Old MATH 320 Exam 3s

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Contents

Chapter 1

General information about these exams

These are the Exam 3s I have given between 2018 and 2024 in Calculus 3 courses. To help give you some guidance on what questions are appropriate, each question on each exam is followed by a section number in parenthesis (like "(3.2)"). That means that question can be solved using material from that section (or from earlier sections) in the 2024 version of my *Vector Calculus Lecture Notes*.

Questions marked with TH were take-home questions where notes, calculators and *Mathematica* were allowed; other questions are closed-note with no calculators allowed.

1.1 Spring 2024 Exam 3

1. (7.5) TH Compute
$$\iiint_E yz\, dV$$
where $E \subseteq \mathbb{R}^3$ is the region of points (x, y, z) satisfying $1 \leq x^2 + y^2 \leq 4$, $y \geq 0$, $0 \leq z \leq x^2 + y^2$.

2. (8.5) TH Verify that Green's Theorem holds (by working out both sides of the formula in Green's Theorem and seeing that they work out to the same thing) in the situation where $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is the vector field $\mathbf{f}(x, y) = (x^2 y, x^2 - 2y)$ and $E \subseteq \mathbb{R}^2$ is the triangle with vertices $(0, 0)$, $(2, 4)$, and $(0, 4)$.

3. Compute each iterated integral:

   a) (7.3) $\displaystyle\int_0^5 \int_1^2 \left(12x^2 + 6y\right)\, dy\, dx$ b) (7.3) $\displaystyle\int_0^2 \int_x^{3x} 8xy\, dy\, dx$

4. Compute each iterated integral:

   a) (7.5) $\displaystyle\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2 + y^2)^{5/2}\, dy\, dx$ b) (7.3) $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\, dy\, dx$

5. (7.6) Compute the volume of the region in $\mathbb{R}^3$ consisting of the points (x, y, z) satisfying $0 \leq y \leq 1$, $0 \leq x \leq y$, and $x^2 \leq z \leq y^2$.

6. Let $\mathbf{f}(x, y, z) = (4xy, 2x^2 - 6yz^3, -9y^2 z^2)$.

   a) (8.2) Compute div $\mathbf{f}(1, 2, -1)$.

   b) (8.2) Show that curl $\mathbf{f} = \mathbf{0}$.

   c) (8.6) Find a potential function of $\mathbf{f}$.

   d) (8.6) Evaluate $\displaystyle\int_\gamma \mathbf{f} \cdot d\mathbf{s}$, where γ is parameterized by $\mathbf{x}(t) = (2\cos t, 2\sin t, \cos t + \sin t)$ for $0 \leq t \leq \pi$.

7. (8.4) The ceiling of an auditorium is modeled by the graph of $z = y^2 - \frac{1}{3}x^2 + 5$. If a drape is hung from the ceiling to the floor of the auditorium (i.e. the xy-plane) so that the drape is directly above the line segment on the floor going from $(0, 0)$ to $(3, 4)$, what is the area of the drape?

Solutions

1. In cylindrical coordinates, the region E can be described with the inequalities $1 \le r \le 2$, $0 \le \theta \le \pi$ and $0 \le z \le r^2$. Thus the integral becomes

$$\begin{aligned}
\iiint_E yz\,dV &= \int_0^\pi \int_1^2 \int_0^{r^2} (r\sin\theta) zr\,dz\,dr\,d\theta \\
&= \int_0^\pi \int_1^2 \int_0^{r^2} r^2 z \sin\theta\,dz\,dr\,d\theta \\
&= \int_0^\pi \int_1^2 \left[\frac{1}{2} r^2 \sin\theta z^2\right]_0^{r^2} dr\,d\theta \\
&= \int_0^\pi \int_1^2 \frac{1}{2} r^2 \sin\theta (r^2)^2\,dr\,d\theta \\
&= \int_0^\pi \int_1^2 \frac{1}{2} r^6 \sin\theta\,dr\,d\theta \\
&= \int_0^\pi \left[\frac{1}{14} r^7 \sin\theta\right]_1^2 d\theta \\
&= \int_0^\pi \frac{1}{14}(2^7-1)\sin\theta\,d\theta \\
&= \int_0^\pi \frac{127}{14}\sin\theta\,d\theta \\
&= -\frac{127}{14}\cos\theta\Big|_0^\pi = -\frac{127}{14}(-1-1) = \boxed{\frac{127}{7}}.
\end{aligned}$$

2. Green's Theorem asserts that if $\mathbf{f}(x,y) = (M,N)$, then $\iint_E (N_x - M_y)\,dA = \oint_{\partial E} M\,dx + N\,dy$. We start by working out the left-hand side of Green's Theorem:

$$\begin{aligned}
\iint_E (N_x - M_y)\,dA &= \int_0^4 \int_0^{y/2} \left[(x^2-2y)_x - (x^2y)_y\right] dx\,dy \\
&= \int_0^4 \int_0^{y/2} \left[2x - x^2\right] dx\,dy \\
&= \int_0^4 \int_0^{y/2} \left[x^2 - \frac{1}{3}x^3\right]_0^{y/2} dy \\
&= \int_0^4 \left(\frac{y^2}{4} - \frac{1}{24}y^3\right) dy \\
&= \left[\frac{1}{12}y^3 - \frac{1}{96}y^4\right]_0^4 = \frac{64}{12} - \frac{256}{96} = \frac{16}{3} - \frac{8}{3} = \boxed{\frac{8}{3}}.
\end{aligned}$$

Now for the right-hand side of Green's Theorem. Note that the boundary of E is the sum of three line segments:

- γ_1, the diagonal of the triangle, which is parameterized by $(2t, 4t)$ for $0 \leq t \leq 1$; on this path, $\mathbf{x}'(t) = (2, 4)$ so $dx = 2\,dt$ and $dy = 4\,dt$;
- γ_2, the top side of the triangle, which is parameterized from right to left by $(2-2t, 4)$ for $0 \leq t \leq 1$; on this path, $\mathbf{x}'(t) = (-2, 0)$ so $dx = -2\,dt$ and $dy = 0$;
- and γ_3, the left side of the triangle, which is parametrized by $(0, 4-4t)$ for $0 \leq t \leq 1$; on this path; $\mathbf{x}'(t) = (0, -4)$ so $dx = 0$ and $dy = -4dt$.

Therefore

$$\begin{aligned}
\oint_{\partial E} (M\,dx + N\,dy) &= \int_{\gamma_1} \left[(x^2y)\,dx + (x^2 - 2y)\,dy\right] \\
&\quad + \int_{\gamma_2} \left[(x^2y)\,dx + (x^2 - 2y)\,dy\right] \\
&\quad\quad + \int_{\gamma_3} \left(x^2y\right)dx + (x^2 - 2xy)\,dy\Big] \\
&= \int_0^1 \left[(2t)^2(4t)(2\,dt) + ((2t)^2 - 2(4t))(4\,dt)\right] \\
&\quad + \int_0^1 \left[(2-2t)^2(4)(-2\,dt) + ((2-2t)^2 - 4)(0)\right] \\
&\quad\quad + \int_0^1 \left[0^2(4-4t)(0) + (0^2 - 2(4-4t))(-4\,dt)\right] \\
&= \int_0^1 (32t^3 + 16t^2 - 32t)\,dt + \int_0^1 -32(1-t)^2\,dt + \int_0^1 32(1-t)\,dt \\
&= \left[8t^4 + \frac{16}{3}t^3 - 16t^2\right]_0^1 + \left[\frac{32}{3}(1-t)^3\right]_0^1 + \left[-16(1-t)^2\right]_0^1 \\
&= -\frac{8}{3} - \frac{32}{3} + 16 = \boxed{\frac{8}{3}}.
\end{aligned}$$

3. a)

$$\begin{aligned}
\int_0^5 \int_1^2 \left(12x^2 + 6y\right) dy\,dx &= \int_0^5 \left[12x^2y + 3y^2\right]_1^2 dx \\
&= \int_0^5 \left[24x^2 + 12 - 12x^2 - 3\right] dx \\
&= \int_0^5 \left(12x^2 + 9\right) dx \\
&= \left[4x^3 + 9x\right]_0^5 = 4(125) + 9(5) = \boxed{545}.
\end{aligned}$$

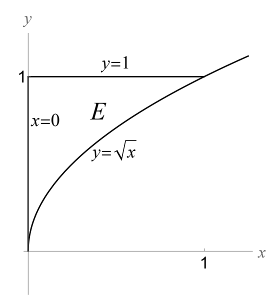
b)

$$\int_0^2 \int_x^{3x} 8xy\,dy\,dx = \int_0^2 \left[4xy^2\right]_x^{3x}\,dx$$
$$= \int_0^2 \left[4x(3x)^2 - 4x(x^2)\right]\,dx$$
$$= \int_0^2 \left[36x^3 - 4x^3\right]\,dx$$
$$= \int_0^2 32x^3 = 8x^4\Big|_0^2 = 8(2^4) = \boxed{128}.$$

4. a) Change this integral to polar coordinates (this is a double integral over a circle of radius 2):

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2+y^2)^{5/2}\,dy\,dx = \int_0^{2\pi} \int_0^2 r^5 r\,dr\,d\theta$$
$$= \int_0^{2\pi} \int_0^2 r^6\,dr\,d\theta$$
$$= 2\pi \cdot \frac{1}{7}r^7\Big|_0^2 = 2\pi\frac{2^7}{7} = \boxed{\frac{256}{7}\pi}.$$

b) Change the order of integration first; this is a double integral over the region E shown here:



and E can also be described by $0 \le y \le 1$, $0 \le x \le y^2$. This means

$$\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\,dy\,dx = \int_0^1 \int_0^{y^2} e^{y^3}\,dx\,dy = \int_0^1 \left[e^{y^3}x\right]_0^{y^2}\,dy = \int_0^1 e^{y^3}y^2\,dy.$$

Now use the u-substitution $u = y^3$, $du = 3y^2\,dy$ to get

$$\int_0^1 e^u \frac{1}{3}\,du = \frac{1}{3}e^u\Big|_0^1 = \boxed{\frac{1}{3}(e-1)}.$$

5. This volume is given by

$$\begin{aligned}\iiint_E 1\,dV &= \int_0^1 \int_0^y \int_{x^2}^{y^2} 1\,dz\,dx\,dy \\ &= \int_0^1 \int_0^y (y^2 - x^2)\,dx\,dy \\ &= \int_0^1 \left[y^2 x - \frac{1}{3}x^3\right]_0^y dy = \int_0^1 \frac{2}{3}y^3\,dy = \left[\frac{1}{6}y^4\right]_0^1 = \boxed{\frac{1}{6}}.\end{aligned}$$

6. a) By direct calculation,

$$\begin{aligned}\text{div}\,\mathbf{f}(x,y,z) &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \\ &= (4xy)_x + (2x^2 - 6yz^3)_y + (-9y^2z^2)_z \\ &= 4y - 6z^3 - 18y^2 z.\end{aligned}$$

Therefore div $\mathbf{f}(1,2,-1) = 4(2) - 6(-1)^3 - 18(2^2)(-1) = 8 + 6 + 72 = \boxed{86}$.

b) By direct calculation,

$$\begin{aligned}\text{curl}\,\mathbf{f} &= ((f_3)_y - (f_2)_z, (f_1)_z - (f_3)_x, (f_2)_x - (f_1)_y) \\ &= \left(-18yz^2 + 18yz^2, 0 - 0, 4x - 4x\right) = \boxed{(0,0,0)}.\end{aligned}$$

c) To find a potential function $f : \mathbb{R}^3 \to \mathbb{R}$, i.e. a function f so that $\mathbf{f} = \nabla f$, we set

$$\begin{cases} f_x = 4xy & \Rightarrow f(x,y,z) = 2x^2y + A(y,z) \\ f_y = 2x^2 - 6yz^3 & \Rightarrow f(x,y,z) = 2x^2y - 3y^2z^3 + B(x,z) \\ f_z = -9y^2z^2 & \Rightarrow f(x,y,z) = -3y^2z^3 + C(x,y) \end{cases}$$

To reconcile all these, let $A(y,z) = -3y^2z^3$, $B(x,z) = 0$ and $C(x,y) = 2x^2y$. That makes $f(x,y,z) = \boxed{2x^2y - 3y^2z^3}$.

d) Since curl $\mathbf{f} = \mathbf{0}$, $\mathbf{f}$ is conservative and therefore has path-independent line integrals. So by the FTLI,

$$\int_\gamma \mathbf{f} \cdot d\mathbf{s} = \int_\gamma \nabla f \cdot d\mathbf{s} = f(\mathbf{x}(b)) - f(\mathbf{x}(a))$$

where f is as in part (c), $\mathbf{x}(a) = \mathbf{x}(0) = (2\cos 0, 2\sin 0, \cos 0 + \sin 0) = (2,0,1)$ and $\mathbf{x}(b) = (2\cos\pi, 2\sin\pi, \cos\pi + \sin\pi) = (-2,0,-1)$. Thus the answer is

$$\begin{aligned}\int_\gamma \nabla f \cdot d\mathbf{s} &= f(\mathbf{x}(b)) - f(\mathbf{x}(a)) = f(-2,0,-1) - f(2,0,1) \\ &= \left[2(2^2)(0) - 3(0^2)(-1)\right] - \left[2(2^2)0 - 3(0^2)(1)\right] = \boxed{0}.\end{aligned}$$

7. This area is given by the line integral $\int_\gamma f\,ds$ where $f(x,y) = y^2 - \frac{1}{3}x^2 + 5$ and γ is the line segment parametrized by $\mathbf{x}(t) = (3t, 4t)$ for $0 \le t \le 1$. This integral is

$$\begin{aligned}
\int_\gamma f\,ds &= \int_0^1 f(\mathbf{x}(t))||\mathbf{x}'(t)||\,dt \\
&= \int_0^1 f(3t,4t)||(3,4)||\,dt \\
&= \int_0^1 \left((4t)^2 - \frac{1}{3}(3t)^2 + 5\right)\sqrt{3^2+4^2}\,dt \\
&= 5\int_0^1 \left(13t^2+5\right)\,dt = 5\left[\frac{13}{3}t^3 + 5t\right]_0^1 = 5\left[\frac{13}{3}+5\right] = 5\left(\frac{28}{3}\right) = \boxed{\frac{140}{3}}.
\end{aligned}$$

1.2 Fall 2021 Exam 3

1. (7.6) TH Find the volume of the solid in $\mathbb{R}^3$ consisting of the points lying inside the cylinder $(x-1)^2+y^2=1$, above the xy-plane, and under the graph of $f(x,y)=\sqrt{x^2+y^2}$.

2. (8.5) TH Compute the area enclosed by the curve parametrized by

$$\mathbf{x}(t) = (t^2 - t^4, t^2 - t^8)$$

   where $0 \le t \le 1$.

3. Throughout this problem, let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field

$$\mathbf{f}(x,y,z) = (2x - z, 3z, -2xy + 3z^3).$$

   a) (8.2) Compute the divergence of $\mathbf{f}$ at the point $(1,2,3)$.
   b) (8.2) What does the sign of your answer to part (a) tell you about behavior of the vector field at $(1,2,3)$?
   c) (8.2) Compute curl $\mathbf{f}$.
   d) (8.4) Rewrite the line integral $\int_\gamma \mathbf{f} \cdot d\mathbf{s}$ as a Riemann integral, where γ is the straight line segment starting at the origin and ending at $(-5,-2,1)$. (The only variable allowed in your answer is t.)

4. Let $A = [0,5] \times [0,2]$ and let $B = [0,5] \times [2,4]$, and suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that $\iint_A f(x,y)\, dA = 7$ and $\iint_B f(x,y) = -2$.

   a) (7.2) Compute $\iint_{A\cup B} f(x,y)\, dA$.
   b) (7.2) Compute $\iint_{A\cap B} f(x,y)\, dA$.
   c) (7.2) Compute $\iint_A 2f(x,y)\, dA$.
   d) (7.2) Compute $\iint_A [f(x,y)+3]\, dA$.
   e) (7.4) Compute the average value of f on A.

5. (7.5) Compute

$$\iiint_E y\, dV$$

   where E is the triangular pyramid in $\mathbb{R}^3$ consisting of points (x,y,z) satisfying $x \ge 0$, $y \ge 0$, $z \ge 0$ and $2x + 2y + z \le 2$.

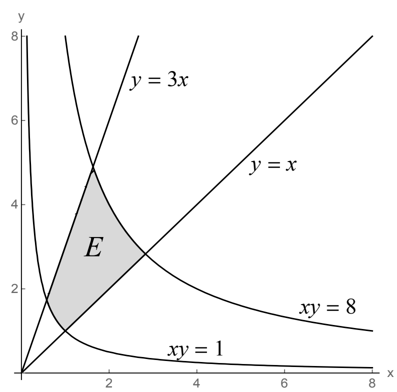
6. Throughout this problem, let $f(x,y) = x^2 + 3y^2$.

   a) (7.3) Compute $\iint_E f(x,y)\, dA$, where E is the rectangle with vertices $(0,0)$, $(3,0)$, $(0,1)$ and $(3,1)$.

b) (8.4) Compute $\int_\gamma f\,ds$, where γ is the circle of radius 2 centered at the origin, oriented counterclockwise.

7. (7.3) Compute $\iint_E e^x\,dA$, where E is the triangle with vertices $(0,0)$, $(2,0)$ and $(4,2)$.

8. (7.5) Compute
$$\iint_E y^2\,dA$$
where E is the region pictured below:



Solutions

1. The base of the solid is a circle of radius 1 centered at $(1,0)$, which has polar equation $r = 2\cos\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Using this, the solid can be described in cylindrical coordinates by the inequalities $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $0 \le r \le 2\cos\theta$, $0 \le z \le r$. So the volume of the solid E is

$$\begin{aligned}\iiint_E 1\,dV &= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\int_0^r r\,dz\,dr\,d\theta \\ &= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} \frac{8}{3}\cos^3\theta\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} \frac{8}{3}\cos^2\theta\cos\theta\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} \frac{8}{3}(1-\sin^2\theta)\cos\theta\,d\theta.\end{aligned}$$

Now use the u-sub $u = \sin\theta$, $du = \cos\theta\,d\theta$ to get

$$\begin{aligned}\int_{-1}^{1} \frac{8}{3}(1-u^2)\,du &= \frac{8}{3}\left[u - \frac{1}{3}u^3\right]_{-1}^{1} \\ &= \frac{8}{3}\left[1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)\right] = \frac{8}{3}\left[\frac{4}{3}\right] = \boxed{\frac{32}{9}}.\end{aligned}$$

2. Use Green's Theorem with $\mathbf{f}(x,y) = (0,x)$ to get

$$area(E) = \oint_{\partial E} x\,dy.$$

Since the curve is parametrized by $x = t^2 - t^4$, $y = t^2 - t^8$, we have $dy = (2t - 8t^7)\,dt$ so we obtain

$$\begin{aligned}area(E) &= \int_0^1 (t^2 - t^4)(2t - 8t^7)\,dt \\ &= \int_0^1 (2t^3 - 2t^5 - 8t^9 + 8t^{11})\,dt \\ &= \left[\frac{1}{2}t^4 - \frac{1}{3}t^6 - \frac{4}{5}t^{10} + \frac{2}{3}t^{12}\right]_0^1 \\ &= \frac{1}{2} - \frac{1}{3} - \frac{4}{5} + \frac{2}{3} = \boxed{\frac{1}{30}}.\end{aligned}$$

3. a) div $\mathbf{f}(x,y,z) = (f_1)_x + (f_2)_y + (f_3)_z = 2 + 0 + 9z^2$, so div $\mathbf{f}(1,2,3) = 2 + 9(3^2) = \boxed{83}$.

b) Since div $\mathbf{f}(1,2,3)$ is positive, the arrow in the picture of f ending at $(1,2,3)$ is shorter than the arrow in the picture of f starting at $(1,2,3)$, i.e. the vector field has more net flow out of $(1,2,3)$ than in, i.e. the vector field is "spreading out" at $(1,2,3)$.

c) By the usual formula for curl,

$$\begin{aligned}\nabla \times \mathbf{f} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - z & 3z & -2xy + 3z^3 \end{pmatrix} \\ &= (-2x - 3, -1 - (-2y), 0 - 0) \\ &= \boxed{(-2x - 3, 2y - 1, 0)}.\end{aligned}$$

d) γ is parametrized by $\mathbf{x}(t) = (-5t, -2t, t)$ for $0 \le t \le$; we have $d\mathbf{s} = (-5, -2, 1)\, dt$ so the integral becomes

$$\begin{aligned}\int_\gamma \mathbf{f} \cdot d\mathbf{s} &= \int_0^1 (2x - z, 3z, -2xy + 3z^3) \cdot (-5, -2, 1)\, dt \\ &= \int_0^1 (-10t - t, 3t, -2(-5t)(-2t) + 3t^3) \cdot (-5, -2, 1)\, dt \\ &= \int_0^1 (55t - 6t + 3t^3 - 20t^2)\, dt \\ &= \boxed{\int_0^1 (49t - 20t^2 + 3t^3)\, dt}.\end{aligned}$$

4. a) $\iint_{A \cup B} f(x,y)\, dA = \iint_A f(x,y)\, dA + \iint_B f(x,y)\, dA = 7 + (-2) = \boxed{5}$.

b) $\iint_{A \cap B} f(x,y)\, dA = \boxed{0}$ since $A \cap B$ has zero area.

c) $\iint_A 2f(x,y)\, dA = 2 \iint_A f(x,y)\, dA = 2(7) = \boxed{14}$.

d) $\iint_A [f(x,y) + 3]\, dA = \iint_A f(x,y)\, dA + \iint_A 3\, dA = 7 + 3 \cdot area(A) = 7 + 3(5)2 = \boxed{37}$.

e) The average value of f on A is $\frac{1}{area(A)} \iint_A f(x,y)\, dA = \frac{1}{10}(7) = \boxed{\frac{7}{10}}$.

5. The solid E can be described by the inequalities $0 \le y \le 1$, $0 \le x \le 1 - y$,

$0 \le z \le 2 - 2x - 2y$, so the integral is

$$\begin{aligned}
\iiint_E y\,dV &= \int_0^1 \int_0^{1-y} \int_0^{2-2x-2y} y\,dz\,dx\,dy \\
&= \int_0^1 \int_0^{1-y} (2y - 2xy - 2y^2)\,dx\,dy \\
&= \int_0^1 \left[2xy - x^2y - 2xy^2\right]_0^{1-y}\,dy \\
&= \int_0^1 \left[2(1-y)y - (1-y)^2y - 2(1-y)y^2\right]\,dy \\
&= \int_0^1 \left[y^3 - 2y^2 + y\right]\,dy \\
&= \left[\frac{1}{4}y^4 - \frac{2}{3}y^3 + \frac{1}{2}y^2\right]_0^1 = \frac{1}{4} - \frac{3}{2} + \frac{1}{2} = \boxed{\frac{1}{12}}.
\end{aligned}$$

6. a) Apply Fubini's Theorem:

$$\begin{aligned}
\iint_E f(x,y)\,dA &= \int_0^3 \int_0^1 (x^2 + 3y^2)\,dy\,dx \\
&= \int_0^3 \left[x^2y + y^3\right]_0^1\,dx \\
&= \int_0^3 \left[x^2 + 1\right]\,dx \\
&= \left[\frac{1}{3}x^3 + x\right]_0^3 = \frac{1}{3}(27) + 3 - 0 = \boxed{12}.
\end{aligned}$$

b) Here, parametrize γ by $\mathbf{x}(t) = (2\cos t, 2\sin t)$ for $0 \le t \le 2\pi$. We have $||\mathbf{x}'(t)|| = ||(-2\sin t, 2\cos t)|| = \sqrt{4\sin^2 t + 4\cos^2 t} = \sqrt{4} = 2$, so the integral becomes

$$\begin{aligned}
\int_\gamma f\,ds &= \int_0^{2\pi} (4\cos^2 t + 12\sin^2 t)2\,dt \\
&= \int_0^{2\pi} (8 + 16\sin^2 t)\,dt \\
&= \int_0^{2\pi} (8 + 8(1 - \cos 2t)\,dt \\
&= \int_0^{2\pi} (16 - 8\cos 2t)\,dt \\
&= [16t - 4\sin 2t]_0^{2\pi} = \boxed{32\pi}.
\end{aligned}$$

7. The triangle can be described by the inequalities $0 \le y \le 2$, $2y \le x \le y + 2$,

so the integral becomes

$$\begin{aligned}\iint_F f(x,y)\,dA &= \int_0^2 \int_{2y}^{y+2} e^x\,dx\,dy \\ &= \int_0^2 \left[e^{y+2} - e^{2y}\right]\,dy \\ &= \left[e^{y+2} - \frac{1}{2}e^{2y}\right]_0^2 \\ &= e^4 - \frac{1}{2}e^4 - \left[e^2 - \frac{1}{2}\right] = \boxed{\frac{1}{2}e^4 - e^2 + \frac{1}{2}}.\end{aligned}$$

8. Use the change of variable $(x,y) \overset{\varphi}{\mapsto} (u,v)$ given by $u = xy$, $v = \frac{y}{x}$, so that $\varphi(E) = \{(u,v) : 1 \le u \le 8, 1 \le v \le 3\}$. The Jacobian of φ is

$$J(\varphi) = \det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \det\begin{pmatrix} y & x \\ \frac{-y}{x^2} & \frac{1}{x} \end{pmatrix} = \frac{y}{x} + \frac{y}{x} = \frac{2y}{x} = 2v.$$

Next, solve for x and y in terms of u and v: $v = \frac{y}{x}$ gives $vx = y$, so $u = xvx$ so $x = \sqrt{\frac{u}{v}}$ and $y = vx = \sqrt{uv}$. So $y^2 = uv$ and the change of variable formula therefore gives

$$\begin{aligned}\iint_E y^2\,dA &= \int_1^8 \int_1^3 uv\frac{1}{|2v|}\,dv\,du \\ &= \int_1^8 \int_1^3 \frac{1}{2}u\,dv\,du \quad \text{(since } v > 0 \text{ we can disregard the } |\cdot|\text{)} \\ &= \int_1^8 \left[\frac{1}{2}uv\right]_1^3\,du \\ &= \int_1^8 u\,du = \frac{1}{2}u^2\Big|_1^8 = 32 - \frac{1}{2} = \boxed{\frac{63}{2}}.\end{aligned}$$

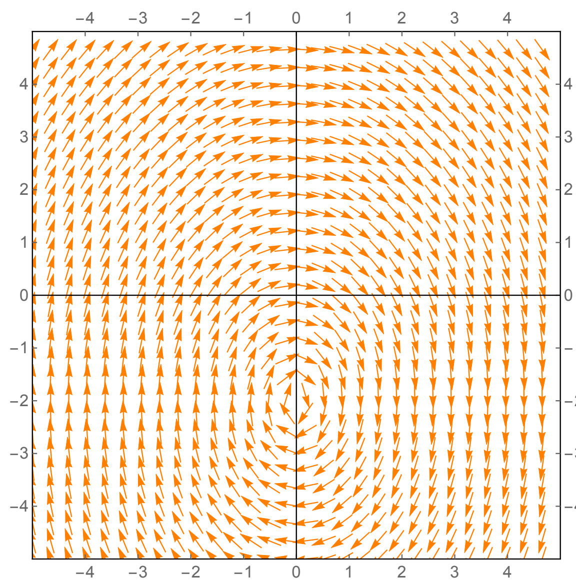
1.3 Spring 2021 Exam 3

1. (7.5) TH Compute the volume of the solid consisting of the points $(x, y, z) \in \mathbb{R}^3$ lying above the xy-plane, lying outside the cone $z^2 = x^2 + y^2$, but inside the sphere $x^2 + y^2 + z^2 = 4$.

2. (7.3) TH Compute the exact value of this iterated integral:

$$\int_1^2 \int_{1/2}^{1/x} 2xe^{\left(y+\frac{1}{y}\right)}\, dy\, dx.$$

3. The picture of some unknown vector field $\mathbf{f}$ on $\mathbb{R}^2$ is shown here:



   a) (8.1) On the picture above, sketch the flow line to $\mathbf{f}$ passing through the point $(1, 0)$.
   b) Use the picture to answer the following questions :
      i. (8.2) Is div $\mathbf{f}(3, 2)$ positive, negative or zero?
      ii. (8.2) If you thought of $\mathbf{f}$ as a vector field on $\mathbb{R}^3$ by setting its z-coordinate equal to 0, would the z-coordinate of curl $\mathbf{f}(0, 0, 0)$ be positive, negative or zero?
      iii. (8.4) Let γ be the line segment with initial point $(0, 3)$ and terminal point $(3, 0)$. Is $\int_\gamma \mathbf{f} \cdot d\mathbf{s}$ positive, negative or zero?

4. (8.2) Compute the divergence of the vector field $\mathbf{f}(x, y) = (3e^{2x-y}, 4e^{x+5y})$.

5. (8.5) Compute the area of the region in $\mathbb{R}^2$ bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{25} = 1$.

6. (7.4) Compute $\int_0^2 \int_y^2 \int_0^1 (6y + 8xz)\ dz\, dx\, dy$.

7. (8.4) Compute $\int_\gamma f\, ds$, where $f(x, y) = 4xy$ and γ is the line segment starting at $(1, 1)$ and ending at $(5, -2)$.

8. (7.5) Compute $\int_0^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} x^2\, dy\, dx$.

9. (7.2) Compute the volume of the solid consisting of the points $(x, y, z) \in \mathbb{R}^3$ satisfying the inequalities $0 \le z \le 12x^2y$, $x^2 \le y \le x$, and $0 \le x \le 1$.

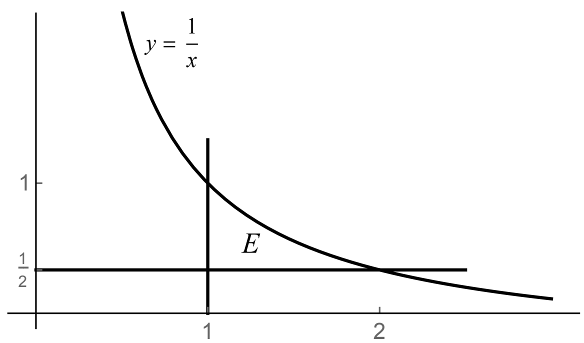
10. (7.3) Compute $\iint_E y^2\, dA$, where E is the triangle with vertices $(6, 0)$, $(6, 3)$ and $(0, 3)$.

Solutions

1. The solid S can be described in spherical coordinates as $0 \leq \rho \leq 2$ (since the sphere is $\rho = 2$), $0 \leq \theta \leq 2\pi$ and $\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}$ (since the cone is $\varphi = \frac{\pi}{4}$ and the xy-plane is $\varphi = \frac{\pi}{2}$). So the volume is

$$\begin{aligned}\iiint_S 1\,dV &= \int_0^{2\pi}\int_0^2\int_{\pi/4}^{\pi/2} \rho^2 \sin\varphi\,d\varphi\,d\rho\,d\theta \\ &= \int_0^{2\pi}\int_0^2 -\rho^2\cos\varphi\Big|_{\pi/4}^{\pi/2}\,d\rho\,d\theta \\ &= \int_0^{2\pi}\int_0^2 -\rho^2\left(0 - \frac{\sqrt{2}}{2}\right)d\rho\,d\theta \\ &= \int_0^{2\pi} \frac{\sqrt{2}}{2}\cdot\frac{\rho^3}{3}\Big|_0^2\,d\theta \\ &= \int_0^{2\pi} \frac{4}{3}\sqrt{2}\,d\theta \\ &= 2\pi\cdot\frac{4}{3}\sqrt{2} = \boxed{\frac{8}{3}\pi\sqrt{2}}.\end{aligned}$$

2. This integral is $\iint_E 2xe^{\left(y+\frac{1}{y}\right)}\,dA$, where E is as shown here:



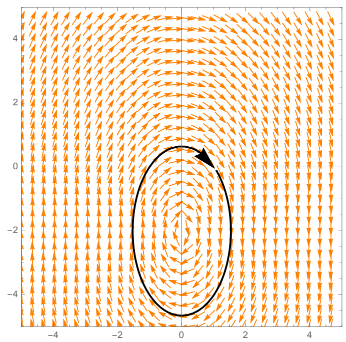
The curve $y = \frac{1}{x}$ can also be written as $x = \frac{1}{y}$. So by using Fubini's theorem to reverse the order of integration, we see that the given double integral is also

$$\begin{aligned}\int_{1/2}^1\int_1^{1/y} 2xe^{\left(y+\frac{1}{y}\right)}\,dx\,dy &= \int_{1/2}^1 \left[x^2 e^{\left(y+\frac{1}{y}\right)}\right]_1^{1/y}\,dy \\ &= \int_{1/2}^1 \left(\frac{1}{y^2} - 1\right)e^{\left(y+\frac{1}{y}\right)}\,dy.\end{aligned}$$

Now perform the u-substitution $u = y + \frac{1}{y}$, $du = \left(1 - \frac{1}{y^2}\right)dy$ to get

$$\int_{5/2}^2 -e^u\,du = -e^u\Big|_{5/2}^2 = \boxed{-e^2 + e^{5/2}}.$$

3. a) The flow line is an ellipse:



b) Use the picture to answer the following questions :

i. div $\mathbf{f}(3,2)$ is $\boxed{\text{zero}}$, since the arrows entering and leaving $(3,2)$ appear to have the same length.

ii. curl $\mathbf{f}(0,0,0)$ is $\boxed{\text{negative}}$, since the vector field rotates clockwise (or by the right-hand rule).

iii. Since γ predominantly goes in the same direction as $\mathbf{f}$, $\int_\gamma \mathbf{f} \cdot d\mathbf{s}$ is $\boxed{\text{positive}}$.

4. Write the coordinates of $\mathbf{f}$ as f_1 and f_2. Then

$$\text{div } \mathbf{f}(x,y) = (f_1)_x + (f_2)_y = \boxed{6e^{2x-y} + 20e^{x+5y}}.$$

5. Let E be the region whose area we want; the boundary ∂E is parametrized by $x = 3\cos t, y = 5\sin t$ for $0 \leq t \leq 2\pi$. This means $dx = -3\sin t\, dt$ and $dy = 5\cos t\, dt$. So by the area formula coming from Green's Theorem,

$$\begin{aligned}
area(E) = \iint_E 1\, dA &= \frac{1}{2}\oint_{\partial E} (x\, dy - y\, dx) \\
&= \frac{1}{2}\int_0^{2\pi} ((3\cos t)dy - (5\sin t)\, dx) \\
&= \frac{1}{2}\int_0^{2\pi} (3\cos t(5\cos t) - 5\sin t(-3\sin t))\ dt \\
&= \frac{1}{2}\int_0^{2\pi} \left(15\cos^2 t + 15\sin^2 t\right)\ dt \\
&= \frac{1}{2}\int_0^{2\pi} 15\, dt = \frac{1}{2}(15)2\pi = \boxed{15\pi}.
\end{aligned}$$

6. This is a direct computation:

$$\begin{aligned}\int_0^2 \int_y^2 \int_0^1 (6y + 8xz)\; dz\, dx\, dy &= \int_0^2 \int_y^2 \left[6yz + 4xz^2\right]_0^1 \; dx\, dy \\ &= \int_0^2 \int_y^2 (6y + 4x)\; dx\, dy \\ &= \int_0^2 \left[6yx + 2x^2\right]_y^2 \; dy \\ &= \int_0^2 \left[(12y + 8) - (6y^2 + 2y^2)\right]\; dy \\ &= \int_0^2 (-8y^2 + 12y + 8)\, dy \\ &= \left[-\frac{8}{3}y^3 + 6y^2 + 8y\right]_0^2 \\ &= -\frac{64}{3} + 24 + 16 = \boxed{\frac{56}{3}}.\end{aligned}$$

7. Parametrize γ by $\mathbf{x}(t) = (x(t), y(t)) = (1 + 4t, 1 - 3t)$ for $0 \le t \le 1$. Then, $||\mathbf{x}'(t)|| = ||\,(4, -3)\,|| = 5$. So the line integral is

$$\begin{aligned}\int_\gamma f\, ds &= \int_0^1 f\,(1 + 4t, 1 - 3t)\; ||\mathbf{x}'(t)||\, dt \\ &= \int_0^1 4(1 + 4t)(1 - 3t)5\, dt \\ &= \int_0^1 20(1 + t - 12t^2)\, dt \\ &= 20t + 10t^2 - 80t^3\Big|_0^1 = 20 + 10 - 80 = \boxed{-50}.\end{aligned}$$

8. Change to polar coordinates (this is the right-half of a circle of radius 5 centered at the origin):

$$\begin{aligned}\int_0^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} x\, dy\, dx &= \int_{-\pi/2}^{\pi/2} \int_0^5 (r\cos\theta) r\, dr\, d\theta \\ &= \int_{-\pi/2}^{\pi/2} \int_0^5 r^2 \cos\theta\, dr\, d\theta \\ &= \int_{-\pi/2}^{\pi/2} \frac{1}{3} r^3 \cos\theta\Big|_0^5 \; d\theta \\ &= \frac{125}{3} \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta \\ &= \frac{125}{3}\, \sin\theta|_{-\pi/2}^{\pi/2} = \frac{125}{3}(1 - (-1)) = \boxed{\frac{250}{3}}.\end{aligned}$$

9. This is a direct computation, starting with either a triple integral in the first line of this solution, or the double integral in the third line:

$$\begin{aligned}\iiint_S 1\,dV &= \int_0^1 \int_{x^2}^x \int_0^{12x^2y} 1\,dz\,dy\,dx \\ &= \int_0^1 \int_{x^2}^x z\Big|_0^{12x^2y}\,dy\,dx \\ &= \int_0^1 \int_{x^2}^x 12x^2y\,dy\,dx \\ &= \int_0^1 6x^2y^2\Big|_{x^2}^x\,dx \\ &= \int_0^1 (6x^4 - 6x^6)\,dx \\ &= \frac{6}{5}x^5 - \frac{6}{7}x^7\Big|_0^1 = \frac{6}{5} - \frac{6}{7} = \boxed{\frac{12}{35}}.\end{aligned}$$

10. The region is bounded by the vertical line $x = 6$, the horizontal line $y = 3$ and the diagonal line $x + 2y = 6$, i.e. $x = 6 - 2y$, i.e. $y = 6 - \frac{1}{2}x$. This integral can be done in either order, but I'll do it $dx\,dy$ to avoid using a slope of $-\frac{1}{2}$ in the line.

$$\begin{aligned}\int_0^3 \int_{6-2y}^6 y^2\,dx\,dy &= \int_0^3 y^2x\Big|_{6-2y}^6\,dy \\ &= \int_0^3 \left[6y^2 - y^2(6-2y)\right]\,dy \\ &= \int_0^3 2y^3\,dy = \frac{1}{2}y^4\Big| 0^3 = \boxed{\frac{81}{2}}.\end{aligned}$$

1.4 Fall 2020 Exam 3

NOTE: This exam did not cover Chapter 8. In Fall 2020, that chapter was skipped due to disruptions to the course schedule related to the COVID-19 pandemic.

1. (7.5) TH Compute
$$\iint_E x^2 \, dA$$
where E is the region shown bounded by the x-axis, the lines $x + y = 1$, $x + y = 3$, and $y = 4x$.

2. (7.6) TH Compute the volume of the solid consisting of the points $(x, y, z) \in \mathbb{R}^3$ lying above the xy-plane, outside the cylinder $x^2 + y^2 = 16$ but inside the sphere $x^2 + y^2 + z^2 = 36$.

3. (7.2) Suppose $g : \mathbb{R}^2 \to \mathbb{R}$ is a function such that
$$\int_0^2 \int_0^2 g(x, y) \, dy \, dx = 5; \qquad \int_0^2 \int_0^4 g(x, y) \, dy \, dx = 8;$$
$$\int_0^4 \int_0^2 g(x, y) \, dy \, dx = 3; \qquad \int_0^4 \int_0^4 g(x, y) \, dy \, dx = 10.$$
Use this given information to evaluate each quantity:
   a) $\int_0^2 \int_0^4 g(x, y) \, dx \, dy$
   b) $\int_2^4 \int_2^4 g(x, y) \, dy \, dx$
   c) $\int_0^4 \int_0^2 3g(x, y) \, dx \, dy$
   d) $\int_0^4 \int_2^2 g(x, y) \, dy \, dx$
   e) $\int_0^2 \int_0^2 (2 + 5g(x, y)) \, dy \, dx$

4. (7.5) Compute
$$\iiint_E (x^2 + y^2 + z^2)^2 \, dV$$
where E is the sphere of radius 2 centered at the origin.

5. Let $f(x, y) = x + 2y$.
   a) (7.3) Compute
   $$\iint_E f(x, y) \, dA$$
   where E is the rectangle with vertices $(0, 0)$, $(4, 0)$, $(0, 2)$ and $(4, 2)$.

b) (7.3) Compute

$$\iint\limits_D f(x,y)\,dA$$

where D is the triangle with vertices $(0,0)$, $(0,2)$ and $(2,2)$.

6. (7.4) Compute the iterated integral:

$$\int_0^4 \int_0^1 \int_0^2 xz\,dx\,dy\,dz$$

7. (7.5) Compute the iterated integral:

$$\int_0^4 \int_{\sqrt{x}}^2 \left(y^3+2\right)^{3/2}\,dy\,dx$$

Solutions

1. Let $(u,v) = \varphi(x,y)$ where $u = x+y$ and $v = y/x$. Thus the region E can be described as the set of (u,v) satisfying $1 \le u \le 3$ and $0 \le v \le 4$. Next,

$$J(\varphi) = \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \det \begin{pmatrix} 1 & 1 \\ \frac{-y}{x^2} & \frac{1}{x} \end{pmatrix} = \frac{1}{x} + \frac{y}{x^2} = \frac{x+y}{x^2} = \frac{u}{x^2}.$$

We next need to back-solve for x and y in terms of u and v. Since $v = y/x$, $y = vx$ so $u = x + vx = x(1+v)$. That means $x = \frac{u}{1+v}$ and $y = vx = \frac{uv}{1+v}$. So by the change of variable formula, the integral is

$$\begin{aligned}
\int_0^4 \int_1^3 x^2 \frac{1}{|J(\varphi)|}\, du\, dv &= \int_0^4 \int_1^3 x^2 \left(\frac{x^2}{u}\right) du\, dv \\
&= \int_0^4 \int_1^3 \frac{x^4}{u}\, du\, dv \\
&= \int_0^4 \int_1^3 \frac{\left(\frac{u}{1+v}\right)^4}{u}\, du\, dv \\
&= \int_0^4 \int_1^3 \frac{u^3}{(1+v)^4}\, du\, dv \\
&= \int_0^4 \left[\frac{u^4}{4}(1+v)^{-4}\right]_1^3 dv \\
&= \int_0^4 20(1+v)^{-4}\, dv \\
&= \frac{-20}{3}(1+v)^{-3}\Big|_0^4 = \frac{-20}{3}\left[\frac{1}{125} - 1\right] = \frac{-20}{3} \cdot \frac{-124}{125} = \boxed{\frac{496}{75}}.
\end{aligned}$$

2. Let S denote the solid described in the problem. In cylindrical coordinates, this solid consists of the points (r,θ,z) with $0 \le \theta \le 2\pi$, $4 \le r \le 6$ and

$0 \le z \le \sqrt{36 - r^2}$. So the volume can be evaluated as follows:

$$\begin{aligned} V = \iiint_S 1\,dV &= \int_0^{2\pi} \int_4^6 \int_0^{\sqrt{36-r^2}} r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi} \int_4^6 rz\Big|_0^{\sqrt{36-r^2}}\,dr\,d\theta \\ &= \int_0^{2\pi} \int_4^6 r\sqrt{36-r^2}\,dr\,d\theta \\ &\quad \text{(Here, use the } u\text{-sub } u = 36 - r^2,\ du = -2r\,dr) \\ &= \int_0^{2\pi} \int_{20}^0 \frac{-1}{2}\sqrt{u}\,du\,d\theta \\ &= \int_0^{2\pi} \frac{-1}{3} u^{3/2}\Big|_{20}^0\,d\theta \\ &= \int_0^{2\pi} \frac{1}{3}(20)^{3/2}\,d\theta = \boxed{\frac{2\pi}{3}(20)^{3/2}}. \end{aligned}$$

3. a) By Fubini's Theorem, the order of integration can be reversed over a rectangle like $[0,4]\times[0,2]$, so we get $\int_0^2 \int_0^4 g(x,y)\,dx\,dy = \int_0^4 \int_0^2 g(x,y)\,dy\,dx = \boxed{3}$.

   b) By additivity, we have

$$\begin{aligned} 10 = \int_0^4 \int_0^4 g(x,y)\,dy\,dx &= \int_0^2 \int_0^4 g(x,y)\,dy\,dx + \int_2^4 \int_0^4 g(x,y)\,dy\,dx \\ &= 8 + \int_2^4 \int_0^4 g(x,y)\,dy\,dx. \end{aligned}$$

   Therefore $\int_2^4 \int_0^4 g(x,y)\,dy\,dx = 10 - 8 = 2$. Again using additivity, we have

$$\begin{aligned} 3 = \int_0^4 \int_0^2 g(x,y)\,dy\,dx &= \int_0^2 \int_0^2 g(x,y)\,dy\,dx + \int_2^4 \int_0^2 g(x,y)\,dy\,dx \\ &= 5 + \int_2^4 \int_0^2 g(x,y)\,dy\,dx \end{aligned}$$

   so $\int_2^4 \int_0^2 g(x,y)\,dy\,dx = 3 - 5 = -2$. Using additivitiy a third time, we get

$$\begin{aligned} \int_2^4 \int_0^4 g(x,y)\,dy\,dx &= \int_2^4 \int_0^2 g(x,y)\,dy\,dx + \int_2^4 \int_2^4 g(x,y)\,dy\,dx \\ 2 &= -2 + \int_2^4 \int_2^4 g(x,y)\,dy\,dx \end{aligned}$$

   so $\int_2^4 \int_2^4 g(x,y)\,dy\,dx = \boxed{4}$.

   c) $\int_0^4 \int_0^2 3g(x,y)\,dx\,dy = 3\int_0^2 \int_0^4 g(x,y)\,dy\,dx = 3(8) = \boxed{24}$.

d) Since the upper and lower limits on the inside integral are the same, we obtain $\int_0^4 \int_2^2 g(x,y)\,dy\,dx = \boxed{0}$.

e) $\int_0^2 \int_0^2 (2+5g(x,y))\,dy\,dx = \int_0^2 \int_0^2 2\,dy\,dx + 5\int_0^2 \int_0^2 g(x,y)\,dy\,dx = 2(2)2 + 5(5) = 8 + 25 = \boxed{33}$.

4. In spherical coordinates, E is the set of points with $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$ and $0 \le \rho \le 2$. So this integral is

$$\begin{aligned}
\iiint_E (x^2+y^2+z^2)^2\,dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^2 (\rho^2)^2 \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^2 \rho^6 \sin\varphi\,d\rho\,d\varphi\,d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \frac{1}{7}\rho^7 \sin\varphi\Big|_0^2\,d\varphi\,d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \frac{1}{7}(2)^7 \sin\varphi\,d\varphi\,d\theta \\
&= \int_0^{2\pi} \frac{-1}{7}(2)^7 \cos\varphi\Big|_0^{\pi}\,d\theta \\
&= \int_0^{2\pi} \frac{-1}{7}(2)^7(-1-1)\,d\theta \\
&= \int_0^{2\pi} \frac{2^8}{7}\,d\theta \\
&= \boxed{\frac{2^9}{7}\pi}.
\end{aligned}$$

5. a) Notice E can be described by the inequalities $0 \le x \le 4$ and $0 \le y \le 2$. Therefore, by Fubini's Theorem, we have

$$\begin{aligned}
\iint_E (x+2y)\,dA &= \int_0^4 \int_0^2 (x+2y)\,dy\,dx \\
&= \int_0^4 \left[xy+y^2\right]_0^2\,dx = \int_0^4 [2x+4]\,dx = \left[x^2+4x\right]_0^4 = \boxed{32}.
\end{aligned}$$

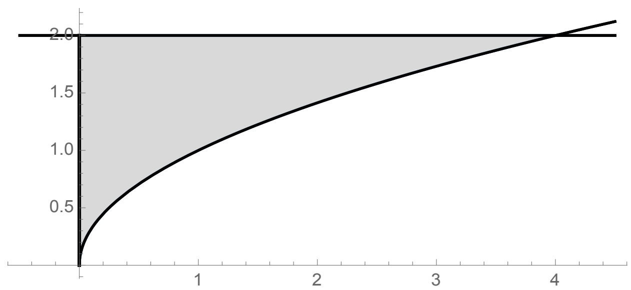
b) Notice D can be described by the inequalities $0 \le y \le 2$ and $0 \le x \le y$. Therefore, by Fubini's Theorem, we have

$$\begin{aligned}
\iint_D f(x,y)\,dA &= \int_0^2 \int_0^y (x+2y)\,dx\,dy = \int_0^2 \left[\frac{1}{2}x^2+2xy\right]_0^y\,dy \\
&= \int_0^2 \left[\frac{5}{2}y^2\right]\,dy = \left[\frac{5}{6}y^3\right]_0^2 = \boxed{\frac{20}{3}}.
\end{aligned}$$

6. Compute this directly:

$$\int_0^4 \int_0^1 \int_0^2 xz\,dx\,dy\,dz = \int_0^4 \int_0^1 \left[\frac{1}{2}x^2 z\right]_0^2 dy\,dz$$
$$= \int_0^4 \int_0^1 2z\,dy\,dz = \int_0^4 [2yz]_0^1\,dz = \int_0^4 2z\,dz = z^2|_0^4 = \boxed{16}.$$

7. You need to reverse the order of integration, because there's no way to come up with an antiderivative of $(y^3+2)^{3/2}$ with respect to y. A picture of the region over which you are integrating is the shaded region below (where the curve is $y = \sqrt{x}$):



This region can also be described as the set of (x, y) satisfying $0 \le y \le 2$, $0 \le x \le y^2$, so after reversing the order of the integrals we get

$$\int_0^4 \int_{\sqrt{x}}^2 \left(y^3+2\right)^{3/2} dy\,dx = \int_0^2 \int_0^{y^2} \left(y^3+2\right)^{3/2} dx\,dy$$
$$= \int_0^2 x\left(y^3+2\right)^{3/2}\Big]_0^{y^2} dy$$
$$= \int_0^2 y^2\left(y^3+2\right)^{3/2} dy$$

(Here, use the u-sub $u = y^3+2$, $du = 3y^2\,dy$)

$$= \int_2^{10} \frac{1}{3}u^{3/2}\,dy$$
$$= \frac{2}{15}u^{5/2}\Big|_2^{10} = \boxed{\frac{2}{15}\left(10^{5/2} - 2^{5/2}\right)}.$$

1.5 Spring 2018 Exam 3

1. (8.2) Find the curl of the vector field
$$\mathbf{f}(x,y,z) = \left(2x^2 - z, 3xz + y^2, 4y^2 + z\right)$$
at the point $(2,1,-3)$.

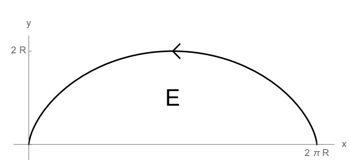
2. (7.3) Let $E = [0,2] \times [1,3]$. Compute the double integral
$$\iint_E (4x + 2y)\, dA.$$

3. (8.4) Compute the line integral
$$\int_\gamma \mathbf{f} \cdot d\mathbf{s}$$
where $\mathbf{f}(x,y) = (-y^2, 2x^2)$ and γ is the straight line from the origin to the point $(2,1)$.

4. (7.30 Compute the iterated integral
$$\int_0^1 \int_x^1 \sqrt{y^2+1}\, dy\, dx.$$

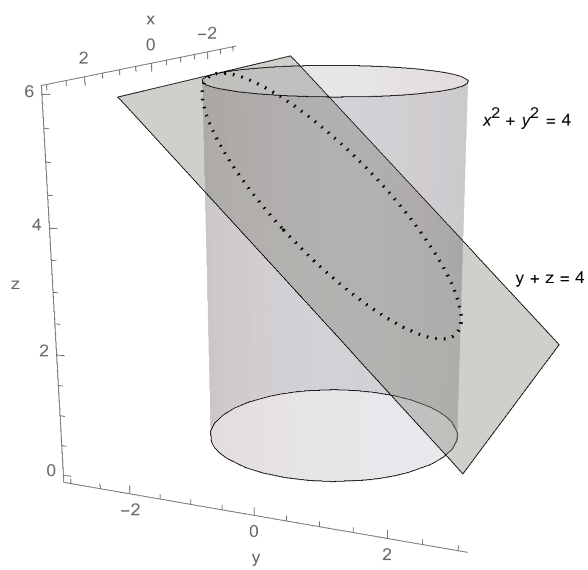
5. (7.5) Compute the double integral
$$\iint_E x\, dA$$
where E is the "pizza-slice" shaped region $\{(x,y) : 0 \le y \le x, x^2 + y^2 \le 9\}$.

6. (8.5) Let $R > 0$ be a constant. A **cycloid** is a curve parameterized (from right to left, as shown in the picture below) by the parametric equations
$$\begin{cases} x(t) &= R(2\pi + \sin t - t) \\ y(t) &= R(1 - \cos t) \end{cases}$$
Find the area of the region E consisting of points under the cycloid and above the x-axis.



Hint: One or more of the following integral facts may be useful:
$$\int \sin^2\theta\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C \qquad \int \cos^2\theta\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$$

7. (7.6) Find the volume of the solid in $\mathbb{R}^3$ consisting of points above the xy-plane, inside the cylinder $x^2 + y^2 = 4$, and below the plane $z + y = 4$. Here is a picture:



Solutions

1. Write $\mathbf{f} = (M, N, P) = (2x^2 - z, 3xz + y^2, 4y^2 + z)$. Then

$$\text{curl}\,\mathbf{f} = (P_y - N_z, M_z - P_x, N_x - M_y) = (8y - 3x, -1 - 0, 3z - 0)\,.$$

Therefore $\text{curl}\,\mathbf{f}(2, 1, -3) = (8(1) - 3(2), -1, 3(-3)) = (2, -1, -9)$.

2. Since E is a rectangle, we have

$$\begin{aligned}\iint_E (4x + 2y)\,dA &= \int_0^2 \int_1^3 (4x + 2y)\,dy\,dx \\ &= \int_0^2 \left[4xy + y^2\right]_1^3 dx \\ &= \int_0^2 [12x + 9 - 4x - 1]\;dx \\ &= \int_0^2 (8x + 8)\,dx \\ &= \left[4x^2 + 8x\right]_0^2 = 16 + 16 = 32.\end{aligned}$$

3. Parameterize the line segment γ by $x(t) = 2t$, $y(t) = t$ for $0 \le t \le 1$. That means $dx = 2\,dt$ and $dy = dt$. Therefore

$$\begin{aligned}\int_\gamma \mathbf{f} \cdot d\mathbf{s} &= \int_\gamma -y^2\,dx + 2x^2\,dy \\ &= \int_0^1 -(t)^2\,2\,dt + 2(2t)^2\,dt \\ &= \int_0^1 (-2t^2 + 8t^2)\,dt \\ &= \int_0^1 6t^2\,dt = 2t^3\Big|_0^1 = 2.\end{aligned}$$

4. First, change the order of integration. Let E be the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 1)$; then

$$\begin{aligned}\int_0^1 \int_x^1 \sqrt{y^2 + 1}\,dy\,dx &= \iint_E \sqrt{y^2 + 1}\,dA \\ &= \int_0^1 \int_0^y \sqrt{y^2 + 1}\,dx\,dy \\ &= \int_0^1 \left[x\sqrt{y^2 + 1}\right]_0^y dy \\ &= \int_0^1 y\sqrt{y^2 + 1}\,dy.\end{aligned}$$

Now use the substitution $u = y^2 + 1$. Thus $du = 2y\,dy$ so $\frac{1}{2}du = y\,dy$. As for the limits of integration, when $y = 0$, $u = 0^2 + 1 = 1$ and when $y = 1$, $u = 1^2 + 1 = 2$. So the integral becomes

$$\int_1^2 \frac{1}{2}\sqrt{u}\,du = \int_1^2 \frac{1}{2}u^{1/2}\,du = \frac{1}{3}u^{3/2}\Big|_1^2 = \frac{1}{3}\left(2^{3/2} - 1\right).$$

5. Change the integral to polar coordinates, since $E = \{(r,\theta) : 0 \le r \le 3, 0 \le \theta \le \frac{\pi}{4}\}$. Therefore

$$\begin{aligned}\iint_E x\,dA &= \int_0^{\pi/4}\int_0^3 (r\cos\theta)r\,dr\,d\theta \\ &= \int_0^{\pi/4}\int_0^3 r^2\cos\theta\,dr\,d\theta \\ &= \int_0^{\pi/4}\left[\frac{1}{3}r^3\cos\theta\right]_0^3 d\theta \\ &= \int_0^{\pi/4} 9\cos\theta\,d\theta \\ &= 9\sin\theta|_0^{\pi/4} = 9\left(\frac{\sqrt{2}}{2}\right) - 0 = \frac{9\sqrt{2}}{2}.\end{aligned}$$

6. By Green's Theorem, the area is

$$area(E) = \oint_{\partial E} -y\,dx = \int_{\gamma_1} -y\,dx + \int_{\gamma_2} -y\,dx$$

where γ_1 is the cycloid and γ_2 is the line segment across the bottom of E (running from $(0,0)$ to $(2\pi R, 0)$).

Now γ_2 is parameterized by $x(t) =$ something, $y(t) = 0$ so $dy = 0$. This means

$$\int_{\gamma_2} -y\,dx = \int_0^1 0\text{ (something) }dt = 0,$$

so all we really have to compute is $\int_{\gamma_1} -y\,dx$. To do this integral, use the given parameterization and first compute $dx = R(\cos t - 1)\,dt$. Therefore

$$\begin{aligned}area(E) = \oint_{\partial E} -y\,dx &= \int_{\gamma_1} -y\,dx \\ &= \int_0^{2\pi} -R(1-\cos t)R(\cos t - 1)\,dt \\ &= R^2\int_0^{2\pi}(1-\cos t)^2\,dt \\ &= R^2\int_0^{2\pi}(1 - 2\cos t + \cos^2 t)\,dt.\end{aligned}$$

Using the given integration fact, this becomes

$$R^2\left[t - 2\sin t + \frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = R^2\left[(2\pi - 0 + \pi + 0) - (0)\right] = 3\pi R^2.$$

7. We can do this with either a double integral in polar coordinates, or a triple integral in cylindrical coordinates.

   Double integral solution: let D be the disk of radius 2 centered at the origin; we want

$$\begin{aligned}
\iint_D (4-y)\,dA &= \int_0^2 \int_0^{2\pi} (4-y)r\,d\theta\,dr \\
&= \int_0^2 \int_0^{2\pi} (4 - r\sin\theta)r\,d\theta\,dr \\
&= \int_0^2 \int_0^{2\pi} (4r - r^2\sin\theta)\,d\theta\,dr \qquad (*) \\
&= \int_0^2 \left[4r\theta + r^2\cos\theta\right]_0^{2\pi}\,dr \\
&= \int_0^2 \left[(8\pi r + r^2) - (0 + r^2)\right]\,dr \\
&= \int_0^2 8\pi r\,dr = 4\pi r^2\Big|_0^2 = 16\pi.
\end{aligned}$$

   Triple integral solution: in cylindrical coordinates, the solid E whose volume we want is

$$\{(r,\theta,z) : 0 \le r \le 2, 0 \le \theta \le 2\pi, z \le 4 - y = 4 - r\sin\theta\}.$$

   Therefore its volume is

$$\begin{aligned}
vol(E) = \iiint_E 1\,dV &= \int_0^2 \int_0^{2\pi} \int_0^{4-r\sin\theta} r\,dz\,d\theta\,dr \\
&= \int_0^2 \int_0^{2\pi} [rz]_0^{4-r\sin\theta}\,d\theta\,dr \\
&= \int_0^2 \int_0^{2\pi} (4r - r^2\sin\theta)\,d\theta\,dr \qquad (*)
\end{aligned}$$

   This is the same integral as in the double integral solution, and is evaluated the same way to get 16π.

   Non-calculus solution: if you slice through the solid horizontally along the plane $z = 4$, the portion of the solid above the plane can be flipped over and placed on the rest of the solid to obtain a cylinder with constant height 4. Thus the volume of the solid is the same as the volume of a cylinder with radius 2 and height 4, which is $V = \pi r^2 h = \pi 2^2(4) = 16\pi$.