Mathematical prerequisites for Calculus III

So you want to take (and succeed in) multivariable calculus? Here is an outline of some of the elementary mathematics that comes before Calculus III. The less of it you know, the more difficult you will find multivariable calculus.

A. TRIGONOMETRIC IDENTITIES

Basic Identities: $\csc x = \frac{1}{\sin x}$; $\sec x = \frac{1}{\cos x}$; $\cot x = \frac{1}{\tan x}$; $\tan x = \frac{\sin x}{\cos x}$; $\cot x = \frac{\cos x}{\sin x}$

Odd-Even Identities: $\sin(-x) = -\sin x$; $\cos(-x) = \cos x$; $\tan(-x) = -\tan x$

Pythagorean Identities: $\sin^2 x + \cos^2 x = 1$; $\sec^2 x = \tan^2 x + 1$; $\csc^2 x = \cot^2 x + 1$

Double-Angle Identities: $\sin 2x = 2\sin x \cos x$; $\cos 2x = \cos^2 x - \sin^2 x$

Half-Angle Identities: $\sin^2 x = \frac{1-\cos 2x}{2}$; $\cos^2 x = \frac{1+\cos 2x}{2}$

You should also know how to evaluate all the trig functions at values like $\frac{n\pi}{4}$ or $\frac{n\pi}{6}$ where n is any integer.

B. HYPERBOLIC TRIG FUNCTIONS

Definitions: $\cosh x = \frac{e^x+e^{-x}}{2}$; $\sinh x = \frac{e^x-e^{-x}}{2}$

Hyperbolic Pythagorean Identity: $\cosh^2 x - \sinh^2 x = 1$

C. CONIC SECTIONS

Ellipses: The equation of an ellipse in standard form is given by

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

The graph of such an ellipse is centered at the point (h, k) and passes through the points $(h, k+b)$, $(h, k-b)$, $(h-a, k)$ and $(h+a, k)$. Note that these points are above, below, to the left of, and to the right of the center respectively. The ellipse looks like an "egg" passing through these 4 points.

Circles: If $a = b$ in the ellipse equation given above, then the equation represents a circle. It can be rewritten as

$$(x-h)^2 + (y-k)^2 = r^2;$$

this circle is centered at the point (h, k) and has radius r.

Parabolas: Every parabola can be written as one of the following two equations:

$$y - k = a(x-h)^2 \text{ or } x - h = a(y-k)^2.$$

In the first case the parabola opens up if $a > 0$ and opens down if $a < 0$, and for the second type of equation the parabola opens to the right if $a > 0$ and to the left if $a < 0$. In either case, the vertex (turning point) of the parabola is at the point (h, k).

Hyperbolas: The graph of a hyperbola looks like the graph of two parabolas placed back to back. The equation of a hyperbola in standard form is given by

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \text{ or } \frac{(y-k)^2}{b^2} - \frac{(x-k)^2}{a^2} = 1.$$

In the first case, the pieces of the hyperbola open to the left and right and their turning points are at $(h-a, k)$ and $(h+a, k)$. For the second type of equation, the pieces of the hyperbola open up and down and their turning points are at $(h, k+b)$ and $(h, k-b)$.

D. FUNCTIONS AND LIMITS

1. What is the domain of a function? How do you find the domain of a function?
2. When do limits exist? When do they not exist?
3. What does it mean for a function to be continuous?
4. What are some examples of functions which are not continuous at some point $x = c$?
5. Evaluating limits by plugging in
6. Evaluating limits using an algebra trick (factoring or conjugating)
7. Evaluating limits using L'Hopital's Rule
8. Evaluating limits using the Squeeze Theorem

E. DIFFERENTIATION RULES

Constant Functions: $\frac{d}{dx}c = 0$

Power Rule: $\frac{d}{dx}x^n = nx^{n-1}$ for any rational number $n \neq 0$.

Trig Functions: $\frac{d}{dx}\sin x = \cos x$; $\frac{d}{dx}\cos x = -\sin x$; $\frac{d}{dx}\tan x = \sec^2 x$; $\frac{d}{dx}\cot x = -\csc^2 x$; $\frac{d}{dx}\sec x = \sec x \tan x$; $\frac{d}{dx}\csc x = -\csc x \cot x$

Logarithmic Functions: $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Exponential Functions: $\frac{d}{dx}(e^x) = e^x$; more generally $\frac{d}{dx}(a^x) = (\ln a)(a^x)$ for $a > 0$

Inverse Trig Functions: $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$; $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{x^2+1}$

Hyperbolic Functions: $\frac{d}{dx}(\sinh x) = \cosh x$; $\frac{d}{dx}(\cosh x) = \sinh x$

Sum Rule: $\frac{d}{dx}(f+g) = \frac{d}{dx}f + \frac{d}{dx}g$

Difference Rule: $\frac{d}{dx}(f-g) = \frac{d}{dx}f - \frac{d}{dx}g$

Constant Multiple Rule: $\frac{d}{dx}(cf) = c \cdot \frac{d}{dx}f$

Product Rule: $\frac{d}{dx}(f \cdot g) = g \cdot (\frac{d}{dx}f) + f \cdot (\frac{d}{dx}g)$

Quotient Rule: $\frac{d}{dx}(\frac{f}{g}) = \frac{(\frac{d}{dx}f)\cdot g - (\frac{d}{dx}g)\cdot f}{g^2}$

Chain Rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ i.e. $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

F. DIFFERENTIATION THEORY

Given a function f, the *derivative* of f, denoted f', is the function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

wherever the above limit exists (there are other equivalent definitions of the derivative and there are other notations for the derivative). We call a function *differentiable* if the above limit exists. The derivative gives the slope of the line tangent to the graph of f at the point $(x, f(x))$. Consequently the tangent line to f at the point where $(x_0, f(x_0))$ has equation

$$L(x) = f(x_0) + f'(x_0)(x - x_0).$$

This tangent line is called the *linearization* of f at x_0; it is the best linear approximation to f at that point (and is the first-order Taylor polynomial for f about x).

A function whose graph is "smooth" has a derivative where it is defined. Derivatives fail to exist at points where the function is discontinuous or at "cusps" or "sharp points" in the graph. A typical example is the graph of $f(x) = |x|$; this function is not differentiable at 0.

If an object is moving along a line and its position is given by some function f, then f' gives the velocity of that object and f'' gives the acceleration of that object. More generally, the derivative of a function $y = f(x)$ gives the rate of change of y with respect to x. If y is not a function of x, this rate of change $\frac{dy}{dx}$ can still be found via *implicit differentiation*.

The derivative allows one to approximate values to functions by use of *differentials*. In particular $f(x + dx) \approx f(x) + dy$ where $dy = f'(x)dx$.

G. INTEGRATION RULES

$\int K\,dx = Kx + C$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ for any real number $n \neq -1$

$\int \sin x\,dx = -\cos x + C$ (more generally, $\int \sin kx\,dx = -\frac{1}{k}\cos kx + C$)

$\int \cos x\,dx = \sin x + C$ (more generally, $\int \cos kx\,dx = \frac{1}{k}\sin kx + C$)

$\int \sec^2 x\,dx = \tan x + C$

$\int \csc^2 x\,dx = -\cot x + C$

$\int \sec x \tan x\,dx = \sec x + C$

$\int \csc x \cot x\,dx = -\csc x + C$

$\int e^x\,dx = e^x + C$ (more generally, $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$)

$\int \frac{1}{x}\,dx = \ln|x| + C$

$\int \ln x\,dx = x\ln x - x + C$

$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$

$\int \sinh x\,dx = \cosh x + C$

$\int \cosh x\,dx = \sinh x + C$

Integration by Substitution: $\int f(g(x))g'(x)\,dx = \int f(u)\,du$ by setting $u = g(x)$

Integration by Parts: $\int udv = uv - \int vdu$

Other Integration Techniques: these include partial fraction decomposition, inverse trigonometric substitutions, etc.

H. SOME CALCULUS THEOREMS

Max-Min Existence Theorem (Extreme Value Theorem): If a function f is cts on a closed interval $[a, b]$, then f has an absolute maximum and an absolute minimum on that interval.

First Derivative Theorem for Extrema: If a function f has a local extremum at an interior point c of its domain, then either $f'(c) = 0$ or $f'(c)$ DNE.

Critical Point Theorem: All local extrema of a function f must occur at endpoints of the domain of f and/or critical points of f.

Monotonicity Test (First Derivative Test for Monotone Functions): If f is cts on $[a, b]$ and differentiable on (a, b), then

- $f'(x) > 0 \Leftrightarrow f$ is increasing
- $f'(x) < 0 \Leftrightarrow f$ is decreasing

Second Derivative Test: Suppose f'' is cts on an open interval containing $x = c$ where c is a critical point of f. Then:

- $f''(c) < 0 \Leftrightarrow f$ has a local max at $x = c$
- $f''(c) > 0 \Leftrightarrow f$ has a local min at $x = c$
- $f''(c) = 0 \Leftrightarrow$ test is inconclusive

Fundamental Theorem of Calculus Part I: Let f be continuous on $[a, b]$. Then the function $F(x) = \int_a^x f(t)dt$ is continuous and differentiable on $[a, b]$ and $F'(x) = f(x)$.

Fundamental Theorem of Calculus Part II: Let f be continuous on $[a, b]$. Then $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f on $[a, b]$.