12 PM Section:

1. Calculate the indefinite integral

$$\int \left\langle \frac{2}{e^t}, \sqrt{t+2}, \frac{2}{5}\sinh t \right\rangle dt.$$

Solution:

$$\begin{aligned} \int \left\langle \frac{2}{e^t}, \sqrt{t+2}, \frac{2}{5}\sinh t \right\rangle dt &= \int \left\langle 2e^{-t}, (t+2)^{1/2}, \frac{2}{5}\sinh t \right\rangle dt \\ &= \left\langle -2e^{-t}, \frac{2}{3}(t+2)^3 2, \frac{2}{5}\cosh t \right\rangle + \overrightarrow{c}. \end{aligned}$$

2. Sketch a rough graph of each of the following in 2-dimensional space:

   (a) $r = 2\sin\theta$
   Solution: This is a circle of radius 1 centered at $(0, 1)$:

   (b) The $xy-$trace of $x^2 - y^2 + z^2 = 1$
   Solution: To find the $xy-$trace, set $z = 0$ in the equation to obtain $x^2 - y^2 = 1$ which is a hyperbola centered at the origin opening in the $x-$direction:

   (c) $(x-2)^2 + 9y^2 = 9$
   Solution: Divide through the entire equation by 9 to obtain $\frac{(x-2)^2}{9} + \frac{y^2}{1} = 1$ which is an ellipse centered at $(2, 0)$:

3. Sketch a rough graph of each of the following in 3-dimensional space:

   (a) $y = x^2$
   Solution: This is a cylinder parallel to the $z-$axis:

(b) $y = x^2 + z^2$
Solution: This is an elliptic paraboloid opening toward the positive $y-$axis:

(c) $y^2 = x^2 - z^2$
Solution: This equation can be rewritten as $x^2 - y^2 - z^2 = 0$; the graph is a cone opening toward the positive and negative $x-$axis:

4. Suppose a particle's position at time t is given by the vector-valued function $\overrightarrow{r}(t) = \langle \sin t, 3\cos t, t^3 + 1\rangle$, where $-\infty < t < \infty$.

(a) Sketch the graph of $\overrightarrow{r}(t)$ as seen from the perspective of the $z-$axis.
Solution: To do this, consider only the $x-$ and $z-$ components of the function. They are: $\langle \sin t, 3\cos t\rangle$ which are the parametric equations of an ellipse centered at the origin:

(b) Sketch the graph of $\overrightarrow{r}(t)$ as seen from the usual perspective. You should give the coordinates of one point that the graph passes through.
Solution: At $t = 0$, the point passes through the point whose coordinates are given by $\overrightarrow{r}(0) = \langle 0, 3, 1\rangle$. As t increases so does z since z is a monotone increasing function of t; in the $x-$ and $y-$ directions we see rotation by part (a). So the graph is:

(c) Find the particle's velocity and acceleration when $t = 0$.
Solution: $\overrightarrow{v}(t) = \overrightarrow{r}'(t) = \langle \cos t, -3\sin t, 3t^2 \rangle$ so $\overrightarrow{v}(0) = \langle 1, 0, 0 \rangle$. As for the acceleration, $\overrightarrow{a}(t) = \overrightarrow{v}'(t) = \langle -\sin t, -3\cos t, 6t \rangle$ so $\overrightarrow{a}(0) = \langle 0, -3, 0 \rangle$.

(d) Sketch a picture which shows the local behavior of $\overrightarrow{r}(t)$ at the point where $t = 0$. Include and label the velocity, and acceleration vectors in the appropriate places on your graph.
Solution: At $t = 0$, the graph passes through the point $(0, 3, 1)$ from part (b) and the velocity and acceleration vectors are known from part (c). Draw these vectors and indicate that the graph $(0, 3, 1)$, is tangent to $\overrightarrow{v}(0) = \langle 1, 0, 0 \rangle$ and bends toward $\overrightarrow{a}(0) = \langle 0, -3, 0 \rangle$:

(e) Find the curvature of the graph of $\overrightarrow{r}(t)$ at the point $(0, 3, 1)$.
Solution: The point corresponds to when $t = 0$ so we have

$$\begin{aligned}
\kappa &= \frac{||\overrightarrow{v}(0) \times \overrightarrow{a}(0)||}{||\overrightarrow{v}(0)||^3} \\
&= \frac{||\langle 1, 0, 0 \rangle \times \langle 0, -3, 0 \rangle||}{||\langle 1, 0, 0 \rangle||^3} \\
&= \frac{||\langle 0, 0, -3 \rangle||}{1^3} \\
&= \frac{3}{1} = 3.
\end{aligned}$$

(f) Write a definite integral which gives the length of the portion of the graph of $\overrightarrow{r}(t)$ between the points $(0, 3, 1)$ and $(1, 0, \frac{\pi^3}{8} + 1)$.
Solution: At the first point, $t = 0$ and at the second point, since $z = \frac{\pi^3}{8} + 1$, we have $t = \pi/2$. So

$$\begin{aligned}
s = \int_0^{\pi/2} ||\overrightarrow{r}'(t)||\, dt &= \int_0^{\pi/2} ||\left\langle \cos t, -3\sin t, 3t^2 \right\rangle||\, dt \\
&= \int_0^{\pi/2} \sqrt{\cos^2 t + 9\sin^2 t + 9t^4}\, dt.
\end{aligned}$$

5. Consider the two lines:

$$\frac{x-2}{6} = \frac{y+1}{-4} = \frac{z}{2} \quad \text{and} \quad \frac{x+3}{-9} = \frac{y-2}{6} = \frac{z+1}{-3}.$$

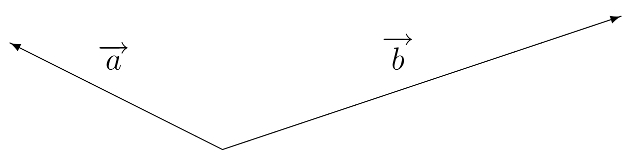
(a) Are the two lines parallel, perpendicular, or neither?
Solution: The direction vectors for the lines are $\langle 6, -4, 2\rangle$ and $\langle -9, 6, -3\rangle$. The first vector is equal to (-2/3) times the second vector so the vectors are multiples of one another. Hence the lines are parallel.

(b) Write the equation of the plane containing the two lines.
Solution: First, the vector $\overrightarrow{a} = \langle 6, -4, 2\rangle$ is in the plane since it is the direction vector for the first line. Second, the vector $\overrightarrow{b} = \langle -5, 3, -1\rangle$ is in the plane; this vector comes from subtracting the coordinates of $(2, -1, 0)$ from $(-3, 2, -1)$ (these are points on the respective lines). To find a normal vector to the plane, take a cross product:

$$\overrightarrow{n} = \overrightarrow{a} \times \overrightarrow{b} = \langle 2, 4, 2\rangle$$

So the equation of the plane, using the normal vector $\overrightarrow{n}$ and the point $(2, -1, 0)$, is $2(x - 2) + 4(y + 1) + 2z = 0$.

6. Consider the following two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$:



(a) Suppose $\overrightarrow{v} = \mathbf{proj}_{\overrightarrow{b}}\overrightarrow{a}$. On the picture above, draw and label the vector $\overrightarrow{v}$.
Solution: See above.

(b) Let $\overrightarrow{w} = \overrightarrow{a} - \overrightarrow{v}$. On the picture above, draw and label the vector $\overrightarrow{w}$.
Solution: See above.

(c) What appears to be the relationship between the two vectors $\overrightarrow{v}$ and $\overrightarrow{w}$?
Solution: It appears that they are orthogonal.

(d) (6 pts) Prove that the relationship you described in part (c) holds.
Solution: To show that two vectors are orthogonal, take their dot product

(it should come out to zero):

$$\begin{aligned}
\overrightarrow{v} \cdot \overrightarrow{w} &= \mathbf{proj}_{\overrightarrow{b}} \overrightarrow{a} \cdot (\overrightarrow{a} - \mathbf{proj}_{\overrightarrow{b}} \overrightarrow{a}) \\
&= \left(\mathbf{proj}_{\overrightarrow{b}} \overrightarrow{a}\right) \cdot \overrightarrow{a} - \left(\mathbf{proj}_{\overrightarrow{b}} \overrightarrow{a}\right) \cdot \left(\mathbf{proj}_{\overrightarrow{b}} \overrightarrow{a}\right) \\
&= \left(\frac{a \cdot b}{b \cdot b}\right) \overrightarrow{b} \cdot \overrightarrow{a} - \left(\frac{a \cdot b}{b \cdot b}\right) \overrightarrow{b} \cdot \left(\frac{a \cdot b}{b \cdot b}\right) \overrightarrow{b} \\
&= \left(\frac{a \cdot b}{b \cdot b}\right) (b \cdot a) - \left(\frac{a \cdot b}{b \cdot b}\right)^2 (b \cdot b) \\
&= \frac{(a \cdot b)^2}{b \cdot b} - \frac{(a \cdot b)^2}{b \cdot b} \\
&= 0.
\end{aligned}$$

so in fact the vectors are orthogonal (in the above calculation, all a and b should have arrows over them).

9 AM Section:

1. (9 pts) Calculate the indefinite integral

$$\int \left\langle 4e^t,\ \sqrt[3]{2-t},\ 2\cosh t \right\rangle \, dt.$$

Solution:

$$\begin{aligned}
\int \left\langle 4e^t,\ \sqrt[3]{2-t},\ 2\cosh t \right\rangle \, dt &= \int \left\langle 4e^t,\ (2-t)^{1/3},\ 2\cosh t \right\rangle \, dt \\
&= \left\langle 4e^t, -\frac{3}{4}(2-t)^{4/3}, 2\sinh t \right\rangle + \overrightarrow{c}
\end{aligned}$$

2. (9 pts) Sketch a rough graph of each of the following in 2-dimensional space:

   (a) $r = 3$
   Solution: This is a circle of radius 3 centered at the origin:

   (b) The $z = 4$ trace of $x^2 + y^2 - z^2 = 0$
   Solution: To find the $z = 4$ trace, set $z = 4$ in the above equation to get $x^2 + y^2 = 16$. This is a circle of radius 4 centered at the origin:

(c) $\begin{cases} x = 2\cosh t - 2 \\ y = \sinh t \end{cases}$
Solution: These are parametric equations for the right-half of a hyperbola centered at $(-2, 0)$ opening in the $x-$direction. The graph is:

3. Sketch a rough graph of each of the following in 3-dimensional space:

(a) $1 = x^2 + z^2$
Solution: This is a cylinder parallel to the $y-$axis:

(b) $y^2 = x^2 + z^2$
Solution: This can be written as $x^2 - y^2 + z^2 = 0$ which is a cone opening to the positive and negative $y-$axis:

(c) $y^2 = x^2 + z^2 + 1$
Solution: This can be rewritten as $y^2 - x^2 - z^2 = 1$ which is a hyperboloid of two sheets opening to the positive and negative $y-$axis:

4. Suppose a bug is flying around in three-dimensional space such that its velocity at any time t is given by $\overrightarrow{v}(t) = \langle 2\sqrt{2}, -2\sin 4t, 2\cos 4t\rangle$; suppose that at time 0 the bug is at the point $(0, \frac{1}{2}, 0)$.

(a) Where is the bug at time $3\pi/4$?
Solution: The bug's position $\overrightarrow{r}(t)$ is given by

$$\overrightarrow{r}(t) = \int \overrightarrow{v}(t)\,dt = \left\langle 2\sqrt{2}t + c_1, \frac{1}{2}\cos 4t + c_2, \frac{1}{2}\cos 4t + c_3 \right\rangle;$$

to find the constants use the initial condition $\overrightarrow{r}(0) = (0, \frac{1}{2}, 0)$ to obtain $c_1 = c_2 = c_3 = 0$. Then at time $3\pi/4$ the bug is at

$$\overrightarrow{r}(\frac{3\pi}{4}) = \left\langle 2\sqrt{2}(\frac{3\pi}{4}), \frac{1}{2}\cos(3\pi), \frac{1}{2}\sin(3\pi) \right\rangle = \left\langle \frac{3\sqrt{2}\pi}{2}, -\frac{1}{2}, 0 \right\rangle.$$

(b) Show that the bug flies with constant speed.
Solution: The bug's speed is

$$\begin{aligned} ||\overrightarrow{v}(t)|| &= ||\left\langle 2\sqrt{2}, -2\sin 4t, 2\cos 4t \right\rangle|| \\ &= \sqrt{(2\sqrt{2})^2 + (-2\sin 4t)^2 + (2\cos rt)^2} \\ &= \sqrt{8 + 4\sin^2 4t + 4\cos^2 4t} \\ &= \sqrt{8+4} = \sqrt{12} \end{aligned}$$

which is a constant not depending on t.

(c) How far does the bug travel between time 1 and time 9?
Solution: The length of the bug's path is

$$s = \int_1^9 ||\overrightarrow{v}(t)||\,dt = \int_1^9 \sqrt{12}\,dt = 8\sqrt{12}.$$

(We used the calculation from part (b) where we found the magnitude of the velocity.)

(d) Find the tangential component of the bug's acceleration when $t = 0$.
Solution:

$$a_T = \frac{d}{dt}(||\overrightarrow{r}'(t)||) = \frac{d}{dt}(\sqrt{12}) = 0.$$

(Again, use the calculation from part (b).)
Alternate solution: First find $\overrightarrow{a}(t) = \overrightarrow{v}'(t) = \langle 0, -8\cos 2t, -8\sin 2t\rangle$ and notice that $\overrightarrow{a}(0) = \langle 0, -8, 0\rangle$. Next,

$$\overrightarrow{T}(0) = \frac{\overrightarrow{v}(0)}{||\overrightarrow{v}(0)||} = \frac{\langle 2\sqrt{2}, 0, 2\rangle}{\sqrt{12}} = \left\langle \frac{2}{\sqrt{6}}, 0, \frac{2}{\sqrt{12}} \right\rangle.$$

Finally, $a_T = \overrightarrow{a}(0) \cdot \overrightarrow{T}(0) = \langle 0, -8, 0 \rangle \cdot \left\langle \frac{2}{\sqrt{6}}, 0, \frac{2}{\sqrt{12}} \right\rangle = 0.$

(e) Find the normal component of the bug's acceleration when $t = 0$.
Solution:

$$\begin{aligned} a_N = \frac{||\overrightarrow{v}(0) \times \overrightarrow{a}(0)||}{||\overrightarrow{v}(0)||} &= \frac{||\left\langle 0, -8, 0 \right\rangle \times \left\langle 2\sqrt{2}, 0, 2 \right\rangle||}{||\left\langle 2\sqrt{2}, 0, 2 \right\rangle||} \\ &= \frac{||\left\langle 16, 0, -16\sqrt{2} \right\rangle||}{\sqrt{12}} \\ &= \frac{16||\left\langle 1, 0, -\sqrt{2} \right\rangle||}{\sqrt{12}} \\ &= \frac{16\sqrt{3}}{\sqrt{12}} = \frac{16}{\sqrt{4}} = 8. \end{aligned}$$

Alternate solution:

$$\begin{aligned} a_N = ||\overrightarrow{a}(0) \times \overrightarrow{T}(0)|| &= ||\left\langle 0, -8, 0 \right\rangle \times \left\langle \frac{2}{\sqrt{6}}, 0, \frac{2}{\sqrt{12}} \right\rangle|| \\ &= ||\left\langle \frac{-16}{\sqrt{12}}, 0, \frac{16}{\sqrt{6}} \right\rangle|| \\ &= \sqrt{\frac{16^2}{12} + \frac{16^2}{6}} \\ &= 16\sqrt{\frac{1}{12} + \frac{1}{6}} \\ &= 16\sqrt{\frac{1}{4}} = 16\left(\frac{1}{2}\right) = 8. \end{aligned}$$

A third solution: Since $a_T = 0$, all of the acceleration lies in the direction of $\overrightarrow{N}$. Therefore $a_N = ||\overrightarrow{a}(0)|| = ||\left\langle 0, -8, 0 \right\rangle|| = 8$.

5. Suppose $\overrightarrow{a}$ and $\overrightarrow{b}$ are two vectors such that $\overrightarrow{a} \cdot \overrightarrow{a} = 3$, $\overrightarrow{a} \cdot \overrightarrow{b} = 6$, and $\overrightarrow{b} \cdot \overrightarrow{b} = 16$. Find the measure of the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$.
Solution: First recall that $\overrightarrow{v} \cdot \overrightarrow{v} = ||\overrightarrow{v}||^2$ for any vector $\overrightarrow{v}$ so we have $||\overrightarrow{a}|| = \sqrt{3}$ and $||\overrightarrow{b}|| = 4$. Now,

$$\begin{aligned} \overrightarrow{a} \cdot \overrightarrow{b} &= ||\overrightarrow{a}||\,||\overrightarrow{b}|| \cos\theta \\ 6 &= (\sqrt{3})\,(4)\,\cos\theta \\ \Rightarrow \cos\theta &= \frac{6}{4\sqrt{3}} = \frac{\sqrt{3}}{2}. \\ \Rightarrow \theta &= \frac{\pi}{6}. \end{aligned}$$

6. Find the curvature of the graph of $y = e^x$ at the point $(\ln 2, 2)$.
Solution: Place this graph in three-dimensions by setting $\overrightarrow{r}(t) = \langle t, e^t, 0\rangle$. Then $\overrightarrow{r}'(t) = \langle 1, e^t, 0\rangle$ and $\overrightarrow{r}''(t) = \langle 0, e^t, 0\rangle$. So

$$\kappa = \frac{||\overrightarrow{r}'(\ln 2) \times \overrightarrow{r}''(\ln 2)||}{||\overrightarrow{r}'(\ln 2)||^3} = \frac{\langle 1, 2, 0\rangle \times \langle 0, 2, 0\rangle}{||\langle 1, 2, 0\rangle||} = \frac{||\langle 0, 0, 2\rangle||}{(\sqrt{5})^3} = \frac{2}{5^{3/2}}.$$

7. Consider the two lines:

$$\begin{cases} x = t - 2 \\ y = 3 \\ z = 2t + 1 \end{cases} \qquad \text{and} \qquad \frac{x+1}{-2} = \frac{y-3}{6} = z - 3.$$

(a) Are the two lines parallel, perpendicular, or neither?
Solution: The direction vectors for the two lines are $\langle 1, 0, 2\rangle$ and $\langle -2, 6, 1\rangle$ respectively. Since $\langle 1, 0, 2\rangle \cdot \langle -2, 6, 1\rangle = 0$, the lines are perpendicular (we can assume they intersect since question (b) assumes they lie in the same plane).

(b) Write the equation of the plane containing the two lines.
Solution: The two direction vectors above lie in the plane; to find a normal vector to the plane take their cross product:

$$\overrightarrow{n} = \langle 1, 0, 2\rangle \times \langle -2, 6, 1\rangle = \langle -12, -5, 6\rangle$$

So using this normal vector and the point $(2, 3, -1)$ (which the first point passes through), we have the equation $-12(x-2) - 5(y-3) + 6(z+1) = 0$.