Old MATH 320 Exam 2s

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Contents

Chapter 1

General information about these exams

These are the Exam 2s I have given between 2018 and 2024 in Calculus 3 courses. To help give you some guidance on what questions are appropriate, each question on each exam is followed by a section number in parenthesis (like “(3.2)”). That means that question can be solved using material from that section (or from earlier sections) in the 2024 version of my *Vector Calculus Lecture Notes*.

Questions marked with TH were take-home questions where notes, calculators and *Mathematica* were allowed; other questions are closed-note with no calculators allowed.

1.1 Spring 2024 Exam 2

1. (4.2) Compute the total derivative of the function $\mathbf{f} : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$\mathbf{f}(w, x, y, z) = \left(e^{3w+2xy} + 5e^{-y}, e^{wxy} - 4e^{2y-yz}\right).$$

2. (4.4) Suppose $3xz^4 - 5wx^2y^3 + 4w^5x^4z^3 = 17$. Compute $\dfrac{dx}{dy}$.

3. (4.5) Compute the normal equation of the tangent plane to the surface

$$x^2y + 2xy^3 - 3x^3z = 9$$

   at the point $(1, 2, 3)$.

4. (4.3) Write parametric equations for the line tangent to the function

$$\mathbf{f}(t) = \left(\frac{3}{t}, \frac{4}{t^2+1}, 8t^2\right)$$

   at the point where $t = 1$.

5. Throughout this problem, let $g(x, y) = \sin(4y - x)$.

   a) (4.2) Compute $\dfrac{\partial g}{\partial y}(\pi, \pi)$.

   b) (4.2) What is the geometric interpretation of the quantity you computed in part (a)?

   c) (4.2) Compute g_{xxy}.

   d) (4.5) Compute $D_{\mathbf{u}}g(4, 1)$, where $\mathbf{u} = \left(\frac{3}{5}, -\frac{4}{5}\right)$.

6. (5.1) An alien lives on a planet where the acceleration due to gravity is 12 m/sec^2 (as opposed to the 9.8 m/sec^2 that it is on Earth). The alien fires a projectile due north from a height 20 m, with an initial velocity $\left(\frac{15}{4}, \frac{25}{4}, 7\right)$ m/sec. In addition to gravity, the projectile experiences acceleration $(2t, 6t, 0)$ m/sec^2 coming from wind (where t is the number of seconds after the balloon is launched). Calculate the point where the projectile lands on the ground.

7. A hummingbird is flying in $\mathbb{R}^3$ so that at time t (in min), its position, in m, is $\mathbf{x}(t) = (t^2 - 1, 2t^3 - 23t, t^4 - 6t^2 + 5)$.

   a) (5.1) Compute the velocity of the hummingbird at time 2.

   b) (5.1) Compute the acceleration of the hummingbird at time 2.

c) (5.1) How fast is the hummingbird flying at time 2?

d) (5.2) What is the rate of change of the hummingbird's speed at time 2?

8. (6.1) Find the four critical points of the function

$$f(x, y) = 2x^2 - y^3 - 3x^2y + 3y.$$

Classify each of the four critical points as a local minimum, local maximum, or saddle.

9. (6.3) Determine the maximum value of the function $f(x, y, z) = x + y - z$, subject to the constraint $x^2 + y^2 + z^2 = 243$.

Solutions

1. Writing the components of $\mathbf{f}$ as (f_1, f_2), we have

$$D\mathbf{f} = \begin{pmatrix} (f_1)_w & (f_1)_x & (f_1)_y & (f_1)_z \\ (f_2)_w & (f_2)_x & (f_2)_y & (f_2)_z \end{pmatrix}$$
$$= \boxed{\begin{pmatrix} 3e^{3w+2xy} & 2ye^{3w+2xy} & 2xe^{3w+2xy} - 5e^{-y} & 0 \\ xye^{wxy} & wye^{wxy} & wxe^{wxy} - 4(2-z)e^{2y-yz} & 4ye^{2y-yz} \end{pmatrix}}.$$

2. Let $f(w,x,y,z) = 3xz^4 - 5wx^2y^3 + 4w^5x^4z^3$, so that we have the equation $f(w,x,y,z) = 17$. By the implicit differentiation formula,

$$\frac{dx}{dy} = -\frac{f_y}{f_x} = -\frac{-15wx^2y^2}{3z^4 - 10wxy^3 + 16w^5x^3z^3} = \boxed{\frac{15wx^2y^2}{3z^4 - 10wxy^3 + 16w^5x^3z^3}}.$$

3. Let $f(x,y,z) = x^2y + 2xy^3 - 3x^3z$ so that the surface is the level surface to f at height 9. A normal vector to the plane tangent to the surface is $\mathbf{n} = \nabla f(1,2,3)$, so we compute the gradient:

$$\nabla f = (f_x, f_y, f_z) = (2xy + 2y^3 - 9x^2z, x^2 + 6xy^2, -3x^3)$$

Therefore

$$\mathbf{n} = \nabla f(1,2,3) = (2(1)(2) + 2(2^3) - 9(1^2)3, 1^2 + 6(1)(2^2), -3(3^3)) = (-7, 25, -3)$$

and the equation of the tangent plane is

$$\mathbf{n} \cdot (\mathbf{x} - (1,2,3)) = 0,$$

i.e. $-7(x-1) + 25(y-2) - 3(z-3) = 0$, which rearranges to $\boxed{-7x + 25y - 3z = 34}$.

4. The tangent line is the linearization of $\mathbf{f}$, which has equation

$$\begin{aligned} \mathbf{L}(t) &= \mathbf{f}(1) + D\mathbf{f}(1)(t-1) \\ &= \left(\frac{3}{1}, \frac{4}{1^2+1}, 8(1^2)\right) + \left(-\frac{3}{t^2}, -\frac{4(2t)}{(t^2+1)^2}, 16t\right)\Bigg|_{t=1} (t-1) \\ &= (3,2,8) + (-3,-2,16)(t-1) \end{aligned}$$

which, written coordinate-wise, is

$$\boxed{\begin{cases} x &= 3 - 3(t-1) \\ y &= 2 - 2(t-1) \\ z &= 8 + 16(t-1) \end{cases}}.$$

5. a) Differentiate with respect to y to get $\dfrac{\partial g}{\partial y} = 4\cos(4y - x)$. Then $\dfrac{\partial g}{\partial y}(\pi, \pi) = 4\cos(4\pi - \pi) = 4\cos 3\pi = 4(-1) = \boxed{-4}$.

   b) This partial derivative is the **slope of the line tangent to the surface** $z = g(x, y)$ **at the point** (π, π)**, parallel to the** y**-axis**.

   c) Differentiate twice with respect to x, then with respect to y: $g_x = -\cos(4y - x)$; $g_{xx} = -\sin(4y - x)$ and finally, $g_{xxy} = \boxed{-4\cos(4y - x)}$.

   d) First, find the gradient: $\nabla g = (g_x, g_y) = (-\cos(4y - x), 4\cos(4y - x))$ so $\nabla g(4, 1) = (-\cos(4(1) - 4), 4\cos(4(1) - 4)) = (-\cos 0, 4\cos 0) = (-1, 4)$. Then, the directional derivative is

$$D_{\mathbf{u}} g(4, 1) = \nabla g(4, 1) \cdot \mathbf{u} = (-1, 4) \cdot \left(\frac{3}{5}, -\frac{4}{5}\right) = -\frac{3}{5} - \frac{16}{5} = \boxed{-\frac{19}{5}}.$$

6. We are given that the acceleration is the gravity plus the wind, so $\mathbf{a}(t) = (2t, 6t, -12)$. Integrate this to get the velocity:

$$\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = \int (2t, 6t, -12)\, dt = (t^2, 3t^2, -12t) + \mathbf{C}.$$

To find $\mathbf{C}$, observe that the initial velocity is $\left(\frac{15}{4}, \frac{25}{4}, 7\right) = \mathbf{v}(0) = \mathbf{C}$, so $\mathbf{C} = \left(\frac{15}{4}, \frac{25}{4}, 7\right)$. Therefore

$$\begin{aligned} \mathbf{v}(t) = \int \mathbf{a}(t)\, dt = \int (2t, 6t, -12)\, dt &= (t^2, 3t^2, -12t) + \left(\frac{15}{4}, \frac{25}{4}, 7\right) \\ &= \left(t^2 + \frac{15}{4}, 3t^2 + \frac{25}{4}, -12t + 7\right). \end{aligned}$$

To get the position, integrate again:

$$\begin{aligned} \mathbf{x}(t) = \int \mathbf{v}(t)\, dt &= \int \left(t^2 + \frac{15}{4}, 3t^2 + \frac{25}{4}, -12t + 7\right) dt \\ &= \left(\frac{1}{3}t^3 + \frac{15}{4}t, t^3 + \frac{25}{4}t, -6t^2 + 7t\right) + \mathbf{C}. \end{aligned}$$

To find $\mathbf{C}$, observe that the initial position is $(0, 0, 20) = \mathbf{x}(0) = \mathbf{C}$, so $\mathbf{C} = (0, 0, 20)$. That makes the position

$$\begin{aligned} \mathbf{x}(t) &= \left(\frac{1}{3}t^3 + \frac{15}{4}t, t^3 + \frac{25}{4}t, -6t^2 + 7t\right) + (0, 0, 20) \\ &= \left(\frac{1}{3}t^3 + \frac{15}{4}t, t^3 + \frac{25}{4}t, -6t^2 + 7t + 20\right). \end{aligned}$$

When the projectile lands, the z-coordinate of $\mathbf{x}(t)$ is 0, so we have the equation

$$\begin{aligned} -6t^2 + 7t + 20 &= 0 \\ -(3t+4)(2t-5) &= 0 \end{aligned}$$

So $t = \dfrac{5}{2}$ and $t = -\dfrac{4}{3}$. The negative t makes no sense, so the projectile lands at time $t = \dfrac{5}{2}$. The position at this time is

$$\begin{aligned} \mathbf{x}\left(\frac{5}{2}\right) &= \left(\frac{1}{3}\left(\frac{5}{2}\right)^3 + \frac{15}{4}\left(\frac{5}{2}\right), \left(\frac{5}{2}\right)^3 + \frac{25}{4}\left(\frac{5}{2}\right), -6\left(\frac{5}{2}\right)^2 + 7\left(\frac{5}{2}\right) + 20\right) \\ &= \left(\frac{125}{24} + \frac{75}{8}, \frac{125}{8} + \frac{125}{8}, -\frac{75}{2} + \frac{35}{2} + 20\right) \\ &= \left(\frac{350}{24}, \frac{125}{4}, 0\right) = \boxed{\left(\frac{175}{12}, \frac{125}{4}, 0\right) \text{ m}}. \end{aligned}$$

7. a) The bird's velocity is $\mathbf{v}(t) = \mathbf{x}'(t) = (2t, 6t^2 - 23, 4t^3 - 12t)$, so its velocity at time 2 is $\mathbf{v}(2) = (2(2), 6(2^2) - 23, 4(2^3) - 12(2)) = \boxed{(4, 1, 8) \text{ m/min}}$.

   b) The bird's acceleration is $\mathbf{a}(t) = \mathbf{x}''(t) = (2, 12t, 12t^2 - 12)$, so its acceleration at time 2 is $\mathbf{a}(2) = (2, 12(2), 12(2^2) - 12) = \boxed{(2, 24, 36) \text{ m/min}^2}$.

   c) This is the bird's speed, which is $||\mathbf{v}(2)|| = ||(4, 1, 8)|| = \sqrt{4^2 + 1^2 + 8^2} = \sqrt{81} = \boxed{9 \text{ m/min}}$.

   d) This is the tangential component of the bird's acceleration, which at time 2 is

$$a_T(2) = \mathbf{a}(2) \cdot \mathbf{T}(2) = \frac{\mathbf{a}(2) \cdot \mathbf{v}(2)}{||\mathbf{v}(2)||} = \frac{(2, 24, 36) \cdot (4, 1, 8)}{9} = \boxed{\frac{320}{9} \text{ m/min}^2}.$$

8. First, compute the gradient: $\nabla f = (f_x, f_y) = (4x - 6xy, -3y^2 - 3x^2 + 3)$. Set the gradient equal to $\mathbf{0}$ to get

$$\begin{cases} 0 = 4x - 6xy = 2x(2 - 3y) \\ 0 = -3y^2 - 3x^2 + 3 \end{cases}$$

If $2x = 0$, then $x = 0$ and by substituting in the second equation, we get $-3y^2 + 3 = 0$, i.e. $y^2 = 1$, i.e. $y = \pm 1$. This gives two critical points: $(0, 1)$ and $(0, -1)$. If $2 - 3y = 0$, then $y = \frac{2}{3}$ and by plugging into the second equation, we get $0 = -3\left(\dfrac{4}{9}\right) - 3x^2 + 3$, i.e. $0 = -3x^2 + \frac{5}{3}$, i.e. $x^2 = \frac{5}{9}$, i.e. $x = \pm\frac{\sqrt{5}}{3}$. All together, we have four critical points: $(0, 1)$, $(0, -1)$, $\left(\frac{\sqrt{5}}{3}, \frac{2}{3}\right)$ and $\left(-\frac{\sqrt{5}}{3}, \frac{2}{3}\right)$.

To classify these, use the Hessian:

$$Hf(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 4-6y & -6x \\ -6x & -6y \end{pmatrix}.$$

$$\begin{aligned}
Hf(0,1) = \begin{pmatrix} -2 & 0 \\ 0 & -6 \end{pmatrix} &\text{ has positive determinant and negative trace} \\
&\Rightarrow Hf(0,1) < 0 \Rightarrow \boxed{(0,1) \text{ is a local max}} \\
Hf(0,-1) = \begin{pmatrix} 10 & 0 \\ 0 & 6 \end{pmatrix} &\text{ has positive determinant and trace} \\
&\Rightarrow Hf(0,-1) > 0 \Rightarrow \boxed{(0,-1) \text{ is a local min}} \\
Hf\left(\frac{\sqrt{5}}{3}, \frac{2}{3}\right) = \begin{pmatrix} 0 & -2\sqrt{5} \\ -2\sqrt{5} & -4 \end{pmatrix} &\text{ has negative determinant} \\
&\Rightarrow Hf\left(\frac{\sqrt{5}}{3}, \frac{2}{3}\right) \text{ is neither pos. def. nor neg. def.} \\
&\Rightarrow \boxed{\left(\frac{\sqrt{5}}{3}, \frac{2}{3}\right) \text{ is a saddle}} \\
Hf\left(-\frac{\sqrt{5}}{3}, \frac{2}{3}\right) = \begin{pmatrix} 0 & 2\sqrt{5} \\ 2\sqrt{5} & -4 \end{pmatrix} &\text{ has negative determinant} \\
&\Rightarrow Hf\left(-\frac{\sqrt{5}}{3}, \frac{2}{3}\right) \text{ is neither pos. def. nor neg. def.} \\
&\Rightarrow \boxed{\left(\frac{-\sqrt{5}}{3}, \frac{2}{3}\right) \text{ is a saddle}}.
\end{aligned}$$

9. Use Lagrange's method: let $g(x,y,z) = x^2 + y^2 + z^2$ so that the constraint is $g(x,y,z) = 243$. Then, setting $\nabla f = \lambda \nabla g$ together with the constraint, we have

$$\begin{cases} 1 &= \lambda(2x) \\ 1 &= \lambda(2y) \\ -1 &= \lambda(2z) \\ 243 &= x^2 + y^2 + z^2 \end{cases}$$

From the first three equations, we get $x = \frac{1}{2\lambda}$, $y = \frac{1}{2\lambda}$ and $z = -\frac{1}{2\lambda}$ so $y = x$ and $z = -x$. Plugging into the constraint gives $243 = x^2 + x^2 + (-x)^2 = 3x^2$, i.e. $81 = x^2$ so $x = \pm 9$. When $x = 9$, $y = 9$ and $z = -9$ giving the point $(9, 9, -9)$. When $x = -9$, $y = -9$ and $z = 9$ so we get the point $(-9, -9, 9)$.

Test these two points to determine the maximum value of the utility f:

$$f(-9,-9,9) = -9 + (-9) - 9 = -27$$
$$f(9,9,-9) = 9 + 9 - (-9) = \boxed{27} \leftarrow \text{ maximum value}$$

1.2 Fall 2021 Exam 2

1. (5.1) TH A water balloon is launched from ground level, directly eastward, at an angle of $\frac{\pi}{3}$ to the horizontal, with an initial velocity of $80\sqrt{3}$ ft/sec. In addition to gravity, the water balloon experiences acceleration of magnitude $12t$ ft/sec^2 coming from a wind blowing north (where t is the number of seconds after the balloon is launched). Calculate the x- and y-coordinates of the point where the balloon lands.

2. (6.2) TH Compute the absolute maximum value and absolute minimum value of the function

$$f(x,y) = 3y^2 - 2xy - 2x + 4y$$

on the square D with vertices $(-2,-2)$, $(2,-2)$, $(2,2)$ and $(-2,2)$.

3. Throughout this problem, let $g(x,y,z) = e^{x-3y} + 2e^{4x-3z} - ze^{2x-y}$.

   a) (4.5) Compute the gradient of g.

   b) (4.5) Compute the directional derivative of g at the origin, in the direction $\mathbf{u} = \left(\frac{-1}{\sqrt{19}}, \frac{3}{\sqrt{19}}, \frac{-3}{\sqrt{19}}\right)$. *Note:* $\mathbf{u}$ is a unit vector.

   c) (4.5) What is the smallest possible value of the directional derivative of g at the origin, if you allow yourself to choose any direction?

   d) (4.2) Compute $\dfrac{\partial^6 g}{\partial^2 x \partial z \partial x \partial y \partial x}$.

4. Parts (a)-(c) of this question are not related to one another.

   a) (4.2) Compute the total derivative of $\mathbf{h} : \mathbb{R}^2 \to \mathbb{R}^2$, if $\mathbf{h}(x,y) = (\sin 2x \cos y, \sin xy)$.

   b) (4.5) Write a normal equation of the plane tangent to the surface $2x^2 + 3y^2 + z^4 = 27$ at the point $(2,-1,2)$.

   c) (4.4) Suppose $xyz^2 + 4x^3z = 4$. Compute $\left.\frac{dy}{dx}\right|_{x=1,y=-1,z=2}$.

5. Throughout this problem, suppose that an object is moving in $\mathbb{R}^3$ so that its position at time t is $(t^{-1}, \ln t, 2t^2)$.

   a) (5.1) Compute the velocity of the object at time 1.

   b) (5.1) Compute the acceleration of the object at time 1.

   c) (5.4) Compute the curvature of the path the object travels at the point corresponding to time 1.

6. (6.1) Find all critical points of the function $f(x,y) = 9x^2 + 2y^2 - 3x^2y$. Classify each critical point as a local maximum, a local minimum, or a saddle, using appropriate reasoning.

7. (6.3) Find the maximum value of $f(x, y) = x^2y$, subject to the constraint $x^2 + y^4 = 5$.

Solutions

1. The acceleration of the balloon is the acceleration due to gravity plus the acceleration due to the wind, which is $\mathbf{a}(t) = (0, 12t, 0) + (0, 0, -32) = (0, 12t, -32)$. We also know the initial velocity is $\mathbf{v}(0) = \left(80\sqrt{3}\cos\frac{\pi}{3}, 0, 80\sqrt{3}\sin\frac{\pi}{3}\right) = (40\sqrt{3}, 0, 120)$ and the initial position is $\mathbf{x}(0) = (0, 0, 0)$.

   Integrate the acceleration to get the velocity:

   $$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int (0, 12t, -32)\,dt = (0, 6t^2, -32t) + \mathbf{C}.$$

   By plugging in $t = 0$, we see $\mathbf{C} = \mathbf{v}(0) = (40\sqrt{3}, 0, 120)$, so the velocity is

   $$\mathbf{v}(t) = (40\sqrt{3}, 6t^2, -32t + 120).$$

   Now integrate the velocity to get the position:

   $$\mathbf{x}(t) = \int \mathbf{v}(t)\,dt = \int (40\sqrt{3}, 6t^2, -32t + 120)\,dt = (40\sqrt{3}t, 2t^3, -16t^2 + 120t) + \mathbf{C}.$$

   By plugging in $t = 0$, we see $\mathbf{C} = \mathbf{x}(0) = (0, 0, 0)$, so the position is

   $$\mathbf{x}(t) = (40\sqrt{3}t, 2t^3, 16t^2 + 120t).$$

   To find when the balloon lands, set the z-coordinate of the position equal to 0 and solve for t:

   $$0 = 16t^2 + 120t = 8t(2t + 15) \Rightarrow t = 0, t = \frac{15}{2}.$$

   We know $t = 0$ is when the balloon launches, so the balloon lands at time $t = \frac{15}{2}$. The position at this time is

   $$\mathbf{x}\left(\frac{15}{2}\right) = \left(40\sqrt{3}\left(\frac{15}{2}\right), 2\left(\frac{15}{2}\right)^3, 0\right) = \boxed{\left(300\sqrt{3}, \frac{3375}{4}, 0\right)}.$$

2. First, find the critical points of f: the gradient is $\nabla f(x, y) = (f_x, f_y) = (-2y - 2, 6y - 2x + 4)$; setting the gradient equal to $\mathbf{0}$ gives

   $$\begin{cases} -2y - 2 &= 0 \\ 6y - 2x + 4 &= 0 \end{cases} \Rightarrow (x, y) = (-1, -1).$$

   Next, we parametrize each of the four pieces of the boundary and find the boundary critical points (BCPs):

   TOP: This is parametrized by $(t, 2)$ for $-2 \leq t \leq 2$, so our function is $f(t, 2) = 12 - 4t - 2t + 8 = 20 - 6t$. Differentiate to get $f'(t) = -6$ which is never zero, so no BCPs here.

BOTTOM: This is parametrized by $(t, -2)$ for $-2 \le t \le 2$, so our function is $f(t, -2) = 12 + 4t - 2t - 8 = 4 + 2t$. Differentiate to get $f'(t) = 2$ which is never zero, so no BCPs here.

LEFT: This is parametrized by $(-2, t)$ for $-2 \le t \le 2$, so our function is $f(-2, t) = 3t^2 + 4t + 4 + 4t = 3t^2 + 8t + 4$. Differentiate to get $f'(t) = 6t + 8$; set $f'(t) = 0$ and solve for t to get $t = \frac{-4}{3}$. This gives the BCP $(-2, \frac{-4}{3})$.

RIGHT: This is parametrized by $(2, t)$ for $-2 \le t \le 2$, so our function is $f(2, t) = 3t^2 - 4t - 4 + 4t = 3t^2 - 4$. Differentiate to get $f'(t) = 6t$; set $f'(t) = 0$ and solve for t to get $t = 0$. This gives the BCP $(2, 0)$.

Finally, take the CP, the two BCPs, and the four corners of the square and test all of them in the utility:

Point	Test
CP $(-1, -1)$	$f(-1, -1) = 3 - 2 + 2 - 4 = -1$
BCP $(-2, \frac{-4}{3})$	$f(-2, \frac{-4}{3}) = \frac{16}{3} - \frac{16}{3} + 4 - \frac{16}{3} = -\frac{4}{3}$
BCP $(2, 0)$	$f(2, 0) = \boxed{-4} \longleftarrow$ ABS MIN
CORNER $(2, 2)$	$f(2, 2) = 12 - 8 - 4 + 8 = 8$
CORNER $(2, -2)$	$f(2, -2) = 12 + 8 - 4 - 8 = 8$
CORNER $(-2, 2)$	$f(-2, 2) = 12 + 8 + 4 + 8 = \boxed{32} \longleftarrow$ ABS MAX
CORNER $(-2, -2)$	$f(-2, -2) = 12 - 8 + 4 - 8 = 0$

3. a) Compute the partial derivatives:

$$\begin{aligned} \nabla g(x, y, z) &= (g_x, g_y, g_z) \\ &= \boxed{\left(e^{x-3y} + 8e^{4x-3z} - 2ze^{2x-y}, -3e^{x-3y} + ze^{2x-y}, -6e^{4x-3z} - e^{2x-y}\right)}. \end{aligned}$$

b) Take the dot product of the gradient and $\mathbf{u}$:

$$\begin{aligned} D_{\mathbf{u}}g(0, 0, 0) &= \nabla g(0, 0, 0) \cdot \mathbf{u} \\ &= (1 + 8 - 0, -3, -6 - 1) \cdot \left(\frac{-1}{\sqrt{19}}, \frac{3}{\sqrt{19}}, \frac{-3}{\sqrt{19}}\right) \\ &= \frac{1}{\sqrt{19}}(9, -3, -7) \cdot (-1, 3, -3) \\ &= \frac{1}{\sqrt{19}}(-9 - 9 + 21) = \boxed{\frac{3}{\sqrt{19}}}. \end{aligned}$$

c) The smallest possible value of $D_{\mathbf{u}}g(0, 0, 0)$ is

$$-||\nabla g(0, 0, 0)|| = -||(9, -3, -7)|| = -\sqrt{9^2 + (-3)^2 + (-7)^2} = \boxed{-\sqrt{139}}.$$

d) By Clairaut's Theorem, we can do these partial derivatives in any order. That means

$$\begin{aligned}\frac{\partial^6 g}{\partial x^2 \partial z \partial x \partial y \partial x} &= \left(e^{x-3y} + 2e^{4x-3z} - ze^{2x-y}\right)_{xyxzxx} \\ &= \left(e^{x-3y} + 2e^{4x-3z} - ze^{2x-y}\right)_{zyxxxx} \\ &= \left(0 - 6e^{4x-3z} - e^{2x-y}\right)_{yxxxx} \\ &= \left(e^{2x-y}\right)_{xxxx} = \boxed{16e^{2x-y}}.\end{aligned}$$

4. a) Compute the partial derivatives of each component function:

$$D\mathbf{h}(x, y) = \begin{pmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{pmatrix} = \boxed{\begin{pmatrix} 2\cos 2x \cos y & -\sin 2x \sin y \\ y \cos xy & x \cos xy \end{pmatrix}}.$$

b) Let $f(x, y, z) = 2x^2 + 3y^2 + z^4$. A normal vector to the level surface at height 27 is $\nabla f(2, -1, 2) = (8, -6, 32)$, so the normal equation of the plane is

$$\boxed{8(x - 2) - 6(y + 1) + 32(z - 2) = 0}.$$

c) Let $f(x, y, z) = xyz^2 + 4x^3z$. Then, by the implicit differentiation formula, $\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-(yz^2+12x^2z)}{xz^2}$. At the given point $(1, -1, 2)$, this is

$$\left.\frac{dy}{dx}\right|_{x=1,y=-1,z=2} = \frac{-(-4 + 24)}{4} = \boxed{-5}.$$

5. a) $\mathbf{v}(t) = \mathbf{x}'(t) = (-t^{-2}, t^{-1}, 4t)$. At time 1, this velocity is $\mathbf{v}(1) = \boxed{(-1, 1, 4)}$.

b) $\mathbf{a}(t) = \mathbf{v}'(t) = (2t^{-3}, -t^{-2}, 4)$. At time 1, this acceleration is $\mathbf{a}(1) = \boxed{(2, -1, 4)}$.

c) The curvature is given by

$$\begin{aligned}\kappa(1) &= \frac{||\mathbf{v}(1) \times \mathbf{a}(1)||}{||\mathbf{v}(1)||^3} \\ &= \frac{||(-1, 1, 4) \times (2, -1, 4)||}{||(-1, 1, 4)||^3} = \frac{||(8, 12, -1)||}{(\sqrt{18})^3} = \boxed{\frac{\sqrt{209}}{18\sqrt{18}}}.\end{aligned}$$

6. Find the critical points by setting the gradient equal to zero:

$$\nabla f(x, y) = 0 \Rightarrow \begin{cases} 18x - 6xy &= 0 \\ 4y - 3x^2 &= 0 \end{cases}$$

From the second equation, $y = \frac{3}{4}x^2$. Plugging this into the first equation, we get $18x - \frac{9}{2}x^3 = 0$, i.e. $\frac{9}{2}x(4 - x^2) = 0$, i.e. $\frac{9}{2}x(2 - x)(2 + x) = 0$. This gives $x = 0$, $x = 2$ and $x = -2$, which have respective y-values $y = 0$, $y = 3$ and $y = 3$. So there are three critical points: $(0, 0)$, $(2, 3)$ and $(-2, 3)$. Test these using the Hessian:

$$Hf(x, y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 18 - 6y & -6x \\ -6x & 4 \end{pmatrix}.$$

Now $Hf(0, 0) = \begin{pmatrix} 18 & 0 \\ 0 & 4 \end{pmatrix}$ has positive determinant and positive trace, so $Hf(0, 0) > 0$, making $\boxed{(0, 0) \text{ a local minimum}}$.

$Hf(2, 3) = \begin{pmatrix} 0 & -12 \\ -12 & 4 \end{pmatrix}$ has negative determinant, making $\boxed{(2, 3) \text{ a saddle}}$.

Finally, $Hf(-2, 3) = \begin{pmatrix} 0 & 12 \\ 12 & 4 \end{pmatrix}$ has negative determinant, making $\boxed{(-2, 3) \text{ a saddle}}$.

7. Use Lagrange's method; the equation $\nabla f = \lambda \nabla g$, together with the constraint $x^2 + y^4 = 5$, gives the system

$$\begin{cases} 2xy &= \lambda(2x) \\ x^2 &= \lambda(4y^3) \\ x^2 + y^4 &= 5 \end{cases}$$

From the first equation, either $x = 0$ or $y = \lambda$. If $x = 0$, then from the second equation $\lambda = 0$ (which is impossible since λ is never zero in Lagrange's method) or $y = 0$. But if $x = 0$ and $y = 0$, the constraint isn't satisfied. This rules out $x = 0$ and leaves us with $y = \lambda$.

Substituting into the second equation, we get $x^2 = 4y^4$ and substituting this for x^2 in the constraint gives $4y^4 + y^4 = 5$, i.e. $5y^4 = 5$, i.e. $y^4 = 1$, i.e. $y = \pm 1$.

For either value of y, $x^2 = 4y^4$ so $x^2 = 4$ so $x = \pm 2$. This gives four candidate points: $(2, 1)$, $(2, -1)$, $(-2, 1)$ and $(-2, -1)$. Test these to find the maximum value:

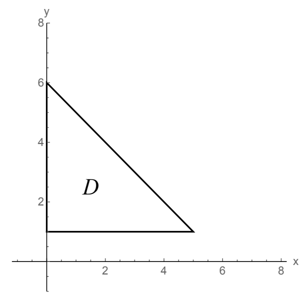
$$\begin{aligned} f(\pm 2, 1) &= (\pm 2)^2(1) = \boxed{4} \longleftarrow \text{ MAX} \\ f(\pm 2, -1) &= (\pm 2)^2(-1) = -4 \end{aligned}$$

1.3 Spring 2021 Exam 2

1. (6.2) TH Determine the absolute maximum and absolute minimum values obtained by the function

$$f(x,y) = 2y^2 - x^2 + 4x - 12y$$

on the region D, which is the triangle with vertices $(0,1)$, $(5,1)$ and $(0,6)$ shown below:



2. TH Suppose an object is moving in $\mathbb{R}^4$, so that its position (in meters) at time t (in seconds) is

$$\mathbf{x}(t) = (4\cos t, \sin 3t, 4\sin t, \cos 3t).$$

a) (5.1) Compute and simplify the speed of the object at time t.

b) (5.2) What is the tangential component of the object's acceleration at time t?

c) (5.4) Compute the curvature of the path the object travels at time t.

3. Let S be the surface consisting of the points $(x,y,z) \in \mathbb{R}^3$ satisfying the equation

$$2x^3 + x^2y - xyz^2 - 3y^2z = 12.$$

a) (4.5) Write the normal equation of the plane that is tangent to S at the point $(1,2,-1)$.

b) (4.5) Use your answer to part (a) to estimate the value of x so that $(x, 1.8, -.9)$ is on the surface.

4. The parts of this question are unrelated to one another.

a) (4.1) Compute the total derivative of the function g, where

$$\mathbf{g}(x,y,z) = (x\sin(yz), y + \cos(yz)).$$

b) (4.5) Let $h(x,y) = x^2 - 2y^2$. Compute the direction in which h is increasing most rapidly at the point $(3,1)$.

c) (4.4) Suppose $y^2e^{4x} - x^2e^y = 8$. Compute $\frac{dy}{dx}$.

5. A dog is running around a yard, so that its velocity at time t, in km/min, is

$$\mathbf{v}(t) = \left(\sqrt{t^2-1}, \sqrt{3t^2+1}\right).$$

a) (5.1) Compute the speed of the dog at time 2.

b) (5.1) Compute the distance the dog travels from time 1 to time 2.

6. (6.1) Find the two critical points of the function $f(x,y) = y^3 + 3xy + x^2 + 5x$. Classify each critical point as a local maximum, local minimum or saddle.

7. (6.3) Find the point (x,y) where $f(x,y) = x - 2y$ is maximized, subject to the constraint $x^2 + 5y^2 = 1$.

Solutions

1. First, find critical points of f: $\nabla f(x,y) = (-2x+4, 4y-12) = (0,0)$ when $(x,y) = (2,3)$.

   Second, we parameterize the bottom of the triangle by $(t,1)$ where $0 \le t \le 5$. Substitute into the utility to get $f(t,1) = 2 - t^2 + 4t - 12 = -t^2 + 4t - 10$. Differentiate to get $f'(t) = -2t + 4$; setting this equal to 0 and solving for t gives $t = 2$, yielding the boundary critical point $(2,1)$.

   Third, we parameterize the left-side of the triangle by $(0,t)$ where $1 \le t \le 6$. Substitute into the utility to get $f(0,t) = 2t^2 - 12y$; differentiate to get $f'(t) = 4t - 12$. Setting this equal to 0, we get $t = 3$, yielding boundary critical point $(0,3)$.

   Fourth, we parametrize the diagonal side of the triangle; this line has equation $y = 6 - x$, so we parametrize it by $(t, 6-t)$. Substitute into the utility to get $f(t,6-t) = 2(6-t)^2 - t^2 + 4t - 12(6-t) = 2(36-12t+t^2) - t^2 + 4t - 72 + 12t = t^2 - 8t$. Differentiate to get $f'(t) = 2t - 8$; set equal to 0 to get boundary critical point $t = 4$, i.e. $(4,2)$.

   Last, we test the critical point, the boundary critical points, and the corners, by plugging all of them into the utility:

   $$\begin{array}{ll} \text{CP } (2,3): & f(2,3) = 2(9) - 4 + 8 - 36 = -14 \\ \text{BCP } (2,1): & f(2,1) = 2 - 4 + 8 - 12 = -6 \\ \text{BCP } (0,3): & f(0,3) = 18 - 36 = \boxed{-18} \longleftarrow \text{ ABS MIN} \\ \text{BCP } (4,2): & f(4,2) = 2(2^2) - 4^2 + 4(4) - 12(2) = 8 - 24 = -16 \\ \text{CORNER } (0,1): & f(0,1) = 2 - 12 = -10 \\ \text{CORNER } (0,6): & f(0,6) = 2(36) - 12(6) = \boxed{0} \longleftarrow \text{ ABS MAX} \\ \text{CORNER } (5,1): & f(5,1) = 2 - 25 + 20 - 12 = -15 \end{array}$$

2. a) The speed is

   $$\begin{aligned} ||\mathbf{v}(t)|| = ||\mathbf{x}'(t)|| &= ||(-4\sin t, 3\cos 3t, 4\cos t, -3\sin 3t)|| \\ &= \sqrt{16\sin^2 t + 9\cos^2 3t + 16\cos^2 t + 9\sin^2 3t} \\ &= \sqrt{16+9} = \boxed{5 \text{ m/sec}}. \end{aligned}$$

   b) $a_T(t) = \frac{ds}{dt} = \frac{d}{dt}||\mathbf{v}(t)|| = \frac{d}{dt}(5) = \boxed{0 \text{ m/sec}^2}$.

   c) First, the acceleration is

   $$\mathbf{a}(t) = \mathbf{x}''(t) = (-4\cos t, -9\sin 3t, -4\sin t, -9\cos 3t),$$

and the norm of the acceleration is

$$||\mathbf{a}(t)|| = \sqrt{16\cos^2 t + 81\sin^2 3t + 16\sin^2 t + 81\cos^2 3t} = \sqrt{16+81} = \sqrt{97}.$$

Next, the normal component of the acceleration is

$$a_N(t) = \sqrt{||\mathbf{a}(t)||^2 - a_T(t)^2} = \sqrt{97-0} = \sqrt{97}\,\text{m/sec}^2.$$

Last, the normal component is the curvature, times the speed squared, so we have

$$\begin{aligned} a_N(t) &= \kappa(t)\left(\frac{ds}{dt}\right)^2 \\ \sqrt{97} &= \kappa(t)(5)^2 \\ \boxed{\frac{1}{25}\sqrt{97}\,\text{m}^{-1}} &= \kappa(t). \end{aligned}$$

3. a) Let $f(x,y,z) = 2x^3 + x^2y - xyz^2 - 3y^2z$. S is the level surface to f at height 12, so a normal vector $\mathbf{n}$ to this surface is given by $\nabla f(1,2,-1)$. Observe

$$\nabla f(x,y,z) = (6x^2 + 2xy - yz^2, x^2 - xz^2 - 6yz, -2xyz - 3y^2)$$

so

$$\mathbf{n} = \nabla f(1,2,-1) = (6+4-2, 1-1+12, 4-12) = (8,12,-8).$$

Let $\mathbf{p} = (1,2,-1)$; the normal equation of the plane is $\mathbf{n}\cdot(\mathbf{x}-\mathbf{p}) = 0$, i.e.

$$8(x-1) + 12(y-2) - 8(z+1) = 0$$

which simplifies to

$$\boxed{2x + 3y - 2z = 10}.$$

b) Plug in $(x, 1.8, -.9)$ to the tangent plane to get $2x + 3(1.8) - 2(-.9) = 10$, i.e. $2x + 7.2 = 10$, i.e. $\boxed{x = 1.4}$.

4. a) Write $\mathbf{g} = (g_1, g_2)$. Compute the partials and arrange them in a 2×3 matrix:

$$Dg(x,y,z) = \begin{pmatrix} (g_1)_x & (g_1)_y & (g_1)_z \\ (g_2)_x & (g_2)_y & (g_2)_z \end{pmatrix} = \boxed{\begin{pmatrix} \sin(yz) & xz\cos(yz) & xy\cos(yz) \\ 0 & 1 - z\sin(yz) & -y\sin(yz) \end{pmatrix}}.$$

b) The direction in which h is increasing most rapidly at the point $(3,1)$ is

$$\nabla h(3,1) = (2x, -4y)|_{(3,1)} = (2(3), -4(1)) = \boxed{(6,-4)}.$$

c) Let $f(x,y) = y^2e^{4x} - x^2e^y$. Since f is constant,

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \boxed{\frac{-(4y^2e^{4x} - 2xe^y)}{2ye^{4x} - x^2e^y}}.$$

5. a) $||\mathbf{v}(2)|| = ||(\sqrt{3}, \sqrt{13})|| = \sqrt{3+13} = \boxed{4 \text{ km/min}}$.

   b) This is an arc length calculation:

$$\begin{aligned}\int_1^2 ||\mathbf{v}(t)||\,dt &= \int_1^2 ||(\sqrt{t^2-1}, \sqrt{3t^2+1})||\,dt \\ &= \int_1^2 \sqrt{(\sqrt{t^2-1})^2 + (\sqrt{3t^2+1})^2}\,dt \\ &= \int_1^2 \sqrt{t^2-1+3t^2+1}\,dt \\ &= \int_1^2 \sqrt{4t^2}\,dt \\ &= \int_1^2 2t\,dt = t^2\Big|_1^2 = \boxed{3 \text{ km}}.\end{aligned}$$

6. To find the CPs, set $\nabla f(x,y) = \mathbf{0}$. This gives

$$\begin{cases} 3y + 2x + 5 &= 0 \\ 3y^2 + 3x &= 0 \end{cases}$$

From the second equation, $x = -y^2$ so substituting into the first equation gives $3y - 2y^2 + 5 = 0$, i.e. $2y^2 - 3y - 5 = 0$. This factors as $(2y-5)(y+1) = 0$, so $y = \frac{5}{2}$ or $y = -1$. When $y = \frac{5}{2}$, $x = -y^2 = -\frac{25}{4}$ and when $y = -1$, $x = -y^2 = -1$ so we get the two critical points $\left(-\frac{25}{4}, \frac{5}{2}\right)$ and $(-1,-1)$. To classify these points, use the Hessian:

$$Hf(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 6y \end{pmatrix}.$$

Now $Hf(\left(-\frac{25}{4}, \frac{5}{2}\right) = \begin{pmatrix} 2 & 3 \\ 3 & 15 \end{pmatrix}$ has positive trace and determinant, so is positive definite. Therefore $\boxed{\left(-\frac{25}{4}, \frac{5}{2}\right) \text{ is a local minimum}}$.

Also, $Hf(-1,-1) = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$ has negative determinant, so is neither positive definite nor negative definite, so $\boxed{(-1,-1) \text{ is a saddle}}$.

Find the two critical points of the function $f(x,y) = y^3 + 3xy + x^2 + 5x$. Classify each critical point as a local maximum, local minimum or saddle.

7. Use Lagrange's method: set $g(x,y) = x^2 + 5y^2$; then $\nabla f = \lambda \nabla g$ gives

$$\begin{cases} 1 &= \lambda(2x) \\ -2 &= \lambda(10y) \end{cases}$$

From the first equation, $\lambda = \frac{1}{2x}$. Substituting into the second equation, we get $-2 = \frac{1}{2x}(10y) = \frac{5y}{x}$, so $y = -\frac{2}{5}x$. Substituting into the constraint gives

$$x^2 + 5\left(-\frac{2}{5}x\right)^2 = 1 \;\Rightarrow\; x^2 + \frac{4}{5}x^2 = 1 \;\Rightarrow\; \frac{9}{5}x^2 = 1 \;\Rightarrow\; x = \pm\frac{\sqrt{5}}{3}.$$

From $y = -\frac{2}{5}x$, we get the two candidate points $\left(\frac{\sqrt{5}}{3}, -\frac{2\sqrt{5}}{15}\right)$ and $\left(-\frac{\sqrt{5}}{3}, \frac{2\sqrt{5}}{15}\right)$. Test these:

$$f\left(\frac{\sqrt{5}}{3}, -\frac{2\sqrt{5}}{15}\right) = \frac{\sqrt{5}}{3} - 2\left(-\frac{2\sqrt{5}}{15}\right) = \frac{3}{5}\sqrt{5};$$
$$f\left(-\frac{\sqrt{5}}{3}, \frac{2\sqrt{5}}{15}\right) = -\frac{\sqrt{5}}{3} - 2\left(\frac{2\sqrt{5}}{15}\right) = -\frac{3}{5}\sqrt{5}.$$

Therefore f is maximized at the point $\boxed{\left(\frac{\sqrt{5}}{3}, -\frac{2\sqrt{5}}{15}\right)}$.

1.4 Fall 2020 Exam 2

1. (4.3) TH Compute the linearization of $\mathbf{f}(x,y,z) = (e^{2\sin x+5y-3z}, e^{xz-z})$ at the origin, and use that linearization to estimate $\mathbf{f}\left(\frac{1}{10}, \frac{3}{10}, -\frac{1}{5}\right)$.

2. (6.2) TH Determine the absolute minimum value obtained by the function

$$f(x,y) = (x-1)^2 + (y-2)^2 + 2xy$$

   on the region $D = [0,3] \times [0,3]$.

3. (4.5) Write the normal equation of the plane tangent to the ellipsoid $2x^2 + 5y^2 + z^2 = 31$ at the point $(-1,2,-3)$.

4. Let $f(x,y,z) = 3x^2yz^4 - 2xz^2 + 3x^2y^5$.

   a) (4.1) If you wrote the total derivative of f as a matrix, what size would that matrix be?

   b) (4.5) Compute $\nabla f(x,y,z)$.

   c) (4.2) Compute $\frac{\partial^3 f}{\partial x \partial z \partial x}$.

   d) (4.5) Compute $D_{\mathbf{u}}f(1,0,-1)$, where $\mathbf{u}$ is in the direction $(2,1,-2)$.

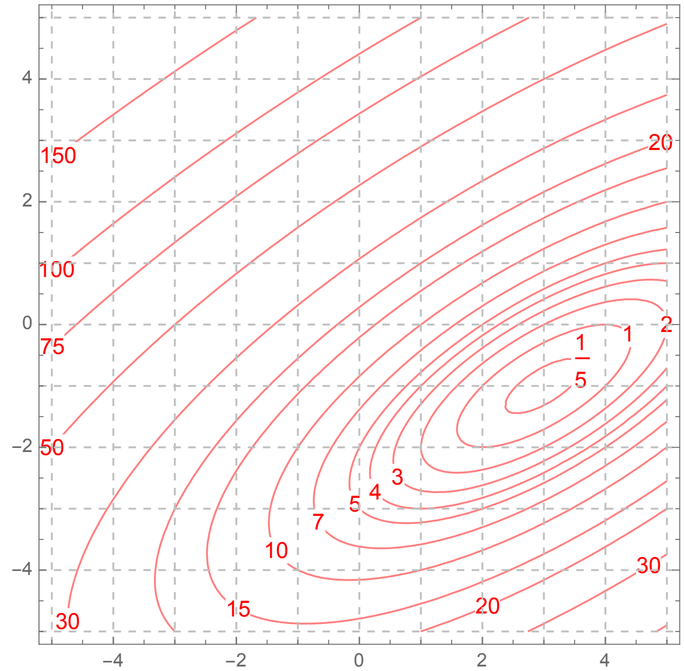
5. A bee is flying around $\mathbb{R}^3$ so that its position at time t, measured in seconds, is $\mathbf{x}(t) = (\cos 2t, 3t, \sin 2t)$ ft.

   a) (5.1) Compute the distance travelled by the bee from time 0 to time π.

   b) (5.1) Compute the acceleration of the bee at time 0.

   c) (5.4) Compute the curvature of the bee's flight path at time 0.

6. The contour plot of an unknown function $h : \mathbb{R}^2 \to \mathbb{R}$ is pictured below.



Use the contour plot to answer the following questions:

a) (6.1) Find the coordinates of all local minima of h, if there are any. (If there aren't any local minima in this viewing window, say so.)

b) (6.1) Find the coordinates of all saddles of h, if there are any. (If there aren't any saddles in this viewing window, say so.)

c) (4.2) Estimate $h_y(1, -4)$.

d) (4.5) Give the coordinates (x, y) of a point where $\nabla h(x, y)$ points southwest.

e) (4.5) Let $\mathbf{u} = \left(-\frac{3}{5}, \frac{4}{5}\right)$. Is $D_{\mathbf{u}}h(2, 3)$ positive, negative or zero?

7. (6.1) Find all critical points of the function $f(x, y) = x^3 - 2xy^2 - 6x$. Classify each critical point as a local maximum, local minimum or saddle.

8. (6.3) Find the minimum value of $f(x, y, z) = x \ln x + y \ln y + z \ln z$, subject to the constraint $x + y + z = 1$.

Solutions

1. First, we compute the linearization of $\mathbf{f}$. To do this, we first need the total derivative of $\mathbf{f}$ at $(0,0,0)$:

$$D\mathbf{f}(x,y,z) = \begin{pmatrix} (2\cos x)e^{2\sin x+5y-3z} & 5e^{2\sin x+5y-3z} & -3e^{2\sin x+5y-3z} \\ ze^{xz-z} & 0 & (x-1)e^{xz-z} \end{pmatrix}$$

so

$$D\mathbf{f}(0,0,0) = \begin{pmatrix} (2\cos 0)e^0 & 5e^0 & -3e^0 \\ 0e^0 & 0 & (0-1)e^0 \end{pmatrix} = \begin{pmatrix} 2 & 5 & -3 \\ 0 & 0 & -1 \end{pmatrix}.$$

Therefore the linearization of $\mathbf{f}$ at the origin is

$$\begin{aligned} \mathbf{L}(x,y,z) &= \mathbf{f}(0,0,0) + D\mathbf{f}(0,0,0)\begin{pmatrix} x-0 \\ y-0 \\ z-0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 & 5 & -3 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x-0 \\ y-0 \\ z-0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2x+5y-3z \\ -z \end{pmatrix} \\ &= \begin{pmatrix} 1+2x+5y-3z \\ 1-z \end{pmatrix} \\ &= (1+2x+5y-3z, 1-z). \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{f}\left(\frac{1}{10}, \frac{3}{10}, -\frac{1}{5}\right) \approx \mathbf{L}\left(\frac{1}{10}, \frac{3}{10}, -\frac{1}{5}\right) &= \left(1+2(\frac{1}{10})+5(\frac{3}{10})-3(-\frac{1}{5}), 1-(-\frac{1}{5})\right) \\ &= \boxed{\left(\frac{33}{10}, \frac{6}{5}\right)}. \end{aligned}$$

2. First, find the critical points of f: the gradient of f is $\nabla f(x,y) = (2(x-1)+2y, 2(y-2)+2x)$; set this equal to $\mathbf{0}$ and solve for x and y:

$$\begin{cases} 2(x-1)+2y &= 0 \\ 2(y-2)+2x &= 0 \end{cases} \Rightarrow \begin{cases} 2x+2y &= 2 \\ 2x+2y &= 4 \end{cases} \Rightarrow 2 = 4 \Rightarrow \text{ no solution}$$

This means that f has no critical points. Next, we find boundary critical points. Notice that the boundary ∂D consists of four pieces, so we have to find boundary critical points along each piece:

Left edge of ∂D: $x = 0, 0 \le y \le 3$: $f(0,y) = 1+(y-2)^2$, so $f'(0,y) = 2(y-2)$. Set this equal to 0 and solve for y to get $y = 2$, which is the point $(0,2)$.

Right edge of ∂D: $x = 3, 0 \le y \le 3$: $f(3,y) = 4 + (y-2)^2 + 6y$, so $f'(2,y) = 2(y-2)+6 = 2y+2$. Set this equal to 0 and solve for y to get $y = -1$, which isn't in D, so we throw it out.

Top of ∂D: $y = 3, 0 \le x \le 3$: $f(x,3) = (x-1)^2 + 1 + 6x$, so $f'(x,3) = 2(x-1)+6 = 2x+4$. Set this equal to 0 and solve for x to get $x = -2$, which isn't in D, so we throw it out.

Bottom of ∂D: $y = 0, 0 \le x \le 3$: $f(x,0) = (x-1)^2+4$, so $f'(x,0) = 2(x-1) = 2x-2$. Set this equal to 0 and solve for x to get $x = 1$, which is the point $(1,0)$.

Last, we test the critical points, the boundary critical points and the corners in the function f:

CP	none	
BCP	$(0,2)$	$f(0,2) = 1$
BCP	$(1,0)$	$f(1,0) = 4$
CORNER	$(0,0)$	$f(0,0) = 5$
CORNER	$(3,0)$	$f(3,0) = 8$
CORNER	$(0,3)$	$f(0,3) = 2$
CORNER	$(3,3)$	$f(3,3) = 23$

Therefore the absolute minimum value is $\boxed{1}$, occurring at $(0,2)$.

3. Define $f(x,y,z) = 2x^2 + 5y^2 + z^2$ so that the ellipsoid is the level surface at height 31. Then a normal vector to the tangent plane is $\mathbf{n} = \nabla f(-1,2,-3) = (4x, 10y, 2z)|_{(-1,2,-3)} = (-4,20,-6)$. So the tangent plane has equation

$$\begin{aligned} \mathbf{n}\cdot(\mathbf{x}-\mathbf{p}) &= 0 \\ (-4,20,-6)\cdot(x+1, y-2, z+3) &= 0 \\ -4(x+1)+20(y-2)-6(z+3) &= 0 \\ -4x+20y-6z &= 62. \end{aligned}$$

4. a) Since $f:\mathbb{R}^3 \to \mathbb{R}$, $Df(x,y,z)$ is 1×3.

   b) $\nabla f(x,y,z) = (f_x, f_y, f_z) = (6xyz^4 - 2z^2 + 6xy^5, 3x^2z^4 + 15x^2y^4, 12x^2yz^3 - 4xz)$.

   c) $\frac{\partial^3 f}{\partial x \partial z \partial x} = f_{xzx} = (6xyz^4 - 2z^2 + 6xy^5)_{zx} = (24xyz^3 - 4z)_x = 24yz^3$.

   d) First, we need a unit vector for the direction, so set $\mathbf{u} = \frac{(2,1,-2)}{||(2,1,-2)||} = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$. Next, the gradient at $(1,0,-1)$ is (using our answer to part (b))

$$(6xyz^4 - 2z^2 + 6xy^5, 3x^2z^4 + 15x^2y^4, 12x^2yz^3 - 4xz)|_{(1,0,-1)} = (-2,3,4).$$

Therefore

$$D_{\mathbf{u}}f(1,0,-1) = \nabla f(1,0,-1)\cdot \mathbf{u} = (-2,3,4)\cdot\left(\frac{2}{3},\frac{1}{3},-\frac{2}{3}\right) = \boxed{-3}.$$

5. A bee is flying around $\mathbb{R}^3$ so that its position at time t, measured in seconds, is $\mathbf{x}(t) = (\cos 2t, 3t, \sin 2t)$ ft.

   a) The speed of the bee is

   $$||\mathbf{v}(t)|| = ||\mathbf{x}'(t)|| = ||(-2\sin 2t, 3, 2\cos 2t)|| = \sqrt{4\sin^2 2t + 9 + 4\cos^2 2t} = \sqrt{13},$$

   so the distance travelled by the bee from time 0 to time π is

   $$\int_0^\pi ||\mathbf{v}(t)||\,dt = \int_0^\pi \sqrt{13}\,dt = t\sqrt{13}\Big|_0^\pi = \boxed{\pi\sqrt{13}\text{ ft}}.$$

   b) $\mathbf{a}(t) = \mathbf{x}''(t) = (-4\cos 2t, 0, -4\sin 2t)$, so $\mathbf{a}(0) = \boxed{(-4,0,0)\text{ ft/sec}^2}$.

   c) First, notice that the velocity at time 0 is $\mathbf{v}(0) = (-2\sin 0, 3, 2\cos 0) = (0,3,2)$. Therefore, applying our work in part (a) when we figured the speed, the curvature at time 0 is

   $$\begin{aligned}\kappa(0) = \frac{||\mathbf{v}(0)\times\mathbf{a}(0)||}{||\mathbf{v}(0)||^3} &= \frac{||(0,3,2)\times(-4,0,0)||}{\left(\sqrt{13}\right)^3}\\ &= \frac{||(0,-8,12)||}{13\sqrt{13}}\\ &= \frac{\sqrt{8^2+12^2}}{13\sqrt{13}}\\ &= \frac{\sqrt{208}}{13\sqrt{13}} = \boxed{\frac{4}{13}\text{ ft}^{-1}}.\end{aligned}$$

6. a) h has one local minimum in the viewing window at $\boxed{(3,-1)}$.

   b) h has $\boxed{\text{no saddles}}$ in the viewing window.

   c) $h_y(1,-4) \approx \frac{h(1,-3)-h(1,-4)}{-3-(-4)} = \frac{4-10}{1} = \boxed{-6}$. (Answers may vary, but should be between -6 and -10.)

   d) Anywhere where the direction of greatest increase in h is southwest works; for example, $\boxed{(-1,-4)}$ or $\boxed{(0,-3)}$.

   e) $D_{\mathbf{u}}h(2,3)$ is $\boxed{\text{positive}}$ because if you move from point $(2,3)$ in the direction $\mathbf{u}$, the values of h will increase.

7. Find all critical points of the function $f(x, y) = x^3 - 2xy^2 - 6x$. Classify each critical point as a local maximum, local minimum or saddle.

   Start by setting the gradient equal to $\mathbf{0}$ to find critical points: $\nabla f(x, y) = (3x^2 - 2y^2 - 6, -4xy)$, so we get the system

$$\begin{cases} 3x^2 - 2y^2 - 6 &= 0 \\ -4xy &= 0 \end{cases} \Rightarrow \quad x = 0 \text{ or } y = 0$$

   If $x = 0$, then by substituting into the first equation we get $-2y^2 - 6 = 0$ so $y^2 = -3$, which has no solution. However, if $y = 0$, then by substituting into the first equation we get $3x^2 - 6 = 0$ so $x^2 = 2$ so $x = \pm\sqrt{2}$. Thus there are two critical points: $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$.

   To classify the critical points, compute the Hessian:

$$Hf(x, y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6x & -4y \\ -4y & -4x \end{pmatrix}.$$

   Therefore:

$$Hf(\sqrt{2}, 0) = \begin{pmatrix} 6\sqrt{2} & 0 \\ 0 & -4\sqrt{2} \end{pmatrix}; \det Hf(\sqrt{2}, 0) = -48 < 0 \Rightarrow \boxed{(\sqrt{2}, 0) \text{ is a saddle}}.$$

$$Hf(-\sqrt{2}, 0) = \begin{pmatrix} -6\sqrt{2} & 0 \\ 0 & 4\sqrt{2} \end{pmatrix}; \det Hf(-\sqrt{2}, 0) = -48 < 0 \Rightarrow \boxed{(-\sqrt{2}, 0) \text{ is a saddle}}.$$

8. Use Lagrange's method, with constraint $g(x, y, z) = 1$ for $g(x, y, z) = x + y + z$. $\nabla f(x, y, z) = (f_x, f_y, f_z) = (\ln x + 1, \ln y + 1, \ln z + 1)$ and $\nabla g(x, y, z) = (1, 1, 1)$, so we obtain

$$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} \ln x + 1 &= \lambda(1) \\ \ln y + 1 &= \lambda(1) \\ \ln z + 1 &= \lambda(1) \end{cases}$$
$$\text{Constraint} \Rightarrow \quad x + y + z = 1$$

   It is apparent from the first three equations that $\ln x + 1 = \ln y + 1 = \ln z + 1$, so $\ln x = \ln y = \ln z$ and therefore $x = y = z$. From the constraint, we conclude $x = y = z = \frac{1}{3}$. Answering the question that was asked, we see that the minimum value of the utility f is

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3} \ln \frac{1}{3} + \frac{1}{3} \ln \frac{1}{3} + \frac{1}{3} \ln \frac{1}{3} = \boxed{\ln \frac{1}{3}}.$$

1.5 Spring 2018 Exam 2

1. a) (4.2) Find the total derivative of the function
$$\mathbf{f}(x, y, z) = \left(\frac{y}{x - z}, 2x^2 y\right).$$
   b) (4.3) Find an equation of the plane tangent to the surface
$$2x^3 z + (x - y)^2 - yz = 2$$
   at the point $(1, 3, 2)$.
   c) (4.4) Suppose $\sin(xy) + y^4 - 3x^5 y = 0$. Find $\frac{dy}{dx}$.

2. Let $f(x, y, z) = e^{xz+y}$.
   a) (4.3) Use linearization to estimate $f\left(\frac{1}{5}, \frac{1}{5}, \frac{-1}{5}\right)$.
   b) (4.5) Find the direction in which f is increasing most rapidly at the point $(\ln 2, \ln 3, 0)$.
   c) (4.2) Compute $\frac{\partial^3 f}{\partial x \partial y \partial x}$.

3. Suppose that the position of an object at time t is $\mathbf{x}(t) = (e^t + e^{-t}, e^t - e^{-t}, e^{2t})$.
   a) (5.1) Find the acceleration of the object at time t.
   b) (5.4) Find the curvature of the object's path at time $t = 0$.
   c) (5.1) Write a definite integral which, when evaluated, will compute the distance the object travels from time 2 to time 10. Your integral should not contain letters/variables other than t.

4. (6.1) Find the critical points of the function
$$f(x, y) = x^4 - 16xy + 8y^2 + 4.$$
   Classify, with justification, each critical point as a local maximum, local minimum, or saddle.

5. (6.2 or 6.3) Find the dimensions of the rectangular box with the greatest volume, if the box sits on the xy-plane with one vertex at the origin and the opposite vertex lying on the surface $z = 4 - x^2 - 4y^2$.

Solutions

1. a)

$$Df(x,y,z) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{-y}{(x-z)^2} & \frac{1}{x-z} & \frac{y}{(x-z)^2} \\ 4xy & 2x^2 & 0 \end{pmatrix}$$

b) Let $f(x,y,z) = 2x^3z+(x-y)^2-yz$. The gradient of f is $\nabla f = (6x^2z+2(x-y), -2(x-y)-z, 2x^3-y)$ and $\nabla f(1,3,2) = (12+2(-2), -2(-2)-2, 2-3) = (8,2,-1)$. The equation of the plane tangent to f at $(1,3,2)$ is therefore

$$\begin{aligned} (8,2,-1)\cdot(x-1, y-3, z-2) &= 0 \\ 8(x-1)+2(y-3)-(z-2) &= 0 \\ 8x+2y-z &= 12. \end{aligned}$$

c) Let $F(x,y) = \sin(xy)+y^4-3x^5y$. Then

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(y\cos xy - 15x^4y)}{x\cos xy + 4y^3 - 3x^5} = \frac{-y\cos xy + 15x^4y}{x\cos xy + 4y^3 - 3x^5}.$$

2. a) First, let $\mathbf{a} = (0,0,0)$. Next, the total derivative of f at $\mathbf{a}$ is

$$Df(\mathbf{a}) = \begin{pmatrix} ze^{xz+y} & e^{xz+y} & xe^{xz+y} \end{pmatrix}|_{(x,y,z)=(0,0,0)} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}.$$

Therefore, letting $\mathbf{x} = \left(\frac{1}{5}, \frac{1}{5}, \frac{-1}{5}\right)$, we have

$$\begin{aligned} f(\mathbf{x} &= f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x}-\mathbf{a}) \\ &= 1 + \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{-1}{5} \end{pmatrix} \\ &= 1 + \frac{1}{5} \\ &= \frac{6}{5}. \end{aligned}$$

b) This is $\nabla f(\ln 2, \ln 3, 0) = (ze^{xz+y}, e^{xz+y}, xe^{xz+y})|_{(x,y,z)=(\ln 2, \ln 3, 0)} = (0, 3, 3\ln 2)$.

c) $\frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}e^{xz+y}\right)\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(ze^{xz+y}\right)\right] = \frac{\partial}{\partial x}\left[ze^{xz+y}\right] = z^2e^{xz+y}$.

3. a) First, the velocity is $\mathbf{v}(t) = \mathbf{x}'(t) = (e^t - e^{-t}, e^t + e^{-t}, 2e^{2t})$. Thus the acceleration is $\mathbf{a}(t) = \mathbf{x}''(t) = (e^t + e^{-t}, e^t - e^{-t}, 4e^{2t})$.

b) At $t = 0$, the velocity is $(0, 2, 2)$ and the acceleration is $(2, 0, 4)$. Thus the curvature is

$$\begin{aligned}\kappa = \frac{||\mathbf{x}' \times \mathbf{x}''||}{||\mathbf{x}'||^3} &= \frac{||\mathbf{v} \times \mathbf{a}||}{||\mathbf{v}||^3} \\ &= \frac{||(8, 4, -4)||}{(\sqrt{0^2 + 2^2 + 2^2})^3} \\ &= \frac{\sqrt{8^2 + 4^2 + (-4)^2}}{(\sqrt{8})^3} \\ &= \frac{\sqrt{96}}{(\sqrt{8})^3} = \frac{\sqrt{12}}{8} = \frac{\sqrt{3}}{4}.\end{aligned}$$

c) The distance traveled, a.k.a. arc length, is

$$\begin{aligned}s &= \int_a^b ||\mathbf{v}(t)||\, dt \\ &= \int_2^{10} ||\left(e^t - e^{-t}, e^t + e^{-t}, 2e^{2t}\right)||\, dt \\ &= \int_2^{10} \sqrt{(e^t - e^{-t})^2 + (e^t + e^{-t})^2 + (2e^{2t})^2}\, dt \\ &= \int_2^{10} \sqrt{e^{2t} - 2 + e^{-2t} + e^{2t} + 2 + e^{-2t} + 4e^{4t}}\, dt \\ &= \int_2^{10} \sqrt{2e^{2t} + 2e^{-2t} + 4e^{4t}}\, dt.\end{aligned}$$

4. First, $\nabla f = (4x^3 - 16y, -16x + 16y)$. Setting $\nabla f = (0, 0)$, we get $4x^3 - 16y = 0$ and $-16x + 16y = 0$. From the second of these equations, $x = y$; substituting into the first equation we get $4x^3 - 16x = 0$, i.e. $4x(x^2 - 4) = 0$, i.e. $x = 0$, $x = 2$ and $x = -2$. Thus there are three critical points: $(0, 0)$, $(2, 2)$ and $(-2, -2)$.

   To test the critical points, compute the Hessian:

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x^2 & -16 \\ -16 & 16 \end{pmatrix}.$$

   Observe $Hf(2, 2) = Hf(-2, -2) = \begin{pmatrix} 48 & -16 \\ -16 & 16 \end{pmatrix}$ which is positive definite by the minors test, making $(2, 2)$ and $(-2, -2)$ local minima.

   Also $Hf(0, 0) = \begin{pmatrix} 0 & -16 \\ -16 & 16 \end{pmatrix}$. Since the determinant of the first principal minor is zero, the normal test doesn't work. However, you can show this matrix is neither positive definite nor negative definite by definition, via testing against some vectors:

$$(0,1)^T Hf(0,0)(0,1) = 16 \Rightarrow Hf(0,0) \text{ is not negative definite}$$

$$(1,1)^T Hf(0,0)(1,1) = -16 \Rightarrow Hf(0,0) \text{ is not positive definite}$$

Therefore $(0,0)$ is the location of a saddle.

5. Let $f(x,y,z) = xyz$ and let $g(x,y,z) = x^2 + 4y^2 + z$. We need to maximize f subject to $g(x,y,z) = 4$ so we can use Lagrange multipliers. The system $\nabla f = \lambda \nabla g$ becomes

$$\begin{cases} yz &= \lambda 2x \\ xz &= \lambda 8y \\ xy &= \lambda \end{cases}$$

Substituting the third equation into the second, we get $xz = xy(8y)$, i.e. $z = 8y^2$. Substituting the third equation into the first, we get $yz = xy(2x)$, i.e. $z = 2x^2$. Therefore $8y^2 = 2x^2$, so $4y^2 = x^2$ so $2y = x$. Finally, substitute $x = 2y$ and $z = 8y^2$ into the constraint $g(x,y,z) = 4$ to get

$$4y^2 + 4y^2 + 8y^2 = 4 \Rightarrow 16y^2 = 4 \Rightarrow y = \frac{1}{2}.$$

Thus $x = 2y = 1$ and $z = 8y^2 = 2$ so the dimensions of the largest box are $1 \times \frac{1}{2} \times 2$.