) = e^{2x^3}$.

55. Compute $f^{(42)}(0)$, if $f(x) = e^{2x^3}$.

56. Compute $f^{(30)}(0)$, if $f(x) = \sum_{n=0}^{\infty} \dfrac{x^n}{(4n)!}$.

57. Compute $f^{(30)}(0)$, if $f(x) = \sum_{n=0}^{\infty} \dfrac{x^{4n+2}}{(4n)!}$.

58. Suppose the third-order Taylor polynomial (centered at 0) of some unknown function f is given by $P_3(x) = 2 - x - \frac{x^2}{3} + 2x^3$. Determine $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$.

59. Determine whether the series $\sum_{n=1}^{\infty} \left(e^{1/\sqrt{n}} - 1\right)$ converges or diverges.

60. Determine whether the series $\sum_{n=1}^{\infty} \left(\cos(\frac{1}{n}) - 1\right)$ converges or diverges.

61. Determine whether the series $\sum_{n=1}^{\infty} \ln\left(1 - \frac{1}{n}\right)$ converges or diverges.

In Exercises 62-79, compute the sum of each of the following series (you may assume without proof that each series converges):

62. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$

63. $2 + 2 + \frac{2}{2!} + \frac{2}{3!} + \frac{2}{4!} + \frac{2}{5!} + \ldots$

64. $\frac{\pi}{4} - \frac{\pi^2}{4^2 2!} + \frac{\pi^4}{4^4 4!} - \frac{\pi^6}{4^6 6!} + \ldots$

65. $1 - 3 + \frac{9}{2!} - \frac{27}{3!} + \frac{81}{4!} - \frac{3^5}{5!} + \ldots$

66. $1 + e + \frac{e^2}{2} + \frac{e^3}{3!} + \frac{e^4}{4!} + \ldots$

67. $2 - \frac{2\pi^2}{2!} + \frac{2\pi^4}{4!} - \frac{2\pi^6}{6!} + \ldots$

68. $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - \ldots$

69. $\frac{100}{2!} - \frac{10000}{4!} + \frac{10^6}{6!} - \frac{10^8}{8!} + \ldots$

70. $1 - \frac{\pi^2}{2^2 3!} + \frac{\pi^4}{2^4 5!} - \frac{\pi^6}{2^6 7!} + \frac{\pi^8}{2^8 9!} - \ldots$

71. $\frac{1}{2} - \frac{1}{2^2 \cdot 2} + \frac{1}{2^3 \cdot 3} - \frac{1}{2^4 \cdot 4} + \ldots$

72. $\sum_{n=0}^{\infty} \frac{(-1)^n}{7^n n!}$

73. $\sum_{n=2}^{\infty} \frac{3^n}{n!}$

74. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$

75. $\sum_{n=0}^{\infty} \frac{(-3\pi)^n}{2^n(2n+1)!}$

76. $\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{n3^n}$

77. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n\sqrt{3}}$

78. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n5^n}$

79. $\sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n4^n}$

80. Suppose f is an unknown function with $f'(x) = rf(x)$ and $f(0) = y_0$, where r and y_0 are constants.

a) Determine the Taylor series of f (your answer will be in terms of r and y_0).

b) Identify the function f you wrote in part (a) as a common function of x.

81. Suppose y is a function of x so that $y' + 2xy = 0$, where $y(0) = 3$.

    a) Compute the fourth Taylor polynomial of y.

    b) Write the Taylor series of y in Σ notation.

    c) Identify the Taylor series you wrote in part (b) as a function of x.

82. Suppose f is an unknown function with $f''(x) - xf(x) = 0$, where $f(0) = 1$ and $f'(0) = -1$.

    a) Compute the fifth Taylor polynomial of f.

    b) Use your answer to part (a) to estimate $f\left(\frac{1}{2}\right)$.

    c) Use your answer to part (a) to estimate $\displaystyle\int_0^1 f(x)\,dx$.

83. Suppose f is a function of x so that $x^2 f'''(x) = f(x)$, where $f''(0) = 12$. Compute the fifth Taylor polynomial of f.

84. Suppose y is a function of x so that $y'' - 2xy' + y = 0$, where $y(0) = 1$ and $y'(0) = 1$. Compute the fourth Taylor polynomial of y.

Exercises from Section 8.3

In Exercises 85-90,

a) Compute the radius of convergence of the given power series.

b) In Exercises 85-89, determine the interval of convergence of the series (i.e. determine the values of x for which the series converges). Omit this part in Exercise 90.

85. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2}(x+1)^n$

86. $\displaystyle\sum_{n=2}^{\infty} \frac{4^n}{n}(x-1)^n$

87. $\displaystyle\sum_{n=0}^{\infty} \frac{5}{2^n}(x-3)^n$

88. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}(x-1)^n$

89. $\displaystyle\sum_{n=3}^{\infty} \frac{n!}{2^n}(x+4)^n$

90. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}x^n$

Exercises from Section 8.4

91. Let f be a periodic function with period T.

    When we computed the Fourier coefficients a_0, c_n and s_n of f, we integrated all our functions from 0 to T (over one period of T). In fact, we can also integrate these functions from $-T/2$ to $T/2$ to obtain the same constants, i.e. we can also get a_0, c_n and s_n by

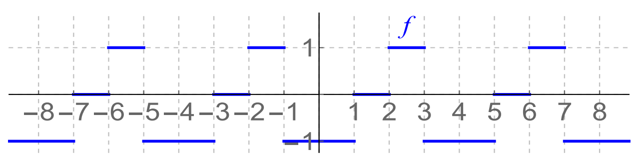
$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\,dx, \quad c_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos\left(\frac{2\pi n}{T}x\right)\,dx$$

$$\text{and} \quad s_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin\left(\frac{2\pi n}{T}x\right)\,dx.$$

    To justify this, show that for any periodic function g of period T,

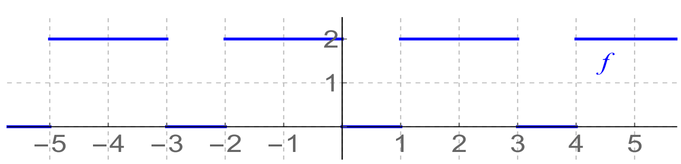
$$\int_0^T g(x)\,dx = \int_{-T/2}^{T/2} g(x)\,dx.$$

92. Let f be the function whose graph is shown below:



    a) Compute (by hand) the Fourier constant coefficient a_0 of f.
    b) Compute (by hand) the Fourier cosine coefficients c_n of f.
    c) Compute (by hand) the Fourier sine coefficients s_n of f.
    d) What is the fifth harmonic of f?
    e) Write the third Fourier polynomial of f.
    f) Use *Mathematica* to graph the third Fourier polynomial of f.

93. Let f be the function whose graph is shown below:



    a) Compute (by hand) the Fourier constant coefficient a_0 of f.
    b) Compute (by hand) the Fourier cosine coefficients c_n of f.
    c) Compute (by hand) the Fourier sine coefficients s_n of f.

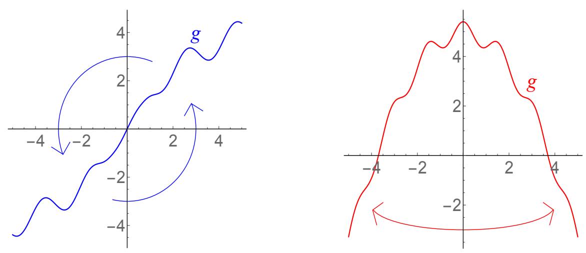
d) Write the second Fourier polynomial of f.

e) Use *Mathematica* to graph the second Fourier polynomial of f.

94. Use *Mathematica* to compute the Fourier series of the *triangular wave function* which is obtained by taking the graph of $|x|$ between $x = -1$ and $x = 1$ and extending it periodically with period 2.

95. Use *Mathematica* to compute the Fourier series of the *full-wave rectifier*, which is the function $f(x) = |\sin x|$.

96. An **odd** function is one whose graph is unchanged when you rotate it $180°$ about the origin (see the picture below at left). Equivalently, a function g is called odd if $g(-x) = -g(x)$ for all x.



An **even** function is one whose graph is symmetric about the y-axis (see the picture above at right). Algebraically, g is even if $g(-x) = g(x)$ for all x.

a) Classify each function as even, odd, or neither:

i. $g(x) = \tan x$	v. $g(x) = x$	ix. $g(x) = \arcsin x$
ii. $g(x) = \sin x$	vi. $g(x) = \frac{1}{x}$	x. $g(x) = 2x + 3$
iii. $g(x) = \cos x$	vii. $g(x) = \sqrt{x}$	xi. $g(x) = x^4$
iv. $g(x) = \lvert x \rvert$	viii. $g(x) = 5$	xii. $g(x) = \arctan x$

b) Show that for any odd function g and any number $T > 0$, $\displaystyle\int_{-T/2}^{T/2} g(x) = 0$.

c) The product of an even function and an odd function must be what kind of function? Explain.

d) Suppose f is an odd function. What must be true about the Fourier constant coefficient a_0 of that function? Explain (as a hint, look at part (b) of this problem and the result of Problem 91).

e) Suppose f is an odd function. What must be true about the Fourier cosine coefficients c_n of that function? Explain.

f) Suppose f is an even function. What must be true about the Fourier sine coefficients s_n of that function? Explain.

97. Establish Fact 2 from the last part of Section 8.4, which says

$$\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\,dx = 0.$$

98. Establish Fact 3 from the last part of Section 8.4, which says

$$\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi n}{T}x\right)\,dx = 0.$$

Hint: To do this, first rewrite the integrand using the "double angle" trig identity

$$\sin Ax\cos Ax = \frac{1}{2}\sin 2Ax,$$

then use the Linear Replacement Principle.

99. Establish Fact 4 from the last part of Section 8.4, which says

$$\int_0^T \cos\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi k}{T}x\right)\,dx = \begin{cases} \frac{T}{2} & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

Hint: First, rewrite the integrand using another "product to sum" trig identity, and then split the integral into two terms, using the Linear Replacement Principle on each term.

$$\cos Ax\cos Bx = \frac{1}{2}\left[\cos(A-B)x + \cos(A+B)x\right].$$

100. Establish Fact 5 from the last part of Section 8.4, which says

$$\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\sin\left(\frac{2\pi k}{T}x\right)\,dx = \begin{cases} \frac{T}{2} & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

Hint: First, rewrite the integrand using yet another "product to sum" trig identity:

$$\sin Ax\sin Bx = \frac{1}{2}\left[\cos(A-B)x - \cos(A+B)x\right].$$

Answers

1. a) $\rho = \lim_{n\to\infty} \frac{\left|\frac{x^{n+1}}{(n+1)!}\right|}{\left|\frac{x^n}{n!}\right|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$

   b) converges absolutely for all x

2. a) $\rho = \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}x^{2(n+1)+1}}{(2(n+1)+1)!}\right|}{\left|\frac{(-1)^n x^{2n}}{(2n+1)!}\right|} = \lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{|x|^{2n+1}} = \lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} =$
0

   b) converges absolutely for all x

3. $\sum_{n=0}^{\infty} 3x^n = 3 + 3x + 3x^2 + 3x^3 + 3x^4 + ...$

4. $\sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + ...$

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - ...$

6. $\sum_{n=0}^{\infty} (n+1)(n+2)x^n = 2 + 6x + 12x^2 + 20x^3 + 30x^4 + ...$

7. $\sum_{n=0}^{\infty} x^{3n} = 1 + x^3 + x^6 + x^9 + ...$

8. $\sum_{n=0}^{\infty} \left(\frac{-5}{2}\right)^n x^n = 1 - \frac{5}{2}x + \left(\frac{5}{2}\right)^2 x^2 - \left(\frac{5}{2}\right)^3 x^3 + \left(\frac{5}{2}\right)^4 x^4 - ...$

9. $\sum_{n=0}^{\infty} \frac{3}{2}\left(\frac{-1}{2}\right)^n x^n = \frac{3}{2} - \frac{3}{2}\cdot\frac{1}{2}x + \frac{3}{2}\left(\frac{1}{2}\right)^2 x^2 - \frac{3}{2}\left(\frac{1}{2}\right)^3 x^3 + \frac{3}{2}\left(\frac{1}{2}\right)^4 x^4 - ...$

10. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - ...$

11. $\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n}}{(2n)!} x^{2n} = 1 - \frac{4^2}{2!}x^2 + \frac{4^4}{4!}x^4 - \frac{4^6}{6!}x^6 + ...$

12. $\sum_{n=1}^{\infty} \frac{-2^n}{n} x^n = -2x - \frac{4}{2}x^2 - \frac{8}{3}x^3 - \frac{16}{4}x^4 - \frac{32}{5}x^5 - ...$

13. $\sum_{n=0}^{\infty} (-2)^n x^n = 1 - 2x + 4x^2 - 8x^3 + 16x^4 - ...$

14. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n} = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + ...$

15. $\sum_{n=1}^{\infty} \frac{-1}{n} x^{2n} = -x^2 - \frac{1}{2}x^4 - \frac{1}{3}x^6 - \frac{1}{4}x^8 \frac{1}{5}x^{10} - \frac{1}{6}x^{12} + ...$

16. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!} x^{2n+2} = \frac{3}{1!}x^2 - \frac{3^3}{3!}x^4 + \frac{3^5}{5!}x^6 - \frac{3^7}{7!}x^8 + \frac{3^9}{9!}x^{10} - ...$

17. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+1} = x - x^3 + \frac{x^5}{2!} - \frac{x^7}{3!} + \frac{x^9}{4!} - ...$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+3} = -\frac{1}{3!}x^7 + \frac{1}{5!}x^{11} - \frac{1}{7!}x^{15} + \frac{1}{9!}x^{19} - ...$

19. $\displaystyle\sum_{n=0}^{\infty} \frac{2}{(2n)!} x^{2n} = 2 + x^2 + \frac{2}{4!}x^4 + \frac{2}{6!}x^6 + ...$

20. $\displaystyle\sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + ...$

21. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{4n+3} = x^3 - \frac{x^7}{3} + \frac{x^{11}}{5} - \frac{x^{15}}{7} + ...$

22. $\displaystyle\sum_{n=0}^{\infty} 2(-1)^n x^{2n} = 2 - 2x^2 + 2x^4 - 2x^6 + 2x^8 - ...$

23. $\displaystyle 2 + \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{1}{(2n)!} - \frac{2}{(2n+2)!} \right) x^{2n+2} = 2 + \left(1 - \frac{2}{2!}\right) x^2 - \left(\frac{1}{2!} - \frac{2}{4!}\right) x^4 + ...$

24. 1

25. 0

26. $\dfrac{-1}{3}$

27. $\dfrac{1}{4}$

28. $\dfrac{1}{3}$

29. -72

30. 0

31. $\dfrac{-1}{6}$

32. $-\infty$

33. $\dfrac{1}{2}$

34. $\dfrac{-6}{25}$

35. $\dfrac{3}{10}$

36. $\dfrac{49}{25}$

37. $\dfrac{1}{6}$

38. $\dfrac{5}{4}$

39. $\dfrac{15}{64}$

40. $\dfrac{60}{625}$

41. $\dfrac{31}{16}$

42. $\dfrac{2}{27}$

43. $\dfrac{301}{256}$

44. $\dfrac{9}{20}$

45. $\dfrac{1}{24}$

46. $\dfrac{23}{15}$

47. $\dfrac{4}{15}$

48. $\dfrac{1}{5}$

49.
   a) $\dfrac{31}{360}$
   b) you would use parts 8 times.

   b) $f(2) \approx \dfrac{4}{3}$
   c) $\dfrac{32}{15}$
   d) -6

50.
   a) $P_3(x) = 2 + 4x - 3x^2 + \frac{1}{3}x^4$

51. -120

52. $\dfrac{-36!}{18!}$

53. $\dfrac{-2^{52}100!}{50}$

54. 0

55. $\dfrac{2^{14}42!}{14!}$

56. $\dfrac{30!}{120!}$

57. 870

58. $f(0) = 2$;
    $f'(0) = -1$;
    $f''(0) = \frac{-2}{3}$;
    $f'''(0) = 12$.

59. diverges

60. converges

61. diverges

62. $\dfrac{\pi}{4}$

63. $2e$

64. $\dfrac{\sqrt{2}}{2}$

65. e^{-3}

66. e^e

67. -2

68. $\cos 1$

69. $\cos 10 - 1$

70. $\dfrac{2}{\pi}$

71. $\ln\left(\dfrac{1}{2}\right)$

72. $e^{-1/7}$

73. $e^3 - 4$

74. $\cos 1$

75. $\sqrt{\dfrac{2}{3\pi}}\sin\sqrt{\dfrac{3\pi}{2}}$

76. $2\ln\dfrac{5}{3}$

77. $\dfrac{\pi}{6}$

78. $-\ln\dfrac{6}{5}$

79. $\ln\dfrac{5}{4} - \dfrac{7}{32}$

80. a) $f(x) = \sum_{n=0}^{\infty} y_0 \dfrac{r^n}{n!} x^n$
    b) $f(x) = y_0 e^{rx}$

81. a) $P_4(x) = 3 - 3x^2 + \dfrac{3}{2}x^4$
    b) $f(x) = \sum_{n=0}^{\infty} \dfrac{3(-1)^n}{n!} x^{2n}$
    c) $f(x) = 3e^{-x^2}$

82. a) $P_5(x) = 1 - x + \dfrac{1}{6}x^3 + \dfrac{1}{12}x^4$
    b) $\dfrac{31}{64}$
    c) $\dfrac{19}{40}$

83. $6x^2 + x^3 + \dfrac{1}{24}x^4 + \dfrac{1}{1440}x^5$

84. $1 + x - \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3 - \dfrac{1}{8}x^4$

85. a) $R = 1$
    b) $[0, 2]$

86. a) $R = \frac{1}{4}$
    b) $\left[\frac{3}{4}, \frac{5}{4}\right)$

87. a) $R = 4$
    b) $[-4, 4)$

88. a) $R = 2$
    b) $(1, 5)$

89. a) $R = 1$
    b) $[0, 2)$

90. a) $R = 0$

91. By additivity of integrals, $\int_{-T/2}^{T/2} g(x)\,dx = \int_{-T/2}^{0} g(x)\,dx + \int_{0}^{T/2} g(x)\,dx$. On the first interval, use the periodicity of g to rewrite it as $\int_{-T/2}^{0} g(x+T)\,dx$; then use the u-sub $u = x + T$ to rewrite it as $\int_{T/2}^{T} g(u)\,du$. So all together, we have

$$\begin{aligned}
\int_{-T/2}^{T/2} g(x)\,dx &= \int_{-T/2}^{0} g(x)\,dx + \int_{0}^{T/2} g(x)\,dx \\
&= \int_{T/2}^{T} g(u)\,du + \int_{0}^{T/2} g(x)\,dx \\
&= \int_{T/2}^{T} g(x)\,dx + \int_{0}^{T/2} g(x)\,dx \\
&\qquad \text{(since the name of the variable in the first integral doesn't matter)} \\
&= \int_{0}^{T} g(x)\,dx \text{ (by additivity of integrals).}
\end{aligned}$$

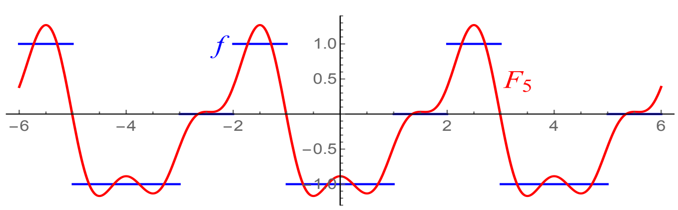
92. a) $a_0 = -\dfrac{1}{4}$.

b) $c_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-3}{n\pi} & \text{if } n = 1, 5, 9, 13, \ldots \\ \frac{3}{n\pi} & \text{if } n = 3, 7, 11, 15, \ldots \end{cases}$

c) $s_n = \begin{cases} \frac{-1}{n\pi} & \text{if } n \text{ is odd} \\ \frac{1}{n\pi} & \text{if } n \text{ is even} \end{cases}$

d) $\dfrac{-3}{5\pi}\cos\dfrac{5\pi x}{2} - \dfrac{1}{5\pi}\sin\dfrac{5\pi x}{2}$

e) $F_3(x) = -\dfrac{1}{4} - \dfrac{3}{\pi}\cos\dfrac{\pi x}{2} - \dfrac{1}{\pi}\sin\dfrac{\pi x}{2} + \dfrac{1}{\pi}\sin \pi x - \dfrac{1}{\pi}\cos\dfrac{3\pi x}{2} - \dfrac{1}{3\pi}\sin\dfrac{3\pi x}{2}$

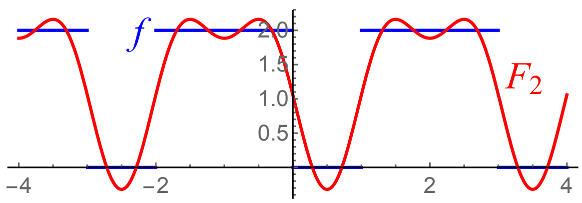
f) 

93. a) $a_0 = \dfrac{4}{3}$

b) $c_n = \dfrac{-2}{n\pi} \sin \dfrac{2\pi n}{3}$

c) $s_n = \dfrac{2}{n\pi}\left(\cos \dfrac{2\pi n}{3} - 1\right)$

d) $F_2(x) = \dfrac{4}{3} - \dfrac{\sqrt{3}}{\pi} \cos \dfrac{2\pi x}{3} + \dfrac{\sqrt{3}}{2\pi} \cos \dfrac{4\pi x}{3} - \dfrac{3}{\pi} \sin \dfrac{2\pi x}{3} - \dfrac{3}{2\pi} \sin \dfrac{4\pi x}{3}$

e) 

94. $f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} \dfrac{4\left(-2 + 2\cos\frac{n\pi}{2} + n\pi \sin\frac{n\pi}{2}\right)}{n^2\pi^2} \cos(\pi n x)$

95. $f(x) = \dfrac{2}{\pi} + \displaystyle\sum_{n=1}^{\infty} \dfrac{4\left(-2 + n\pi \sin\frac{n\pi^2}{4}\right)}{-4 + n^2\pi^2} \cos(2nx)$

96. a)

i. odd	iv. even	vii. neither	x. neither
ii. odd	v. odd	viii. even	xi. even
iii. even	vi. odd	ix. odd	xii. odd

b) By additivity of integrals, $\displaystyle\int_{-T/2}^{T/2} g(x)\,dx = \int_{-T/2}^{0} g(x)\,dx + \int_{0}^{T/2} g(x)\,dx$.
Use the u-sub $u = -x$ on the first integral to get

$$\int_{T/2}^{0} g(-x)(-\,dx) = -\int_{T/2}^{0} g(-x)\,dx.$$

Since g is odd, the first integral is

$$-\int_{T/2}^{0} -g(x)\,dx = \int_{T/2}^{0} g(x)\,dx = -\int_{0}^{T/2} g(x)\,dx.$$

All together, we have

$$\int_{-T/2}^{T/2} g(x)\,dx = \int_{-T/2}^{0} g(x)\,dx + \int_{0}^{T/2} g(x)\,dx = -\int_{0}^{T/2} g(x)\,dx + \int_{0}^{T/2} g(x)\,dx = \boxed{0}.$$

c) An even function times an odd function is **odd**. To see why, let f be even and g be odd. Then $(fg)(-x) = f(-x)g(-x) = f(x)[-g(x)] = -f(x)g(x) = -fg(x)$.

d) If f is odd, then by part (b) and Problem 91, a_0 must be $\boxed{0}$.

e) If f is odd, then since $\cos\left(\frac{2\pi n}{T}x\right)$ is even, $g(x) = f(x)\cos\left(\frac{2\pi n}{T}x\right)$ is odd by part (c). So by part (b) and # 91 applied to g, $c_n = \int_{-T/2}^{T/2} g(x)\,dx = \boxed{0}$.

f) If f is even, then since $\sin\left(\frac{2\pi n}{T}x\right)$ is odd, $g(x) = f(x)\sin\left(\frac{2\pi n}{T}x\right)$ is odd by part (c). So by part (b) and # 91 applied to g, $s_n = \int_{-T/2}^{T/2} g(x)\,dx = \boxed{0}$.

97. Use the Linear Replacement Principle:

$$\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\,dx = -\frac{T}{2\pi n}\cos\left(\frac{2\pi n}{T}x\right)\Big|_0^T = -\frac{T}{2\pi n}\cos(2\pi n) + \frac{T}{2\pi n}\cos 0 = -1+1 = \boxed{0}.$$

98. First, use the product to sum identity, then use the Linear Replacement Principle:

$$\begin{aligned}\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi n}{T}x\right)\,dx &= \int_0^T \frac{1}{2}\sin\left(\frac{4\pi n}{T}x\right)\,dx \\ &= -\frac{1}{2}\cdot\frac{T}{4\pi n}\cos\left(\frac{4\pi n}{T}x\right)\Big|_0^T \\ &= \frac{-T}{8\pi n}\left[\cos 4\pi n - \cos 0\right] = \frac{-T}{8\pi n}[1-1] = \boxed{0}.\end{aligned}$$

99. First, use the product to sum identity:

$$\int_0^T \cos\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi k}{T}x\right)\,dx = \int_0^T \frac{1}{2}\left[\cos\left(\frac{2\pi(n-k)}{T}x\right) + \cos\left(\frac{2\pi(n+k)}{T}x\right)\right]\,dx$$

If $n \neq k$, then we can split the integral as

$$\begin{aligned}&\frac{1}{2}\int_0^T \cos\left(\frac{2\pi(n-k)}{T}x\right)\,dx + \frac{1}{2}\int_0^T \cos\left(\frac{2\pi(n+k)}{T}x\right)\,dx \\ &= \frac{T}{4\pi(n-k)}\sin\left(\frac{2\pi(n-k)}{T}x\right)\Big|_0^T + \frac{T}{4\pi(n+k)}\sin\left(\frac{2\pi(n+k)}{T}x\right)\Big|_0^T \\ &= \frac{T}{4\pi(n-k)}\left[\sin 2\pi(n-k) - \sin 0\right] + \frac{T}{4\pi(n+k)}\left[\sin 2\pi(n+k) - \sin 0\right] \\ &= \frac{T}{4\pi(n-k)}[0-0] + \frac{T}{4\pi(n+k)}[0-0] = \boxed{0}.\end{aligned}$$

If $n = k$, the second integral is still zero. But the first term is

$$\frac{1}{2}\int_0^T \cos 0x\,dx = \frac{1}{2}\int_0^T 1\,dx = \frac{1}{2}[T-0] = \boxed{\frac{T}{2}}.$$

100. This is exactly the same as Exercise 99, except that the red $+$ signs become $-$ signs, due to the different "product to sum" identity.

Appendix A

Additional review material

Old MATH 230 exams are available on my website:

http://mcclendonmath.com/230.html

I recommend using the old exams as your primary review material, but if you need more practice, this appendix contains some additional problems.

A.1 Additional review exercises for Exam 1

In Exercises 1-13, compute each integral.

1. $\displaystyle\int \left(3\sec x\tan x + 5\sin x - \frac{\sec^2 x}{2}\right) dx$
2. $\displaystyle\int \frac{\sin(1/x)}{x^2}\, dx$
3. $\displaystyle\int x\sin 3x\, dx$
4. $\displaystyle\int_{\pi/4}^{\pi/3} \cot x\, dx$
5. $\displaystyle\int \sin^4 x\cos^3 x\, dx$
6. $\displaystyle\int \sec^2 x\tan^5 x\, dx$
7. $\displaystyle\int \frac{4+x}{(x-2)^2}\, dx$
8. $\displaystyle\int \frac{3x-2}{x+3}\, dx$
9. $\displaystyle\int \frac{19-x}{x^2+2x-15}\, dx$
10. $\displaystyle\int \frac{e^x}{e^{2x}-4}\, dx$

    Hint: Start with the $u-$sub $u = e^x$.
11. $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\, dx$

12. $\displaystyle\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

13. $\displaystyle\int x^3 \ln x\,dx$

In Exercises 14-19, determine, with justification, whether each improper integral converges or diverges.

14. $\displaystyle\int_3^\infty \frac{\ln x}{x}\,dx$

15. $\displaystyle\int_2^\infty xe^{-4x}\,dx$

16. $\displaystyle\int_1^5 \frac{1}{\sqrt{5-x}}\,dx$

17. $\displaystyle\int_2^\infty \frac{1}{x^3+4x+1}\,dx$

18. $\displaystyle\int_2^\infty \frac{x^3+\sin x+2}{x^4}\,dx$

19. $\displaystyle\int_0^\infty x^5 e^{-2x}\,dx$

20. Use the method of undetermined coefficients to find a function f so that $f''(x) - 8f'(x) + 15f(x) = 12e^{2x}$.

Some sample *Mathematica* questions:

21. For each problem, you are given a problem that a student was trying to solve on *Mathematica*, and what the student typed in. What they typed in is WRONG; explain why their code is wrong.

    a) Student wants to find the sine of $\pi/6$ and types in `Sin(Pi/6)`
    b) Student wants to find $\log 7$ and types in `Log[7]`
    c) Student wants to solve the equation $x^2 + 3x = 7$ and types in `Solve[x^2 + 3x = 7, x]`
    d) Student wants to define function $f(x) = x^2$ and types in `f[x] = x^2`
    e) Student wants to define function $f(x) = e^{2x}$ and types in `f[x_] = e^(2x)`
    f) Student wants to evaluate $\frac{32+9}{63-17}$ and types in `[32+9]/[63-17]`
    g) Student wants to define function $f(x) = \frac{x-1}{x+1}$ and types in `f[x_] = x-1/x+1`
    h) Student wants a decimal approximation to $\int_1^2 e^{-2/x^2}\,dx$ and types in `N[Integrate[E^(-2/x^2), {x, 1, 2}]]`

22. In each part of this problem, you are given some code in *Mathematica* (the code works). Determine what output *Mathematica* will give you.

    a) `f[x_] = x^2 + x; f[3]`
    b) `Cos[2 Pi/3]`
    c) `g[x_] = 1/x-1; g[x+1]`
    d) `Solve[x+3 ==5, x]`
    e) `Factor[x^2 - 4, x]`
    f) `D[x^2, x]`
    g) `Integrate[x, {x, 2, 4}]`
    h) `Limit[1/x, x -> 3]`

23. Suppose you typed in the following command into *Mathematica*:

```
Plot[x^3 Log[x^2 + 1], {x, -3, 5}, PlotRange -> {0,4}]
```

a) What function is being plotted? (Write the function in hand-written notation, not *Mathematica* syntax.)
b) What $x-$value will be at the left edge of the graph?
c) What $y-$value will be at the top of the graph?

24. Write the *Mathematica* code which will accomplish each of the following tasks:

a) Find the exact value of $\cos(\frac{\pi}{8})$
b) Define the function $f(x) = \cos(e^x - 1)$
c) Find the indefinite integral of $f(x) = (x^2 + 1)^5$
d) Find the definite integral of $f(x) = 2\sec x$ from $x = 0$ to $x = \pi/3$
e) Find a decimal approximation to the definite integral of $f(x) = \sqrt{x^2 + 1}$ from $x = 0$ to $x = 4$
f) Find the third derivative of $f(x) = e^{x^2} \sin x$
g) Compute $5 + 7 + 9 + 11 + ... + 1755$
h) Find $f'(6)$, given that $f(x) = 2^x$

Answers

1. $3\sec x - 5\cos x + \frac{1}{2}\tan x + C$ (just do it).

2. $\cos\left(\frac{1}{x}\right) + C$ (use the u-sub $u = \frac{1}{x}$).

3. $\frac{-1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$ (use parts with $r = x$, $ds = \sin 3x\,dx$).

4. $\frac{1}{2}\ln\frac{3}{2}$ (this can also be written as $\frac{1}{2}\ln 3 - \frac{1}{2}\ln 2$)
(write $\cot x$ as $\frac{\cos x}{\sin x}$, then use the u-sub $u = \sin x$).

5. $\dfrac{\sin^5 x}{5} - \dfrac{\sin^7 x}{7} + C$ (rewrite as $\int \sin^4 x(1 - \sin^2 x)\cos x\,dx$ then use the u-sub $u = \sin x$).

6. $\frac{1}{6}\tan^6 x + C$ (use the u-sub $u = \tan x$).

7. $\dfrac{-6}{x-2} + \ln|x - 2| + C$ (either use the u-sub $u = x - 2$ or use partial fractions).

8. $3(x+3) - 11\ln|x+3| + C$ (use the u-sub $u = x+3$).
   (the answer $3x - 11\ln|x+3| + C$ is also correct)
9. $2\ln|x-3| - 3\ln|x+5| + C$ (use partial fractions).
10. $\frac{1}{4}\ln|e^x - 2| - \frac{1}{4}\ln|e^x + 2| + C$ (after the u-sub in the hint, use partial fractions).
11. $-\sqrt{1-x^2} + C$ (use the u-sub $u = 1 - x^2$).
12. $\frac{-1}{3}x^2\sqrt{1-x^2} - \frac{2}{3}\sqrt{1-x^2} + C$ (use the u-sub $u = 1 - x^2$).
13. $\dfrac{1}{4}x^4\ln x - \dfrac{1}{16}x^4 + C$ (use parts with $r = \ln x$, $ds = x^3\,dx$).
14. diverges (compute it directly)
15. converges (compute it directly; it converges to $\frac{9}{16}e^{-8}$)
16. converges (compute it directly; it converges to 4)
17. converges (by the Comparison Test; compare the integrand with $\frac{1}{x^3}$)
18. diverges (by the Comparison Test; compare the integrand with $\frac{x^3}{x^4} = \frac{1}{x}$)
19. converges (it is a Gamma integral that converges to $\frac{15}{8}$)
20. $f(x) = 4e^{2x}$.
21. a) Used parentheses instead of brackets: command should have been `Sin[Pi/6]`.
    b) `Log` computes natural logarithm, not logarithm base 10.
    c) Equation inside the `Solve` command needs two equal signs, not one.
    d) Missing underscore after the `x`: command should have been `f[x_] = x^2`.
    e) e is capital `E`, not lowercase `e`: should have been `f[x_] = E^(2x)`.
    f) Used brackets instead of parentheses: should have been `(32+9)/(63-17)`.
    g) Forgot parentheses: should have been `f[x_] = (x-1)/(x+1)`.
    h) `NIntegrate[... ]` is much faster than `N[Integrate[ ...]`.
22. a) 12
    b) $-1/2$
    c) $\frac{1}{x+1} - 1$
    d) 2
    e) $(x-2)(x+2)$
    f) $2x$
    g) 6 (this is $\int_2^4 x\,dx$)
    h) $\frac{1}{3}$

23. a) $x^3 \ln(x^2+1)$ b) -3 c) 4

24. There could be more than one answer to some of these questions:

   a) Cos[Pi/8]
   b) f[x_] = Cos[E^x - 1]
   c) Integrate[(x^2+1)^5, x]
      (you could also do ∫ (x^2+1)^5 dx, etc.)
   d) Integrate[2 Sec[x], {x, 0, Pi/3}]
      (you could also do $\int_0^{\mathrm{Pi}/3}$ 2 Sec[x] dx, etc.)
   e) NIntegrate[Sqrt[x^2+1], {x, 0, 4}]
      (you could also do NIntegrate[$\sqrt{\text{x^2+1}}$, {x, 0, 4}] , etc.)
   f) D[E^(x^2) Sin[x], {x, 3}]
      (you could also do f[x_] = E^(x^2) Sin[x]; f"'[x] , etc.)
   g) Sum[x, {x, 5, 1755, 2}]
   h) f[x_] = 2^x; f'[6]

A.2 Additional review exercises for Exam 2

1. Let Q be the region in the $xy-$plane bounded by the graphs of $y + x = 0$, $2y - x = 0$ and $y = \sqrt{x+1} + 1$. (The corner points of this region are $(0,0)$, $(-1,1)$ and $(8,4)$; you might be responsible for solving for these on the exam.)

   a) Write an expression involving one or more integrals with respect to x which gives the area of Q.

   b) Write an expression involving one or more integrals with respect to y which gives the area of Q.

2. Let R be the region in the $xy-$plane which lies below the graph of $y = 8x$ and above the graph of $y = e^{-x}$ between $x = 1$ and $x = 4$.

   a) Write an expression involving one or more integrals (with respect to any variable you want) which gives the volume of the solid obtained by revolving R around the $x-$axis.

   b) Write an expression involving one or more integrals (with respect to any variable you want) which gives the volume of the solid obtained by revolving R around the $y-$axis.

c) Write an expression involving one or more integrals (with respect to any variable you want) which gives the volume of the solid whose base is R and whose cross-sections parallel to the $y-$axis are semicircles with diameter in the $xy-$plane.

3. Let S be the region in the $xy-$plane bounded by the graphs of $y = \ln x$, $y = 0$ and $x = e^2$. Write an integral with respect to x, and a different integral with respect to y, which gives the volume of the solid obtained by revolving S around each of these lines:

$$y = 8 \qquad y = -2 \qquad x = 13 \qquad x = -5 \qquad x = e^2$$

4. Write an integral which gives the length of the curve $y = x^{2/3}$ from $x = 0$ to $x = 8$.

5. Suppose a 20-foot rod is made from a material whose density is $x(30-x)$ g/ft where x is the number of feet from the left end of the rod.

   a) What is the mass of the rod?

   b) How far from the left end of the rod is its center of mass?

6. Find the centroid of the region bounded by the $x-$axis and the portion of the graph of $y = \cos x$ from $x = -\pi/2$ to $x = \pi/2$. You may leave your answer in terms of unevaluated integrals if you like.

7. Suppose a slab of metal has shape bounded by the graphs $y = x$ and $y = \frac{1}{6}x^2$, and that its density at any point (x, y) is $x + 1$ kg/sq unit. Find the center of mass of the slab; you may leave your answer in terms of unevaluated integrals if you like.

8. Compute the moments of inertia about the x- and y-axes for the region of points in the xy-plane lying above the graph of $y = x^2 - 1$, to the left of the y-axis, and below the graph of $y = 3$.

9. Suppose X is a continuous random variable with density function

$$f(x) = \begin{cases} cx^5 & \text{if } 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

   where c is some unknown constant.

   a) Determine the value of c.

   b) Compute the expected value of X.

10. Suppose X is a continuous random variable with density function

$$f(x) = \begin{cases} 2e^{-2x} & \text{if } x \geq 0 \\ 0 & \text{else} \end{cases}$$

a) Compute the probability that $X \geq 5$.

b) If b is so that $P(X \leq b) = \frac{3}{4}$, what is the value of b?

c) Compute the expected value of X.

Answers

1. a) $\displaystyle\int_{-1}^{0} \left[\sqrt{x+1} + 1 - (-x)\right] \, dx + \int_{0}^{8} \left[\sqrt{x+1} + 1 - \frac{x}{2}\right] \, dx$

   b) $\displaystyle\int_{0}^{1} [2y - (-y)] \, dy + \int_{1}^{4} \left[2y - ((y-1)^2 - 1)\right] \, dy$

2. a) $\displaystyle\int_{1}^{4} \pi \left[(8x)^2 - (e^{-x})^2\right] \, dx$

   b) $\displaystyle\int_{1}^{4} 2\pi x \left[8x - e^{-x}\right] \, dx$

   c) $\displaystyle\int_{1}^{4} \pi \frac{1}{2} \left[\frac{8x - e^{-x}}{2}\right]^2 \, dx$

3. • Around $y = 8$:

$$V = \int_{1}^{e^2} \pi \left[8^2 - (8 - \ln x)^2\right] \, dx \; = \; \int_{0}^{2} 2\pi(8-y) \left[e^2 - e^y\right] \, dy$$

   • Around $y = -2$:

$$V = \int_{1}^{e^2} \pi \left[(\ln x + 2)^2 - (0 - -2)^2\right] \, dx \; = \; \int_{0}^{2} 2\pi(y+2) \left[e^2 - e^y\right] \, dy$$

   • Around $x = 13$:

$$V = \int_{1}^{e^2} 2\pi(13 - x) \left[(\ln x) - 0\right] \, dx \; = \; \int_{0}^{2} \pi \left[(13 - e^y)^2 - (13 - e^2)^2\right] \, dy$$

   • Around $x = -5$:

$$V = \int_{1}^{e^2} 2\pi(x + 5) \left[(\ln x) - 0\right] \, dx \; = \; \int_{0}^{2} \pi \left[(e^2 + 5)^2 - (e^y + 5)^2\right] \, dy$$

   • Around $x = e^2$:

$$V = \int_{1}^{e^2} 2\pi(e^2 - x) \left[(\ln x) - 0\right] \, dx \; = \; \int_{0}^{2} \pi \left[(e^2 - e^y)^2\right] \, dy$$

4. $s = \int_0^8 \sqrt{1 + \left(\frac{2}{3}x^{-1/3}\right)^2}\, dx$

5. a) $M = \int_0^{20} x(30 - x)\, dx = \frac{10000}{3}$ g.

   b) $M_0 = \int_0^{20} x \cdot x(30 - x)\, dx = 40000$ so $\overline{x} = 12$ feet from the left of the rod.

6. The centroid is $(\overline{x}, \overline{y})$ where

$$\overline{x} = 0 \text{ (by symmetry) and } \overline{y} = \frac{M_x}{M} = \frac{\int_{-\pi/2}^{\pi/2} \frac{1}{2} \cos^2 x\, dx}{\int_{-\pi/2}^{\pi/2} \cos x\, dx}.$$

7. The center of mass is $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{\int_0^6 x(x - \frac{1}{6}x^2)(x+1)\, dx}{\int_0^6 (x - \frac{1}{6}x^2)(x+1)\, dx} \text{ and } \overline{y} = \frac{\int_0^6 \frac{1}{2}\left[x^2 - (\frac{1}{6}x)^2)^2\right](x+1)\, dx}{\int_0^6 (x - \frac{1}{6}x^2)(x+1)\, dx}.$$

8. $I_y = \frac{64}{15}$; $I_x = \frac{1712}{105}$.

9. a) $c = 6$

   b) $EX = \frac{6}{7}$

10. a) e^{-10}

    b) $b = \ln 2$

    c) $EX = \frac{1}{2}$

A.3 Additional review exercises for Exam 3

1. Compute the third partial sum of the harmonic series.

2. Write each of these series in $\Sigma-$notation.

   a) $2 - \frac{4}{3!} + \frac{8}{6!} - \frac{16}{9!} + \frac{32}{12!} - ...$

   b) $3 + \frac{5}{4} + \frac{7}{7} + \frac{9}{10} + \frac{11}{13} + \frac{13}{16} + ...$

3. Rewrite the series $\sum_{n=2}^{\infty} \frac{3n+1}{4^{n-1}}$ so that its starting index is 0; write your answer in $\Sigma-$notation.

4. Determine, with justification, whether each of the following series converges absolutely, converges conditionally, or diverges:

   a) $\sum_{n=1}^{\infty} \frac{3}{2^n + n^2 + 6}$

   b) $\sum_{n=0}^{\infty} \frac{\cos(\pi n)}{5n\sqrt{n}}$

   c) $\sum_{n=0}^{\infty} \frac{(-1)^n (3n)!}{(n!)^3}$

   d) $\sum_{n=0}^{\infty} \left(\frac{2}{n+3} - \frac{4}{5^n} \right)$

   e) $\sum_{n=0}^{\infty} \frac{3n-2}{2n+1}$

   f) $\sum_{n=2}^{\infty} (-1)^n n^{-1/3}$

5. Write the Taylor series of the function $f(x) = \sin(2x^2)$.

6. Let $f(x) = x^3 \ln(x^3 + 1)$.

   a) Write the Taylor series of f in Σ notation.

   b) Write the eighth Taylor polynomial of $f(x)$.

   c) Use the eighth Taylor polynomial to estimate $f(\frac{1}{2})$.

7. Determine the interval of convergence of each power series:

   a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot n}{n+2} (x-1)^n$

   b) $\sum_{n=0}^{\infty} \frac{2n}{3^n} x^n$

   c) $\sum_{n=0}^{\infty} (n+2)!(x-2)^n$

8. Suppose the power series $f(x) = \sum_{n=0}^{\infty} a_n (x+3)^n$ converges conditionally when $x = 0$.

   a) Does this power series converge or diverge when $x = 2$?

   b) Does this power series converge or diverge when $x = -5$?

9. Suppose $f(x) = x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \frac{x^5}{25} + ...$ Compute $f^{(12)}(0)$, the twelfth derivative of f at zero.

10. Estimate $\cos .3$ by approximating a suitable function with its third Taylor polynomial.

11. Estimate $\int_0^1 x \arctan x \, dx$ by replacing the integrand with its third Taylor polynomial.

12. Evaluate $\lim_{x \to 0} \frac{\cos(4x^5) - 1}{x^{10}}$ without using L'Hôpital's Rule.

13. Compute the exact sum of each series; simplify your answers.

    a) $\sum_{n=1}^{\infty} \frac{-3}{5^{n-1}}$

    b) $\sum_{n=0}^{\infty} \frac{3 \cdot 4^n}{2 \cdot 5^{2n-1}}$

    c) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n!}$

    d) $\sum_{n=0}^{\infty} \frac{1}{5^n n!}$

    e) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{3^{2n+1}(2n+1)!}$

    f) $\sum_{n=4}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{2}{3}\right)^n$

    g) $3 - \frac{3^3}{3} + \frac{3^5}{5} - \frac{3^7}{7} + \frac{3^9}{9} - \frac{3^{11}}{11} + ...$

    h) $2 - 1 + \frac{1}{2} - \frac{1}{4} + ... - \frac{1}{2^{20}}$

    i) $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - ...$

14. Suppose y is a function with $y(0) = 0$, $y'(0) = 1$ and $xy'' + 2x^2y' = y$. Write the fifth Taylor polynomial of y.

15. Let f be the piecewise constant function

$$f(x) = \begin{cases} 0 & \text{if } x \in ..., [-4,-2), [0,2), [4,6), [8,10), ... \\ 6 & \text{if } x \in ..., [-6,-4), [-2,0), [2,4), [6,8), ... \end{cases}$$

Compute the third Fourier polynomial of f.

16. a) Why is calculus required to add up an infinite list of numbers?
    b) What does it mean (precisely) for a series to *converge*?
    c) What does it mean (precisely) for a series to *converge absolutely*?
    d) What does it mean for a series to *converge conditionally*?
    e) Why do we care whether or not knowing if a series absolutely converges (rather than just knowing if it converges)?
    f) What is the difference between the terms *power series* and *Taylor series*?
    g) What is meant by *uniqueness of power series*?

Answers

1. $\frac{11}{6}$
2. There are many different correct answers to each part of this question.

   a) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1}}{(3n)!}$ b) $\sum_{n=0}^{\infty} \frac{2n+3}{3n+1}$
3. $\sum_{n=0}^{\infty} \frac{3n+7}{4^{n+1}}$
4. a) converges absolutely (positive series; Comparison Test)
   b) converges absolutely (p-series Test (the Alt. Series test also applies to tell you the series converges))
   c) diverges (Ratio Test, $\rho = 27 > 1$)
   d) diverges (divergent (harmonic) $-$ convergent (geometric with $r = \frac{1}{5}$))
   e) diverges (nth Term Test)
   f) converges conditionally (converges by AST; converges conditionally by p-series test)
5. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{4n+2}$
6. a) $x^6 - \frac{x^9}{2} + \frac{x^{12}}{3} - \frac{x^{15}}{4} + \frac{x^{18}}{5} - \frac{x^{21}}{6} + \ldots$
   b) $P_8(x) = x^6$
   c) $\left(\frac{1}{2}\right)^6$
7. a) $(0, 2)$
   b) $(-3, 3)$

c) $\{2\}$ (i.e. the series converges only when $x = 2$ since $R = 0$)

8. a) The series diverges when $x = 2$.
   b) The series converges (absoutely) when $x = -5$.

9. $f^{(12)}(0) = \frac{12!}{144}$

10. $\cos .3 \approx \frac{191}{200}$

11. $\frac{1}{3}$

12. -8

13. a) $\frac{-15}{4}$
    b) $\frac{10}{7}$
    c) e^{-1}
    d) $e^{1/5}$
    e) $\frac{\sqrt{3}}{2}$
    f) $\ln \frac{5}{3} - \frac{44}{81}$
    g) $\arctan 3$
    h) $\frac{4}{3}\left(1 - 2^{-22}\right)$
    i) $\cos 1$

14. $P_5(x) = x + \frac{1}{2}x^2 - \frac{1}{4}x^3 - \frac{9}{48}x^4 + \frac{21}{320}x^5$

15. $F_3(x) = 3 - \frac{12}{\pi}\sin\frac{\pi x}{2} - \frac{4}{\pi}\sin\frac{3\pi x}{2}$

16. a) Addition is formally a binary operation (it has two inputs and one output). If you try to add an infinite list of numbers two at a time, you never run out of numbers in the list. So to add the infinite list, you need to take a limit, which means you need calculus.
    b) $\sum a_n$ converges to L if $\lim_{N\to\infty} S_N = L$, where S_N is the N^{th} partial sum of the series.
    c) $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.
    d) $\sum a_n$ converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.
    e) The terms of a series can be rearranged legally if and only if the series absolutely converges. So if we know whether or not a series converges absolutely, we know if we can rerarrange or regroup terms.
    f) A power series is any function of the form $\sum a_n(x-a)^n$. A Taylor series is a specific kind of power series associated to a function f; it is defined by $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ and its main theoretical attribute is that the Taylor series of f centered at a is the only power series centered at a which can be equal to the function f.
    g) The uniqueness of power series is the principle that if you have two power series centered at a which are equal as functions, all the coefficients of the two power series have to be equal. In symbols, this means that if $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$, then $a_n = b_n$ for all n.

A.4 Additional review exercises for the Final Exam

1. Evaluate each integral:

   a) $\int x \sin \pi x \, dx$

   b) $\int_0^{\pi/3} \cos x \sin^2 x \, dx$

   c) $\int \cot 4x \, dx$

   d) $\int \frac{4x^2 + 6x - 12}{x^3 - 4x} \, dx$

   e) $\int \frac{3}{x^2 + 1} \, dx$

   f) $\int_0^1 x e^x \, dx$

   g) $x\sqrt{x-3}\, dx$

   h) $\int \frac{x+4}{x^2 + 3x + 2} \, dx$

2. Determine whether or not each improper integrals converges or diverges:

   a) $\int_0^{\infty} x^4 e^{-x} \, dx$

   b) $\int_0^{\infty} x e^{-x^2} \, dx$

   c) $\int_1^{\infty} \frac{3x}{x^3 + 1} \, dx$

   d) $\int_0^2 \frac{3}{\sqrt[5]{x^2}} \, dx$

3. Compute the area of the "triangular"-shaped region in the first quadrant bounded by the graphs of $y = x^2$, $x + y = 2$ and the $x-$axis.

4. Write integals which give the volume of the solid formed by revolving the region of the previous problem around each of the following lines (you should be able to write integrals with respect to either x or y):

   a) the $x-$axis

   b) the line $x = -2$

   c) the line $x = 5$

   d) the line $y = 2$

5. Compute the length of the curve $y = 4x^{3/2} + 1$ between $x = 1$ and $x = 2$.

6. Compute the center of mass of a rod of length 3 ft, where the density of the rod at a point x ft from the left-end of the rod is $\rho(x) = x^2 + x$ mg/ft.

7. Let R be the region in the xy plane above the line $y = x - 3$ and below the curve $y = \sqrt[4]{x-3}$. (The left- and right-most points of this region are $(3, 0)$ and $(4, 1)$.)

   a) Determine the centroid of R.

   b) Compute the moment of inertia of R about the y-axis.

8. Suppose the amount of time, in months, that a lightbulb lasts is a continuous random variable whose density function is

$$f(x) = \begin{cases} Ce^{-x/9} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} .$$

a) Determine the value of C.
b) Compute the probability that the lightbulb lasts at least 4 months.
c) Compute the expected lifespan of a lightbulb.

9. State precisely what it means for an infinite series to *converge*.

10. Compute the sum of each series:

a) $\displaystyle\sum_{n=2}^{\infty}\left(\frac{2}{3}\right)^n$

b) $\displaystyle\sum_{n=0}^{\infty}\frac{3\cdot 2^{3(n+1)}}{5\cdot 11^{n-1}}$

c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\pi^{2n}}{(2n)!}$

d) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{(2n+1)4^{2n+1}}$

11. Determine, with appropriate justification, whether each series converges absolutely, converges conditionally, or diverges:

a) $\displaystyle\sum_{n=2}^{\infty}\left[\frac{4}{n^3}+\left(\frac{1}{4}\right)^n+\frac{3}{n}\right]$

b) $\displaystyle\sum_{n=0}^{\infty}\frac{5n}{n^3+3}$

c) $\displaystyle\sum\frac{5^n}{(2n)!}$

d) $\displaystyle\sum\frac{(-1)^{n+1}}{4n}$

12. Determine the interval of convergence of each power series:

a) $\displaystyle f(x)=\sum_{n=0}^{\infty}\frac{1}{4^n}(x-1)^n$

b) $\displaystyle g(x)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4 n!}(x+3)^n$

c) $\displaystyle h(x)=\sum_{n=0}^{\infty}n^n x^n$

13. Write the Taylor series of each function:

a) $\displaystyle f(x)=\frac{4}{2-3x}$

b) $g(x)=x^2e^{-x}$

c) $h(x)=\arctan(2x^3)$

d) $\displaystyle k(x)=\frac{2}{(1-x)^3}$

14. Estimate $\arctan\frac{1}{3}$ by using the fifth Taylor polynomial for an appropriate function.

15. Estimate $\displaystyle\int_0^1\cos x^3\,dx$ by replacing the integrand with its seventh Taylor polynomial.

16. Evaluate $\displaystyle\lim_{x\to 0}\frac{\ln(x^2+1)-\sin x^2}{e^{x^4}-1}$ without using L'Hôpital's Rule.

17. Let f be an unknown function with $f(0)=2$, $f'(0)=\frac{1}{2}$, $f''(0)=-3$ and $f'''(0)=6$. Estimate $f(-1)$ using the third Taylor polynomial of f.

Answers

1. a) Use parts with $r = x$, $ds = \sin \pi x$ to get $\dfrac{-1}{\pi} \cos \pi x + \dfrac{1}{\pi^2} \sin \pi x + C$.

   b) Use u-sub $u = \sin x$ to get $\dfrac{\sqrt{3}}{8}$.

   c) Rewrite as $\frac{\cos 4x}{\sin 4x}$; then the u-sub $u = \sin 4x$ to get $\frac{1}{4} \ln |\sin 4x| + C$.

   d) Use partial fractions to get $2 \ln |x - 2| + 3 \ln |x| - \ln |x + 2| + C$.

   e) Just write the answer to get $3 \arctan x + C$.

   f) Use integration by parts to get 1.

   g) Use the u-sub $u = x - 3$ to get

   $$\frac{-12}{5}\sqrt{x-3} - \frac{2}{5}x\sqrt{x-3} + \frac{2}{5}x^2\sqrt{x-3} + C.$$

   h) Use partial fractions to get $3 \ln |x + 1| - 2 \ln |x + 2| + C$.

2. they all converge (use the Comparison Test on (c))

3. $\dfrac{5}{6}$

4. a) $\displaystyle\int_0^1 \pi(x^2)^2\,dx + \int_1^2 \pi(2-x)^2\,dx$

   or $\displaystyle\int_0^1 2\pi y(2 - y - \sqrt{y})\,dy$

   b) $\displaystyle\int_0^1 2\pi(x+2)[x^2]\,dx + \int_1^2 2\pi(x+2)(2-x)\,dx$

   or $\displaystyle\int_0^1 \pi[(2-y+2)^2 - (\sqrt{y}+2)^2]\,dy$

   c) $\displaystyle\int_0^1 2\pi(5-x)[x^2]\,dx + \int_1^2 2\pi(5-x)(2-x)\,dx$

   or $\displaystyle\int_0^1 \pi[(5-\sqrt{y})^2 - (5-(2-y))^2]\,dy$

   d) $\displaystyle\int_0^1 \pi[2^2 - (2-x^2)^2]\,dx + \int_1^2 \pi[2^2 - (2-(2-x))^2]\,dx$

   or $\displaystyle\int_0^1 2\pi(2-y)(2-y-\sqrt{y})\,dy$

5. $\dfrac{1}{54}\left(37^{3/2} - 1\right)$

6. $\overline{x} = \dfrac{13}{6}$ ft from the left end of the rod.

7. a) $(\overline{x}, \overline{y}) = \left(\frac{91}{27}, \frac{5}{9}\right)$ b) $I_y = \frac{2671}{780}$

8. a) $C = \frac{1}{9}$ b) $e^{-4/9}$ c) $EX = 9$ months.

9. $\sum a_n$ converges to L if $\lim\limits_{N\to\infty} S_N = L$, where S_N is the N^{th} partial sum of the series.

10. a) $\frac{4}{3}$ b) $\frac{512}{5}$ c) -2 d) 1

11. a) diverges
    b) converges absolutely
    c) converges absolutely
    d) converges conditionally

12. a) $(-3, 5)$ b) $(-\infty, \infty)$ c) $\{0\}$

13. a) $\sum\limits_{n=0}^{\infty} 2\left(\frac{3}{2}\right)^n x^n$ b) $\sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{n!}$ c) $\sum\limits_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{2n+1} x^{6n+3}$

14. $\frac{1}{3} - \frac{(1/3)^3}{3} = \frac{26}{81}$

15. $\frac{13}{14}$

16. $\frac{-1}{2}$

17. 2

Appendix B

Mathematica reference

B.1 What is *Mathematica*?

Mathematica is an extremely useful and powerful software package / programming language invented by a mathematician named Stephen Wolfram. Early versions of *Mathematica* came out in the late 1980s and early 1990s; as of 2023, the most recent version available to you is *Mathematica* 13.

Mathematica does symbolic manipulation of mathematical expressions; it solves all kinds of equations; it has a library of important functions from mathematics which it recognizes while doing computations; it does $2-$ and $3-$dimensional graphics; it has a built-in word processor tool; it works well with Java and C++; etc. One thing it doesn't do is prove theorems, so it is less useful for a theoretical mathematician than it is for an engineer or college student.

A bit about how *Mathematica* works: When you use the *Mathematica* program, you are actually running two programs. The "front end" of *Mathematica* is the part that you type on and the part you see. The "kernel" is the part of *Mathematica* that actually does the calculations. If you type in 2 + 2 and hit [ENTER] (actually [SHIFT]+[ENTER]; see below), the front end "sends" that information to the kernel which actually does the computation. The kernel then "sends" the result back to the front end, which displays 4 on the screen.

About *Mathematica* notebooks and cells: The actual files that *Mathematica* produces that you can edit and save are called *notebooks* and carry the file designation `*.nb`; they take up little space and can easily be saved to Google docs or on a flash drive, or emailed to yourself if you want them somewhere you can retrieve them.

Suggestion: when saving any file, include the date in the file name (so it is easier to remember which file you are supposed to be open).

A *Mathematica* notebook is broken into *cells*. A cell can contain text, input, or output. A cell is indicated by a dark blue, right bracket (a "]") on the right-hand side of the notebook. To select a cell, click that bracket. This highlights the "]" in blue. Once selected, you can cut/copy/paste/delete cells as you would highlighted blocks of text in a Word document.

To change the formatting of a cell, select the cell, then click "Format / Style" and select the style you want. You may want to play around with this to see what the various styles look like. There are three particularly important styles:

- **input:** this is the default style for new cells you type
- **output:** this is the default style for cells the kernel produces from your commands
- **text:** changing a cell to text style allows you to make comments in between the calculations

To execute an input cell, put the cursor anywhere in the cell and hit [SHIFT]+[ENTER] (or the [ENTER] on the numeric keypad at the far-right edge of the keyboard). The [ENTER] next to the apostrophe key (a.k.a. [RETURN]) gives you only a carriage return.

B.2 Important general concepts re: *Mathematica* syntax

Executing mathematical commands: To execute an input cell, put the cursor anywhere in the cell and hit [SHIFT]+[ENTER] (or the [ENTER] on the numeric keypad at the far-right edge of the keyboard). The [ENTER] next to the apostrophe key (a.k.a. [RETURN]) gives you only a carriage return.

Multiplication: use a star or a space: 2 * 3 or 2 3 will multiply numbers; a x means a times x; ax means the variable ax (in *Mathematica*, variables do not have to be named after one letter; they can be named by words or other strings of characters as well).

Parentheses: used for grouping only. Parentheses mean "times" in *Mathematica*.

Brackets: used to enclose all functions and *Mathematica* commands. For example, to evaluate a function $f(x)$, you would type f[x]; for $\sin x$ you type Sin[x]; etc.. Brackets mean "of" in *Mathematica* and cannot be used for multiplication.

Capitalization: All *Mathematica* commands and built-in functions begin with capital letters. For example, to find the sine of π, typing sin(pi) does you no good (this would be the variable "sin" times the variable "pi"). The correct syntax is Sin[Pi].

Spaces: *Mathematica* commands do not have spaces in them; for example, the inverse function of sine is ArcSin, not Arc Sin or Arcsin.

Pallettes: Lots of useful commands are available on the Basic Math Assistant Pallette, which can be brought up by clicking "Pallettes / Basic Math Assistant" on the toolbar. If you click on a button in the pallette, what you see appears in the cell.

Commands *Mathematica* knows: Sqrt, Sin, Cos, Tan, Csc, Cot, Sec, ArcSin, ArcCos, ArcTan, ArcCsc, ArcSec, ArcCot, ! (for factorial). It knows what Pi and E are (but not pi or e).

Logarithms: Log[ ] means natural logarithm (base e); Log10[ ] means common logarithm (base 10).

% refers to the last output (like ANS on a TI-calculator).

Exact answers versus decimal approximations: *Mathematica* gives exact answers for everything if possible. If you need a decimal approximation, click "numerical value" or use the command N[ ]. For example, N[Pi] spits out 3.14159...

To solve an equation: make sure there are two equals signs ("==") in your equation.

Getting help from the program: To get help on a command, type ? followed by the command you don't understand (with no space between the ? and the command).

To export graphics: Once *Mathematica* produces a graphic, you can right-click the graphic, and select "Copy Graphic". Then you can go in a Word document or a PowerPoint, and paste the graphic. You can subsequently resize it and/or move it around as you see fit.

Troubleshooting: For a command to run correctly, you usually want everything in your command to be black. If anything is purple or red, that suggests where the problem is. Variables that don't have values should be blue. Next, check that everything is capitalized appropriately. Next, check that you aren't missing a space if you are trying to multiply two variables. Next, if you are using variables in your code, try clearing the variables by executing something like Clear[x] (if your variable is x). Then re-run the command that is giving you trouble.

If *Mathematica* freezes up in the middle of a calculation and you see "Running..." at the top of your screen, click "Evaluation / Abort Evaluation" on the toolbar. If this doesn't help, kill the program and restart it.

To get help: Email me, and attach your *Mathematica* file to your email. I can troubleshoot things pretty quickly if the file is attached. If the file isn't attached, it is hard for me to figure out what you are doing wrong. Alternatively, seek assistance from another math major who has experience with *Mathematica*.

B.3 *Mathematica* quick reference guides

General tasks

TASK	MATHEMATICA SYNTAX
To call the preceding output	%
To get a decimal approximation to the preceding output	N[%] (or click numerical value)

Algebraic manipulations

TASK	MATHEMATICA SYNTAX
To factor an expression	Factor[ ]
To multiply out an expression (i.e. FOIL an expression)	Expand[ ]
Partial fraction decomposition	Apart[ ]
To combine rational terms (i.e. "undo" a partial fraction decomp)	Together[ ]
To simplify an answer	Simplify[ ] (or FullSimplify[ ])

Solving equations

GOAL	MATHEMATICA SYNTAX
Find exact solution(s) to equation of form $lhs = rhs$ (assuming the variable is x)	Solve[lhs == rhs, x] (two equals signs) (works only with polynomials or other relatively "easy" equations)
Find decimal approx. to solutions of equation $lhs = rhs$	NSolve[lhs == rhs, x] (two equals signs) (works only with "easy" equations)
Find decimal approx. to solutions of equation $lhs = rhs$	FindRoot[lhs == rhs, {x, $guess$}] (two equals signs)
Solve two (or more) equations together, like $\begin{cases} lhs_1 = rhs_1 \\ lhs_2 = rhs_2 \end{cases}$ (assuming variables are x and y)	Solve[{lhs_1==rhs_1, lhs_2==rhs_2}, {x,y}]

Precalculus operations

	Expression	Mathematica syntax
Special Symbols	e	E (not e) (or use Basic Math Assistant pallette)
	π	Pi (or use Basic Math Assistant)
	∞	Infinity (or use Basic Math Assistant) (or type [Esc] inf [Esc])
	$i = \sqrt{-1}$	I (not i) (or use Basic Math Assistant)
Arithmetic	$3 + 4x$	3 + 4x
	$5 - 27$	5 - 27
	$12x$	12x or 12 x or 12 * x
	xy	x y (don't forget the space)
	$\frac{x}{y}$	x/y (or use Basic Math Assistant pallette) (or type [CTRL]+/ to get $\frac{\square}{\square}$)
	$\sqrt{32}$	Sqrt[32] (or use Basic Math Assistant) (or type [CTRL]+2 for the $\sqrt{\ }$ sign)
	$\sqrt[4]{40}$	40^(1/4) (or use Basic Math Assistant)
	$\lvert x-3\rvert$	Abs[x-3]
	30! (factorial)	30!
Trig	$\sin\pi$	Sin[Pi]
	$\cos(x(y+1))$	Cos[x(y+1)]
	$\cos 60°$	Cos[60 Degree] (or use Basic Math Assistant)
	$\cot\left(\frac{2\pi}{3}+\frac{3\pi}{4}\right)$	Cot[2 Pi/3 + 3 Pi/4]
	$\sin^2 x$	Sin[x]^2 (not Sin^2[x])
	$\arctan 1$	ArcTan[1]
Exps / Logs	$\ln 3$	Log[3]
	$\log_6 63$	Log[6,63]
	$\log 18$	Log10[18] or Log[10, 18]
	2^{7y}	2^(7y) (or use Basic Math Assistant) (or type [CTRL]+6 to get $\square^{\square}$)
	e^{x-5+x^2}	E^(x-5+x^2) or Exp[x-5+x^2] (or use Basic Math Assistant)

Defining functions

Class of function	Syntax to define function
Calculus 1 function $f : \mathbb{R} \to \mathbb{R}$ $x \xmapsto{f} y$	f[x_] = *formula* (one equals sign, underscore after the x)
Ex: $f(x) = 3\cos(x^{2-x})$	f[x_] = 3 Cos[x^(2-x)]

Algebraic operations on functions

All these commands assume you have previously defined the function(s) as outlined above.

Expression	Mathematica syntax
Generate table of values for f	Table[{x, f[x]}, {x, *xmin*, *xmax*, *step*}] (put //TableForm after this command to format the output in a table)
$f(x+3)$	f[x+3]
$xf(2x) - x^2 f(x)$	x f[2x] - x^2 f[x] (spaces important)
Composition $(f \circ g)(x)$	f[g[x]]
Addition $(f+g)(x)$	f[x] + g[x]
Multiplication $(fg)(x)$	f[x] g[x]
Powers $f^n(x)$ **Ex:** $\sin^2 x$	(f[x])^n (or just f[x]^n) Sin[x]^2

Graphs

The basic command to graph a function is Plot[$f(x)$, {x, $xmin$, $xmax$}]; the examples below describe how to adapt the Plot[ ] command:

GOAL	HOW TO ADAPT THE Plot[ ] COMMAND
Plot multiple graphs at once	Plot[{$formula$, $formula$, ..., $formula$}, {x, $xmin$, $xmax$}]
Plot the graph of $f(x) = formula$ with range of $y-$values specified	Plot[$formula$, {x, $xmin$, $xmax$}, PlotRange -> {$ymin$, $ymax$}]
Plot the graph of $f(x) = formula$ with x- and y-axes on same scale	Plot[$formula$, {x,$xmin$,$xmax$}, PlotRange -> $ymin$,$ymax$, AspectRatio -> Automatic]
Plot the graph of $f(x) = formula$ with a red, dashed curve	Plot[$formula$, {x,$xmin$,$xmax$}, PlotStyle -> {Red, Dashed}]

Single-variable calculus

EXPRESSION	MATHEMATICA SYNTAX
$\lim_{x\to 4} f(x)$	Limit[f[x], x -> 4]
$f'(3)$	f'[3]
$h'(x)$	D[h[x], x]
$\frac{d}{dx}(\cos x)$	D[Cos[x], x]
$g'''(x)$	g' ' '[x] or D[g[x], {x,3}]
$\int x^2\,dx$	Integrate[x^2, x] (or use Basic Math Assistant pallette) **Note:** answer will be missing the "$+C$"
$\int_2^5 \cos x\,dx$	For an exact answer: Integrate[Cos[x], {x, 2, 5}] (or use Basic Math Assistant) For a decimal approximation: NIntegrate[Cos[x], {x, 2, 5}]
$\sum_{k=1}^{12} f(k)$	Sum[f[k], {k, 1, 12}] (or use Basic Math Assistant)
$\sum_{n=3}^{\infty} blah$	Sum[$blah$, {n, 3, Infinity}] (or use Basic Math Assistant)

B.4 More on solving equations with *Mathematica*

There are three methods to solve an equation using *Mathematica*. They have something in common: to solve an equation, the equation **must be typed with two equals signs** where the $=$ is. (A single equal sign is used in *Mathematica* to assign values to variables, which doesn't apply in the context of solving equations.)

The Solve command

To solve an equation of the form $lhs = rhs$, execute

$$\text{Solve}[lhs\ == \ rhs,\ variable]$$

where $variable$ is the name of the variable you want to solve for. For example, to solve $x^2 - 2x - 7 = 0$ for x, execute `Solve[x^2 - 2x - 7 == 0, x]`.

You can solve an equation for one variable in terms of others: for example, `Solve[a x + b == c, x]` solves for x in terms of a, b and c.

WARNING: The advantage of the Solve command is that it gives exact answers (no decimals); this can be a pro or con (as sometimes the exact answers are horrible to write down). The disadvantage is that it only works on polynomial, rational and other "easy" equations. It won't work on equations that mix-and-match trigonometry and powers of x like $x^2 = \cos x$.

The NSolve command

NSolve works exactly like Solve, except that it gives decimal approximations to the solutions. It has the same drawback as Solve in that it only works on reasonably "easy" equations. The syntax is

$$\text{NSolve}[lhs\ == \ rhs,\ variable]$$

The FindRoot command

To find decimal approximations to equations that are too hard for the Solve and NSolve commands, use FindRoot. This executes a numerical algorithm to estimate a solution to an equation. The good news is that this command always works; the bad news is that it requires an initial "guess" as to what the solution is (usually you determine the initial guess by graphing both sides of the equation and seeing roughly where the graphs cross). For example, to find a solution to $x^2 = \cos x$ near $x = 1$, execute

```
FindRoot[x^2 == Cos[x], {x, 1}]
```

and to find a solution to the same equation near $x = -1$, execute

```
FindRoot[x^2 == Cos[x], {x, -1}]
```

(these probably won't give the same solution). The general syntax for this command is

FindRoot[lhs == rhs,{$variable$, $guess$}]

Index