12 PM Section:

1. Find the rectangular coordinates of the point whose spherical coordinates are $(12, 2\pi/3, \pi/6)$.

2. Suppose that a particle is moving in three-dimensional space and that at time 0, you know the following information:

$$\begin{aligned} \overrightarrow{r}(0) &= \langle 2,0,0\rangle \\ \overrightarrow{v}(0) = \overrightarrow{r}'(0) &= \langle 1,-2,2\rangle \\ \overrightarrow{a}(0) = \overrightarrow{r}''(0) &= \langle 0,3,0\rangle \end{aligned}$$

   Find the tangential and normal components of the particle's acceleration at time 0.

3. Find the following limits or state that they do not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} \qquad \lim_{(x,y)\to(0,0)} \frac{2x\sqrt{y}}{y-x^2}$$

4. Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial x}{\partial y}$ if

$$\cos(yz-x) = x^2e^y.$$

5. Suppose $f_x(x,y) = 3x^2y^2+y-2$ and $f_y(x,y) = 2x^3y+x-y$. What is a possible rule for $f(x,y)$?

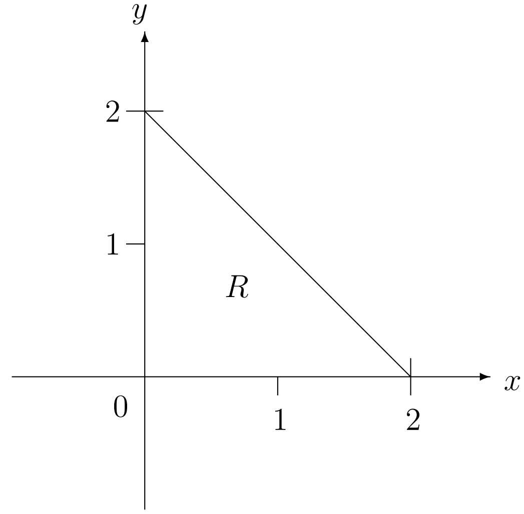
6. Find the directional derivative of the function $f(x,y) = x^2\sin(2y)$ at $(1, \frac{\pi}{2})$ in the direction of $\langle 3,-4\rangle$.

7. Find the equation of the tangent plane to the surface $y^4 - x^2 = 7z^4$ at the point $(3,2,-1)$.

8. Find all critical points for the function $f(x,y) = 12xy - x^3 - y^3$. Classify each critical point as a local maximum, local minimum, or saddle point.

9 AM Section:

1. Find the rectangular coordinates of the point whose spherical coordinates are $(8, 3\pi/4, \pi/3)$.
2. Given $f(x,y) = x^3y - \sin x \cos y$, find f_{xyx}.
3. Find the directional derivative of the function $f(x,y) = x^2 \sin(3y)$ at $(-1, \frac{\pi}{2})$ in the direction of $\langle -5, 12 \rangle$.
4. Find the following limits or state that they do not exist.
$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \qquad \lim_{(x,y)\to(0,0)} \frac{3x^2\sqrt{y}}{y+x^4}$$
5. Find the partial derivatives $\frac{\partial x}{\partial z}$ and $\frac{\partial y}{\partial x}$ if
$$\sin(xy+z) = z^2 e^x.$$
6. Find the linear approximation to the function $f(x,y) = x^2y^4$ at $(2,1)$. Use this linear approximation to estimate $(2.0001)^2(.9998)^4$.
7. Find all critical points for the function $f(x,y) = x^4 + y^4 - 4xy$. Classify each critical point as a local maximum, local minimum, or saddle point.
8. Find the maximum value of $f(x,y) = xy + 2x + y + 1$ on the closed triangular region R indicated below.



Solutions (12 PM Section):

1. We have

$$\begin{aligned} x &= \rho\sin\phi\cos\theta = 12\sin\frac{2\pi}{3}\cos\frac{\pi}{6} = 12\left(\frac{\sqrt{3}}{2}\right)\frac{\sqrt{3}}{2} = 9, \\ y &= \rho\sin\phi\sin\theta = 12\sin\frac{2\pi}{3}\sin\frac{\pi}{6} = 12\left(\frac{\sqrt{3}}{2}\right)\frac{1}{2} = 3\sqrt{3}, \\ z &= \rho\cos\phi = 12\cos\frac{2\pi}{3} = 12\left(\frac{-1}{2}\right) = -6. \end{aligned}$$

Therefore the rectangular coordinates are $(9, 3\sqrt{3}, -6)$.

2. First find the unit tangent vector $\overrightarrow{T}(0)$:

$$\overrightarrow{T}(0) = \frac{\overrightarrow{v}(0)}{||\overrightarrow{v}(0)||} = \frac{\langle 1,-2,2\rangle}{||\langle 1,-2,2\rangle||} = \frac{\langle 1,-2,2\rangle}{3} = \left\langle \frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right\rangle.$$

Now you can calculate the components of acceleration:

$$a_T = \overrightarrow{a}\cdot\overrightarrow{T} = \langle 0,3,0\rangle \cdot \left\langle \frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right\rangle = -2.$$

$$a_N = ||\overrightarrow{a}\times\overrightarrow{T}|| = ||\langle 0,3,0\rangle \times \left\langle \frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right\rangle|| = ||\langle -2,0,1\rangle|| = \sqrt{5}.$$

3. For the first limit, use polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = \lim_{r\to 0}\frac{r^4\cos^2\theta\sin^2\theta}{r^2} = \lim_{r\to 0} r^2\cos^2\theta\sin^2\theta = 0.$$

For the second limit, use paths. First along the $x-$axis, i.e. where $y = 0$, we have

$$\lim_{(x,0)\to(0,0)} \frac{2x\sqrt{0}}{0-x^2} = \lim_{x\to 0}\frac{0}{-x^2} = 0.$$

Next, along the path $y = x^2$, we have

$$\lim_{(x,x^2)\to(0,0)} \frac{2x\sqrt{x^2}}{x^2-x^2} = \lim_{x\to 0}\frac{2x^2}{0} = \infty \neq 0.$$

Since the limits along the paths do not match, we see that the original limit does not exist.

4. First subtract x^2e^y from both sides to obtain the equation $\cos(yz-x)-x^2e^y = 0$; let $F(x,y,z) = \cos(yz-x) - x^2e^y$. Then:

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-[-\sin(yz-x)(-1) - 2xe^y]}{-y\sin(yz-x)} = \frac{\sin(yz-x) - 2xe^y}{y\sin(yz-x)}.$$

$$\frac{\partial x}{\partial y} = \frac{-F_y}{F_x} = \frac{-[-z\sin(yz-x) - x^2e^y]}{-\sin(yz-x)(-1) - 2xe^y} = \frac{z\sin(yz-x) + x^2e^y}{\sin(yz-x) - 2xe^y}.$$

5. Find a possible $f(x,y)$ by integration:

$$f(x,y) = \int f_x(x,y)dx = \int (3x^2y^2 + y - 2)\,dx = x^3y^2 + xy - 2x + C(y)$$

$$f(x,y) = \int f_y(x,y)dy = \int (2x^3y + x - y)\,dx = x^3y^2 + xy - \frac{y^2}{2} + C(x)$$

To make the answers match, let $C(y) = -\frac{y^2}{2}$ and let $C(x) = -2x$. Then we get

$$f(x,y) = x^3y^2 + xy - \frac{y^2}{2} - 2x.$$

6. First find a unit vector in the direction of $\langle 3,-4\rangle$:

$$\overrightarrow{u} = \frac{\langle 3,-4\rangle}{||\langle 3,-4\rangle||} = \frac{\langle 3,-4\rangle}{5} = \left\langle \frac{3}{5}, \frac{-4}{5}\right\rangle.$$

Now find the gradient:

$$\nabla f(1,\frac{\pi}{2}) = \left\langle 2x\sin(2y), 2x^2\cos(2y)\right\rangle|_{(1,\frac{\pi}{2})} = \langle 2(1)0, 2(1)(-1)\rangle = \langle 0,-2\rangle.$$

Finally,

$$D_{\overrightarrow{u}}f = \nabla f \cdot \overrightarrow{u} = \langle 0,-2\rangle \cdot \left\langle \frac{3}{5}, \frac{-4}{5}\right\rangle = \frac{8}{5}.$$

7. First, let $w = F(x,y,z) = y^4 - x^2 - 7z^4$ and notice that the given surface is a level surface for $F(x,y,z)$ when $w = 0$. So the gradient of F is orthogonal to this level surface, it is a normal vector to the tangent plane of the surface. So find the gradient:

$$\nabla F(3,2,-1) = \left\langle -2x, 4y^3, -28z^3\right\rangle|_{(3,2,-1)} = \langle -6, 32, 28\rangle.$$

The equation of the tangent plane is then

$$-6(x-3) + 32(y-2) + 28(z+1) = 0.$$

8. First, find the gradient:

$$\nabla f(x,y) = \left\langle 12y - 3x^2, 12x - 3y^2 \right\rangle$$

and notice that this gradient always exists. So the critical points are found by setting $\nabla f = \overrightarrow{0}$:

$$\nabla f = \overrightarrow{0} \Rightarrow \begin{cases} 12y - 3x^2 = 0 \\ 12x - 3y^2 = 0 \end{cases} \Rightarrow \begin{cases} 12y = 3x^2 \\ 12x = 3y^2 \end{cases} \Rightarrow \begin{cases} y = \frac{1}{4}x^2 \\ x = \frac{1}{4}y^2 \end{cases}.$$

Substituting the expression for y from the first equation into the second equation, we get

$$x = \frac{1}{4}\left(\frac{1}{4}x^2\right)^2 = \frac{1}{64}x^4;$$

subtract x from both sides to get

$$\begin{array}{rcl} \frac{1}{64}x^4 - x & = & 0 \\ x^4 - 64x & = & 0 \\ x\left(x^3 - 64\right) & = & 0 \\ \Rightarrow x = 0 & \text{or} & x^3 = 64 \\ \Rightarrow x = 0 & \text{or} & x = 4. \end{array}$$

Finding the $y-$coordinate for these points we see that $y = 0$ when $x = 0$ and $y = 4$ when $x = 4$. So the critical points are $(0,0)$ and $(4,4)$.

To classify the critical points use the discriminant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -6x & 12 \\ 12 & -6y \end{vmatrix} = 36xy - 144.$$

At the critical point $(0,0)$, $D = -144 < 0$ so $(0,0)$ is a saddle point.

At the critical point $(4,4)$, $D = 36(16) - 144 > 0$ and $f_{xx}(4,4) = -24 < 0$ so $(4,4)$ is a local maximum.

Solutions (9 AM Section):

1. We have

$$\begin{aligned}
x &= \rho \sin\phi \cos\theta = 8 \sin\frac{3\pi}{4} \cos\frac{\pi}{3} = 8\left(\frac{\sqrt{2}}{2}\right)\frac{1}{2} = 2\sqrt{2}, \\
y &= \rho \sin\phi \sin\theta = 8 \sin\frac{3\pi}{4} \sin\frac{\pi}{3} = 8\left(\frac{\sqrt{2}}{2}\right)\frac{\sqrt{3}}{2} = 2\sqrt{6}, \\
z &= \rho \cos\phi = 8 \cos\frac{3\pi}{4} = 8\left(\frac{-\sqrt{2}}{2}\right) = -4\sqrt{2}.
\end{aligned}$$

Therefore the rectangular coordinates are $(4\sqrt{2}, 4\sqrt{6}, -4\sqrt{2})$.

2.

$$\begin{aligned}
f_{xyx} = (x^3y - \sin x \cos y)_{xyx} &= (3x^2y - \cos x \cos y)_{yx} \\
&= (3x^2 + \cos x \sin y)_x \\
&= 6x - \sin x \sin y.
\end{aligned}$$

3. First find the gradient:

$$\nabla f(-1, \frac{\pi}{2}) = \left\langle 2x \sin 3y, 3x^2 \cos 3y \right\rangle |_{(-1,\frac{\pi}{2})} = \langle 2(-1)(-1), 3(1)(0) \rangle = \langle 2, 0 \rangle .$$

Next find $\overrightarrow{u}$, a unit vector in the direction of $\langle -5, 12 \rangle$:

$$\overrightarrow{u} = \frac{\langle -5, 12 \rangle}{|| \langle -5, 12 \rangle ||} = \frac{\langle -5, 12 \rangle}{13} = \left\langle \frac{-5}{13}, \frac{12}{13} \right\rangle .$$

Finally,

$$D_{\overrightarrow{u}} f = \nabla f \cdot \overrightarrow{u} = \langle 2, 0 \rangle \cdot \left\langle \frac{-5}{13}, \frac{12}{13} \right\rangle = \frac{-10}{13}.$$

4. Use polar coordinates to rewrite the first limit:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = \lim_{r\to 0} \frac{r^2 \cos^2\theta \sin^2\theta}{r^2} = \lim_{r\to 0} \cos^2\theta \sin^2\theta;$$

this limit clearly depends on θ so the original limit does not exist. For the second limit, use paths. First along the $x-$axis, i.e. where $y = 0$, we have

$$\lim_{(x,0)\to(0,0)} \frac{3x^2\sqrt{0}}{0 + x^4} = \lim_{x\to 0} \frac{0}{x^4} = 0.$$

Next, along the path $y = x^4$, we have

$$\lim_{(x,x^4)\to(0,0)} \frac{3x^2\sqrt{x^4}}{x^4 + x^4} = \lim_{x\to 0} \frac{3x^4}{2x^4} = \frac{3}{2} \neq 0.$$

Since the limits along the paths do not match, we see that the original limit does not exist.

5. First subtract z^2e^x from both sides to obtain the equation $\sin(xy+z)-z^2e^x = 0$; let $F(x,y,z) = \sin(xy+z) - z^2e^x$. Then:

$$\frac{\partial x}{\partial z} = \frac{-F_z}{F_x} = \frac{-[\cos(xy+z) - 2ze^x]}{y\cos(xy+z) - z^2e^x}.$$

$$\frac{\partial y}{\partial x} = \frac{-F_x}{F_y} = \frac{-[y\cos(xy+z) - z^2e^x]}{x\cos(xy+z)}.$$

6. The linear approximation is

$$\begin{aligned} L(x,y) &= f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) \\ &= 4 + 2xy^4|_{(2,1)}(x-2) + 4x^2y^3|_{(2,1)}(y-1) \\ &= 4 + 4(x-2) + 16(y-1). \end{aligned}$$

Now the estimate is

$$\begin{aligned} (2.0001)^2(.9998)^4 = f(2.0001, .9998) &\approx L(2.0001, .9998) \\ &= 4 + 4(2.0001-2) + 16(.9998-1) \\ &= 4 + 4(.0001) + 16(-.0002) \\ &= 4 + .0004 - .0032 \\ &= 4 - .0028 = 3.9972. \end{aligned}$$

7. First, find the gradient:

$$\nabla f(x,y) = \left\langle 4x^3 - 4y, 4y^3 - 4x \right\rangle$$

and notice that this gradient always exists. So the critical points are found by setting $\nabla f = \overrightarrow{0}$:

$$\nabla f = \overrightarrow{0} \Rightarrow \begin{cases} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{cases} \Rightarrow \begin{cases} 4y = 4x^3 \\ 4x = 4y^3 \end{cases} \Rightarrow \begin{cases} y = x^3 \\ x = y^3 \end{cases}.$$

Substituting the expression for y from the first equation into the second equation, we get $x = (x^3)^3 = x^9$ subtract x from both sides to get

$$\begin{aligned} x^9 - x &= 0 \\ x(x^8 - 1) &= 0 \\ &\Rightarrow x = 0 \text{ or } x^8 = 1 \\ &\Rightarrow x = 0, x = -1, x = 1 \end{aligned}$$

Finding the $y-$coordinate for these points (from $y = x^3$) we see that $y = 0$ when $x = 0$, $y = 1$ when $x = 1$ and $y = -1$ when $x = -1$. So the critical points are $(0,0)$, $(1,1)$ and $(-1,-1)$.

To classify the critical points use the discriminant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix} = 144x^2y^2 - 16.$$

At the critical point $(0,0)$, $D = -16 < 0$ so $(0,0)$ is a saddle point.

At the critical point $(1,1)$, $D = 144 - 16 > 0$ and $f_{xx}(1,1) = 12 > 0$ so $(1,1)$ is a local minimum.

At the critical point $(-1,-1)$, $D = 144 - 16 > 0$ and $f_{xx}(-1,-1) = 12 > 0$ so $(-1,-1)$ is a local minimum.

8. *Step 1: Find critical points inside R:* First find the gradient of f: $\nabla f = \langle y + 2, x + 1 \rangle$ which always exists. So the critical points are found by setting $\nabla f = \overrightarrow{0}$ which yields the equations

$$y + 2 = 0, x + 1 = 0$$

having only the one solution $(-1,-2)$ which is not in R. So there are no critical points inside R.

Step 2: Critical points on the boundary segments The boundary consists of three pieces:

- $y = 0, 0 \leq x \leq 2$: For this piece, set $y = 0$ in the equation for f to get a new function:

$$g(x) = f(x,0) = 0 + 2x + 0 + 1 = 2x + 1.$$

  Notice that $g'(x) = 2$ so this function has no critical points.

- $x = 0, 0 \leq y \leq 2$: For this piece, set $x = 0$ in the equation for f to get a new function:

$$h(y) = f(0, y) = 0 + 0 + y + 1 = y + 1.$$

  Since $h'(y) = 1$, this function has no critical points.

- $y = -x + 2, 0 \leq x \leq 2$: For this piece, set $y = -x + 2$ in the equation for f to get a new function:

$$k(x) = f(x, -x + 2) = x(-x + 2) + 2x + (-x + 2) + 1 = -x^2 - 3x + 3.$$

  Take the derivative of this function to get $k'(x) = -2x - 3$. Setting this derivative equal to zero we get $x = 3/2$ as a critical point. The $y-$coordinate of this critical point is $y = -x + 2 = -(3/2) + 2 = 1/2$.

Step 3: Analyze all possible extreme points The possible extreme points are the boundary critical point $(3/2, 1/2)$ and the three corner points $(0, 0)$, $(2, 0)$ and $(0, 2)$. Plug each of these into the function f to see which gives the largest value:

$$\begin{aligned}
f(3/2, 1/2) &= 3/4 + 3 + 1/2 + 1 = 5.25 \\
f(0, 0) &= 0 + 0 + 0 + 1 = 1 \\
f(2, 0) &= 0 + 4 + 0 + 1 = 5 \\
f(0, 2) &= 0 + 0 + 2 + 1 = 3
\end{aligned}$$

So the maximum value of the function on the closed region R is 5.25, occurring at the point $(3/2, 1/2)$.