Old MATH 320 Exam 1s

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Contents

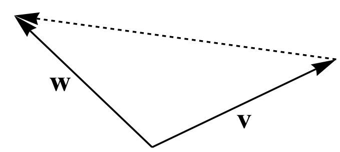
Chapter 1

General information about these exams

These are the exams I have given between 2018 and 2024 in Calculus 3 courses. To help give you some guidance on what questions are appropriate, each question on each exam is followed by a section number in parenthesis (like "(3.2)"). That means that question can be solved using material from that section (or from earlier sections) in the 2024 version of my *Vector Calculus Lecture Notes*.

1.1 Spring 2024 Exam 1

1. a) (2.2) Suppose **v** and **w** are as shown in this diagram:



In terms of **v** and/or **w**, what is the vector indicated by the dashed arrow?

b) (2.7) Write parametric equations for the line in $\mathbb{R}^3$ passing through the points $(2, -5, -3)$ and $(5, -2, -7)$.

c) (2.7) Write the equation of the plane in $\mathbb{R}^3$ that contains the point $(3, -1, 2)$ and contains the line whose parametric equations are

$$\begin{cases} x &= 1 + 2t \\ y &= 3 \\ z &= -8t \end{cases}.$$

2. Let $B = \begin{pmatrix} 3 & 0 & 4 \\ 1 & -2 & 3 \\ 5 & 7 & -3 \end{pmatrix}$.

a) (2.5) Compute the determinant of B.

b) (2.4) Compute the trace of B.

c) (2.4) Compute $B\mathbf{x}$, if $\mathbf{x} = (-1, -4, 2)$.

3. a) (2.4) Let $A = \begin{pmatrix} 4 & 0 \\ -3 & -7 \\ 2 & 5 \\ 1 & -4 \end{pmatrix}$. If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is the function $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$, what are the values of m and n?

b) (2.8) Find a set of polar coordinates which represent the point in $\mathbb{R}^2$ whose Cartesian coordinates are $(4, -4)$.

c) (2.8) Find a set of Cartesian coordinates which represent the point in $\mathbb{R}^3$ whose spherical coordinates are $\left(12, \frac{\pi}{2}, \frac{2\pi}{3}\right)$.

d) (2.9) Let E be the subset of R^3 defined by

$$E = \{(x, y, z) : 1 < x^2 + y^2 + z^2 \leq 4\}.$$

i. Is E bounded?

ii. Is E connected?
iii. Is E compact?

4. (2.8) Throughout this problem, let L, M and N be these subsets of $\mathbb{R}^2$:

$$L = \{(x,y) : x \geq 2\}$$
$$M = \{(r,\theta) : r \geq 0, \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}\}$$
$$N = \{(r,\theta) : 1 \leq r \leq 3\}$$

Sketch a picture of each indicated set:

a) M b) $M \cap N$ c) $L \cup M$

5. Sketch a picture of the points <u>in $\mathbb{R}^3$</u> satisfying each of the following equations:

a) (2.8) $\varphi = \frac{\pi}{4}$
b) (2.8) $\rho = 3$
c) (2.8) $r = 2$
d) (2.8) $\theta = 0$
e) (2.7) $3y - 4z = 12$
f) (2.7) $x + 2y + 4z = 8$

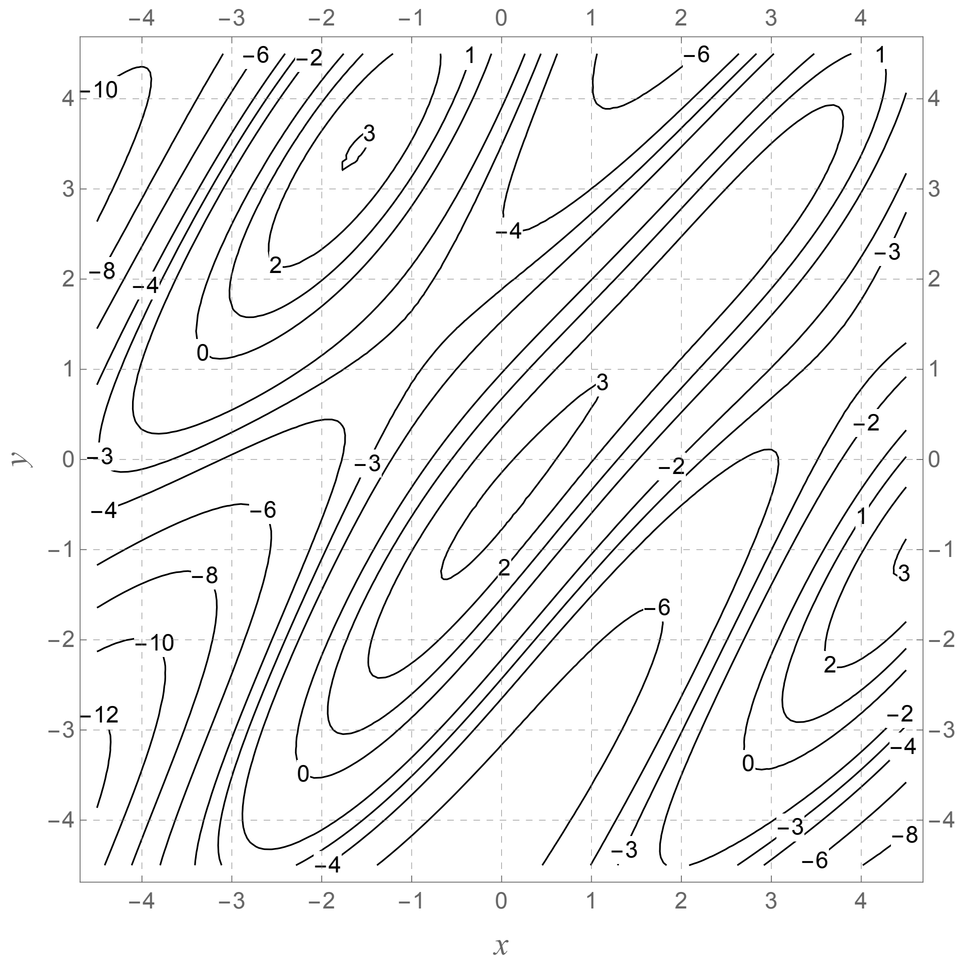
6. Let H be the circle of radius 9, centered at the origin (thought of as a subset of $\mathbb{R}^2$).

a) (3.2) Describe H as the level curve of some function $k : \mathbb{R}^2 \to \mathbb{R}$. Make sure you clearly define the function k and specify the height of the level curve.
b) (3.2) Describe H as the image of a function $\mathbf{h} : \mathbb{R} \to \mathbb{R}^2$.
c) (2.8) Describe H as the graph of a polar function $r = g(\theta)$.
d) (3.2) Describe H as the graph of one or more functions $f : \mathbb{R} \to \mathbb{R}$.

7. (3.5) Compute each of the following two limits:

a) $\lim\limits_{\mathbf{x} \to \mathbf{0}} \frac{x^2+xy}{x^2+y}$
b) $\lim\limits_{(x,y,z) \to (0,0,0)} \frac{x^4}{x^2+y^2+z^2}$

8. (3.2) The contour plot for some unknown function $f : \mathbb{R}^2 \to \mathbb{R}$ is given below:



Use this picture to answer the following questions:

a) Estimate $f(1, 4)$.

b) Estimate any one point (x, y) where $f(x, y) = -2$.

c) At what value of x (between $x = -4$ and $x = 4$) is $f(x, 0)$ maximized?

d) If you are at the point $(3, 2)$, in which direction (written as a vector) would you move to make the value of f decrease as quickly as possible?

Solutions

1. a) The dashed arrow is $\boxed{\mathbf{w}-\mathbf{v}}$, since it runs from the end of $\mathbf{v}$ to the end of $\mathbf{w}$.

   b) A point on the line is$\mathbf{p} = (2,-5,-3)$ and a direction vector is $\mathbf{v} = (5,-2,-7)-(2,-5,-3) = (3,3,-4)$. Thus a set of parametric equations for the line are

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \quad \Rightarrow \quad \boxed{\begin{cases} x &= 2+3t \\ y &= -5+3t \\ z &= -3-4t \end{cases}}$$

   c) A point on the plane is $\mathbf{p} = (3,-1,2)$; two vectors in the plane are $\mathbf{v} = (2,0,-8)$ (a direction vector for the given line) and $\mathbf{w} = (3,-1,2)-(1,3,0) = (2,-4,2)$ (a vector going from a point on the line to the given point in the plane). Therefore a normal vector to the plane is $\mathbf{n} = \mathbf{v} \times \mathbf{w} = (-32,-20,-8)$. Any nonzero multiple of this works, so I'll use $\mathbf{n} = (8,5,2)$.
   Finally, the equation of the plane is

$$\begin{aligned} \mathbf{n}\cdot(\mathbf{x}-\mathbf{p}) &= 0 \\ (8,5,2)\cdot(x-3,y+1,z-2) &= 0 \\ 8x-24+5y+5+2z-4 &= 0 \end{aligned}$$

   and this rearranges into $\boxed{8x+5y+2z=23}$.

2. a) Use the Rule of Sarrus to get $\det B = [3(-2)(-3)+0+4(1)7]-[5(-2)4+7(3)3+0] = [18+28]-[-40+63] = 46-23 = \boxed{23}$.

   b) $tr(B) = 3+(-2)+(-3) = \boxed{-2}$.

   c) $B\mathbf{x} = \begin{pmatrix} 3 & 0 & 4 \\ 1 & -2 & 3 \\ 5 & 7 & -3 \end{pmatrix}\begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 5 \\ 13 \\ -39 \end{pmatrix}}$.

3. a) Since A is 4×2, $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$ defines a function $\mathbb{R}^2 \to \mathbb{R}^4$. Thus $\boxed{m=4}$ and $\boxed{n=2}$.

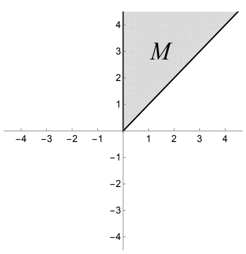
   b) $r = \sqrt{x^2+y^2} = \sqrt{4^2+(-4)^2}\sqrt{32} = 4\sqrt{2}$ and $\theta = \arctan\dfrac{y}{x} = \arctan\dfrac{-4}{4} = \arctan(-1) = -\dfrac{\pi}{4}$. Therefore a set of polar coordinates for the point is $\boxed{\left(4\sqrt{2}, -\frac{\pi}{4}\right)}$.

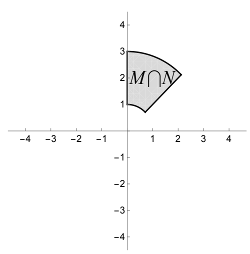
c) We have

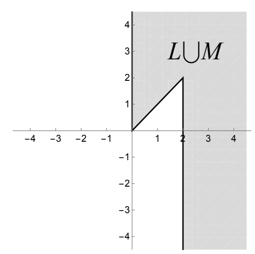
$$x = \rho \sin\phi \cos\theta = 12 \sin\frac{\pi}{2} \cos\frac{2\pi}{3} = 12(1)\left(-\frac{1}{2}\right) = -6$$
$$y = \rho \sin\phi \sin\theta = 12 \sin\frac{\pi}{2} \sin\frac{2\pi}{3} = 12(1)\frac{\sqrt{3}}{2} = 6\sqrt{3}$$
$$z = \rho \cos\phi = 12 \cos\frac{\pi}{2} = 12(0) = 0$$

so the Cartesian coordinates are $\boxed{\left(-6, 6\sqrt{3}, 0\right)}$.

d) i. $\boxed{E \text{ is bounded}}$ (by the sphere of radius 3 centered at the origin).
ii. $\boxed{E \text{ is connected}}$ (it consists of only one connected piece).
iii. $\boxed{E \text{ is not compact}}$ because it is not closed.

4. a) 

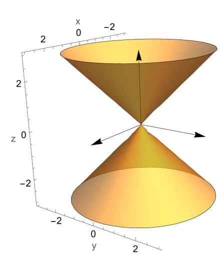
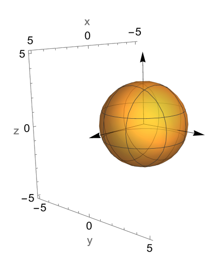
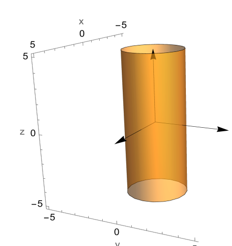
b) 

c) 

5. a) (2.8) $\varphi = \frac{\pi}{4}$ is a cone opening around the z-axis, shown below at left.

b) (2.8) $\rho = 3$ is a sphere of radius 3, centered at the origin. This is shown below in the center.

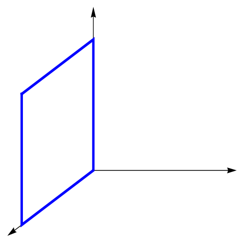
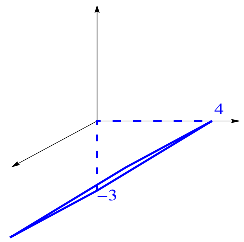
c) (2.8) $r = 2$ is a cylinder of radius 2, opening around the z-axis. This is shown below at right.

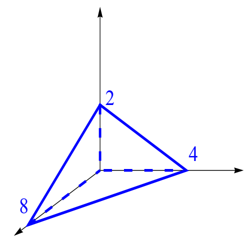




d) (2.8) $\theta = 0$ is the xz-plane, shown below at left.

e) (2.7) $3y - 4z = 12$ is the plane parallel to the x-axis with y-intercept $(0,4,0)$ and z-intercept $(0,0,-3)$, shown below in the middle.

f) (2.7) $x + 2y + 4z = 8$ is the plane with x-intercept $(8,0,0)$, y-intercept $(0,4,0)$ and z-intercept $(0,0,2)$, shown below at right.



6. a) Since H is described by the equation $x^2 + y^2 = 81$, we see that H is the level curve to $\boxed{k(x,y) = x^2 + y^2}$ at $\boxed{\text{height } 81}$.

   b) $H = image(\mathbf{h})$ where $\mathbf{h} : \mathbb{R} \to \mathbb{R}^2$ is $\boxed{\mathbf{h}(t) = (9\cos t, 9\sin t)}$.

   c) H is the graph of the polar function $\boxed{r = 3}$.

   d) Solve the equation $x^2 + y^2 = 81$ for y in terms of x to see that $H = graph(f_1) \cup graph(f_2)$, where $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are $\boxed{f_1(x) = \sqrt{81 - x^2}}$ and $\boxed{f_2(x) = -\sqrt{81 - x^2}}$.

7. a) Along the x-axis, we have

$$\lim_{(x,0)\to(0,0)} \frac{x^2 + xy}{x^2 + y} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

   but along the y-axis, we have

$$\lim_{(0,y)\to(0,0)} \frac{x^2 + xy}{x^2 + y} = \lim_{y\to 0} \frac{0}{y} = 0.$$

   Therefore $\lim_{\mathbf{x}\to\mathbf{0}} \frac{x^2 + xy}{x^2 + y}$ $\boxed{\text{DNE}}$ since limits along different paths going to $(0,0)$ are unequal.
   (This problem could also be done with polar coordinates.)

   b) Use spherical coordinates:

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^4}{x^2 + y^2 + z^2} = \lim_{\rho\to 0} \frac{\rho^4 \sin^4\phi \cos^4\theta}{\rho^2} = \lim_{\rho\to 0} \rho^2 \sin^4\phi \cos^4\theta = \boxed{0}.$$

8. a) $f(1,4) \approx \boxed{-6}$.

   b) Answers can vary here; one possible point is $\boxed{(2,0)}$ since it is on the level curve at height -2.

c) $f(x, 0)$ is maximized at $x \approx \boxed{\frac{1}{3}}$.

d) You need to move southeast, which is in the direction $\boxed{(1, -1)}$.

1.2 Fall 2021 Exam 1

1. a) (2.4) Let $A = \begin{pmatrix} 1 & -3 \\ -2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}$. Compute AB.

   b) (2.4) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be $f(x, y, z) = 3x - y + 6z$. Find a matrix A such that $f(\mathbf{x}) = A\mathbf{x}$.

   c) (2.3) Compute $(3, -4) \cdot (5, 2)$.

   d) (2.8) Find a set of spherical coordinates which represent the point in $\mathbb{R}^3$ whose cylindrical coordinates are $(6, \pi, 6)$.

   e) (2.6) Suppose $\mathbf{v}$ and $\mathbf{w}$ are two vectors in $\mathbb{R}^3$ such that $\mathbf{v} \times \mathbf{w} = (1, 6, -1)$.

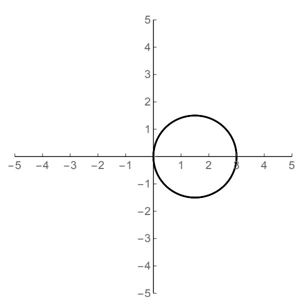
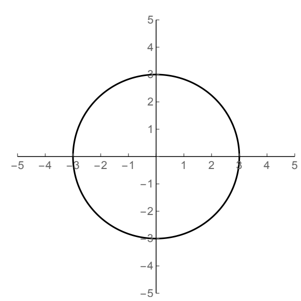
      i. Is this information sufficient to compute $\mathbf{v} \times 3\mathbf{w}$? If so, what is $\mathbf{v} \times 3\mathbf{w}$?

      ii. Is this information sufficient to compute $\mathbf{w} \times \mathbf{v}$? If so, what is $\mathbf{w} \times \mathbf{v}$?

      iii. Is this information sufficient to compute $\mathbf{v} \cdot \mathbf{w}$? If so, what is $\mathbf{v} \cdot \mathbf{w}$?

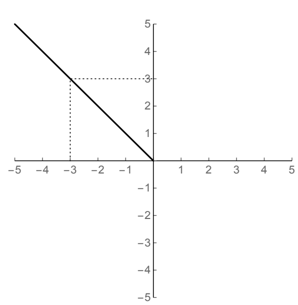
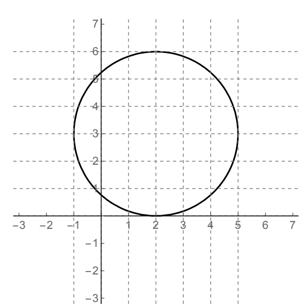
      iv. Is this information sufficient to compute $\mathbf{v} \times (\mathbf{v} + \mathbf{w})$? If so, what is $\mathbf{v} \times (\mathbf{v} + \mathbf{w})$?

2. a) (2.8) Write the polar equation of the circle pictured below, at left.

   b) (2.8) Write the polar equation of the circle pictured above, at right.

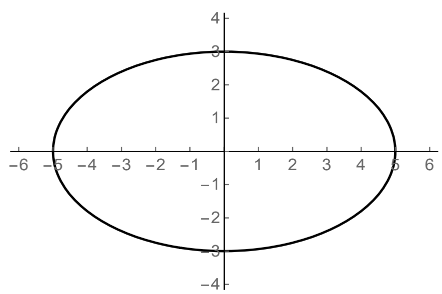
   c) (2.8) Write the polar equation of the (half-)line pictured below, at left.

   d) (2.9) Write the Cartesian equation of the circle pictured above, at right.

3. Sketch a picture of the points in $\mathbb{R}^3$ satisfying each of the following equations:
   a) (2.8) $\varphi = \frac{5\pi}{6}$
   b) (2.8) $\rho = 4$
   c) (2.8) $r = 2$
   d) (2.7) $x + 2y = 6$
   e) (2.7) $x + 2y + 3z = 6$

4. (3.2) Let E be the curve in $\mathbb{R}^2$ pictured below:



   a) Describe E as the level curve of some function $f : \mathbb{R}^2 \to \mathbb{R}$. Make sure you clearly define the function f and specify the height of the level curve.
   b) Which *Mathematica* command would be used to produce a picture of E, in the context of part (a) of this question?
      A. Plot
      B. ContourPlot
      C. ParametricPlot
      D. none of the above
   c) Describe E as the image of a function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^2$.
   d) Which *Mathematica* command would be used to produce a picture of E, in the context of part (c) of this question?
      A. Plot
      B. ContourPlot
      C. ParametricPlot
      D. none of the above

5. a) (2.7) Write parametric equations for the line which is the intersection of the two planes $2x + z = 6$ and $x - 2y + 4z = -1$.
   b) (2.7) Write a normal equation of the plane which contains the three points $(1, -3, 4)$, $(2, -1, 0)$ and $(-1, 4, -1)$.

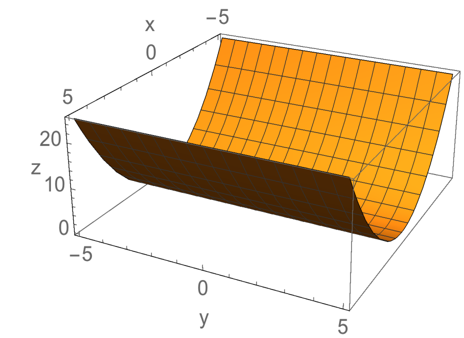
6. (3.5) Compute each of the following two limits:

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x+y}$

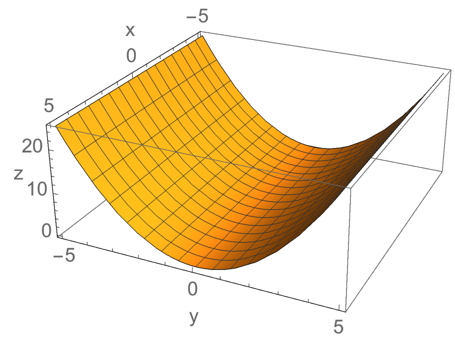
b) $\displaystyle\lim_{\mathbf{x}\to\mathbf{0}} \frac{xyz}{x^2+y^2+z^2}$

7. (3.2) Here are the graphs of four functions f, g, h and $k : \mathbb{R}^2 \to \mathbb{R}$:

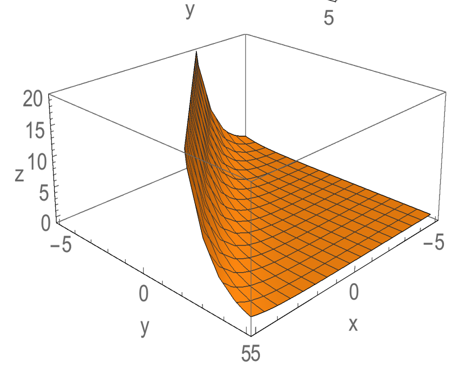
f :



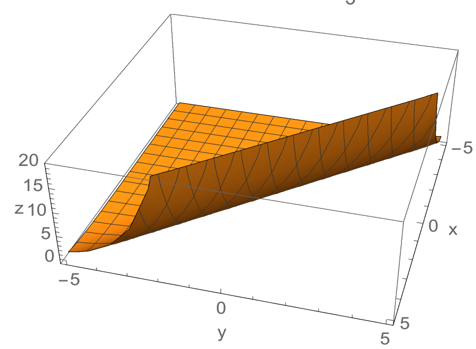
g :



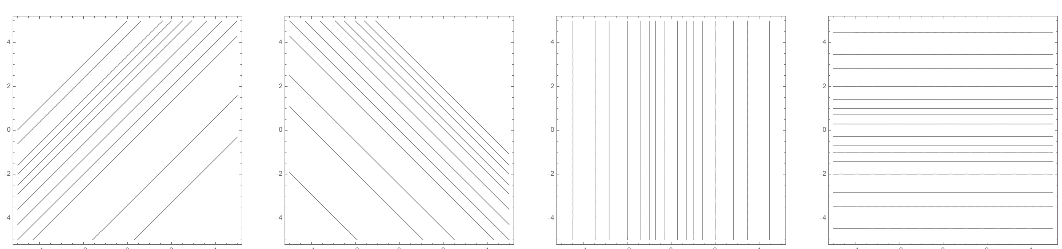
h :



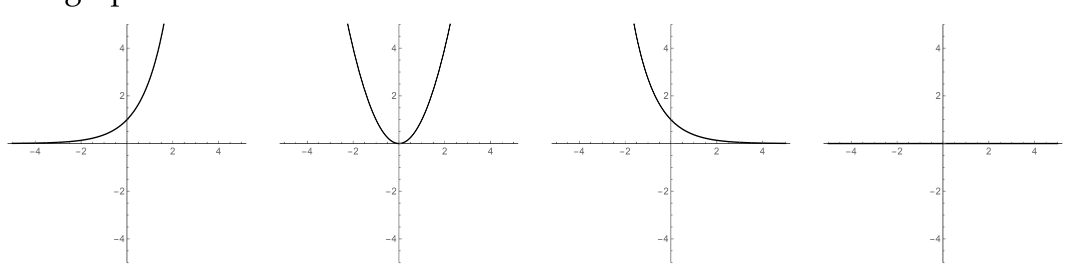
k :



a) For each contour plot below, choose the function (f, g, h or k) which the contour plot represents:



b) For each graph below, choose the function (f, g, h or k) such that the given graph is the graph of the $x = 0$ trace of that function. *Note:* The last graph is the horizontal axis $z = 0$.



Solutions

1. a) $AB = \begin{pmatrix} 1(0)+-3(1) & 1(4)+(-3)3 \\ -2(0)+5(1) & -2(4)+5(3) \end{pmatrix} = \boxed{\begin{pmatrix} -3 & -5 \\ 5 & 7 \end{pmatrix}}$.

   b) Since $f : \mathbb{R}^3 \to \mathbb{R}$, A must be 1×3; in particular we have $\boxed{A = \begin{pmatrix} 3 & -1 & 6 \end{pmatrix}}$.

   c) $(3,-4)\cdot(5,2) = 3(5)+(-4)2 = \boxed{7}$.

   d) We have $\theta = \pi$; $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2} = \sqrt{6^2+6^2} = 6\sqrt{2}$; and $\varphi = \arctan\frac{r}{z} = \arctan\frac{6}{6} = \arctan 1 = \frac{\pi}{4}$, so the spherical coordinates are $\boxed{\left(6\sqrt{2}, \frac{\pi}{4}, \pi\right)}$.

   e) i. $\mathbf{v}\times 3\mathbf{w} = 3(\mathbf{v}\times\mathbf{w}) = 3(1,6,-1) = \boxed{(3,18,-3)}$.

   ii. $\mathbf{w}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{w}) = -(1,6,-1) = \boxed{(-1,-6,1)}$.

   iii. $\mathbf{v}\cdot\mathbf{w}$ cannot be determined from the given information.

   iv. $\mathbf{v}\times(\mathbf{v}+\mathbf{w}) = \mathbf{v}\times\mathbf{v}+\mathbf{v}\times\mathbf{w} = \mathbf{0}+\mathbf{v}\times\mathbf{w} = \mathbf{v}\times\mathbf{w} = \boxed{(1,6,-1)}$.

2. a) $r = 3\cos\theta$

   b) $r = 3$

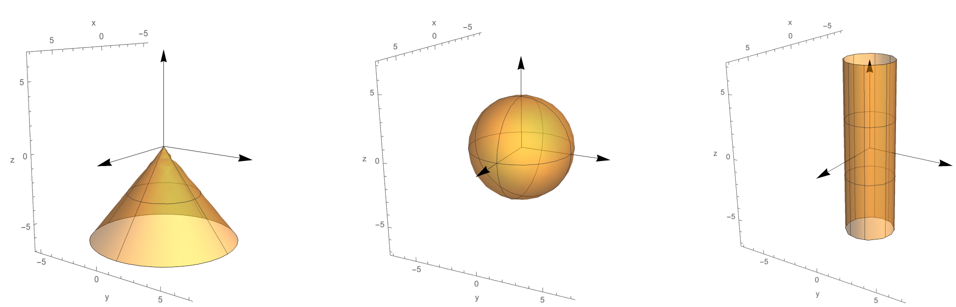
   c) $\theta = \frac{3\pi}{4}$

   d) $(x-2)^2+(y-3)^2 = 9$

3. a) $\varphi = \frac{5\pi}{6}$ is a cone opening downward around the z-axis, pictured below at left.

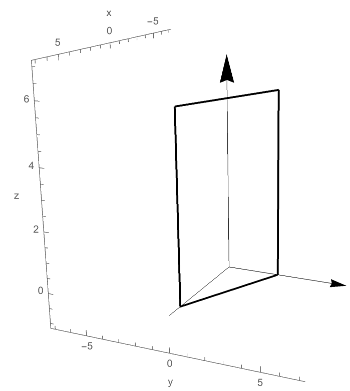
   b) $\rho = 4$ is a sphere of radius 4 centered at $(0,0,0)$, pictured below in the center.

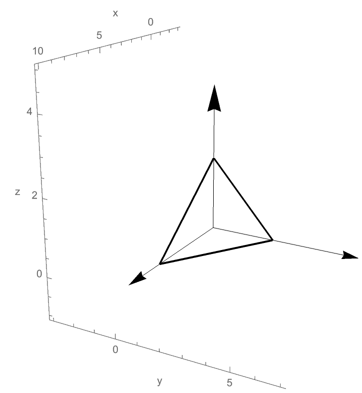
   c) $r = 2$ is a cylinder of radius 2 around the z-axis, pictured below at right.



   d) This plane has intercepts $(6,0,0)$ and $(0,3,0)$ and is parallel to the z-axis, pictured below at left.

e) This plane has intercepts $(6,0,0)$, $(0,3,0)$ and $(0,0,2)$, pictured below at right.





4. a) E is the level curve to $f(x,y) = \frac{x^2}{25} + \frac{y^2}{9}$ at height 1.

   b) B. ContourPlot

   c) E is the image of $\mathbf{f}(t) = (5\cos t, 3\sin t)$ (for $0 \le t \le 2\pi$).

   d) C. ParametricPlot

5. a) From the first equation, we have $z = 6-2x$. Substituting into the second plane, we get $x - 2y + 4(6-2x) = -1$, i.e. $-7x - 2y + 24 = -1$, i.e. $-7x - 2y = -25$. Solve for y to get $y = -\frac{7}{2}x + \frac{25}{2}$. We can then write parametric equations for the line by letting $t = x$, obtaining (by substitution in the equations for y and z in terms of x)

$$\boxed{\begin{cases} x &= t \\ y &= -\frac{7}{2}t + \frac{25}{2} \\ z &= -2t + 6 \end{cases}}$$

   (There are many other correct answers here.)

   b) Two nonparallel vectors in the plane can be found by subtracting pairs of the given points: $\mathbf{v} = (2,-1,0) - (1,-3,4) = (1,2,-4)$ and $\mathbf{w} = (1,-3,4) - (-1,4,-1) = (2,-7,5)$. So a normal vector to the plane is $\mathbf{n} = \mathbf{v} \times \mathbf{w} = (-18,-13,-11)$. Finally, the equation of the plane is

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) &= 0 \\ (-18,-13,-11) \cdot (x-2, y+1, z) &= 0 \\ -18(x-2) - 13(y+1) - 11z &= 0 \\ -18x + 36 - 13y - 13 - 11z &= 0 \end{aligned}$$

   which rearranges into $\boxed{-18x - 13y - 11z = -23}$.
   (Any scalar multiple of this equation is also correct.)

6. a) Factor and cancel:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{(x+y)(x+y)}{x+y} = \lim_{(x,y)\to(0,0)} (x+y) = 0+0 = \boxed{0}.$$

b) Use spherical coordinates:

$$\begin{aligned} \lim_{(x,0,0)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} &= \lim_{\rho\to 0} \frac{\rho^3 \cos\varphi \sin^2\varphi \sin\theta\cos\theta}{\rho^2} \\ &= \lim_{\rho\to 0} \rho \left(\cos\varphi \sin^2\varphi \sin\theta\cos\theta\right) = \boxed{0} \end{aligned}$$

irrespective of the values of φ and/or θ.

7. a) From left to right, these are the contour plots of functions h, k, f and g.

b) From left to right, these are the $x = 0$ traces of functions k, g, h and f.

1.3 Spring 2021 Exam 1

1. a) (2.3) Compute $(3, -4, 1, -2) \cdot (1, 2, 5, -3)$.

b) (2.3) Compute $||(2, -1, 4)||$.

c) (2.4) Compute A^2, where $A = \begin{pmatrix} 1 & -3 \\ 0 & 2 \\ 7 & 5 \end{pmatrix}$.

d) (2.5) Compute $\det \begin{pmatrix} 6 & -4 \\ 3 & 2 \end{pmatrix}$.

e) (2.3) If $\mathbf{v}$ and $\mathbf{w}$ are vectors, each having norm 5, what is the smallest possible value of $\mathbf{v} \cdot \mathbf{w}$?

2. (2.7) Consider the two lines in $\mathbb{R}^3$ given by the following sets of parametric equations:

$$\begin{cases} x &= 7 - 2t \\ y &= 3 + t \\ z &= 5 - 3t \end{cases} \qquad \begin{cases} x &= -2 - t \\ y &= 2 - 5t \\ z &= -3 + 4t \end{cases}$$

These two lines intersect in a point. Find the coordinates of this point.

3. (2.7) Write the normal equation of the plane in $\mathbb{R}^3$ containing the point $(2, -5, -2)$ and the line whose parametric equations are

$$\begin{cases} x &= 1 + 4t \\ y &= 3t \\ z &= 2 \end{cases}$$

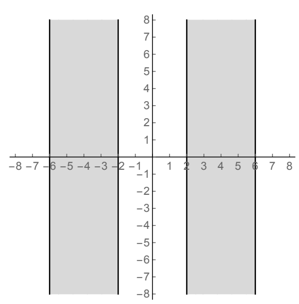
4. Let E be the line in $\mathbb{R}^2$ that has slope 1 and passes through the origin.

   a) (3.2) Describe E as the graph of one or more functions from $\mathbb{R} \to \mathbb{R}$.
   b) (3.2) Describe E as the image of a function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^2$ (i.e. give parametric equations for E).
   c) (3.2) E is the graph of what polar equation?
   d) (3.2) Describe E as the level curve to a function $g : \mathbb{R}^2 \to \mathbb{R}$.

5. Consider the subset F of $\mathbb{R}^2$ defined by

$$F = \{(x, y) \in \mathbb{R}^2 : x^2 \geq 4 \text{ and } x^2 \leq 36\}.$$

   This set F is the shaded region in the picture below:



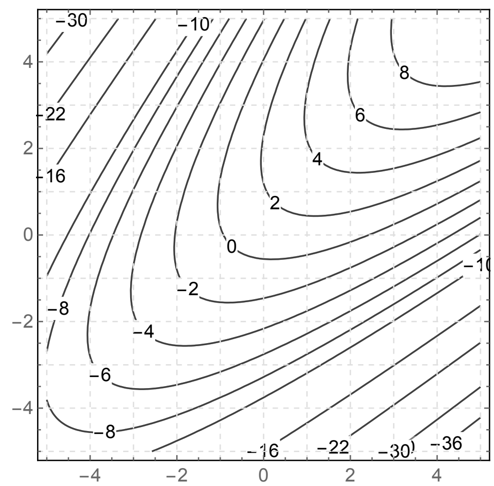
   Answer the following questions about F (no justification is required).

   a) (2.9) Is the set F open?
   b) (2.9) Is the set F closed?
   c) (2.9) Is the set F compact?
   d) (2.9) Is the set F connected?
   e) (2.9) Give the coordinates of any one point which belongs to the boundary of F.
   f) (2.1) Sketch a picture of $F \cup G$ below, if $G = \{(x, y) \in \mathbb{R}^2 : y \leq -4\}$.

6. (3.5) Evaluate the following limit (or state that it does not exist), with proper justification.

$$\lim_{\mathbf{x} \to \mathbf{0}} \frac{x + 2y}{x + 4y}$$

7. (3.2) The contour plot for some unknown function $h : \mathbb{R}^2 \to \mathbb{R}$ is given below.



Use this contour plot to answer these questions:

a) Estimate $h(1, 0)$.

b) Estimate a value of x such that $h(x, 3) = 0$.

c) Which best describes the graph of the $x = 0$ trace of h?
A. A line with positive slope
B. A line with negative slope
C. A parabola that opens upward
D. A parabola that opens downward

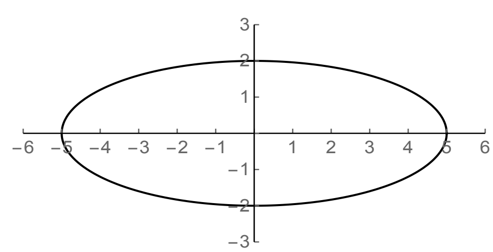
8. Sketch graphs of the following equations. (Each of these pictures should be drawn in $\mathbb{R}^3$.)

a) (3.4) $x^2 - y^2 + z^2 = 1$

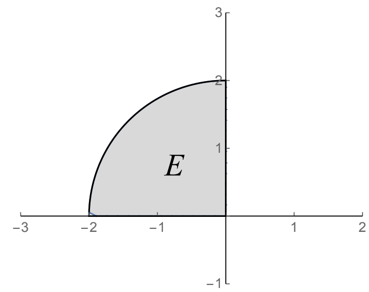
b) (2.7) $2x + 3y = 6$

c) (2.8) $r = 2$ (think of this as an equation in cylindrical coordinates)

9. a) (3.3) Find a set of parametric equations for the curve pictured below:



b) (2.8) Describe the set pictured below by using one or more inequalities, involving polar coordinates:



c) (2.8) Compute the Cartesian coordinates of the point in $\mathbb{R}^3$ whose spherical coordinates are $\left(8, \frac{\pi}{4}, \frac{\pi}{2}\right)$.

Solutions

1. a) $(3,-4,1,-2)\cdot(1,2,5,-3) = 3(1)-4(2)+1(5)-2(-3) = 3-8+5+6 = \boxed{6}$.

   b) $||(2,-1,4)|| = \sqrt{2^2+(-1)^2+4^2} = \boxed{\sqrt{21}}$.

   c) $A^2 = A_{3\times 2}A_{3\times 2}$ $\boxed{\text{DNE}}$, since A is not a square matrix.

   d) $\det\begin{pmatrix} 6 & -4 \\ 3 & 2 \end{pmatrix} = 6(2)-3(-4) = \boxed{24}$.

   e) By either the Cauchy-Schwarz Inequality or the angle formula from dot products, we have $\mathbf{v}\cdot\mathbf{w} \geq -||\mathbf{v}||\,||\mathbf{w}|| = -5(5) = \boxed{-25}$.

2. Change the t to s in the second set of parametric equations, and set the x, y and z-coordinates equal to obtain the system

$$\begin{cases} 7-2t &= -2-s \quad \Rightarrow s = 2t-9 \\ 3+t &= 2-5s \\ 5-3t &= -3+4s \end{cases}$$

   Plug the first equation into the second and solve for t to get $3+t = 2-5(2t-9)$, i.e. $t=4$. Then from the first equation, $s=-1$. Notice that $s=-1$, $t=4$ works in the last equation as well, so the lines intersect in the point (x,y,z) where $s=-1$, i.e. $(-2-(-1), 2-5(-1), -3+4(-1)) = \boxed{(-1,7,-7)}$.

3. One point on the line, which must also be in the plane, is $\mathbf{p} = (1,0,2)$. One vector in the plane can be read off as the direction vector for the given line: $\mathbf{v} = (4,3,0)$. A second vector in the plane can be found by subtracting two points in the plane, one on the line and one not:

$$\mathbf{w} = (2,-5,-2)-\mathbf{p} = (2,-5,-2)-(1,0,2) = (1,-5,-4).$$

   Thus a normal vector to the plane is $\mathbf{n} = \mathbf{v}\times\mathbf{w} = (4,3,0)\times(1,-5,-4) = (-12,16,-23)$. That makes the normal equation of the plane

$$\begin{aligned} \mathbf{n}\cdot(\mathbf{x}-\mathbf{p}) &= 0 \\ (-12,16,-23)\cdot((x,y,z)-(1,0,2)) &= 0 \\ -12(x-1)+16(y-0)-23(z-2) &= 0 \end{aligned}$$

   which simplifies to $\boxed{-12x+16y-23z=-58}$.

4. a) E is the graph of the function $f:\mathbb{R}\to\mathbb{R}$ where $f(x)=x$.

   b) E is the image of $\mathbf{f}:\mathbb{R}\to\mathbb{R}^2$ given by $\mathbf{f}(t) = (t,t)$.

   c) E has polar equation $\theta = \frac{\pi}{4}$.

d) E is the level curve to $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x, y) = y - x$, at height 0.

5. a) F is $\boxed{\text{not open}}$ since it contains at least some of its boundary.

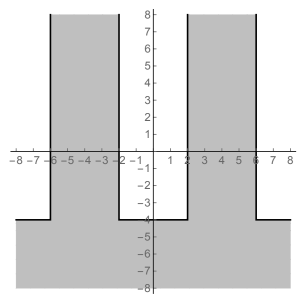
   b) F is $\boxed{\text{closed}}$ since it contains all of its boundary.

   c) F is $\boxed{\text{not compact}}$ since it is not bounded.

   d) F is $\boxed{\text{not connected}}$ since it consists of two "pieces" which do not touch.

   e) Answers can vary here, but any point on the "edge" of F works. These are points with x-coordinate equal to ± 2 or ± 6, like for instance $\boxed{(2, 3)}$.

   f) G is the set of points on or below the horizontal line $y = -4$. Thus the union $F \cup G$ is the set of points in F, or G, or both, as shown here:



6. I'll use paths (although this problem could be done with polar coordinates). Along the x-axis, the limit is

$$\lim_{(x,0)\to(0,0)} \frac{x + 2y}{x + 4y} = \lim_{x\to 0} \frac{x + 2(0)}{x + 4(0)} = \lim_{x\to 0} \frac{x}{x} = 1$$

but along the line $y = x$, the limit is

$$\lim_{(x,x)\to(0,0)} \frac{x + 2y}{x + 4y} = \lim_{x\to 0} \frac{x + 2x}{x + 4x} = \lim_{x\to 0} \frac{3x}{5x} = \frac{3}{5}.$$

Since the limits along different paths are unequal, $\lim_{\mathbf{x}\to\mathbf{0}} \frac{x+2y}{x+4y}$ $\boxed{\text{DNE}}$.

7. a) $h(1, 0) \approx \boxed{1}$, since it is between the level curves at heights 0 and 2.

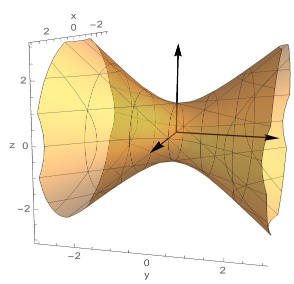
   b) The horizontal line $y = 3$ intersects the level curve at height 0 when x is about $\frac{-1}{2}$, so $h(x, 3) = 0$ when $\boxed{x \approx -\frac{1}{2}}$.

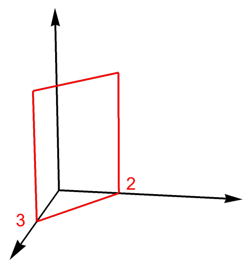
   c) The $x = 0$ trace has z-value -10 when $x \approx -4$, has z-value 2 when $x \approx 1$, but then z-value -4 when $x \approx 4$. Thus this trace starts negative, increases, then decreases, so it is best describe as a parabola that opens downward. This is choice $\boxed{\text{D}}$.

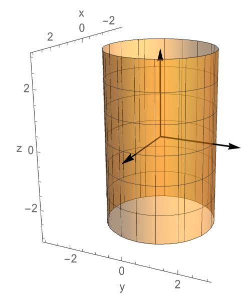
8. a) $x^2 - y^2 + z^2 = 1$ is a hyperboloid of one sheet, opening around the y-axis since the y variable has the $(-)$ sign. This graph is shown below at left.

b) $2x + 3y = 6$ is a plane with x-intercept $(3, 0, 0)$ and y-intercept $(0, 2, 0)$. The plane is parallel to the z-axis since it is missing the z variable, and is shown below in the center.

c) $r = 2$ is a cylinder of radius 2, centered at the origin and opening around the z-axis, as shown below at right.







9. a) This ellipse has parametric equations $\boxed{(5\cos t, 2\sin t)}$ (for $0 \le t \le 2\pi$).

b) This set is $E = \boxed{\left\{(r, \theta) : r \le 2, \frac{\pi}{2} \le \theta \le \pi\right\}}$.

c) We have

$$x = \rho \sin\varphi \cos\theta = 8 \sin\frac{\pi}{4} \cos\frac{\pi}{2} = 8\left(\frac{\sqrt{2}}{2}\right) 0 = 0$$

$$y = \rho \sin\varphi \sin\theta = 8 \sin\frac{\pi}{4} \sin\frac{\pi}{2} = 8\left(\frac{\sqrt{2}}{2}\right) 1 = 4\sqrt{2}$$

$$z = \rho \cos\varphi = 8 \cos\frac{\pi}{4} = 8\left(\frac{\sqrt{2}}{2}\right) = 4\sqrt{2}$$

so the Cartesian coordinates are $\boxed{(0, 4\sqrt{2}, 4\sqrt{2})}$.

1.4 Fall 2020 Exam 1

1. (2.4) Let $A = \begin{pmatrix} -3 & 4 & -7 \\ 3 & 6 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -5 & 3 \\ 1 & 2 \end{pmatrix}$.

   a) Determine which of the two products AB or BA is defined, and compute it.

   b) Determine which of $\det A$ or $\det B$ is defined, and compute it.

   c) Suppose $\mathbf{f}$ is the function defined by $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$. Which of the following best describes how we would indicate the domain and codomain of $\mathbf{f}$? Write the letter of your answer.

   A. $\mathbf{f} : \mathbb{R} \to \mathbb{R}$
   B. $\mathbf{f} : \mathbb{R} \to \mathbb{R}^3$
   C. $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$
   D. $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}$
   E. $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$
   F. $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$

2. (2.6) Let $\mathbf{u} = (2, 3, 4)$ and $\mathbf{v} = (-1, 0, 3)$.

   a) Find a nonzero vector which is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

   b) Is the vector you found in part (a) unique (meaning, is the vector you found the only possible answer to part (a))? Explain.

3. (2.7) Write the normal equation of the plane in $\mathbb{R}^3$ containing the points $(4, -3, 1)$, $(-3, 1, 1)$ and $(4, -2, 8)$.

4. (2.3, 2.6) Suppose $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^3$ with $||\mathbf{v}|| = 9$, $\mathbf{v} \cdot \mathbf{w} = -5$, and $\mathbf{v} \times \mathbf{w} = (1, 2, -4)$.

   For each given quantity below, determine if it is possible to compute that quantity, based on the given information and/or principles developed in class. If it is, compute it. If it isn't, state that quantity is impossible (to compute).

   a) $\mathbf{v} \cdot \mathbf{v}$ b) $\mathbf{v} \times \mathbf{v}$ c) $\mathbf{w} \cdot \mathbf{v}$ d) $\mathbf{w} \times \mathbf{v}$ e) $\mathbf{w} \cdot \mathbf{w}$

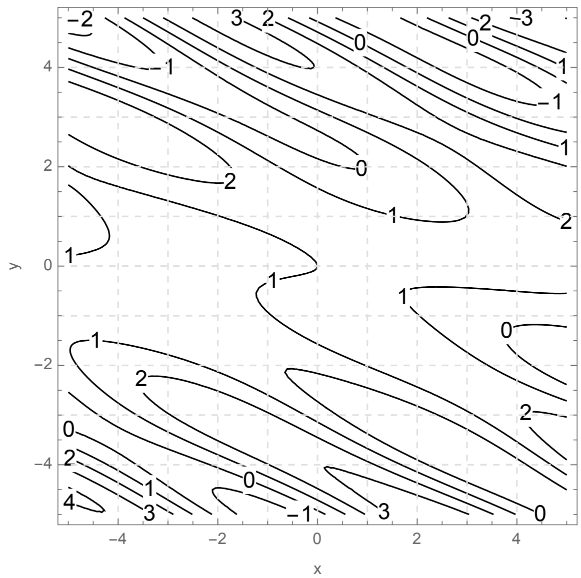
5. (3.2) Let E be the circle in $\mathbb{R}^2$ of radius 2, centered at the origin.

   a) Describe E as the image of a function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^2$ (i.e. give parametric equations for E).

   b) Describe E as the graph of a polar function $r = f(\theta)$.

   c) Describe E as the level curve to a function $g : \mathbb{R}^2 \to \mathbb{R}$.

6. (3.2) Below, you are given a contour plot for some unknown function $h : \mathbb{R}^2 \to \mathbb{R}$.



Use the contour plot to answer the following questions:

a) Estimate $h(4, 3)$.

b) Find a value of y for which $h(1, y) = 2$.

c) If you move in a straight line from $(-2, -3)$ to $(-2, 0)$, are the values of h getting larger, getting smaller, or staying the same?

d) For what x (between -5 and 5) is $h(x, -3)$ maximized?

7. (3.5) Evaluate the following limits (or state that they do not exist), with proper justification.

a) $\lim\limits_{\mathbf{x}\to\mathbf{0}} \frac{x-y}{\sqrt{x^2+y^2}}$

b) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$

8. (2.7, 3.4) Sketch graphs of the following equations:

a) $y^2 = x^2 + z^2$

b) $y = 3$ (this graph should be drawn in $\mathbb{R}^3$)

c) $2x + y + 4z = 8$

9. a) (3.3) Sketch the graph of the conic section whose parametric equations are

$$\begin{cases} x = 4\cos t \\ y = \sin t \end{cases}$$

b) (2.8) Sketch the region of points in $\mathbb{R}^2$ described by the inequalities $0 \leq \theta \leq \frac{\pi}{2}$, $1 \leq r \leq 3$.

c) (2.8) Sketch a picture of the set of points in $\mathbb{R}^3$ which satisfy the equation $r = 3$ in cylindrical coordinates.

Solutions

1. Let $A = \begin{pmatrix} -3 & 4 & -7 \\ 3 & 6 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -5 & 3 \\ 1 & 2 \end{pmatrix}$.

   a) AB is undefined, but $BA = \begin{pmatrix} 24 & -2 & 41 \\ 3 & 16 & -3 \end{pmatrix}$.

   b) $\det A$ is undefined, but $\det B = (-5)2 - 1(3) = -13$.

   c) Since A is 2×3, $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$. This is choice **E**.

2. a) One answer is $\mathbf{u} \times \mathbf{v} = (9, -10, 3)$.

   b) Any (nonzero) multiple of the answer to (a) works, so the answer to (a) is **not unique**.

3. First, subtract pairs of the given points to get two direction vectors for the plane: $\mathbf{v} = (4, -3, 1) - (-3, 1, 1) = (7, -4, 0)$ and $\mathbf{w} = (4, -3, 1) - (4, -2, 8) = (0, -1, -7)$. Then a normal vector to the plane is $\mathbf{n} = \mathbf{v} \times \mathbf{w} = (28, 49, -7)$. (I'll divide through this vector by 7 to keep the numbers small, and set $\mathbf{n} = (4, 7, -1)$.) Now, the normal equation of the plane is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$; any of the given points could be used as $\mathbf{x}_0$. For example, using the first given point we obtain

$$\begin{aligned} (4, 7, -1) \cdot (x - 4, y + 3, z - 1) &= 0 \\ 4(x - 4) + 7(y + 3) - (z - 1) &= 0 \\ 4x + 7y - z + 6 &= 0 \\ 4x + 7y - z &= -6. \end{aligned}$$

4. a) $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2 = 9^2 = 81$.

   b) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ (since any vector $\times$ itself is $\mathbf{0}$).

   c) $\mathbf{w} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = -5$.

   d) $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w} = -(1, 2, -4) = (-1, -2, 4)$.

   e) $\mathbf{w} \cdot \mathbf{w}$ is impossible to compute without more information.

5. a) E is the image of $\mathbf{f}(t) = (2\cos t, 2\sin t)$.

   b) E is the graph of the polar function $r = 2$.

   c) E is the level curve to $g(x, y) = x^2 + y^2$ at height 4.

6. a) $h(4, 3) \approx 0$.

   b) $h(1, y) = 2$ at four places: when $y \approx -5$, $y \approx -4$, $y \approx 3$ and $y \approx 4$.

   c) Values of h are **getting smaller** (from about 2.5 to about $.5$) as you move in a straight line from $(-2, -3)$ to $(-2, 0)$.

d) $h(x, -3)$ is maximized when $x \approx -2$.

7. a) Use the polar coordinates trick (this problem could also be done with paths):

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{x-y}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r\cos\theta - r\sin\theta}{r} = \lim_{r\to 0} (\cos\theta - \sin\theta)\,;$$

when $\theta = 0$ this is $1 - 0 = 1$, but when $\theta = \frac{\pi}{2}$ this is $0 - 1 = -1$. Therefore $\lim\limits_{\mathbf{x}\to\mathbf{0}} \frac{x-y}{\sqrt{x^2+y^2}}$ DNE.

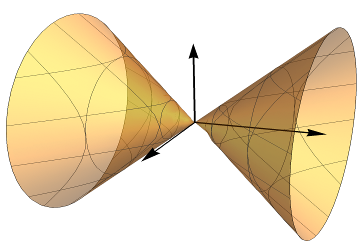
b) Use the polar coordinates trick:

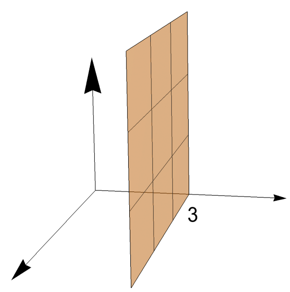
$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2} = \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0.$$

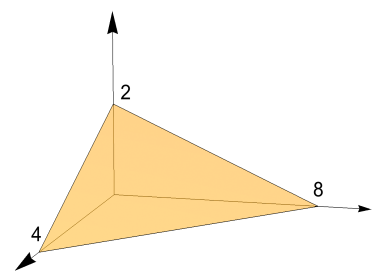
8. a) Rewrite this as $x^2 - y^2 + z^2 = 0$; this is a cone opening around the y-axis, as shown below at left.

b) This is a plane parallel to the x- and z-axes, as shown below in the center.

c) This is a plane with intercepts $(4, 0, 0)$, $(0, 8, 0)$ and $(0, 0, 2)$, as shown below at right. $2x + y + 4z = 8$



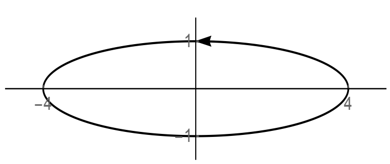


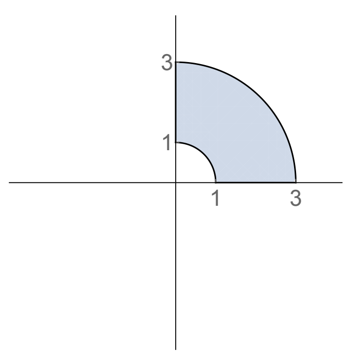


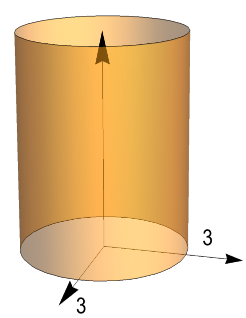
9. a) This is an ellipse traced out counterclockwise, as shown below at left.

b) This region is sketched below, in the center.

c) This is a cylinder of radius 3 around the z-axis, as shown below at right.







1.5 Spring 2018 Exam 1

1. Let $\mathbf{v} = (2, 5, -3)$ and let $\mathbf{w} = 7\mathbf{i} + 2\mathbf{k}$.

   a) (2.3) Find $\mathbf{v} \cdot \mathbf{w}$.

   b) (2.3) Find a vector of length 4 in the opposite direction as $\mathbf{v}$.

   c) (2.3) Find the projection of $\mathbf{w}$ onto $\mathbf{v}$.

   d) (2.3) Suppose $(6, y, 1)$ is orthogonal to $\mathbf{v}$. Find y.

   e) (2.7) Find symmetric equations for the line containing $\mathbf{v}$ and $\mathbf{w}$.

2. a) (2.7) Find the normal equation of the plane containing the points $(2, -1, 4)$, $(3, 3, 5)$ and $(0, -3, -2)$.

   b) (2.7) Find parametric equations of the line which is the intersection of the planes $2x - 3y + 5z = 12$ and $x + 4y - 3z = -5$.

3. a) (2.8) Find Cartesian coordinates of the point whose polar coordinates are $\left(8, \frac{3\pi}{2}\right)$.

   b) Find spherical coordinates of the point whose Cartesian coordinates are $(0, 3, 3)$.

   c) (2.8) Find Cartesian coordinates of the point whose cylindrical coordinates are $\left(4, \frac{\pi}{6}, 2\right)$.

4. (3.3) Sketch graphs of the following equations.

   a) $x^2 = y^2 + z^2$

   b) $x = z^2$ (sketch this as a subset of $\mathbb{R}^3$)

   c) $4x + 3y - 2z = 12$

   d) $x^2 + y^2 - z^2 = 1$

   e) $y = 2$ (sketch this as a subset of $\mathbb{R}^3$)

   f) $\rho = 4$ (this equation is in spherical coordinates)

   g) $r = 6\cos\theta$ (this equation is in polar coordinates)

   h) $r = 6\cos\theta$ (this equation is in cylindrical coordinates)

5. (3.6) Evaluate each of the following limits (if the limit does not exist, say so).

   a) $\lim\limits_{x \to 0} (3x + 5, e^{-x}, 7x^2 - 3)$

   b) $\lim\limits_{(x,y,z) \to (0,0,0)} \frac{2x^2 + yz}{x^2 + y^2 + z^2}$

   c) $\lim\limits_{(x,y) \to (0,0)} \frac{x^2 + 2xy + y^2}{x + y}$

d) $\lim_{\mathbf{x}\to\mathbf{0}} \frac{x-2y}{x+y}$

e) $\lim_{\mathbf{x}\to\mathbf{0}} \mathbf{f}(\mathbf{x})$, where $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\frac{\mathbf{x}}{||\mathbf{x}||}$.

Solutions

1. a) $\mathbf{v} \cdot \mathbf{w} = 2(7) + 5(0) + (-3)2 = 8$.

   b) First, $||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{2^2 + 5^2 + (-3)^2} = \sqrt{38}$. The answer is therefore $-4\frac{\mathbf{v}}{||\mathbf{v}||} = \left(\frac{-8}{\sqrt{38}}, \frac{-20}{\sqrt{38}}, \frac{12}{\sqrt{38}}\right)$.

   c) $\pi_{\mathbf{v}}\mathbf{w} = \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{8}{38}(2, 5, -3) = \left(\frac{8}{19}, \frac{20}{19}, \frac{-12}{19}\right)$.

   d) We have $0 = \mathbf{v} \cdot (6, y, 1) = 2(6) + 5y + (-3)1 = 9 + 5y$. Solving for y, we get $y = \frac{-9}{5}$.

   e) A direction vector for the line is $\mathbf{w}-\mathbf{v} = (7-2, 0-5, 2-(-3)) = (5, -5, 5)$. Therefore, symmetric equations for the line, using the point $(2, 5, -3)$ and the above direction vector, are

   $$\frac{x-2}{5} = \frac{y-5}{-5} = \frac{z+3}{5}.$$

   (Answers may vary in this problem, depending on your choice of direction vector and point on the line.)

2. a) Two vectors in the plane are $\mathbf{v} = (3, 3, 5) - (2, -1, 4) = (1, 4, 1)$ and $\mathbf{w} = (2, -1, 4) - (0, -3, -2) = (2, 2, 6)$. So a normal vector to the plane is $\mathbf{n} = \mathbf{v} \times \mathbf{w} = (22, -4, -6)$. Thus the normal equation of the plane is $22x-4y-6z = d$; to find d, plug in the point $(0, -3, -2)$ to get $d = 22(0) - 4(-3) - 6(-2) = 24$. Thus the normal equation is $22x - 4y - 6z = 24$; dividing through by 2 we get

   $$11x - 2y - 3z = 12.$$

   b) Normal vectors to the two planes can be read off from the normal equations as $\mathbf{n}_1 = (2, -3, 5)$ and $\mathbf{n}_2 = (1, 4, -3)$. The line which is the intersection of the planes must be orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$, so it has direction vector $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = (-11, 11, 11)$.

   Next, find a point on the line by finding a point on both planes. To do this, set $x = 0$ and solve for y and z using the given equations:

   $$\begin{cases} -3y + 5z = 12 \\ 4y - 3z = -5 \end{cases} \quad \Rightarrow y = 1, z = 3.$$

   So $(0, 1, 3)$ is common to both planes, therefore lies on the line of intersection. Using this point and the direction vector from the previous paragraph, we get

   $$\mathbf{x} = \mathbf{p} + t\mathbf{v} \;\Rightarrow\; \begin{cases} x = 0 - 11t \\ y = 1 + 11t \\ z = 3 + 11t \end{cases}$$

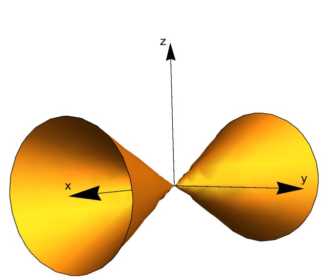
(Answers may vary, depending on the point and direction vector you use.)

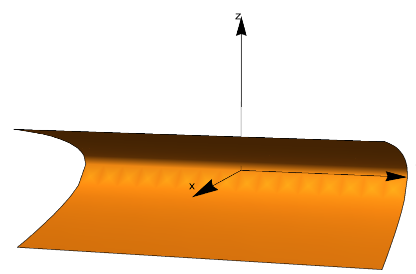
3. a) $x = r\cos\theta = 8\cos\frac{3\pi}{2} = 8(0) = 0$ and $y = r\sin\theta = 8\sin\frac{3\pi}{2} = 8(-1) = -8$, so the Cartesian coordinates are $(x, y) = (0, -8)$.

   b) $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0^2 + 3^2 + 3^2} = 3\sqrt{2}$; $\theta = \frac{\pi}{2}$ since $x = 0$ and $y > 0$; $\varphi = \arctan\frac{z}{\sqrt{x^2+y^2}} = \arctan 1 = \frac{\pi}{4}$, so the spherical coordinates are $(\rho, \varphi, \theta) = \left(3\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2}\right)$.

   c) $x = r\cos\theta = 4\cos\frac{\pi}{6} = 2\sqrt{3}$; $y = r\sin\theta = 4\sin\frac{\pi}{6} = 2$; $z = z = 2$ so the Cartesian coordinates are $(x, y, z) = \left(2\sqrt{3}, 2, 2\right)$.

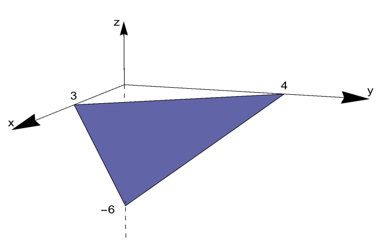
4. a) $x^2 = y^2 + z^2$ is a cone opening around the x-axis, shown below at left:

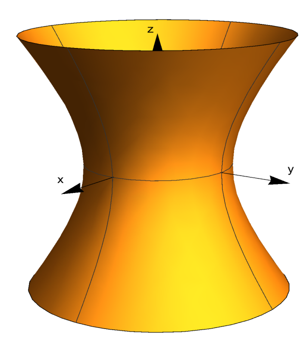




   b) $x = z^2$ is a cylinder parallel to the y-axis; the shape is a parabola opening around the positive x-axis shown above at right:

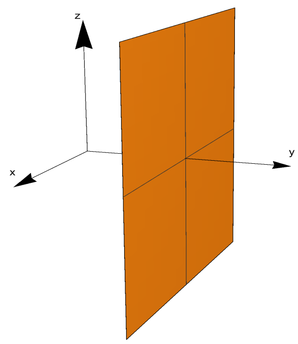
   c) $4x + 3y - 2z = 12$ is a plane with intercepts $(3, 0, 0)$, $(0, 4, 0)$ and $(0, 0, -6)$, shown below at left:

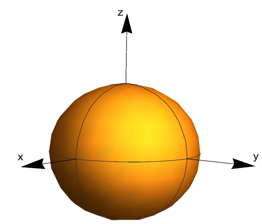




   d) $x^2 + y^2 - z^2 = 1$ is a hyperboloid of one sheet, opening around the z-axis, shown above at right:

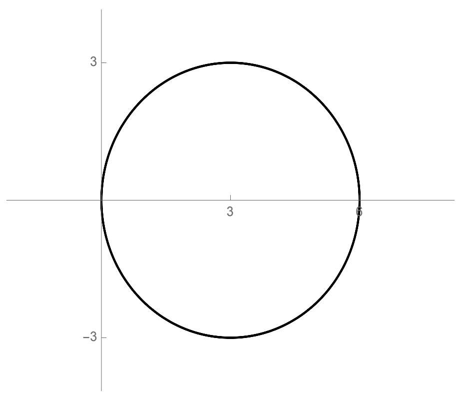
e) $y = 2$ is a plane parallel to the x- and z-axes, shown below at left:

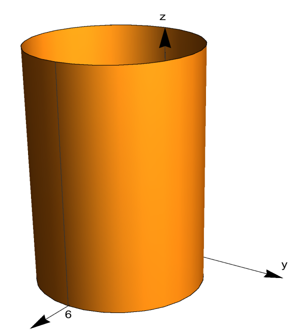




f) $\rho = 4$ is a sphere of radius 4 centered at the origin, shown above at right:

g) $r = 6\cos\theta$, in polar coordinates, is a circle of radius 3 centered at $(3, 0)$ (this graph belongs in $\mathbb{R}^2$), shown below at left:





h) $r = 6\cos\theta$ is a cylinder parallel to the z-axis; the shape of the cylinder is a circle coming from the figure in part (g) of this question. The graph is above at right:

5. a) Plug in 0 to get $\lim\limits_{x\to 0} (3x + 5, e^{-x}, 7x^2 - 3) = (5, -1, -3)$.

b) Change to spherical coordinates:

$$\begin{aligned}
\lim_{(x,y,z)\to(0,0,0)} \frac{2x^2 + yz}{x^2 + y^2 + z^2} &= \lim_{\rho\to 0} \frac{2\rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin\varphi \cos\varphi \sin\theta}{\rho^2} \\
&= \lim_{\rho\to 0} (\sin\varphi) \left(2\sin\varphi\cos^2\theta + \cos\varphi \sin\theta\right) \\
&= \begin{cases} 0 & \text{if } \varphi = 0 \\ 2 & \text{if } \varphi = \frac{\pi}{2} \text{ and } \theta = 0 \end{cases}
\end{aligned}$$

Since the value of the limit depends on φ and θ, the original limit **does not exist**.

(This could also have been shown using paths; along the y-axis and z-axis, the limit is 0, but along the x-axis, the limit is 2.)

c) Factor and cancel:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x+y} = \lim_{(x,y)\to(0,0)} (x+y) = 0.$$

(This could also have been done by changing to polar coordinates.)

d) Use paths: along the x-axis, this limit is

$$\lim_{(x,0)\to(0,0)} \frac{x-2y}{x+y} = \lim_{x\to 0} \frac{x}{x} = 1$$

but along the y-axis, the limit is

$$\lim_{(0,y)\to(0,0)} \frac{x-2y}{x+y} = \lim_{y\to 0} \frac{-2y}{y} = -2.$$

Since the limits are different, the original limit **does not exist**.

(This could also have been done by changing to polar coordinates.)

e) First, writing the function coordinate-wise, this is the limit

$$\lim_{(x,y)\to(0,0)} \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right).$$

Let's look at the first coordinate: changing to polar coordinates, we obtain

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r\cos\theta}{\sqrt{r^2}} = \lim_{r\to 0} \cos\theta.$$

When $\theta = 0$, this is 1, but when $\theta = \frac{\pi}{2}$, this is 0. Since the value depends on θ, the limit

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$$

does not exist; consequently the vector-valued limit

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{\mathbf{x}}{||\mathbf{x}||} = \lim_{(x,y)\to(0,0)} \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$$

also **does not exist**.