1 Setup

Suppose you have an equation (not necessarily a function) relating several variables. An example is

$$3x^2z - x^2y^2 + 2z^3 = -3yz + 5.$$

You want to know what the rate of change of z with respect to x or y is. In other words, what are $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$?

2 Background: implicit differentiation of equations of two variables

Example: Find $\frac{dy}{dx}$ if $x^2y^3 = y + 2$.

Solution: Assume that y is a function of x and take the derivative of both sides. By the Chain Rule, whenever you take a derivative of a "y", you get a $\frac{dy}{dx}$ that appears in the expression. Then solve for $\frac{dy}{dx}$. Here is the written out solution:

$$x^2y^3 = y + 2 \quad (1)$$

$$\frac{d}{dx}(x^2y^3) = \frac{d}{dx}(y + 2) \quad (2)$$

$$2xy^3 + x^2(3y^2)\frac{dy}{dx} = \frac{dy}{dx} \quad (3)$$

$$2xy^3 = (1 - 3x^2y^2)\frac{dy}{dx} \quad (4)$$

$$\frac{2xy^3}{1 - 3x^2y^2} = \frac{dy}{dx} \quad (5)$$

Notice that to get from line (2) to line (3) above, you have to use the product rule to differentiate the expression x^2y^3.

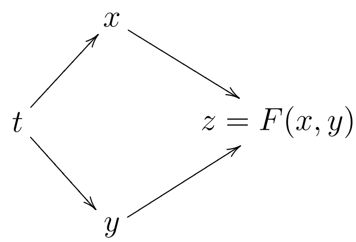
Now, I'm going to come back to this example in a little bit but I want to do a little theory first. Suppose you have an equation

$$F(x, y) = 0$$

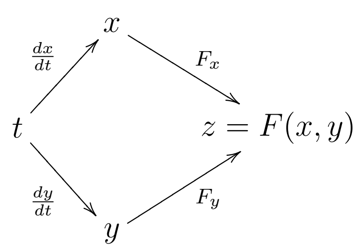
where $F(x, y)$ is some formula of x and y. In other words, I have some function $z = F(x, y)$ and I am looking at the level curve of that function where $z = 0$. Suppose I want to find $\frac{dy}{dx}$ on that level curve. Suppose x and y are functions of some variable t. Then the above equation becomes

$$F(x(t), y(t)) = 0.$$

In other words you have a picture that looks like this:



From what we talked about on Monday, you label the branches of this picture with the appropriate derivatives to get:



By the Chain Rule, we know

$$\frac{dF}{dt} = F_x \cdot \frac{dx}{dt} + F_y \cdot \frac{dy}{dt}.$$

Here is the trick: Let $t = x$. I can do this because t is a variable I made up, so I can let it be whatever I want. Now the previous equation can be written like this:

$$\frac{dF}{dx} = F_x \cdot \frac{dx}{dx} + F_y \cdot \frac{dy}{dx}.$$

What did this accomplish? First of all, remember that we were trying to solve for $\frac{dy}{dx}$; notice that this expression appears. Also, we have a derivative $\frac{dx}{dx}$ in our picture. This just the number 1 (the derivative of any variable with respect to itself is 1). Finally, So our equation is

$$\frac{dF}{dx} = F_x \cdot 1 + F_y \cdot \frac{dy}{dx}.$$

Last, remember that we assumed that we are on a level curve, i.e. $F(x, y) = 0$. This means that

$$\frac{dF}{dx} = \frac{d}{dx}(0) = 0$$

because z does not change if you stay on the level curve where $z = F(x, y) = 0$. This does not mean that the partial derivatives F_x and F_y are zero. The reason is that when you calculate partial derivatives, you can't stay on level curves. They are rates of change of z with respect to x and y irrespective to any level curve. So our equation becomes

$$0 = F_x \cdot 1 + F_y \cdot \frac{dy}{dx}.$$

Solve for $\frac{dy}{dx}$, you get

$$\frac{dy}{dx} = \frac{-F_x}{F_y}.$$

This proves the following theorem:

Theorem. *Suppose* $F(x, y) = 0$*. Then:*

$$\frac{dy}{dx} = \frac{-F_x}{F_y}.$$

Remember that at the beginning of this section we started with an example, namely to find $\frac{dy}{dx}$ if $x^2y^3 = y + 2$. We now do this example again using this theorem. First of all, move all terms to the left-hand side of the equation to get

$$x^2y^3 - y - 2 = 0$$

and let $F(x, y) = x^2y^3 - y - 2$. Now you can use the above theorem to get

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-2xy^3}{3x^2y^2 - 1}.$$

Notice this is the same as what we got before; in fact it is easier because you don't have to solve for $\frac{dy}{dx}$ after differentiation, and you don't have to use the product rule in any complicated fashion.

3 Analogues for multiple variables

The same reasoning as above gives you the following theorem for functions of several variables:

Theorem. *Suppose* $F(x, y, z) = 0$*. Then:*

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}.$$

The example we started with on page 1 was: given

$$3x^2z - x^2y^2 + 2z^3 = -3yz + 5,$$

find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. We use this theorem. First, set one side of this equation equal to zero, i.e.

$$3x^2z - x^2y^2 + 2z^3 + 3yz - 5 = 0$$

and let $F(x,y,z) = 3x^2z - x^2y^2 + 2z^3 + 3yz - 5$. Then:

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(6xz - 2xy^2)}{3x^2 + 6z^2 + 3y} \text{ and}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(2x^2y + 3z)}{3x^2 + 6z^2 + 3y}.$$

The punchline is that to implicitly differentiate an equation, you can use the one of the two theorems described in this handout; this is generally less symbol-crunching than the "old" method of implicit differentiation you already know.