Old MATH 320 Exams

David M. McClendon

Department of Mathematics
Ferris State University

Last updated to include Exams from Fall 2021

Contents

Chapter 1

General information about these exams

These are the exams I have given between 2006 and 2021 in Calculus 3 courses. Unfortunately, most of the older exams do not have solutions typed (and there may be some number of computational errors or typos in the answers that are provided). If you want to check your work on the 2006 and 2007 exams, drop by my office.

Note that I have revised my course several times over the years, and what was on "Exam 1" in past years may not match what is on "Exam 1" now. Second, some of these exams are from a course given at a university on the quarter system, where Calculus 3 was split between two quarter-long courses. As such, these exams only cover half the material of Calculus 3.

To help give you some guidance on what questions are appropriate, each question on each exam is followed by a section number in parenthesis (like "(3.2)"). That means that question can be solved using material from that section (or from earlier sections) in the 2021 version of my *Vector Calculus Lecture Notes*.

Last, my exam-writing style has evolved over the years; generally speaking, the more recent the exam, the more likely you are to see something similar on one of your tests. Last, I did not write all the questions on the exams from 2006 and 2007. You should disregard questions that seem very far from our points of emphasis in class.

Chapter 2

Exams from 2006 to 2007

2.1 Fall 2006 Exam 1

1. (5.1) Calculate the indefinite integral

$$\int \left(4e^t, \frac{3}{t^2+1}, \sin\frac{t}{3}\right) dt.$$

2. (2.7) Determine whether the two lines given below are parallel, intersecting, or skew:

$$\begin{cases} x = -2t - 2 \\ y = t - 3 \\ z = 4 \end{cases} \qquad \begin{cases} x = 2t - 1 \\ y = -t + 1 \\ z = 3t + 1 \end{cases}$$

3. (2.7) Write an equation of the plane containing the point $(3, -3, 1)$ and the line given by the symmetric equations

$$\frac{x-1}{3} = y + 1 = \frac{z}{2}.$$

4. (5.1) Write down a definite integral (you do not need to evaluate the integral) which gives the circumference of the ellipse given by the equation

$$\frac{x^2}{5} + \frac{y^2}{3} = 1.$$

5. (3.4) Sketch a rough graph in $3-$dimensional space of each of the following:

   a) $x^2 = y^2 + z^2$

   b) $y^2 = x^2 + z^2 + 1$

   c) $y^2 + z^2 = 1$

6. (5.1) Consider the vector-valued function $\mathbf{x}(t) = (\cos t, t^3, 4 \sin t)$.

   a) Suppose $\mathbf{x}(t)$ gives the position at time t of a particle moving in $\mathbb{R}^3$. Find the particle's velocity, speed, and acceleration when $t = 0$.

   b) Sketch the image of $\mathbf{x}(t)$. Indicate the velocity and acceleration vectors at $t = 0$ on your graph.

7. (2.7) To say that two planes in $\mathbb{R}^3$ are perpendicular (a.k.a. orthogonal) means heuristically that the two planes meet at right angles. For example, the $xy-$plane is perpendicular to the $xz-$plane.

   a) Here is an incorrect definition of what it means for two planes to be perpendicular:

      Wrong definition: Two planes P and Q are said to be *perpendicular* if every vector in P is perpendicular to every vector in Q.

      Explain why this definition is wrong.

   b) Give a precise and correct mathematical definition of what it means for two planes to be perpendicular (there are many possible answers). Your answer should be a complete sentence that starts "Two planes P and Q are said to be perpendicular if ...".

   c) Verify using the definition you gave in part (b) above that the $xy-$ and $xz-$planes are perpendicular.

Solutions

1. Integrate component-wise to get

$$\int \left(4e^t, \frac{3}{t^2+1}, \sin\frac{t}{3}\right)\,dt \;=\; \left(4e^t, 3\arctan t, -3\cos\frac{t}{3}\right) + \mathbf{C}.$$

2. The direction vectors for the two lines are $(-2,1,4)$ and $(2,-1,3)$ respectively so the lines are clearly not parallel (these vectors are not multiples of one another). To see whether they intersect, replace t with s in the second equation. Then set the expressions for x, y and z equal to each other to obtain the system of equations

$$\begin{cases} -2t-2 = 2s-1 \\ t-3 = -s+1 \\ 4 = 3t+1 \end{cases}$$

From the last equation, $t = 1$. Substituting into the second equation and solving for s, we obtain $s = 3$. But in the first equation, $s = 3$ and $t = 1$ yields $-4 = 5$, a false statement. So this system has no solution for the variables s and t; therefore the lines do not intersect. Thus they are skew.

3. The point $(1,-1,0)$ lies on the given line (choose x and solve for y and z to get this point). So the vector $(3,-3,1)-(1,-1,0) = (2,-2,1)$ lies in the plane and so does the direction vector for the line, namely $(3,1,2)$. Therefore a normal vector to the plane is given by

$$(2,-2,1)\times(3,1,2) = (-5,-1,8).$$

Finally using this normal vector and any point in the plane (I use $(1,-1,0)$), we see the equation of the plane is

$$-5(x-1)-(y+1)+8z = 0.$$

4. A parameterization of this ellipse is $\mathbf{x}(t) = \left(\sqrt{5}\cos t, \sqrt{3}\sin t\right)$ where $0 \le t \le 2\pi$. So the circumference of the ellipse is the length of $\mathbf{x}(t)$, which is given by

$$\begin{aligned} s = \int_0^{2\pi} ||\mathbf{x}'(t)||\,dt \;&=\; \int_0^{2\pi} \sqrt{\left(-\sqrt{5}\sin t\right)^2 + \left(\sqrt{3}\cos t\right)^2}\,dt \\ &=\; \int_0^{2\pi} \sqrt{5\sin^2 t + 3\cos^2 t}\,dt. \end{aligned}$$

   a) This is a cone which opens in the positive and negative directions of the $x-$axis.

   b) This is a hyperboloid of two sheets opening in the direction of the positive and negative $y-$axis.

c) This is a cylinder parallel to the $x-$axis, cross-sections to the cylinder are circles of radius 1.

5. a) Compute these with usual formulas:

$$\begin{aligned}
\mathbf{v}(t) = \mathbf{x}'(t) &= (-\sin t, 3t^2, 4\cos t) \\
\mathbf{v}(0) &= (0, 0, 4) \\
\mathbf{a}(t) = \mathbf{x}''(t) &= (-\cos t, 6t, -4\sin t) \\
\mathbf{a}(0) &= (-1, 0, 0) \\
\text{speed at } t = 0 &= ||\mathbf{v}(0)|| = 4.
\end{aligned}$$

b) This is an elliptical helix (like a Slinky); the graph moves from left to right in the $y-$axis as t increases; the size of the ellipse is always the same (see part (a)). The velocity vector at $t = 0$ points from the point $(1, 0, 0)$ upward to $(1, 0, 4)$ and the acceleration vector at $t = 0$ points from the point $(1, 0, 0)$ to the origin.

6. a) If two planes are perpendicular, then they intersect in a line. Take a nonzero vector that lies in both planes; it cannot be perpendicular to itself.

b) Two planes are **perpendicular** if they have normal vectors which are perpendicular.

c) A normal vector to the $xy-$plane is $< 0, 0, 1 >$; a normal vector to the $xz-$plane is $< 0, 1, 0 >$. These vectors are perpendicular as $< 0, 0, 1 > \cdot < 0, 1, 0 >= 0$.

2.2 Fall 2006 Exam 2

1. (3.5) Find the following limits or state that they do not exist.

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{3x^2+3y^2} \qquad\qquad \lim_{(x,y)\to(0,0)} \frac{y\sqrt{x}}{y^2+x}$$

2. (4.3) Find the equation of the tangent plane to $f(x,y) = x^3y^2$ at $(1,1)$.

3. Consider the function $f(x,y) = x^2\cos y + e^{xy}$. Find the following:

   a) (4.2) $\frac{\partial^3 f}{\partial y \partial^2 x}$

   b) (4.5) The directional derivative of f, in the direction $(2,-1)$, at the point $(2,0)$

4. (5.4) Suppose γ is the graph of the vector-valued function

$$\mathbf{r}(t) = \left(t^5, \frac{5\sqrt{2}}{3}t^3, 5t\right).$$

   a) Find the curvature of γ at the point $(0,0,0)$.

   b) What does the value of the curvature imply about the shape of the graph of $\mathbf{r}(t)$ near $(0,0,0)$?

5. (6.1) Find all critical points for the function $f(x,y) = 2x^4 - x^2 + 3y^2$. Classify each critical point as a local maximum, local minimum, or saddle point.

6. (6.2 or 6.3) Find the maximum volume for a closed rectangular box (with faces parallel to the $xy-$, $xz-$, and $yz-$ planes) inscribed in the ellipsoid

$$\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{16} = 1.$$

   Make sure you justify that you have found the maximum volume.

2.3 Fall 2006 Final Exam

1. Suppose that at time t, the position of a dragonfly is given by $\mathbf{x}(t) = \left(t, t^2, \frac{2}{3}t^3\right)$.

   a) (5.1) Compute the velocity and acceleration of the dragonfly at the time $t = 0$.

   b) (5.4) Compute the curvature of the dragonfly's path at the time $t = 0$.

   c) (5.1) Compute the speed of the dragonfly (as a function of time t).

   d) (5.1) Compute the distance traveled by the dragonfly between times $t = 0$ and $t = 3$.

2. (6.2 or 6.3) An astronomer would like to set up his instruments on the highest ground less than or equal to 2 miles from his home. If the astronomer lives at $(0,0)$ and the height of the countryside is given by $f(x,y) = x^2 + y^2 - 2x$, where should the astronomer set up his instruments?

3. (3.5) Evaluate the following limits, or show that they do not exist. In either case, be sure to justify your answer.

$$\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^2+y^2} \qquad\qquad \lim_{(x,y)\to(0,0)} \frac{x^4y^2}{x^6+y^6}$$

4. (4.3) Consider the function $f(x,y) = x^2y - \ln(x)$.

   a) Compute the linearization of f at $(1,2)$.

   b) Use your answer from part (a) to estimate the value of $f(1.1, 1.9)$.

5. (5.1) An archer shoots an arrow with an initial speed of s_0 at an angle $\pi/4$ from the ground. If gravity acts on the arrow with an acceleration of 10, find the initial speed s_0 required for her to hit a target a distance 100 away.

6. a) (4.4) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \cos(xy^2)$. Suppose that both x and y are functions of s and t, where

$$x(0,0) = 1 \qquad y(0,0) = 0 \qquad x_s(0,0) = 2 \qquad y_s(0,0) = 1.$$

   Compute $\frac{\partial f}{\partial s}(0,0)$.

   b) (4.4) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(\mathbf{x}) = x - y^2$. If $\mathbf{x} = (e^t \sin s, st^2)$, compute $\frac{\partial f}{\partial t}(s,\, t)$.

7. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x,y,z) = x^2 + y^2 - z^2 - 1$.

   a) (4.5) Determine the unit vector in the direction of the maximum rate of increase of f at the point $(1,1,1)$.

b) (4.5) Compute the maximum rate of increase of f at the point $(1, 1, 1)$.

c) (4.3) Let $\mathcal{S}$ be the level surface to f at height 0. Give parametric equations for the normal line to $\mathcal{S}$ at the point $(1, 1, 1)$.

d) (4.3) Find the equation of the tangent plane to $\mathcal{S}$ at the point $(1, 1, 1)$.

8. (5.2) Let $\mathbf{x} : \mathbb{R} \to \mathbb{R}^3$ be C^2. Suppose that when $t = 3$, the norm of $\mathbf{x}''(3)$ is 5 and the angle between $\mathbf{x}'(3)$ and $\mathbf{x}''(3)$ is $\pi/6$. Find the tangential component of acceleration at the time $t = 3$.

9. (2.7) Consider the following four points in space:

$$P = (1, 2, 3), \quad Q = (2, 0, 5), \quad R = (3, 1, -1), \quad S = (0, 4, 2).$$

a) Give parametric equations for the line passing through P and Q.

b) Give parametric equations of the line passing through R and S.

c) Give an equation for the plane passing through the points P and Q, which lies parallel to the line passing through the points R and S.

2.4 Spring 2007 Exam 1

1. (5.1) Calculate the indefinite integral

$$\int \left(\frac{2}{e^t},\ \sqrt{t+2},\ \frac{2}{5}\sin 4t\right)\, dt.$$

2. Sketch a rough graph of each of the following in $\mathbb{R}^2$:

   a) (2.8) $r = 2\sin\theta$

   b) (3.2) The $z = 0$ trace of $x^2 - y^2 + z^2 = 1$

   c) (3.3) $(x-2)^2 + 9y^2 = 9$

3. Sketch a rough graph of each of the following in $\mathbb{R}^3$:

   a) (3.2) $y = x^2$

   b) (3.4) $y = x^2 + z^2$

   c) (3.4) $y^2 = x^2 - z^2$

4. Suppose a particle's position at time t is given by the vector-valued function $\mathbf{x}(t) = (\sin t, 3\cos t, t^3 + 1)$

   a) (5.1) Find the particle's velocity and acceleration when $t = 0$.

   b) (5.1) Sketch a picture which shows the local behavior of $\mathbf{x}(t)$ at the point where $t = 0$. Include and label the velocity, and acceleration vectors in the appropriate places on your graph.

   c) (5,4) Find the curvature of the graph of $\mathbf{x}(t)$ at the point $(0, 3, 1)$.

   d) (5.1) Write a definite integral which gives the length of the portion of the graph of $\mathbf{x}(t)$ between the points $(0, 3, 1)$ and $(1, 0, \frac{\pi^3}{8} + 1)$. The integrand should not contain any limits, derivatives, etc. You do not need to evaluate the integral.

5. (2.7) Consider the two lines whose symmetric equations are

$$\frac{x-2}{6} = \frac{y+1}{-4} = \frac{z}{2} \quad \text{and} \quad \frac{x+3}{-9} = \frac{y-2}{6} = \frac{z+1}{-3}.$$

   a) Are the two lines parallel, perpendicular, or neither?

   b) Write the equation of the plane containing the two lines.

2.5 Spring 2007 Exam 2

1. (2.8) Find the rectangular coordinates of the point whose spherical coordinates are $(12, 2\pi/3, \pi/6)$.

2. (5.2) Suppose that a particle is moving in three-dimensional space and that at time 0, you know the following information:

$$\begin{aligned}\mathbf{x}(0) &= (2,0,0)\\ \mathbf{v}(0) &= (1,-2,2)\\ \mathbf{a}(0) &= (0,3,0)\end{aligned}$$

   Find the tangential and normal components of the particle's acceleration at time 0.

3. (3.5) Find the following limits or state that they do not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} \qquad\qquad \lim_{(x,y)\to(0,0)} \frac{2x\sqrt{y}}{y-x^2}$$

4. (4.4) Find $\frac{dz}{dx}$ and $\frac{dx}{dy}$ if $\cos(yz - x) = x^2e^y$.

5. (4.2 or 8.6) Suppose $f_x(x,y) = 3x^2y^2 + y - 2$ and $f_y(x,y) = 2x^3y + x - y$. What is a possible rule for $f(x,y)$?

6. (4.5) Find the directional derivative of the function $f(x,y) = x^2\sin(2y)$ at $(1, \frac{\pi}{2})$ in the direction of $\mathbf{v} = (3,-4)$.

7. (4.3) Find the equation of the tangent plane to the surface $y^4 - x^2 = 7z^4$ at the point $(3,2,-1)$.

8. (6.2) Find all critical points for the function $f(x,y) = 12xy - x^3 - y^3$. Classify each critical point as a local maximum, local minimum, or saddle point.

Chapter 3

Exams from 2018 to 2021

3.1 Spring 2018 Exam 1

1. Let $\mathbf{v} = (2, 5, -3)$ and let $\mathbf{w} = 7\mathbf{i} + 2\mathbf{k}$.
   a) (2.3) Find $\mathbf{v} \cdot \mathbf{w}$.
   b) (2.3) Find a vector of length 4 in the opposite direction as $\mathbf{v}$.
   c) (2.3) Find the projection of $\mathbf{w}$ onto $\mathbf{v}$.
   d) (2.3) Suppose $(6, y, 1)$ is orthogonal to $\mathbf{v}$. Find y.
   e) (2.7) Find symmetric equations for the line containing $\mathbf{v}$ and $\mathbf{w}$.
2. a) (2.7) Find the normal equation of the plane containing the points $(2, -1, 4)$, $(3, 3, 5)$ and $(0, -3, -2)$.
   b) (2.7) Find parametric equations of the line which is the intersection of the planes $2x - 3y + 5z = 12$ and $x + 4y - 3z = -5$.
3. a) (2.8) Find Cartesian coordinates of the point whose polar coordinates are $\left(8, \frac{3\pi}{2}\right)$.
   b) Find spherical coordinates of the point whose Cartesian coordinates are $(0, 3, 3)$.
   c) (2.8) Find Cartesian coordinates of the point whose cylindrical coordinates are $\left(4, \frac{\pi}{6}, 2\right)$.
4. (3.3) Sketch graphs of the following equations.
   a) $x^2 = y^2 + z^2$
   b) $x = z^2$ (sketch this as a subset of $\mathbb{R}^3$)
   c) $4x + 3y - 2z = 12$

d) $x^2 + y^2 - z^2 = 1$

e) $y = 2$ (sketch this as a subset of $\mathbb{R}^3$)

f) $\rho = 4$ (this equation is in spherical coordinates)

g) $r = 6\cos\theta$ (this equation is in polar coordinates)

h) $r = 6\cos\theta$ (this equation is in cylindrical coordinates)

5. (3.6) Evaluate each of the following limits (if the limit does not exist, say so).

a) $\lim\limits_{x\to 0} (3x+5, e^{-x}, 7x^2-3)$

b) $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{2x^2+yz}{x^2+y^2+z^2}$

c) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x+y}$

d) $\lim\limits_{\mathbf{x}\to\mathbf{0}} \frac{x-2y}{x+y}$

e) $\lim\limits_{\mathbf{x}\to\mathbf{0}} \mathbf{f}(\mathbf{x})$, where $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\frac{\mathbf{x}}{||\mathbf{x}||}$.

Solutions

1. a) $\mathbf{v}\cdot\mathbf{w} = 2(7)+5(0)+(-3)2 = 8$.

   b) First, $||\mathbf{v}|| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{2^2+5^2+(-3)^2} = \sqrt{38}$. The answer is therefore $-4\frac{\mathbf{v}}{||\mathbf{v}||} = \left(\frac{-8}{\sqrt{38}}, \frac{-20}{\sqrt{38}}, \frac{12}{\sqrt{38}}\right)$.

   c) $\pi_{\mathbf{v}}\mathbf{w} = \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{8}{38}(2,5,-3) = \left(\frac{8}{19}, \frac{20}{19}, \frac{-12}{19}\right)$.

   d) We have $0 = \mathbf{v}\cdot(6,y,1) = 2(6)+5y+(-3)1 = 9+5y$. Solving for y, we get $y = \frac{-9}{5}$.

   e) A direction vector for the line is $\mathbf{w}-\mathbf{v} = (7-2, 0-5, 2-(-3)) = (5,-5,5)$. Therefore, symmetric equations for the line, using the point $(2,5,-3)$ and the above direction vector, are

   $$\frac{x-2}{5} = \frac{y-5}{-5} = \frac{z+3}{5}.$$

   (Answers may vary in this problem, depending on your choice of direction vector and point on the line.)

2. a) Two vectors in the plane are $\mathbf{v} = (3,3,5)-(2,-1,4) = (1,4,1)$ and $\mathbf{w} = (2,-1,4)-(0,-3,-2) = (2,2,6)$. So a normal vector to the plane is $\mathbf{n} = \mathbf{v}\times\mathbf{w} = (22,-4,-6)$. Thus the normal equation of the plane is $22x-4y-6z = d$; to find d, plug in the point $(0,-3,-2)$ to get $d = 22(0)-4(-3)-6(-2) = 24$. Thus the normal equation is $22x-4y-6z = 24$; dividing through by 2 we get

   $$11x-2y-3z = 12.$$

   b) Normal vectors to the two planes can be read off from the normal equations as $\mathbf{n}_1 = (2,-3,5)$ and $\mathbf{n}_2 = (1,4,-3)$. The line which is the intersection of the planes must be orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$, so it has direction vector $\mathbf{v} = \mathbf{n}_1\times\mathbf{n}_2 = (-11,11,11)$.

   Next, find a point on the line by finding a point on both planes. To do this, set $x = 0$ and solve for y and z using the given equations:

   $$\begin{cases} -3y+5z = 12 \\ 4y-3z = -5 \end{cases} \quad \Rightarrow y = 1, z = 3.$$

   So $(0,1,3)$ is common to both planes, therefore lies on the line of intersection. Using this point and the direction vector from the previous paragraph, we get

   $$\mathbf{x} = \mathbf{p}+t\mathbf{v} \;\Rightarrow\; \begin{cases} x = 0-11t \\ y = 1+11t \\ z = 3+11t \end{cases}$$

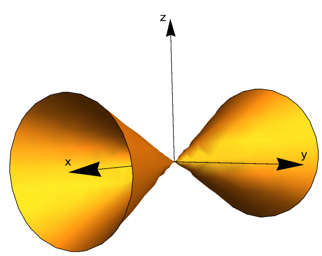
(Answers may vary, depending on the point and direction vector you use.)

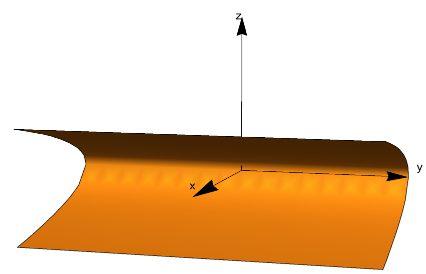
3. a) $x = r\cos\theta = 8\cos\frac{3\pi}{2} = 8(0) = 0$ and $y = r\sin\theta = 8\sin\frac{3\pi}{2} = 8(-1) = -8$, so the Cartesian coordinates are $(x, y) = (0, -8)$.

   b) $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{0^2+3^2+3^2} = 3\sqrt{2}$; $\theta = \frac{\pi}{2}$ since $x = 0$ and $y > 0$; $\varphi = \arctan\frac{z}{\sqrt{x^2+y^2}} = \arctan 1 = \frac{\pi}{4}$, so the spherical coordinates are $(\rho, \varphi, \theta) = \left(3\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2}\right)$.

   c) $x = r\cos\theta = 4\cos\frac{\pi}{6} = 2\sqrt{3}$; $y = r\sin\theta = 4\sin\frac{\pi}{6} = 2$; $z = z = 2$ so the Cartesian coordinates are $(x, y, z) = \left(2\sqrt{3}, 2, 2\right)$.

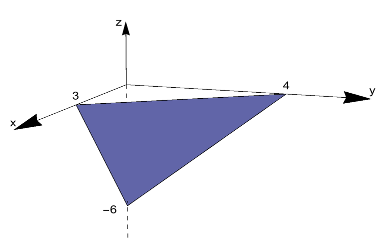
4. a) $x^2 = y^2 + z^2$ is a cone opening around the x-axis, shown below at left:

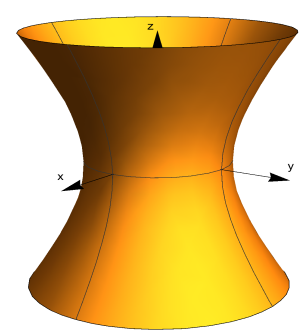




   b) $x = z^2$ is a cylinder parallel to the y-axis; the shape is a parabola opening around the positive x-axis shown above at right:

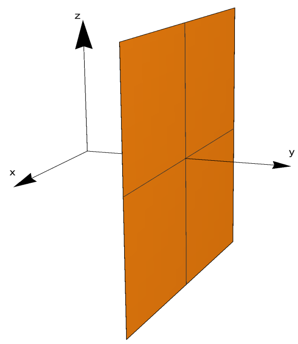
   c) $4x+3y-2z = 12$ is a plane with intercepts $(3, 0, 0)$, $(0, 4, 0)$ and $(0, 0, -6)$, shown below at left:

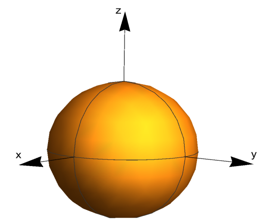




   d) $x^2+y^2-z^2 = 1$ is a hyperboloid of one sheet, opening around the z-axis, shown above at right:

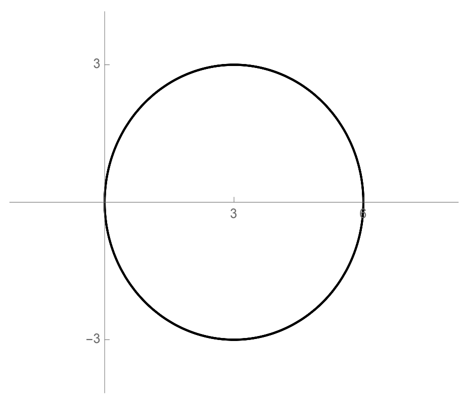
e) $y = 2$ is a plane parallel to the x- and z-axes, shown below at left:

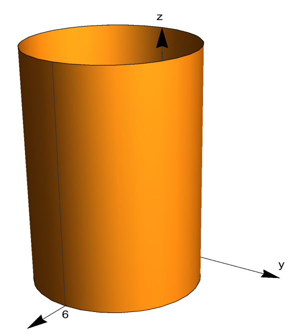




f) $\rho = 4$ is a sphere of radius 4 centered at the origin, shown above at right:

g) $r = 6\cos\theta$, in polar coordinates, is a circle of radius 3 centered at $(3, 0)$ (this graph belongs in $\mathbb{R}^2$), shown below at left:





h) $r = 6\cos\theta$ is a cylinder parallel to the z-axis; the shape of the cylinder is a circle coming from the figure in part (g) of this question. The graph is above at right:

5. a) Plug in 0 to get $\lim\limits_{x\to 0} (3x + 5, e^{-x}, 7x^2 - 3) = (5, -1, -3)$.

b) Change to spherical coordinates:

$$\begin{aligned}
\lim_{(x,y,z)\to(0,0,0)} \frac{2x^2 + yz}{x^2 + y^2 + z^2} &= \lim_{\rho\to 0} \frac{2\rho^2 \sin^2\varphi\cos^2\theta + \rho^2\sin\varphi\cos\varphi\sin\theta}{\rho^2} \\
&= \lim_{\rho\to 0} (\sin\varphi)\left(2\sin\varphi\cos^2\theta + \cos\varphi\sin\theta\right) \\
&= \begin{cases} 0 & \text{if } \varphi = 0 \\ 2 & \text{if } \varphi = \frac{\pi}{2} \text{ and } \theta = 0 \end{cases}
\end{aligned}$$

Since the value of the limit depends on φ and θ, the original limit **does not exist**.

(This could also have been shown using paths; along the y-axis and z-axis, the limit is 0, but along the x-axis, the limit is 2.)

c) Factor and cancel:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x+y} = \lim_{(x,y)\to(0,0)} (x+y) = 0.$$

(This could also have been done by changing to polar coordinates.)

d) Use paths: along the x-axis, this limit is

$$\lim_{(x,0)\to(0,0)} \frac{x-2y}{x+y} = \lim_{x\to 0} \frac{x}{x} = 1$$

but along the y-axis, the limit is

$$\lim_{(0,y)\to(0,0)} \frac{x-2y}{x+y} = \lim_{y\to 0} \frac{-2y}{y} = -2.$$

Since the limits are different, the original limit **does not exist**.

(This could also have been done by changing to polar coordinates.)

e) First, writing the function coordinate-wise, this is the limit

$$\lim_{(x,y)\to(0,0)} \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right).$$

Let's look at the first coordinate: changing to polar coordinates, we obtain

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r\cos\theta}{\sqrt{r^2}} = \lim_{r\to 0} \cos\theta.$$

When $\theta = 0$, this is 1, but when $\theta = \frac{\pi}{2}$, this is 0. Since the value depends on θ, the limit

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$$

does not exist; consequently the vector-valued limit

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{\mathbf{x}}{||\mathbf{x}||} = \lim_{(x,y)\to(0,0)} \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$$

also **does not exist**.

3.2 Fall 2020 Exam 1

1. (2.4) Let $A = \begin{pmatrix} -3 & 4 & -7 \\ 3 & 6 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -5 & 3 \\ 1 & 2 \end{pmatrix}$.

   a) Determine which of the two products AB or BA is defined, and compute it.

   b) Determine which of $\det A$ or $\det B$ is defined, and compute it.

   c) Suppose $\mathbf{f}$ is the function defined by $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$. Which of the following best describes how we would indicate the domain and codomain of $\mathbf{f}$? Write the letter of your answer.

   A. $\mathbf{f} : \mathbb{R} \to \mathbb{R}$
   B. $\mathbf{f} : \mathbb{R} \to \mathbb{R}^3$
   C. $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$
   D. $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}$
   E. $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$
   F. $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$

2. (2.6) Let $\mathbf{u} = (2, 3, 4)$ and $\mathbf{v} = (-1, 0, 3)$.

   a) Find a nonzero vector which is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

   b) Is the vector you found in part (a) unique (meaning, is the vector you found the only possible answer to part (a))? Explain.

3. (2.7) Write the normal equation of the plane in $\mathbb{R}^3$ containing the points $(4, -3, 1)$, $(-3, 1, 1)$ and $(4, -2, 8)$.

4. (2.3, 2.6) Suppose $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^3$ with $||\mathbf{v}|| = 9$, $\mathbf{v} \cdot \mathbf{w} = -5$, and $\mathbf{v} \times \mathbf{w} = (1, 2, -4)$.

   For each given quantity below, determine if it is possible to compute that quantity, based on the given information and/or principles developed in class. If it is, compute it. If it isn't, state that quantity is impossible (to compute).

   a) $\mathbf{v} \cdot \mathbf{v}$ b) $\mathbf{v} \times \mathbf{v}$ c) $\mathbf{w} \cdot \mathbf{v}$ d) $\mathbf{w} \times \mathbf{v}$ e) $\mathbf{w} \cdot \mathbf{w}$

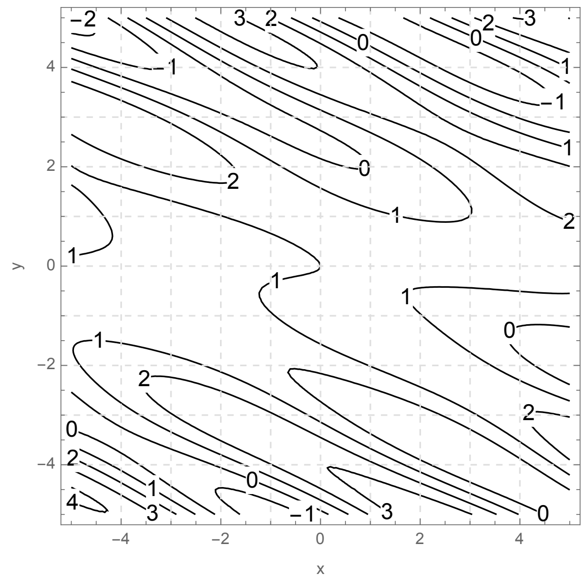
5. (3.2) Let E be the circle in $\mathbb{R}^2$ of radius 2, centered at the origin.

   a) Describe E as the image of a function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^2$ (i.e. give parametric equations for E).

   b) Describe E as the graph of a polar function $r = f(\theta)$.

   c) Describe E as the level curve to a function $g : \mathbb{R}^2 \to \mathbb{R}$.

6. (3.2) Below, you are given a contour plot for some unknown function $h : \mathbb{R}^2 \to \mathbb{R}$.



Use the contour plot to answer the following questions:

a) Estimate $h(4, 3)$.

b) Find a value of y for which $h(1, y) = 2$.

c) If you move in a straight line from $(-2, -3)$ to $(-2, 0)$, are the values of h getting larger, getting smaller, or staying the same?

d) For what x (between -5 and 5) is $h(x, -3)$ maximized?

7. (3.5) Evaluate the following limits (or state that they do not exist), with proper justification.

a) $\lim_{\mathbf{x}\to\mathbf{0}} \frac{x-y}{\sqrt{x^2+y^2}}$

b) $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$

8. (2.7, 3.4) Sketch graphs of the following equations:

a) $y^2 = x^2 + z^2$

b) $y = 3$ (this graph should be drawn in $\mathbb{R}^3$)

c) $2x + y + 4z = 8$

9. a) (3.3) Sketch the graph of the conic section whose parametric equations are

$$\begin{cases} x = 4\cos t \\ y = \sin t \end{cases}$$

b) (2.8) Sketch the region of points in $\mathbb{R}^2$ described by the inequalities $0 \leq \theta \leq \frac{\pi}{2}$, $1 \leq r \leq 3$.

c) (2.8) Sketch a picture of the set of points in $\mathbb{R}^3$ which satisfy the equation $r = 3$ in cylindrical coordinates.

Solutions

1. Let $A = \begin{pmatrix} -3 & 4 & -7 \\ 3 & 6 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -5 & 3 \\ 1 & 2 \end{pmatrix}$.

   a) AB is undefined, but $BA = \begin{pmatrix} 24 & -2 & 41 \\ 3 & 16 & -3 \end{pmatrix}$.

   b) $\det A$ is undefined, but $\det B = (-5)2 - 1(3) = -13$.

   c) Since A is 2×3, $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$. This is choice **E**.

2. a) One answer is $\mathbf{u} \times \mathbf{v} = (9, -10, 3)$.

   b) Any (nonzero) multiple of the answer to (a) works, so the answer to (a) is **not unique**.

3. First, subtract pairs of the given points to get two direction vectors for the plane: $\mathbf{v} = (4, -3, 1) - (-3, 1, 1) = (7, -4, 0)$ and $\mathbf{w} = (4, -3, 1) - (4, -2, 8) = (0, -1, -7)$. Then a normal vector to the plane is $\mathbf{n} = \mathbf{v} \times \mathbf{w} = (28, 49, -7)$. (I'll divide through this vector by 7 to keep the numbers small, and set $\mathbf{n} = (4, 7, -1)$.) Now, the normal equation of the plane is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$; any of the given points could be used as $\mathbf{x}_0$. For example, using the first given point we obtain

$$\begin{aligned} (4, 7, -1) \cdot (x - 4, y + 3, z - 1) &= 0 \\ 4(x - 4) + 7(y + 3) - (z - 1) &= 0 \\ 4x + 7y - z + 6 &= 0 \\ 4x + 7y - z &= -6. \end{aligned}$$

4. a) $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2 = 9^2 = 81$.

   b) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ (since any vector $\times$ itself is $\mathbf{0}$).

   c) $\mathbf{w} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = -5$.

   d) $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w} = -(1, 2, -4) = (-1, -2, 4)$.

   e) $\mathbf{w} \cdot \mathbf{w}$ is impossible to compute without more information.

5. a) E is the image of $\mathbf{f}(t) = (2 \cos t, 2 \sin t)$.

   b) E is the graph of the polar function $r = 2$.

   c) E is the level curve to $g(x, y) = x^2 + y^2$ at height 4.

6. a) $h(4, 3) \approx 0$.

   b) $h(1, y) = 2$ at four places: when $y \approx -5$, $y \approx -4$, $y \approx 3$ and $y \approx 4$.

   c) Values of h are **getting smaller** (from about 2.5 to about $.5$) as you move in a straight line from $(-2, -3)$ to $(-2, 0)$.

d) $h(x,-3)$ is maximized when $x \approx -2$.

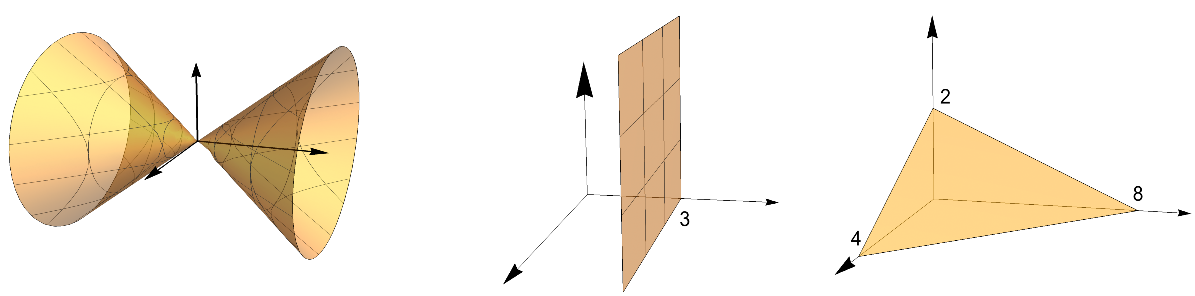
7. a) Use the polar coordinates trick (this problem could also be done with paths):
$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{x-y}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r\cos\theta - r\sin\theta}{r} = \lim_{r\to 0} (\cos\theta - \sin\theta)\,;$$
when $\theta = 0$ this is $1-0=1$, but when $\theta = \frac{\pi}{2}$ this is $0-1=-1$. Therefore $\lim\limits_{\mathbf{x}\to\mathbf{0}} \frac{x-y}{\sqrt{x^2+y^2}}$ DNE.

b) Use the polar coordinates trick:
$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2} = \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0.$$

8. a) Rewrite this as $x^2 - y^2 + z^2 = 0$; this is a cone opening around the y-axis, as shown below at left.

b) This is a plane parallel to the x- and z-axes, as shown below in the center.

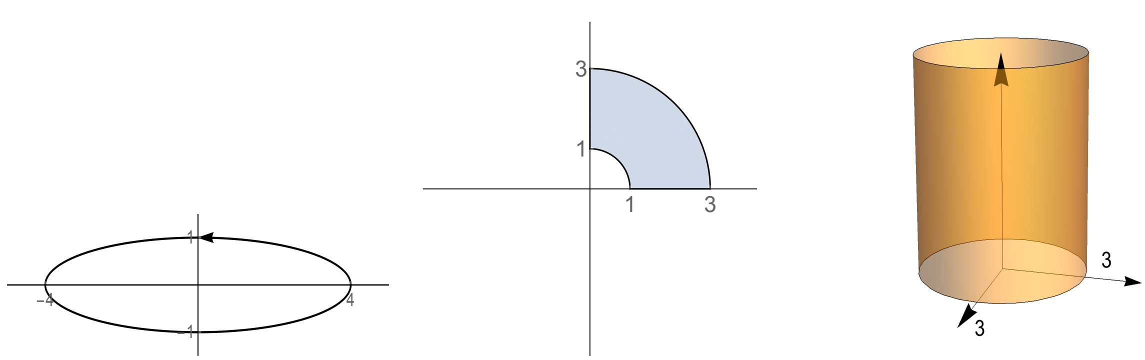
c) This is a plane with intercepts $(4,0,0)$, $(0,8,0)$ and $(0,0,2)$, as shown below at right. $2x + y + 4z = 8$



9. a) This is an ellipse traced out counterclockwise, as shown below at left.

b) This region is sketched below, in the center.

c) This is a cylinder of radius 3 around the z-axis, as shown below at right.



3.3 Spring 2021 Exam 1

1. a) (2.3) Compute $(3,-4,1,-2)\cdot(1,2,5,-3)$.

   b) (2.3) Compute $||(2,-1,4)||$.

   c) (2.4) Compute A^2, where $A = \begin{pmatrix} 1 & -3 \\ 0 & 2 \\ 7 & 5 \end{pmatrix}$.

   d) (2.5) Compute $\det \begin{pmatrix} 6 & -4 \\ 3 & 2 \end{pmatrix}$.

   e) (2.3) If $\mathbf{v}$ and $\mathbf{w}$ are vectors, each having norm 5, what is the smallest possible value of $\mathbf{v}\cdot\mathbf{w}$?

2. (2.7) Consider the two lines in $\mathbb{R}^3$ given by the following sets of parametric equations:

$$\begin{cases} x &= 7-2t \\ y &= 3+t \\ z &= 5-3t \end{cases} \qquad \begin{cases} x &= -2-t \\ y &= 2-5t \\ z &= -3+4t \end{cases}$$

   These two lines intersect in a point. Find the coordinates of this point.

3. (2.7) Write the normal equation of the plane in $\mathbb{R}^3$ containing the point $(2,-5,-2)$ and the line whose parametric equations are

$$\begin{cases} x &= 1+4t \\ y &= 3t \\ z &= 2 \end{cases}$$

4. Let E be the line in $\mathbb{R}^2$ that has slope 1 and passes through the origin.

   a) (3.2) Describe E as the graph of one or more functions from $\mathbb{R}\to\mathbb{R}$.

   b) (3.2) Describe E as the image of a function $\mathbf{f}:\mathbb{R}\to\mathbb{R}^2$ (i.e. give parametric equations for E).

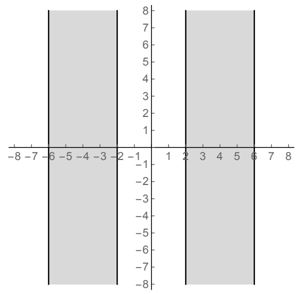
   c) (3.2) E is the graph of what polar equation?

   d) (3.2) Describe E as the level curve to a function $g:\mathbb{R}^2\to\mathbb{R}$.

5. Consider the subset F of $\mathbb{R}^2$ defined by

$$F = \{(x,y)\in\mathbb{R}^2 : x^2 \geq 4 \text{ and } x^2 \leq 36\}.$$

This set F is the shaded region in the picture below:



Answer the following questions about F (no justification is required).

a) (2.9) Is the set F open?

b) (2.9) Is the set F closed?

c) (2.9) Is the set F compact?

d) (2.9) Is the set F connected?

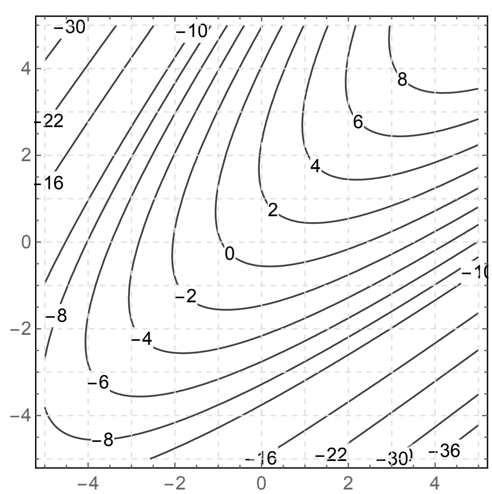
e) (2.9) Give the coordinates of any one point which belongs to the boundary of F.

f) (2.1) Sketch a picture of $F \cup G$ below, if $G = \{(x, y) \in \mathbb{R}^2 : y \leq -4\}$.

6. (3.5) Evaluate the following limit (or state that it does not exist), with proper justification.

$$\lim_{\mathbf{x} \to \mathbf{0}} \frac{x + 2y}{x + 4y}$$

7. (3.2) The contour plot for some unknown function $h : \mathbb{R}^2 \to \mathbb{R}$ is given below.



Use this contour plot to answer these questions:

a) Estimate $h(1,0)$.

b) Estimate a value of x such that $h(x,3)=0$.

c) Which best describes the graph of the $x=0$ trace of h?
A. A line with positive slope
B. A line with negative slope
C. A parabola that opens upward
D. A parabola that opens downward

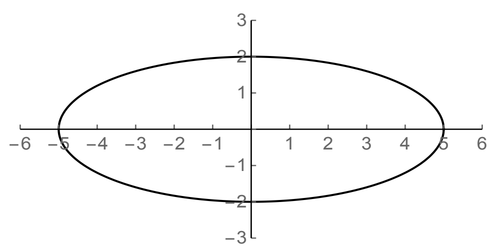
8. Sketch graphs of the following equations. (Each of these pictures should be drawn in $\mathbb{R}^3$.)

a) (3.4) $x^2 - y^2 + z^2 = 1$

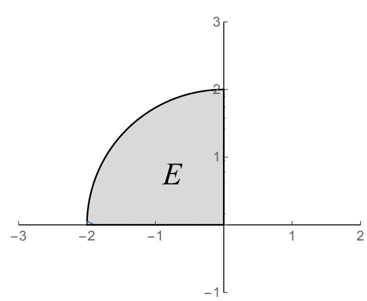
b) (2.7) $2x + 3y = 6$

c) (2.8) $r = 2$ (think of this as an equation in cylindrical coordinates)

9. a) (3.3) Find a set of parametric equations for the curve pictured below:



b) (2.8) Describe the set pictured below by using one or more inequalities, involving polar coordinates:



c) (2.8) Compute the Cartesian coordinates of the point in $\mathbb{R}^3$ whose spherical coordinates are $\left(8, \frac{\pi}{4}, \frac{\pi}{2}\right)$.

Solutions

1. a) $(3,-4,1,-2)\cdot(1,2,5,-3) = 3(1)-4(2)+1(5)-2(-3) = 3-8+5+6 = \boxed{6}$.

   b) $||(2,-1,4)|| = \sqrt{2^2+(-1)^2+4^2} = \boxed{\sqrt{21}}$.

   c) $A^2 = A_{3\times 2}A_{3\times 2}$ $\boxed{\text{DNE}}$, since A is not a square matrix.

   d) $\det\begin{pmatrix} 6 & -4 \\ 3 & 2 \end{pmatrix} = 6(2)-3(-4) = \boxed{24}$.

   e) By either the Cauchy-Schwarz Inequality or the angle formula from dot products, we have $\mathbf{v}\cdot\mathbf{w} \geq -||\mathbf{v}||\,||\mathbf{w}|| = -5(5) = \boxed{-25}$.

2. Change the t to s in the second set of parametric equations, and set the x, y and z-coordinates equal to obtain the system

$$\begin{cases} 7-2t &= -2-s \qquad \Rightarrow s = 2t-9 \\ 3+t &= 2-5s \\ 5-3t &= -3+4s \end{cases}$$

   Plug the first equation into the second and solve for t to get $3+t = 2-5(2t-9)$, i.e. $t=4$. Then from the first equation, $s=-1$. Notice that $s=-1$, $t=4$ works in the last equation as well, so the lines intersect in the point (x,y,z) where $s=-1$, i.e. $(-2-(-1), 2-5(-1), -3+4(-1)) = \boxed{(-1,7,-7)}$.

3. One point on the line, which must also be in the plane, is $\mathbf{p} = (1,0,2)$. One vector in the plane can be read off as the direction vector for the given line: $\mathbf{v} = (4,3,0)$. A second vector in the plane can be found by subtracting two points in the plane, one on the line and one not:

$$\mathbf{w} = (2,-5,-2) - \mathbf{p} = (2,-5,-2) - (1,0,2) = (1,-5,-4).$$

   Thus a normal vector to the plane is $\mathbf{n} = \mathbf{v}\times\mathbf{w} = (4,3,0)\times(1,-5,-4) = (-12,16,-23)$. That makes the normal equation of the plane

$$\begin{aligned} \mathbf{n}\cdot(\mathbf{x}-\mathbf{p}) &= 0 \\ (-12,16,-23)\cdot((x,y,z)-(1,0,2)) &= 0 \\ -12(x-1)+16(y-0)-23(z-2) &= 0 \end{aligned}$$

   which simplifies to $\boxed{-12x+16y-23z=-58}$.

4. a) E is the graph of the function $f:\mathbb{R}\to\mathbb{R}$ where $f(x)=x$.

   b) E is the image of $\mathbf{f}:\mathbb{R}\to\mathbb{R}^2$ given by $\mathbf{f}(t)=(t,t)$.

   c) E has polar equation $\theta = \frac{\pi}{4}$.

d) E is the level curve to $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x, y) = y - x$, at height 0.

5. a) F is $\boxed{\text{not open}}$ since it contains at least some of its boundary.

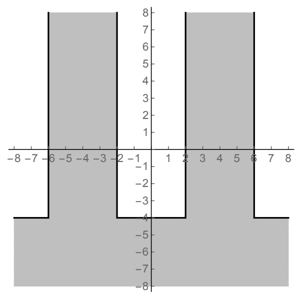
   b) F is $\boxed{\text{closed}}$ since it contains all of its boundary.

   c) F is $\boxed{\text{not compact}}$ since it is not bounded.

   d) F is $\boxed{\text{not connected}}$ since it consists of two "pieces" which do not touch.

   e) Answers can vary here, but any point on the "edge" of F works. These are points with x-coordinate equal to ± 2 or ± 6, like for instance $\boxed{(2,3)}$.

   f) G is the set of points on or below the horizontal line $y = -4$. Thus the union $F \cup G$ is the set of points in F, or G, or both, as shown here:



6. I'll use paths (although this problem could be done with polar coordinates). Along the x-axis, the limit is

$$\lim_{(x,0)\to(0,0)} \frac{x+2y}{x+4y} = \lim_{x\to 0} \frac{x+2(0)}{x+4(0)} = \lim_{x\to 0} \frac{x}{x} = 1$$

but along the line $y = x$, the limit is

$$\lim_{(x,x)\to(0,0)} \frac{x+2y}{x+4y} = \lim_{x\to 0} \frac{x+2x}{x+4x} = \lim_{x\to 0} \frac{3x}{5x} = \frac{3}{5}.$$

Since the limits along different paths are unequal, $\lim\limits_{\mathbf{x}\to\mathbf{0}} \frac{x+2y}{x+4y}$ $\boxed{\text{DNE}}$.

7. a) $h(1, 0) \approx \boxed{1}$, since it is between the level curves at heights 0 and 2.

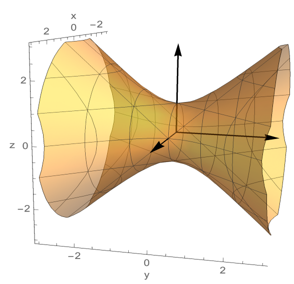
   b) The horizontal line $y = 3$ intersects the level curve at height 0 when x is about $\frac{-1}{2}$, so $h(x, 3) = 0$ when $\boxed{x \approx -\dfrac{1}{2}}$.

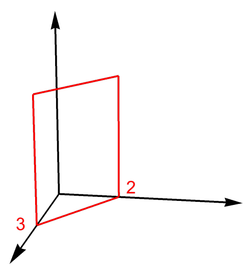
   c) The $x = 0$ trace has z-value -10 when $x \approx -4$, has z-value 2 when $x \approx 1$, but then z-value -4 when $x \approx 4$. Thus this trace starts negative, increases, then decreases, so it is best describe as a parabola that opens downward. This is choice $\boxed{\text{D}}$.

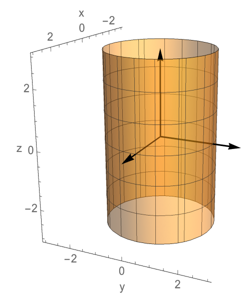
8. a) $x^2 - y^2 + z^2 = 1$ is a hyperboloid of one sheet, opening around the y-axis since the y variable has the $(-)$ sign. This graph is shown below at left.

b) $2x + 3y = 6$ is a plane with x-intercept $(3,0,0)$ and y-intercept $(0,2,0)$. The plane is parallel to the z-axis since it is missing the z variable, and is shown below in the center.

c) $r = 2$ is a cylinder of radius 2, centered at the origin and opening around the z-axis, as shown below at right.







9. a) This ellipse has parametric equations $\boxed{(5\cos t, 2\sin t)}$ (for $0 \le t \le 2\pi$).

b) This set is $E = \boxed{\left\{(r,\theta) : r \le 2, \frac{\pi}{2} \le \theta \le \pi\right\}}$.

c) We have

$$
\begin{aligned}
x &= \rho \sin\varphi\cos\theta = 8\sin\frac{\pi}{4}\cos\frac{\pi}{2} = 8\left(\frac{\sqrt{2}}{2}\right)0 = 0 \\
y &= \rho \sin\varphi\sin\theta = 8\sin\frac{\pi}{4}\sin\frac{\pi}{2} = 8\left(\frac{\sqrt{2}}{2}\right)1 = 4\sqrt{2} \\
z &= \rho\cos\varphi = 8\cos\frac{\pi}{4} = 8\left(\frac{\sqrt{2}}{2}\right) = 4\sqrt{2}
\end{aligned}
$$

so the Cartesian coordinates are $\boxed{(0, 4\sqrt{2}, 4\sqrt{2})}$.

3.4 Fall 2021 Exam 1

1. a) (2.4) Let $A = \begin{pmatrix} 1 & -3 \\ -2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}$. Compute AB.

   b) (2.4) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be $f(x, y, z) = 3x - y + 6z$. Find a matrix A such that $f(\mathbf{x}) = A\mathbf{x}$.

   c) (2.3) Compute $(3, -4) \cdot (5, 2)$.

   d) (2.8) Find a set of spherical coordinates which represent the point in $\mathbb{R}^3$ whose cylindrical coordinates are $(6, \pi, 6)$.

   e) (2.6) Suppose $\mathbf{v}$ and $\mathbf{w}$ are two vectors in $\mathbb{R}^3$ such that $\mathbf{v} \times \mathbf{w} = (1, 6, -1)$.

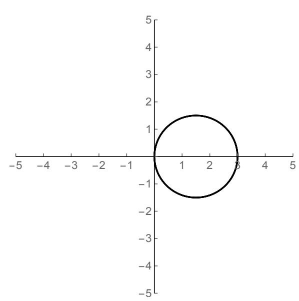
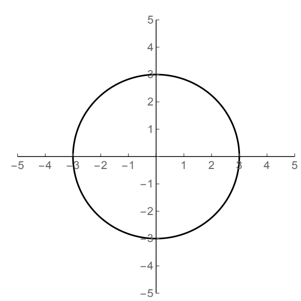
      i. Is this information sufficient to compute $\mathbf{v} \times 3\mathbf{w}$? If so, what is $\mathbf{v} \times 3\mathbf{w}$?

      ii. Is this information sufficient to compute $\mathbf{w} \times \mathbf{v}$? If so, what is $\mathbf{w} \times \mathbf{v}$?

      iii. Is this information sufficient to compute $\mathbf{v} \cdot \mathbf{w}$? If so, what is $\mathbf{v} \cdot \mathbf{w}$?

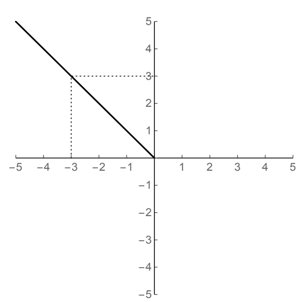
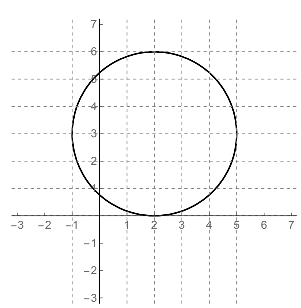
      iv. Is this information sufficient to compute $\mathbf{v} \times (\mathbf{v} + \mathbf{w})$? If so, what is $\mathbf{v} \times (\mathbf{v} + \mathbf{w})$?

2. a) (2.8) Write the polar equation of the circle pictured below, at left.

   b) (2.8) Write the polar equation of the circle pictured above, at right.

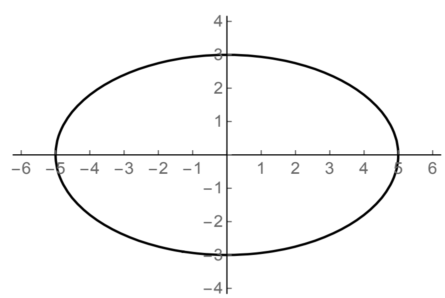
   c) (2.8) Write the polar equation of the (half-)line pictured below, at left.

   d) (2.9) Write the Cartesian equation of the circle pictured above, at right.

3. Sketch a picture of the points in $\mathbb{R}^3$ satisfying each of the following equations:
   a) (2.8) $\varphi = \frac{5\pi}{6}$
   b) (2.8) $\rho = 4$
   c) (2.8) $r = 2$
   d) (2.7) $x + 2y = 6$
   e) (2.7) $x + 2y + 3z = 6$

4. (3.2) Let E be the curve in $\mathbb{R}^2$ pictured below:



   a) Describe E as the level curve of some function $f : \mathbb{R}^2 \to \mathbb{R}$. Make sure you clearly define the function f and specify the height of the level curve.
   b) Which *Mathematica* command would be used to produce a picture of E, in the context of part (a) of this question?
      A. Plot
      B. ContourPlot
      C. ParametricPlot
      D. none of the above
   c) Describe E as the image of a function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^2$.
   d) Which *Mathematica* command would be used to produce a picture of E, in the context of part (c) of this question?
      A. Plot
      B. ContourPlot
      C. ParametricPlot
      D. none of the above

5. a) (2.7) Write parametric equations for the line which is the intersection of the two planes $2x + z = 6$ and $x - 2y + 4z = -1$.
   b) (2.7) Write a normal equation of the plane which contains the three points $(1, -3, 4)$, $(2, -1, 0)$ and $(-1, 4, -1)$.

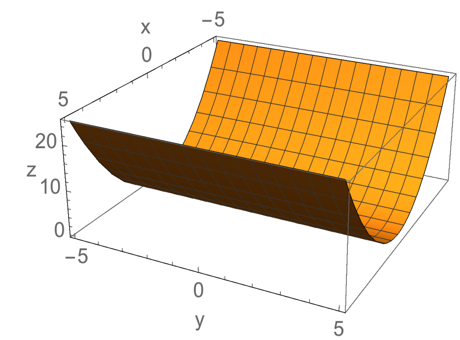
6. (3.5) Compute each of the following two limits:

a) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x+y}$

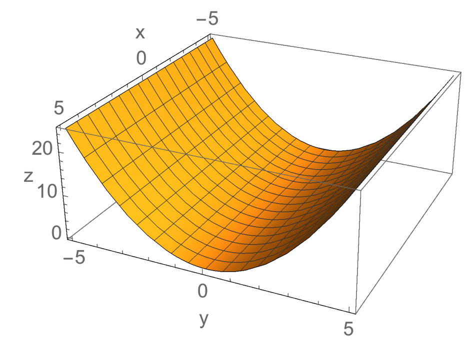
b) $\lim\limits_{\mathbf{x}\to\mathbf{0}} \frac{xyz}{x^2+y^2+z^2}$

7. (3.2) Here are the graphs of four functions f, g, h and $k : \mathbb{R}^2 \to \mathbb{R}$:

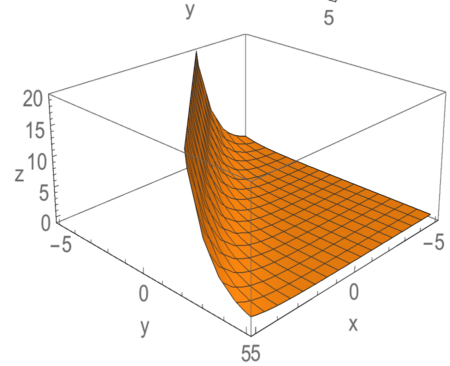
f :



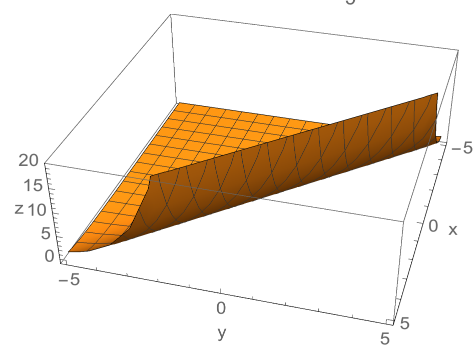
g :



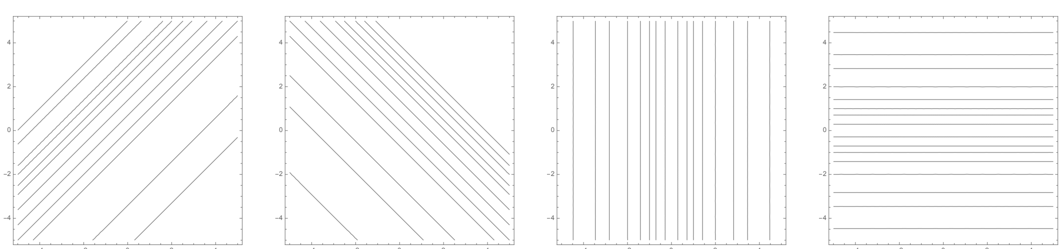
h :



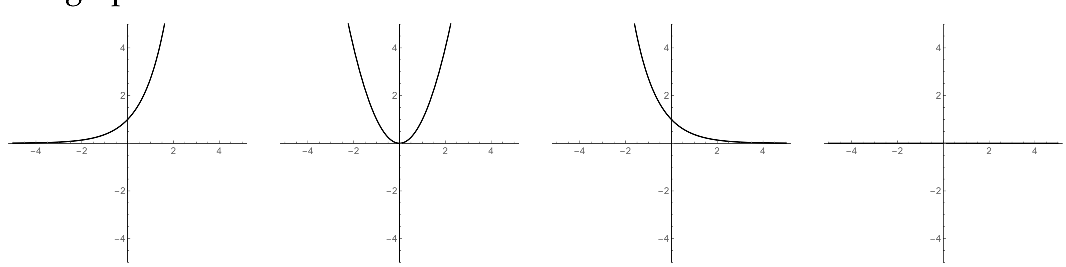
k :



a) For each contour plot below, choose the function (f, g, h or k) which the contour plot represents:



b) For each graph below, choose the function (f, g, h or k) such that the given graph is the graph of the $x = 0$ trace of that function. *Note:* The last graph is the horizontal axis $z = 0$.



Solutions

1. a) $AB = \begin{pmatrix} 1(0) + -3(1) & 1(4) + (-3)3 \\ -2(0) + 5(1) & -2(4) + 5(3) \end{pmatrix} = \boxed{\begin{pmatrix} -3 & -5 \\ 5 & 7 \end{pmatrix}}$.

   b) Since $f : \mathbb{R}^3 \to \mathbb{R}$, A must be 1×3; in particular we have $\boxed{A = \begin{pmatrix} 3 & -1 & 6 \end{pmatrix}}$.

   c) $(3, -4) \cdot (5, 2) = 3(5) + (-4)2 = \boxed{7}$.

   d) We have $\theta = \pi$; $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2} = \sqrt{6^2 + 6^2} = 6\sqrt{2}$; and $\varphi = \arctan \frac{r}{z} = \arctan \frac{6}{6} = \arctan 1 = \frac{\pi}{4}$, so the spherical coordinates are $\boxed{\left(6\sqrt{2}, \frac{\pi}{4}, \pi\right)}$.

   e) i. $\mathbf{v} \times 3\mathbf{w} = 3(\mathbf{v} \times \mathbf{w}) = 3(1, 6, -1) = \boxed{(3, 18, -3)}$.

      ii. $\mathbf{w} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{w}) = -(1, 6, -1) = \boxed{(-1, -6, 1)}$.

      iii. $\mathbf{v} \cdot \mathbf{w}$ cannot be determined from the given information.

      iv. $\mathbf{v} \times (\mathbf{v} + \mathbf{w}) = \mathbf{v} \times \mathbf{v} + \mathbf{v} \times \mathbf{w} = \mathbf{0} + \mathbf{v} \times \mathbf{w} = \mathbf{v} \times \mathbf{w} = \boxed{(1, 6, -1)}$.

2. a) $r = 3\cos\theta$

   b) $r = 3$

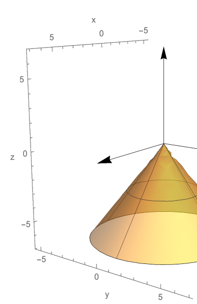
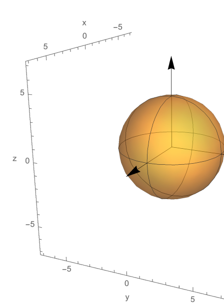
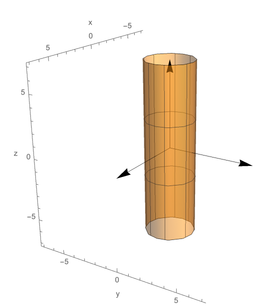
   c) $\theta = \frac{3\pi}{4}$

   d) $(x - 2)^2 + (y - 3)^2 = 9$

3. a) $\varphi = \frac{5\pi}{6}$ is a cone opening downward around the z-axis, pictured below at left.

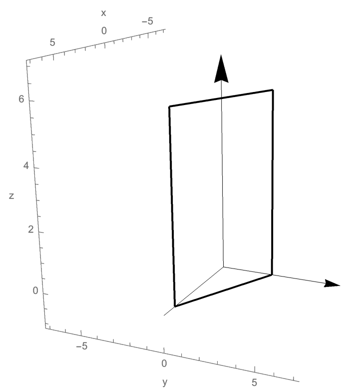
   b) $\rho = 4$ is a sphere of radius 4 centered at $(0, 0, 0)$, pictured below in the center.

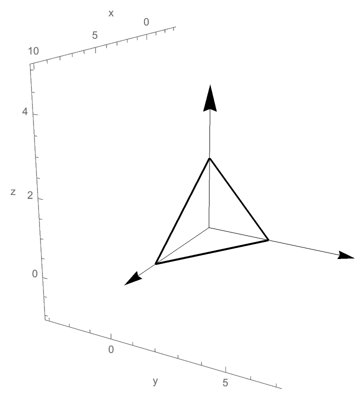
   c) $r = 2$ is a cylinder of radius 2 around the z-axis, pictured below at right.

   d) This plane has intercepts $(6, 0, 0)$ and $(0, 3, 0)$ and is parallel to the z-axis, pictured below at left.

e) This plane has intercepts $(6,0,0)$, $(0,3,0)$ and $(0,0,2)$, pictured below at right.





4. a) E is the level curve to $f(x,y) = \frac{x^2}{25} + \frac{y^2}{9}$ at height 1.

   b) B. ContourPlot

   c) E is the image of $\mathbf{f}(t) = (5\cos t, 3\sin t)$ (for $0 \le t \le 2\pi$).

   d) C. ParametricPlot

5. a) From the first equation, we have $z = 6 - 2x$. Substituting into the second plane, we get $x - 2y + 4(6-2x) = -1$, i.e. $-7x - 2y + 24 = -1$, i.e. $-7x - 2y = -25$. Solve for y to get $y = -\frac{7}{2}x + \frac{25}{2}$. We can then write parametric equations for the line by letting $t = x$, obtaining (by substitution in the equations for y and z in terms of x)

$$\boxed{\begin{cases} x &= t \\ y &= -\frac{7}{2}t + \frac{25}{2} \\ z &= -2t + 6 \end{cases}}$$

   (There are many other correct answers here.)

   b) Two nonparallel vectors in the plane can be found by subtracting pairs of the given points: $\mathbf{v} = (2,-1,0) - (1,-3,4) = (1,2,-4)$ and $\mathbf{w} = (1,-3,4) - (-1,4,-1) = (2,-7,5)$. So a normal vector to the plane is $\mathbf{n} = \mathbf{v} \times \mathbf{w} = (-18,-13,-11)$. Finally, the equation of the plane is

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) &= 0 \\ (-18,-13,-11) \cdot (x-2, y+1, z) &= 0 \\ -18(x-2) - 13(y+1) - 11z &= 0 \\ -18x + 36 - 13y - 13 - 11z &= 0 \end{aligned}$$

   which rearranges into $\boxed{-18x - 13y - 11z = -23}$.
   (Any scalar multiple of this equation is also correct.)

6. a) Factor and cancel:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{(x+y)(x+y)}{x+y} = \lim_{(x,y)\to(0,0)} (x+y) = 0+0 = \boxed{0}.$$

   b) Use spherical coordinates:

$$\begin{aligned}\lim_{(x,0,0)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} &= \lim_{\rho\to 0} \frac{\rho^3 \cos\varphi \sin^2\varphi \sin\theta\cos\theta}{\rho^2} \\ &= \lim_{\rho\to 0} \rho \left(\cos\varphi \sin^2\varphi \sin\theta\cos\theta\right) = \boxed{0}\end{aligned}$$

   irrespective of the values of φ and/or θ.

7. a) From left to right, these are the contour plots of functions h, k, f and g.

   b) From left to right, these are the $x = 0$ traces of functions k, g, h and f.

3.5 Spring 2018 Exam 2

1. a) (4.2) Find the total derivative of the function

$$\mathbf{f}(x,y,z) = \left(\frac{y}{x-z}, 2x^2y\right).$$

   b) (4.3) Find an equation of the plane tangent to the surface

$$2x^3z + (x-y)^2 - yz = 2$$

   at the point $(1,3,2)$.

   c) (4.4) Suppose $\sin(xy) + y^4 - 3x^5y = 0$. Find $\frac{dy}{dx}$.

2. Let $f(x,y,z) = e^{xz+y}$.

   a) (4.3) Use linearization to estimate $f\left(\frac{1}{5}, \frac{1}{5}, \frac{-1}{5}\right)$.

   b) (4.5) Find the direction in which f is increasing most rapidly at the point $(\ln 2, \ln 3, 0)$.

   c) (4.2) Compute $\frac{\partial^3 f}{\partial x \partial y \partial x}$.

3. Suppose that the position of an object at time t is $\mathbf{x}(t) = (e^t + e^{-t}, e^t - e^{-t}, e^{2t})$.

   a) (5.1) Find the acceleration of the object at time t.

   b) (5.4) Find the curvature of the object's path at time $t = 0$.

   c) (5.1) Write a definite integral which, when evaluated, will compute the distance the object travels from time 2 to time 10. Your integral should not contain letters/variables other than t.

4. (6.1) Find the critical points of the function

$$f(x,y) = x^4 - 16xy + 8y^2 + 4.$$

   Classify, with justification, each critical point as a local maximum, local minimum, or saddle.

5. (6.2 or 6.3) Find the dimensions of the rectangular box with the greatest volume, if the box sits on the xy-plane with one vertex at the origin and the opposite vertex lying on the surface $z = 4 - x^2 - 4y^2$.

Solutions

1. a)
$$Df(x,y,z) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{-y}{(x-z)^2} & \frac{1}{x-z} & \frac{y}{(x-z)^2} \\ 4xy & 2x^2 & 0 \end{pmatrix}$$

   b) Let $f(x,y,z) = 2x^3z+(x-y)^2-yz$. The gradient of f is $\nabla f = (6x^2z+2(x-y), -2(x-y)-z, 2x^3-y)$ and $\nabla f(1,3,2) = (12+2(-2), -2(-2)-2, 2-3) = (8,2,-1)$. The equation of the plane tangent to f at $(1,3,2)$ is therefore
$$\begin{aligned} (8,2,-1)\cdot(x-1,y-3,z-2) &= 0 \\ 8(x-1)+2(y-3)-(z-2) &= 0 \\ 8x+2y-z &= 12. \end{aligned}$$

   c) Let $F(x,y) = \sin(xy)+y^4-3x^5y$. Then
$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(y\cos xy - 15x^4y)}{x\cos xy + 4y^3 - 3x^5} = \frac{-y\cos xy + 15x^4y}{x\cos xy + 4y^3 - 3x^5}.$$

2. a) First, let $\mathbf{a} = (0,0,0)$. Next, the total derivative of f at $\mathbf{a}$ is
$$Df(\mathbf{a}) = \begin{pmatrix} ze^{xz+y} & e^{xz+y} & xe^{xz+y} \end{pmatrix}|_{(x,y,z)=(0,0,0)} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}.$$
   Therefore, letting $\mathbf{x} = \left(\frac{1}{5}, \frac{1}{5}, \frac{-1}{5}\right)$, we have
$$\begin{aligned} f(\mathbf{x} &= f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x}-\mathbf{a}) \\ &= 1 + \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{-1}{5} \end{pmatrix} \\ &= 1 + \frac{1}{5} \\ &= \frac{6}{5}. \end{aligned}$$

   b) This is $\nabla f(\ln 2, \ln 3, 0) = (ze^{xz+y}, e^{xz+y}, xe^{xz+y})|_{(x,y,z)=(\ln 2, \ln 3, 0)} = (0,3,3\ln 2)$.

   c) $\frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}e^{xz+y}\right)\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(ze^{xz+y}\right)\right] = \frac{\partial}{\partial x}\left[ze^{xz+y}\right] = z^2e^{xz+y}$.

3. a) First, the velocity is $\mathbf{v}(t) = \mathbf{x}'(t) = (e^t - e^{-t}, e^t + e^{-t}, 2e^{2t})$. Thus the acceleration is $\mathbf{a}(t) = \mathbf{x}''(t) = (e^t + e^{-t}, e^t - e^{-t}, 4e^{2t})$.

b) At $t = 0$, the velocity is $(0, 2, 2)$ and the acceleration is $(2, 0, 4)$. Thus the curvature is

$$\begin{aligned}\kappa = \frac{||\mathbf{x}' \times \mathbf{x}''||}{||\mathbf{x}'||^3} &= \frac{||\mathbf{v} \times \mathbf{a}||}{||\mathbf{v}||^3} \\ &= \frac{||(8, 4, -4)||}{(\sqrt{0^2 + 2^2 + 2^2})^3} \\ &= \frac{\sqrt{8^2 + 4^2 + (-4)^2}}{(\sqrt{8})^3} \\ &= \frac{\sqrt{96}}{(\sqrt{8})^3} = \frac{\sqrt{12}}{8} = \frac{\sqrt{3}}{4}.\end{aligned}$$

c) The distance traveled, a.k.a. arc length, is

$$\begin{aligned}s &= \int_a^b ||\mathbf{v}(t)||\,dt \\ &= \int_2^{10} ||\left(e^t - e^{-t}, e^t + e^{-t}, 2e^{2t}\right)||\,dt \\ &= \int_2^{10} \sqrt{(e^t - e^{-t})^2 + (e^t + e^{-t})^2 + (2e^{2t})^2}\,dt \\ &= \int_2^{10} \sqrt{e^{2t} - 2 + e^{-2t} + e^{2t} + 2 + e^{-2t} + 4e^{4t}}\,dt \\ &= \int_2^{10} \sqrt{2e^{2t} + 2e^{-2t} + 4e^{4t}}\,dt.\end{aligned}$$

4. First, $\nabla f = (4x^3 - 16y, -16x + 16y)$. Setting $\nabla f = (0, 0)$, we get $4x^3 - 16y = 0$ and $-16x + 16y = 0$. From the second of these equations, $x = y$; substituting into the first equation we get $4x^3 - 16x = 0$, i.e. $4x(x^2 - 4) = 0$, i.e. $x = 0$, $x = 2$ and $x = -2$. Thus there are three critical points: $(0, 0)$, $(2, 2)$ and $(-2, -2)$.

   To test the critical points, compute the Hessian:

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x^2 & -16 \\ -16 & 16 \end{pmatrix}.$$

   Observe $Hf(2, 2) = Hf(-2, -2) = \begin{pmatrix} 48 & -16 \\ -16 & 16 \end{pmatrix}$ which is positive definite by the minors test, making $(2, 2)$ and $(-2, -2)$ local minima.

   Also $Hf(0, 0) = \begin{pmatrix} 0 & -16 \\ -16 & 16 \end{pmatrix}$. Since the determinant of the first principal minor is zero, the normal test doesn't work. However, you can show this matrix is neither positive definite nor negative definite by definition, via testing against some vectors:

$$(0,1)^T Hf(0,0)(0,1) = 16 \Rightarrow Hf(0,0) \text{ is not negative definite}$$

$$(1,1)^T Hf(0,0)(1,1) = -16 \Rightarrow Hf(0,0) \text{ is not positive definite}$$

Therefore $(0,0)$ is the location of a saddle.

5. Let $f(x,y,z) = xyz$ and let $g(x,y,z) = x^2 + 4y^2 + z$. We need to maximize f subject to $g(x,y,z) = 4$ so we can use Lagrange multipliers. The system $\nabla f = \lambda \nabla g$ becomes

$$\begin{cases} yz &= \lambda 2x \\ xz &= \lambda 8y \\ xy &= \lambda \end{cases}$$

Substituting the third equation into the second, we get $xz = xy(8y)$, i.e. $z = 8y^2$. Substituting the third equation into the first, we get $yz = xy(2x)$, i.e. $z = 2x^2$. Therefore $8y^2 = 2x^2$, so $4y^2 = x^2$ so $2y = x$. Finally, substitute $x = 2y$ and $z = 8y^2$ into the constraint $g(x,y,z) = 4$ to get

$$4y^2 + 4y^2 + 8y^2 = 4 \Rightarrow 16y^2 = 4 \Rightarrow y = \frac{1}{2}.$$

Thus $x = 2y = 1$ and $z = 8y^2 = 2$ so the dimensions of the largest box are $1 \times \frac{1}{2} \times 2$.

3.6 Fall 2020 Exam 2

Note: Problems 1 and 2 of this exam were open-note and unlimited time; the rest of the exam was closed-note with a time limit of 60 minutes.

1. (4.3) Compute the linearization of $\mathbf{f}(x, y, z) = (e^{2\sin x+5y-3z}, e^{xz-z})$ at the origin, and use that linearization to estimate $\mathbf{f}\left(\frac{1}{10}, \frac{3}{10}, -\frac{1}{5}\right)$.

2. (6.2) Determine the absolute minimum value obtained by the function
$$f(x, y) = (x-1)^2 + (y-2)^2 + 2xy$$
on the region $D = [0, 3] \times [0, 3]$.

3. (4.5) Write the normal equation of the plane tangent to the ellipsoid $2x^2 + 5y^2 + z^2 = 31$ at the point $(-1, 2, -3)$.

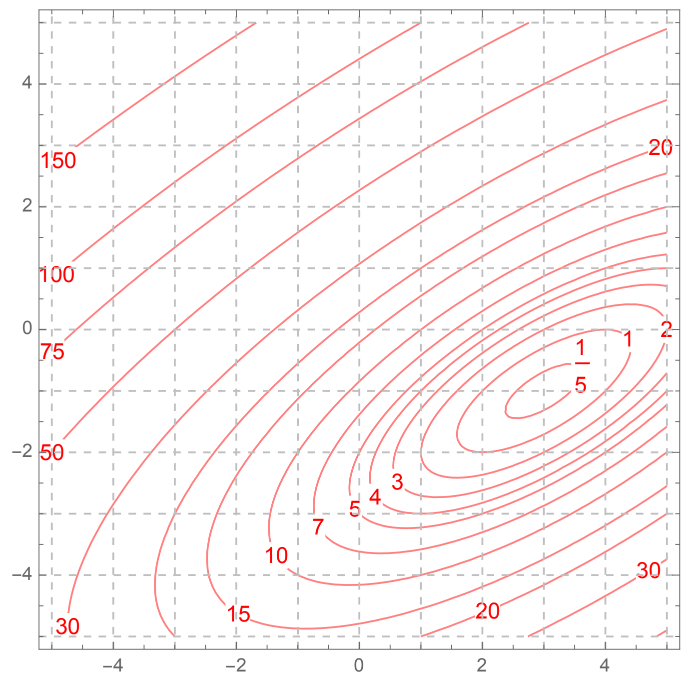
4. Let $f(x, y, z) = 3x^2yz^4 - 2xz^2 + 3x^2y^5$.

   a) (4.1) If you wrote the total derivative of f as a matrix, what size would that matrix be?
   b) (4.5) Compute $\nabla f(x, y, z)$.
   c) (4.2) Compute $\frac{\partial^3 f}{\partial x \partial z \partial x}$.
   d) (4.5) Compute $D_{\mathbf{u}} f(1, 0, -1)$, where $\mathbf{u}$ is in the direction $(2, 1, -2)$.

5. A bee is flying around $\mathbb{R}^3$ so that its position at time t, measured in seconds, is $\mathbf{x}(t) = (\cos 2t, 3t, \sin 2t)$ ft.

   a) (5.1) Compute the distance travelled by the bee from time 0 to time π.
   b) (5.1) Compute the acceleration of the bee at time 0.
   c) (5.4) Compute the curvature of the bee's flight path at time 0.

6. The contour plot of an unknown function $h : \mathbb{R}^2 \to \mathbb{R}$ is pictured below.



Use the contour plot to answer the following questions:

a) (6.1) Find the coordinates of all local minima of h, if there are any. (If there aren't any local minima in this viewing window, say so.)

b) (6.1) Find the coordinates of all saddles of h, if there are any. (If there aren't any saddles in this viewing window, say so.)

c) (4.2) Estimate $h_y(1, -4)$.

d) (4.5) Give the coordinates (x, y) of a point where $\nabla h(x, y)$ points south-west.

e) (4.5) Let $\mathbf{u} = \left(-\frac{3}{5}, \frac{4}{5}\right)$. Is $D_{\mathbf{u}}h(2, 3)$ positive, negative or zero?

7. (6.1) Find all critical points of the function $f(x, y) = x^3 - 2xy^2 - 6x$. Classify each critical point as a local maximum, local minimum or saddle.

8. (6.3) Find the minimum value of $f(x, y, z) = x \ln x + y \ln y + z \ln z$, subject to the constraint $x + y + z = 1$.

Solutions

1. First, we compute the linearization of $\mathbf{f}$. To do this, we first need the total derivative of $\mathbf{f}$ at $(0,0,0)$:

$$D\mathbf{f}(x,y,z) = \begin{pmatrix} (2\cos x)e^{2\sin x+5y-3z} & 5e^{2\sin x+5y-3z} & -3e^{2\sin x+5y-3z} \\ ze^{xz-z} & 0 & (x-1)e^{xz-z} \end{pmatrix}$$

so

$$D\mathbf{f}(0,0,0) = \begin{pmatrix} (2\cos 0)e^0 & 5e^0 & -3e^0 \\ 0e^0 & 0 & (0-1)e^0 \end{pmatrix} = \begin{pmatrix} 2 & 5 & -3 \\ 0 & 0 & -1 \end{pmatrix}.$$

Therefore the linearization of $\mathbf{f}$ at the origin is

$$\begin{aligned} \mathbf{L}(x,y,z) &= \mathbf{f}(0,0,0) + D\mathbf{f}(0,0,0)\begin{pmatrix} x-0 \\ y-0 \\ z-0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 & 5 & -3 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x-0 \\ y-0 \\ z-0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2x+5y-3z \\ -z \end{pmatrix} \\ &= \begin{pmatrix} 1+2x+5y-3z \\ 1-z \end{pmatrix} \\ &= (1+2x+5y-3z, 1-z). \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{f}\left(\frac{1}{10},\frac{3}{10},-\frac{1}{5}\right) \approx \mathbf{L}\left(\frac{1}{10},\frac{3}{10},-\frac{1}{5}\right) &= \left(1+2(\frac{1}{10})+5(\frac{3}{10})-3(-\frac{1}{5}), 1-(-\frac{1}{5})\right) \\ &= \boxed{\left(\frac{33}{10},\frac{6}{5}\right)}. \end{aligned}$$

2. First, find the critical points of f: the gradient of f is $\nabla f(x,y) = (2(x-1)+2y, 2(y-2)+2x)$; set this equal to $\mathbf{0}$ and solve for x and y:

$$\begin{cases} 2(x-1)+2y &= 0 \\ 2(y-2)+2x &= 0 \end{cases} \Rightarrow \begin{cases} 2x+2y &= 2 \\ 2x+2y &= 4 \end{cases} \Rightarrow 2 = 4 \Rightarrow \text{ no solution}$$

This means that f has no critical points. Next, we find boundary critical points. Notice that the boundary ∂D consists of four pieces, so we have to find boundary critical points along each piece:

Left edge of ∂D: $x = 0, 0 \le y \le 3$: $f(0, y) = 1 + (y-2)^2$, so $f'(0, y) = 2(y-2)$. Set this equal to 0 and solve for y to get $y = 2$, which is the point $(0, 2)$.

Right edge of ∂D: $x = 3, 0 \le y \le 3$: $f(3, y) = 4 + (y-2)^2 + 6y$, so $f'(2, y) = 2(y-2) + 6 = 2y + 2$. Set this equal to 0 and solve for y to get $y = -1$, which isn't in D, so we throw it out.

Top of ∂D: $y = 3, 0 \le x \le 3$: $f(x, 3) = (x-1)^2 + 1 + 6x$, so $f'(x, 3) = 2(x-1) + 6 = 2x + 4$. Set this equal to 0 and solve for x to get $x = -2$, which isn't in D, so we throw it out.

Bottom of ∂D: $y = 0, 0 \le x \le 3$: $f(x, 0) = (x-1)^2 + 4$, so $f'(x, 0) = 2(x-1) = 2x - 2$. Set this equal to 0 and solve for x to get $x = 1$, which is the point $(1, 0)$.

Last, we test the critical points, the boundary critical points and the corners in the function f:

CP	none	
BCP	$(0,2)$	$f(0,2) = 1$
BCP	$(1,0)$	$f(1,0) = 4$
CORNER	$(0,0)$	$f(0,0) = 5$
CORNER	$(3,0)$	$f(3,0) = 8$
CORNER	$(0,3)$	$f(0,3) = 2$
CORNER	$(3,3)$	$f(3,3) = 23$

Therefore the absolute minimum value is $\boxed{1}$, occurring at $(0, 2)$.

3. Define $f(x, y, z) = 2x^2 + 5y^2 + z^2$ so that the ellipsoid is the level surface at height 31. Then a normal vector to the tangent plane is $\mathbf{n} = \nabla f(-1, 2, -3) = (4x, 10y, 2z)|_{(-1,2,-3)} = (-4, 20, -6)$. So the tangent plane has equation

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) &= 0 \\ (-4, 20, -6) \cdot (x+1, y-2, z+3) &= 0 \\ -4(x+1) + 20(y-2) - 6(z+3) &= 0 \\ -4x + 20y - 6z &= 62. \end{aligned}$$

4. a) Since $f : \mathbb{R}^3 \to \mathbb{R}$, $Df(x, y, z)$ is 1×3.

   b) $\nabla f(x, y, z) = (f_x, f_y, f_z) = (6xyz^4 - 2z^2 + 6xy^5, 3x^2z^4 + 15x^2y^4, 12x^2yz^3 - 4xz)$.

   c) $\frac{\partial^3 f}{\partial x \partial z \partial x} = f_{xzx} = (6xyz^4 - 2z^2 + 6xy^5)_{zx} = (24xyz^3 - 4z)_x = 24yz^3$.

   d) First, we need a unit vector for the direction, so set $\mathbf{u} = \frac{(2,1,-2)}{||(2,1,-2)||} = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$. Next, the gradient at $(1, 0, -1)$ is (using our answer to part (b))

$$(6xyz^4 - 2z^2 + 6xy^5, 3x^2z^4 + 15x^2y^4, 12x^2yz^3 - 4xz)|_{(1,0,-1)} = (-2, 3, 4).$$

Therefore

$$D_{\mathbf{u}} f(1,0,-1) = \nabla f(1,0,-1) \cdot \mathbf{u} = (-2,3,4) \cdot \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \boxed{-3}.$$

5. A bee is flying around $\mathbb{R}^3$ so that its position at time t, measured in seconds, is $\mathbf{x}(t) = (\cos 2t, 3t, \sin 2t)$ ft.

   a) The speed of the bee is

   $$||\mathbf{v}(t)|| = ||\mathbf{x}'(t)|| = ||(-2\sin 2t, 3, 2\cos 2t)|| = \sqrt{4\sin^2 2t + 9 + 4\cos^2 2t} = \sqrt{13},$$

   so the distance travelled by the bee from time 0 to time π is

   $$\int_0^\pi ||\mathbf{v}(t)||\, dt = \int_0^\pi \sqrt{13}\, dt = t\sqrt{13}\Big|_0^\pi = \boxed{\pi\sqrt{13} \text{ ft}}.$$

   b) $\mathbf{a}(t) = \mathbf{x}''(t) = (-4\cos 2t, 0, -4\sin 2t)$, so $\mathbf{a}(0) = \boxed{(-4,0,0) \text{ ft/sec}^2}$.

   c) First, notice that the velocity at time 0 is $\mathbf{v}(0) = (-2\sin 0, 3, 2\cos 0) = (0,3,2)$. Therefore, applying our work in part (a) when we figured the speed, the curvature at time 0 is

   $$\begin{aligned}\kappa(0) = \frac{||\mathbf{v}(0) \times \mathbf{a}(0)||}{||\mathbf{v}(0)||^3} &= \frac{||(0,3,2) \times (-4,0,0)||}{\left(\sqrt{13}\right)^3} \\ &= \frac{||(0,-8,12)||}{13\sqrt{13}} \\ &= \frac{\sqrt{8^2+12^2}}{13\sqrt{13}} \\ &= \frac{\sqrt{208}}{13\sqrt{13}} = \boxed{\frac{4}{13} \text{ ft}^{-1}}.\end{aligned}$$

6. a) h has one local minimum in the viewing window at $\boxed{(3,-1)}$.

   b) h has $\boxed{\text{no saddles}}$ in the viewing window.

   c) $h_y(1,-4) \approx \frac{h(1,-3)-h(1,-4)}{-3-(-4)} = \frac{4-10}{1} = \boxed{-6}$. (Answers may vary, but should be between -6 and -10.)

   d) Anywhere where the direction of greatest increase in h is southwest works; for example, $\boxed{(-1,-4)}$ or $\boxed{(0,-3)}$.

   e) $D_{\mathbf{u}} h(2,3)$ is $\boxed{\text{positive}}$ because if you move from point $(2,3)$ in the direction $\mathbf{u}$, the values of h will increase.

7. Find all critical points of the function $f(x, y) = x^3 - 2xy^2 - 6x$. Classify each critical point as a local maximum, local minimum or saddle.

   Start by setting the gradient equal to $\mathbf{0}$ to find critical points: $\nabla f(x, y) = (3x^2 - 2y^2 - 6, -4xy)$, so we get the system

$$\begin{cases} 3x^2 - 2y^2 - 6 &= 0 \\ -4xy &= 0 \end{cases} \Rightarrow \quad x = 0 \text{ or } y = 0$$

   If $x = 0$, then by substituting into the first equation we get $-2y^2 - 6 = 0$ so $y^2 = -3$, which has no solution. However, if $y = 0$, then by substituting into the first equation we get $3x^2 - 6 = 0$ so $x^2 = 2$ so $x = \pm\sqrt{2}$. Thus there are two critical points: $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$.

   To classify the critical points, compute the Hessian:

$$Hf(x, y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6x & -4y \\ -4y & -4x \end{pmatrix}.$$

   Therefore:

$$Hf(\sqrt{2}, 0) = \begin{pmatrix} 6\sqrt{2} & 0 \\ 0 & -4\sqrt{2} \end{pmatrix}; \det Hf(\sqrt{2}, 0) = -48 < 0 \Rightarrow \boxed{(\sqrt{2}, 0) \text{ is a saddle}}.$$

$$Hf(-\sqrt{2}, 0) = \begin{pmatrix} -6\sqrt{2} & 0 \\ 0 & 4\sqrt{2} \end{pmatrix}; \det Hf(-\sqrt{2}, 0) = -48 < 0 \Rightarrow \boxed{(-\sqrt{2}, 0) \text{ is a saddle}}.$$

8. Use Lagrange's method, with constraint $g(x, y, z) = 1$ for $g(x, y, z) = x + y + z$. $\nabla f(x, y, z) = (f_x, f_y, f_z) = (\ln x + 1, \ln y + 1, \ln z + 1)$ and $\nabla g(x, y, z) = (1, 1, 1)$, so we obtain

$$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} \ln x + 1 &= \lambda(1) \\ \ln y + 1 &= \lambda(1) \\ \ln z + 1 &= \lambda(1) \end{cases}$$
$$\text{Constraint} \Rightarrow \quad x + y + z = 1$$

   It is apparent from the first three equations that $\ln x + 1 = \ln y + 1 = \ln z + 1$, so $\ln x = \ln y = \ln z$ and therefore $x = y = z$. From the constraint, we conclude $x = y = z = \frac{1}{3}$. Answering the question that was asked, we see that the minimum value of the utility f is

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}\ln\frac{1}{3} + \frac{1}{3}\ln\frac{1}{3} + \frac{1}{3}\ln\frac{1}{3} = \boxed{\ln\frac{1}{3}}.$$

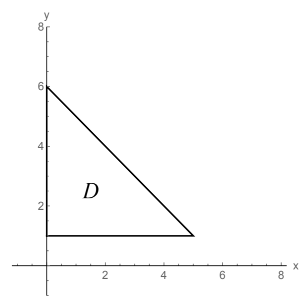
3.7 Spring 2021 Exam 2

Note: Problems 1 and 2 of this exam were open-note and unlimited time; the rest of the exam was closed-note with a time limit of 60 minutes.

1. (6.2) Determine the absolute maximum and absolute minimum values obtained by the function

$$f(x, y) = 2y^2 - x^2 + 4x - 12y$$

on the region D, which is the triangle with vertices $(0, 1)$, $(5, 1)$ and $(0, 6)$ shown below:



2. Suppose an object is moving in $\mathbb{R}^4$, so that its position (in meters) at time t (in seconds) is

$$\mathbf{x}(t) = (4\cos t, \sin 3t, 4\sin t, \cos 3t).$$

   a) (5.1) Compute and simplify the speed of the object at time t.
   b) (5.2) What is the tangential component of the object's acceleration at time t?
   c) (5.4) Compute the curvature of the path the object travels at time t.

3. Let S be the surface consisting of the points $(x, y, z) \in \mathbb{R}^3$ satisfying the equation

$$2x^3 + x^2y - xyz^2 - 3y^2z = 12.$$

   a) (4.5) Write the normal equation of the plane that is tangent to S at the point $(1, 2, -1)$.
   b) (4.5) Use your answer to part (a) to estimate the value of x so that $(x, 1.8, -.9)$ is on the surface.

4. The parts of this question are unrelated to one another.

   a) (4.1) Compute the total derivative of the function g, where

$$\mathbf{g}(x, y, z) = (x\sin(yz), y + \cos(yz)).$$

b) (4.5) Let $h(x, y) = x^2 - 2y^2$. Compute the direction in which h is increasing most rapidly at the point $(3, 1)$.

c) (4.4) Suppose $y^2e^{4x} - x^2e^y = 8$. Compute $\frac{dy}{dx}$.

5. A dog is running around a yard, so that its velocity at time t, in km/min, is

$$\mathbf{v}(t) = \left(\sqrt{t^2 - 1}, \sqrt{3t^2 + 1}\right).$$

   a) (5.1) Compute the speed of the dog at time 2.

   b) (5.1) Compute the distance the dog travels from time 1 to time 2.

6. (6.1) Find the two critical points of the function $f(x, y) = y^3 + 3xy + x^2 + 5x$. Classify each critical point as a local maximum, local minimum or saddle.

7. (6.3) Find the point (x, y) where $f(x, y) = x - 2y$ is maximized, subject to the constraint $x^2 + 5y^2 = 1$.

Solutions

1. First, find critical points of f: $\nabla f(x,y) = (-2x+4, 4y-12) = (0,0)$ when $(x,y) = (2,3)$.

   Second, we parameterize the bottom of the triangle by $(t,1)$ where $0 \leq t \leq 5$. Substitute into the utility to get $f(t,1) = 2 - t^2 + 4t - 12 = -t^2 + 4t - 10$. Differentiate to get $f'(t) = -2t + 4$; setting this equal to 0 and solving for t gives $t = 2$, yielding the boundary critical point $(2,1)$.

   Third, we parameterize the left-side of the triangle by $(0,t)$ where $1 \leq t \leq 6$. Substitute into the utility to get $f(0,t) = 2t^2 - 12y$; differentiate to get $f'(t) = 4t - 12$. Setting this equal to 0, we get $t = 3$, yielding boundary critical point $(0,3)$.

   Fourth, we parametrize the diagonal side of the triangle; this line has equation $y = 6 - x$, so we parametrize it by $(t, 6-t)$. Substitute into the utility to get $f(t,6-t) = 2(6-t)^2 - t^2 + 4t - 12(6-t) = 2(36-12t+t^2) - t^2 + 4t - 72 + 12t = t^2 - 8t$. Differentiate to get $f'(t) = 2t - 8$; set equal to 0 to get boundary critical point $t = 4$, i.e. $(4,2)$.

   Last, we test the critical point, the boundary critical points, and the corners, by plugging all of them into the utility:

$$\begin{array}{ll}
\text{CP } (2,3): & f(2,3) = 2(9) - 4 + 8 - 36 = -14 \\
\text{BCP } (2,1): & f(2,1) = 2 - 4 + 8 - 12 = -6 \\
\text{BCP } (0,3): & f(0,3) = 18 - 36 = \boxed{-18} \longleftarrow \text{ ABS MIN} \\
\text{BCP } (4,2): & f(4,2) = 2(2^2) - 4^2 + 4(4) - 12(2) = 8 - 24 = -16 \\
\text{CORNER } (0,1): & f(0,1) = 2 - 12 = -10 \\
\text{CORNER } (0,6): & f(0,6) = 2(36) - 12(6) = \boxed{0} \longleftarrow \text{ ABS MAX} \\
\text{CORNER } (5,1): & f(5,1) = 2 - 25 + 20 - 12 = -15
\end{array}$$

2. a) The speed is

$$\begin{aligned}
||\mathbf{v}(t)|| = ||\mathbf{x}'(t)|| &= ||(-4\sin t, 3\cos 3t, 4\cos t, -3\sin 3t)|| \\
&= \sqrt{16\sin^2 t + 9\cos^2 3t + 16\cos^2 t + 9\sin^2 3t} \\
&= \sqrt{16+9} = \boxed{5 \text{ m/sec}}.
\end{aligned}$$

   b) $a_T(t) = \frac{ds}{dt} = \frac{d}{dt}||\mathbf{v}(t)|| = \frac{d}{dt}(5) = \boxed{0 \text{ m/sec}^2}$.

   c) First, the acceleration is

$$\mathbf{a}(t) = \mathbf{x}''(t) = (-4\cos t, -9\sin 3t, -4\sin t, -9\cos 3t),$$

and the norm of the acceleration is

$$||\mathbf{a}(t)|| = \sqrt{16\cos^2 t + 81\sin^2 3t + 16\sin^2 t + 81\cos^2 3t} = \sqrt{16+81} = \sqrt{97}.$$

Next, the normal component of the acceleration is

$$a_N(t) = \sqrt{||\mathbf{a}(t)||^2 - a_T(t)^2} = \sqrt{97-0} = \sqrt{97}\,\text{m/sec}^2.$$

Last, the normal component is the curvature, times the speed squared, so we have

$$\begin{aligned} a_N(t) &= \kappa(t)\left(\frac{ds}{dt}\right)^2 \\ \sqrt{97} &= \kappa(t)(5)^2 \\ \boxed{\frac{1}{25}\sqrt{97}\,\text{m}^{-1}} &= \kappa(t). \end{aligned}$$

3. a) Let $f(x,y,z) = 2x^3 + x^2y - xyz^2 - 3y^2z$. S is the level surface to f at height 12, so a normal vector $\mathbf{n}$ to this surface is given by $\nabla f(1,2,-1)$. Observe

$$\nabla f(x,y,z) = (6x^2 + 2xy - yz^2, x^2 - xz^2 - 6yz, -2xyz - 3y^2)$$

so

$$\mathbf{n} = \nabla f(1,2,-1) = (6+4-2, 1-1+12, 4-12) = (8,12,-8).$$

Let $\mathbf{p} = (1,2,-1)$; the normal equation of the plane is $\mathbf{n}\cdot(\mathbf{x}-\mathbf{p}) = 0$, i.e.

$$8(x-1) + 12(y-2) - 8(z+1) = 0$$

which simplifies to

$$\boxed{2x + 3y - 2z = 10}.$$

b) Plug in $(x, 1.8, -.9)$ to the tangent plane to get $2x + 3(1.8) - 2(-.9) = 10$, i.e. $2x + 7.2 = 10$, i.e. $\boxed{x = 1.4}$.

4. a) Write $\mathbf{g} = (g_1, g_2)$. Compute the partials and arrange them in a 2×3 matrix:

$$Dg(x,y,z) = \begin{pmatrix} (g_1)_x & (g_1)_y & (g_1)_z \\ (g_2)_x & (g_2)_y & (g_2)_z \end{pmatrix} = \boxed{\begin{pmatrix} \sin(yz) & xz\cos(yz) & xy\cos(yz) \\ 0 & 1 - z\sin(yz) & -y\sin(yz) \end{pmatrix}}.$$

b) The direction in which h is increasing most rapidly at the point $(3,1)$ is

$$\nabla h(3,1) = (2x, -4y)|_{(3,1)} = (2(3), -4(1)) = \boxed{(6,-4)}.$$

c) Let $f(x,y) = y^2e^{4x} - x^2e^y$. Since f is constant,

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \boxed{\frac{-(4y^2e^{4x} - 2xe^y)}{2ye^{4x} - x^2e^y}}.$$

5. a) $||\mathbf{v}(2)|| = ||(\sqrt{3}, \sqrt{13})|| = \sqrt{3+13} = \boxed{4 \text{ km/min}}$.

b) This is an arc length calculation:

$$\begin{aligned}\int_1^2 ||\mathbf{v}(t)||\,dt &= \int_1^2 ||(\sqrt{t^2-1}, \sqrt{3t^2+1})||\,dt \\ &= \int_1^2 \sqrt{(\sqrt{t^2-1})^2 + (\sqrt{3t^2+1})^2}\,dt \\ &= \int_1^2 \sqrt{t^2-1+3t^2+1}\,dt \\ &= \int_1^2 \sqrt{4t^2}\,dt \\ &= \int_1^2 2t\,dt = t^2\Big|_1^2 = \boxed{3 \text{ km}}.\end{aligned}$$

6. To find the CPs, set $\nabla f(x,y) = \mathbf{0}$. This gives

$$\begin{cases} 3y + 2x + 5 &= 0 \\ 3y^2 + 3x &= 0 \end{cases}$$

From the second equation, $x = -y^2$ so substituting into the first equation gives $3y - 2y^2 + 5 = 0$, i.e. $2y^2 - 3y - 5 = 0$. This factors as $(2y-5)(y+1) = 0$, so $y = \frac{5}{2}$ or $y = -1$. When $y = \frac{5}{2}$, $x = -y^2 = -\frac{25}{4}$ and when $y = -1$, $x = -y^2 = -1$ so we get the two critical points $\left(-\frac{25}{4}, \frac{5}{2}\right)$ and $(-1,-1)$. To classify these points, use the Hessian:

$$Hf(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 6y \end{pmatrix}.$$

Now $Hf(\left(-\frac{25}{4}, \frac{5}{2}\right) = \begin{pmatrix} 2 & 3 \\ 3 & 15 \end{pmatrix}$ has positive trace and determinant, so is positive definite. Therefore $\boxed{\left(-\frac{25}{4}, \frac{5}{2}\right) \text{ is a local minimum}}$.

Also, $Hf(-1,-1) = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$ has negative determinant, so is neither positive definite nor negative definite, so $\boxed{(-1,-1) \text{ is a saddle}}$.

Find the two critical points of the function $f(x,y) = y^3 + 3xy + x^2 + 5x$. Classify each critical point as a local maximum, local minimum or saddle.

7. Use Lagrange's method: set $g(x, y) = x^2 + 5y^2$; then $\nabla f = \lambda \nabla g$ gives

$$\begin{cases} 1 & = \lambda(2x) \\ -2 & = \lambda(10y) \end{cases}$$

From the first equation, $\lambda = \frac{1}{2x}$. Substituting into the second equation, we get $-2 = \frac{1}{2x}(10y) = \frac{5y}{x}$, so $y = -\frac{2}{5}x$. Substituting into the constraint gives

$$x^2 + 5\left(-\frac{2}{5}x\right)^2 = 1 \;\Rightarrow\; x^2 + \frac{4}{5}x^2 = 1 \;\Rightarrow\; \frac{9}{5}x^2 = 1 \;\Rightarrow\; x = \pm\frac{\sqrt{5}}{3}.$$

From $y = -\frac{2}{5}x$, we get the two candidate points $\left(\frac{\sqrt{5}}{3}, -\frac{2\sqrt{5}}{15}\right)$ and $\left(-\frac{\sqrt{5}}{3}, \frac{2\sqrt{5}}{15}\right)$. Test these:

$$f\left(\frac{\sqrt{5}}{3}, -\frac{2\sqrt{5}}{15}\right) = \frac{\sqrt{5}}{3} - 2\left(-\frac{2\sqrt{5}}{15}\right) = \frac{3}{5}\sqrt{5};$$
$$f\left(-\frac{\sqrt{5}}{3}, \frac{2\sqrt{5}}{15}\right) = -\frac{\sqrt{5}}{3} - 2\left(\frac{2\sqrt{5}}{15}\right) = -\frac{3}{5}\sqrt{5}.$$

Therefore f is maximized at the point $\boxed{\left(\frac{\sqrt{5}}{3}, -\frac{2\sqrt{5}}{15}\right)}$.

3.8 Fall 2021 Exam 2

Note: Problems 1 and 2 of this exam were open-note and unlimited time; the rest of the exam was closed-note with a time limit of 60 minutes.

1. (5.1) A water balloon is launched from ground level, directly eastward, at an angle of $\frac{\pi}{3}$ to the horizontal, with an initial velocity of $80\sqrt{3}$ ft/sec. In addition to gravity, the water balloon experiences acceleration of magnitude $12t$ ft/sec^2 coming from a wind blowing north (where t is the number of seconds after the balloon is launched). Calculate the x- and y-coordinates of the point where the balloon lands.

2. (6.2) Compute the absolute maximum value and absolute minimum value of the function
$$f(x,y) = 3y^2 - 2xy - 2x + 4y$$
on the square D with vertices $(-2,-2)$, $(2,-2)$, $(2,2)$ and $(-2,2)$.

3. Throughout this problem, let $g(x,y,z) = e^{x-3y} + 2e^{4x-3z} - ze^{2x-y}$.

   a) (4.5) Compute the gradient of g.

   b) (4.5) Compute the directional derivative of g at the origin, in the direction $\mathbf{u} = \left(\frac{-1}{\sqrt{19}}, \frac{3}{\sqrt{19}}, \frac{-3}{\sqrt{19}}\right)$. *Note:* $\mathbf{u}$ is a unit vector.

   c) (4.5) What is the smallest possible value of the directional derivative of g at the origin, if you allow yourself to choose any direction?

   d) (4.2) Compute $\dfrac{\partial^6 g}{\partial^2 x \partial z \partial x \partial y \partial x}$.

4. Parts (a)-(c) of this question are not related to one another.

   a) (4.2) Compute the total derivative of $\mathbf{h} : \mathbb{R}^2 \to \mathbb{R}^2$, if $\mathbf{h}(x,y) = (\sin 2x \cos y, \sin xy)$.

   b) (4.5) Write a normal equation of the plane tangent to the surface $2x^2 + 3y^2 + z^4 = 27$ at the point $(2,-1,2)$.

   c) (4.4) Suppose $xyz^2 + 4x^3 z = 4$. Compute $\left.\frac{dy}{dx}\right|_{x=1,y=-1,z=2}$.

5. Throughout this problem, suppose that an object is moving in $\mathbb{R}^3$ so that its position at time t is $(t^{-1}, \ln t, 2t^2)$.

   a) (5.1) Compute the velocity of the object at time 1.

   b) (5.1) Compute the acceleration of the object at time 1.

   c) (5.4) Compute the curvature of the path the object travels at the point corresponding to time 1.

6. (6.1) Find all critical points of the function $f(x, y) = 9x^2 + 2y^2 - 3x^2y$. Classify each critical point as a local maximum, a local minimum, or a saddle, using appropriate reasoning.

7. (6.3) Find the maximum value of $f(x, y) = x^2y$, subject to the constraint $x^2 + y^4 = 5$.

Solutions

1. The acceleration of the balloon is the acceleration due to gravity plus the acceleration due to the wind, which is $\mathbf{a}(t) = (0, 12t, 0) + (0, 0, -32) = (0, 12t, -32)$. We also know the initial velocity is $\mathbf{v}(0) = \left(80\sqrt{3}\cos\frac{\pi}{3}, 0, 80\sqrt{3}\sin\frac{\pi}{3}\right) = (40\sqrt{3}, 0, 120)$ and the initial position is $\mathbf{x}(0) = (0, 0, 0)$.

   Integrate the acceleration to get the velocity:

   $$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int (0, 12t, -32)\,dt = (0, 6t^2, -32t) + \mathbf{C}.$$

   By plugging in $t = 0$, we see $\mathbf{C} = \mathbf{v}(0) = (40\sqrt{3}, 0, 120)$, so the velocity is

   $$\mathbf{v}(t) = (40\sqrt{3}, 6t^2, -32t + 120).$$

   Now integrate the velocity to get the position:

   $$\mathbf{x}(t) = \int \mathbf{v}(t)\,dt = \int (40\sqrt{3}, 6t^2, -32t+120)\,dt = (40\sqrt{3}t, 2t^3, -16t^2+120t)+\mathbf{C}.$$

   By plugging in $t = 0$, we see $\mathbf{C} = \mathbf{x}(0) = (0, 0, 0)$, so the position is

   $$\mathbf{x}(t) = (40\sqrt{3}t, 2t^3, 16t^2 + 120t).$$

   To find when the balloon lands, set the z-coordinate of the position equal to 0 and solve for t:

   $$0 = 16t^2 + 120t = 8t(2t + 15) \Rightarrow t = 0, t = \frac{15}{2}.$$

   We know $t = 0$ is when the balloon launches, so the balloon lands at time $t = \frac{15}{2}$. The position at this time is

   $$\mathbf{x}\left(\frac{15}{2}\right) = \left(40\sqrt{3}\left(\frac{15}{2}\right), 2\left(\frac{15}{2}\right)^3, 0\right) = \boxed{\left(300\sqrt{3}, \frac{3375}{4}, 0\right)}.$$

2. First, find the critical points of f: the gradient is $\nabla f(x, y) = (f_x, f_y) = (-2y - 2, 6y - 2x + 4)$; setting the gradient equal to $\mathbf{0}$ gives

   $$\begin{cases} -2y - 2 &= 0 \\ 6y - 2x + 4 &= 0 \end{cases} \Rightarrow (x, y) = (-1, -1).$$

   Next, we parametrize each of the four pieces of the boundary and find the boundary critical points (BCPs):

   TOP: This is parametrized by $(t, 2)$ for $-2 \le t \le 2$, so our function is $f(t, 2) = 12 - 4t - 2t + 8 = 20 - 6t$. Differentiate to get $f'(t) = -6$ which is never zero, so no BCPs here.

BOTTOM: This is parametrized by $(t,-2)$ for $-2 \le t \le 2$, so our function is $f(t,-2) = 12+4t-2t-8 = 4+2t$. Differentiate to get $f'(t) = 2$ which is never zero, so no BCPs here.

LEFT: This is parametrized by $(-2,t)$ for $-2 \le t \le 2$, so our function is $f(-2,t) = 3t^2+4t+4+4t = 3t^2+8t+4$. Differentiate to get $f'(t) = 6t+8$; set $f'(t) = 0$ and solve for t to get $t = \frac{-4}{3}$. This gives the BCP $(-2, \frac{-4}{3})$.

RIGHT: This is parametrized by $(2,t)$ for $-2 \le t \le 2$, so our function is $f(2,t) = 3t^2-4t-4+4t = 3t^2-4$. Differentiate to get $f'(t) = 6t$; set $f'(t) = 0$ and solve for t to get $t = 0$. This gives the BCP $(2,0)$.

Finally, take the CP, the two BCPs, and the four corners of the square and test all of them in the utility:

Point	Test
CP $(-1,-1)$	$f(-1,-1) = 3-2+2-4 = -1$
BCP $(-2,\frac{-4}{3})$	$f(-2,\frac{-4}{3}) = \frac{16}{3} - \frac{16}{3} + 4 - \frac{16}{3} = -\frac{4}{3}$
BCP $(2,0)$	$f(2,0) = \boxed{-4} \longleftarrow$ ABS MIN
CORNER $(2,2)$	$f(2,2) = 12-8-4+8 = 8$
CORNER $(2,-2)$	$f(2,-2) = 12+8-4-8 = 8$
CORNER $(-2,2)$	$f(-2,2) = 12+8+4+8 = \boxed{32} \longleftarrow$ ABS MAX
CORNER $(-2,-2)$	$f(-2,-2) = 12-8+4-8 = 0$

3. a) Compute the partial derivatives:

$$\nabla g(x,y,z) = (g_x, g_y, g_z)$$
$$= \boxed{\left(e^{x-3y} + 8e^{4x-3z} - 2ze^{2x-y}, -3e^{x-3y} + ze^{2x-y}, -6e^{4x-3z} - e^{2x-y}\right)}.$$

b) Take the dot product of the gradient and $\mathbf{u}$:

$$\begin{aligned} D_{\mathbf{u}}g(0,0,0) &= \nabla g(0,0,0) \cdot \mathbf{u} \\ &= (1+8-0, -3, -6-1) \cdot \left(\frac{-1}{\sqrt{19}}, \frac{3}{\sqrt{19}}, \frac{-3}{\sqrt{19}}\right) \\ &= \frac{1}{\sqrt{19}}(9,-3,-7) \cdot (-1,3,-3) \\ &= \frac{1}{\sqrt{19}}(-9-9+21) = \boxed{\frac{3}{\sqrt{19}}}. \end{aligned}$$

c) The smallest possible value of $D_{\mathbf{u}}g(0,0,0)$ is

$$-||\nabla g(0,0,0)|| = -||(9,-3,-7)|| = -\sqrt{9^2+(-3)^2+(-7)^2} = \boxed{-\sqrt{139}}.$$

d) By Clairaut's Theorem, we can do these partial derivatives in any order. That means

$$\begin{aligned}\frac{\partial^6 g}{\partial x^2 \partial z \partial x \partial y \partial x} &= \left(e^{x-3y} + 2e^{4x-3z} - ze^{2x-y}\right)_{xyxzxx} \\ &= \left(e^{x-3y} + 2e^{4x-3z} - ze^{2x-y}\right)_{zyxxxx} \\ &= \left(0 - 6e^{4x-3z} - e^{2x-y}\right)_{yxxxx} \\ &= \left(e^{2x-y}\right)_{xxxx} = \boxed{16e^{2x-y}}.\end{aligned}$$

4. a) Compute the partial derivatives of each component function:

$$D\mathbf{h}(x,y) = \begin{pmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{pmatrix} = \boxed{\begin{pmatrix} 2\cos 2x \cos y & -\sin 2x \sin y \\ y \cos xy & x \cos xy \end{pmatrix}}.$$

b) Let $f(x,y,z) = 2x^2 + 3y^2 + z^4$. A normal vector to the level surface at height 27 is $\nabla f(2,-1,2) = (8,-6,32)$, so the normal equation of the plane is

$$\boxed{8(x-2) - 6(y+1) + 32(z-2) = 0}.$$

c) Let $f(x,y,z) = xyz^2 + 4x^3 z$. Then, by the implicit differentiation formula, $\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-(yz^2+12x^2z)}{xz^2}$. At the given point $(1,-1,2)$, this is

$$\left.\frac{dy}{dx}\right|_{x=1,y=-1,z=2} = \frac{-(-4+24)}{4} = \boxed{-5}.$$

5. a) $\mathbf{v}(t) = \mathbf{x}'(t) = (-t^{-2}, t^{-1}, 4t)$. At time 1, this velocity is $\mathbf{v}(1) = \boxed{(-1,1,4)}$.

b) $\mathbf{a}(t) = \mathbf{v}'(t) = (2t^{-3}, -t^{-2}, 4)$. At time 1, this acceleration is $\mathbf{a}(1) = \boxed{(2,-1,4)}$.

c) The curvature is given by

$$\begin{aligned}\kappa(1) &= \frac{||\mathbf{v}(1) \times \mathbf{a}(1)||}{||\mathbf{v}(1)||^3} \\ &= \frac{||(-1,1,4) \times (2,-1,4)||}{||(-1,1,4)||^3} = \frac{||(8,12,-1)||}{(\sqrt{18})^3} = \boxed{\frac{\sqrt{209}}{18\sqrt{18}}}.\end{aligned}$$

6. Find the critical points by setting the gradient equal to zero:

$$\nabla f(x,y) = 0 \Rightarrow \begin{cases} 18x - 6xy &= 0 \\ 4y - 3x^2 &= 0 \end{cases}$$

From the second equation, $y = \frac{3}{4}x^2$. Plugging this into the first equation, we get $18x - \frac{9}{2}x^3 = 0$, i.e. $\frac{9}{2}x(4 - x^2) = 0$, i.e. $\frac{9}{2}x(2 - x)(2 + x) = 0$. This gives $x = 0$, $x = 2$ and $x = -2$, which have respective y-values $y = 0$, $y = 3$ and $y = 3$. So there are three critical points: $(0, 0)$, $(2, 3)$ and $(-2, 3)$. Test these using the Hessian:

$$Hf(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 18 - 6y & -6x \\ -6x & 4 \end{pmatrix}.$$

Now $Hf(0,0) = \begin{pmatrix} 18 & 0 \\ 0 & 4 \end{pmatrix}$ has positive determinant and positive trace, so $Hf(0,0) > 0$, making $\boxed{(0,0) \text{ a local minimum}}$.

$Hf(2,3) = \begin{pmatrix} 0 & -12 \\ -12 & 4 \end{pmatrix}$ has negative determinant, making $\boxed{(2,3) \text{ a saddle}}$.

Finally, $Hf(-2,3) = \begin{pmatrix} 0 & 12 \\ 12 & 4 \end{pmatrix}$ has negative determinant, making $\boxed{(-2,3) \text{ a saddle}}$.

7. Use Lagrange's method; the equation $\nabla f = \lambda \nabla g$, together with the constraint $x^2 + y^4 = 5$, gives the system

$$\begin{cases} 2xy &= \lambda(2x) \\ x^2 &= \lambda(4y^3) \\ x^2 + y^4 &= 5 \end{cases}$$

From the first equation, either $x = 0$ or $y = \lambda$. If $x = 0$, then from the second equation $\lambda = 0$ (which is impossible since λ is never zero in Lagrange's method) or $y = 0$. But if $x = 0$ and $y = 0$, the constraint isn't satisfied. This rules out $x = 0$ and leaves us with $y = \lambda$.

Substituting into the second equation, we get $x^2 = 4y^4$ and substituting this for x^2 in the constraint gives $4y^4 + y^4 = 5$, i.e. $5y^4 = 5$, i.e. $y^4 = 1$, i.e. $y = \pm 1$.

For either value of y, $x^2 = 4y^4$ so $x^2 = 4$ so $x = \pm 2$. This gives four candidate points: $(2, 1)$, $(2, -1)$, $(-2, 1)$ and $(-2, -1)$. Test these to find the maximum value:

$$\begin{aligned} f(\pm 2, 1) &= (\pm 2)^2(1) = \boxed{4} \longleftarrow \text{ MAX} \\ f(\pm 2, -1) &= (\pm 2)^2(-1) = -4 \end{aligned}$$

3.9 Spring 2018 Exam 3

1. (8.2) Find the curl of the vector field
$$\mathbf{f}(x,y,z) = \left(2x^2 - z, 3xz + y^2, 4y^2 + z\right)$$
at the point $(2,1,-3)$.

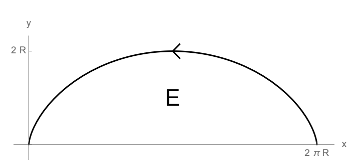
2. (7.3) Let $E = [0,2] \times [1,3]$. Compute the double integral
$$\iint_E (4x + 2y)\, dA.$$

3. (8.4) Compute the line integral
$$\int_\gamma \mathbf{f} \cdot d\mathbf{s}$$
where $\mathbf{f}(x,y) = (-y^2, 2x^2)$ and γ is the straight line from the origin to the point $(2,1)$.

4. (7.30 Compute the iterated integral
$$\int_0^1 \int_x^1 \sqrt{y^2+1}\, dy\, dx.$$

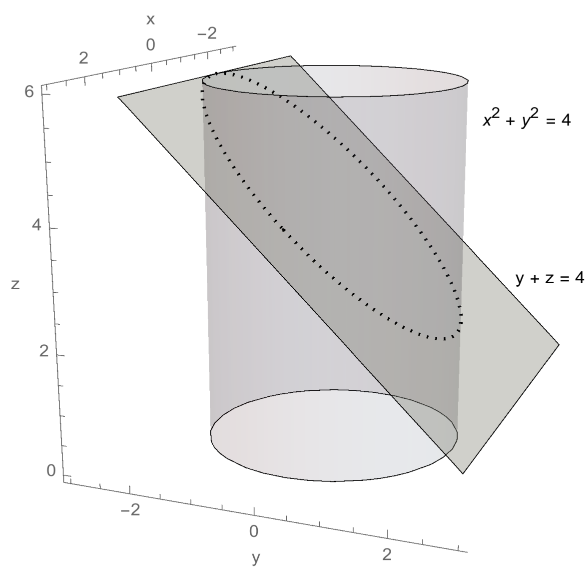
5. (7.5) Compute the double integral
$$\iint_E x\, dA$$
where E is the "pizza-slice" shaped region $\{(x,y) : 0 \le y \le x, x^2 + y^2 \le 9\}$.

6. (8.5) Let $R > 0$ be a constant. A **cycloid** is a curve parameterized (from right to left, as shown in the picture below) by the parametric equations
$$\begin{cases} x(t) & = R(2\pi + \sin t - t) \\ y(t) & = R(1 - \cos t) \end{cases}$$
Find the area of the region E consisting of points under the cycloid and above the x-axis.



Hint: One or more of the following integral facts may be useful:
$$\int \sin^2 \theta\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C \qquad \int \cos^2 \theta\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$$

7. (7.6) Find the volume of the solid in $\mathbb{R}^3$ consisting of points above the xy-plane, inside the cylinder $x^2 + y^2 = 4$, and below the plane $z + y = 4$. Here is a picture:



Solutions

1. Write $\mathbf{f} = (M, N, P) = (2x^2 - z, 3xz + y^2, 4y^2 + z)$. Then

$$\text{curl } \mathbf{f} = (P_y - N_z, M_z - P_x, N_x - M_y) = (8y - 3x, -1 - 0, 3z - 0)\,.$$

Therefore curl $\mathbf{f}(2, 1, -3) = (8(1) - 3(2), -1, 3(-3)) = (2, -1, -9)$.

2. Since E is a rectangle, we have

$$\begin{aligned}\iint_E (4x + 2y)\, dA &= \int_0^2 \int_1^3 (4x + 2y)\, dy\, dx \\ &= \int_0^2 \left[4xy + y^2\right]_1^3 \, dx \\ &= \int_0^2 [12x + 9 - 4x - 1] \, dx \\ &= \int_0^2 (8x + 8)\, dx \\ &= \left[4x^2 + 8x\right]_0^2 = 16 + 16 = 32.\end{aligned}$$

3. Parameterize the line segment γ by $x(t) = 2t$, $y(t) = t$ for $0 \le t \le 1$. That means $dx = 2\, dt$ and $dy = dt$. Therefore

$$\begin{aligned}\int_\gamma \mathbf{f} \cdot d\mathbf{s} &= \int_\gamma -y^2\, dx + 2x^2\, dy \\ &= \int_0^1 -(t)^2\, 2\, dt + 2(2t)^2\, dt \\ &= \int_0^1 (-2t^2 + 8t^2)\, dt \\ &= \int_0^1 6t^2\, dt = 2t^3\Big|_0^1 = 2.\end{aligned}$$

4. First, change the order of integration. Let E be the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 1)$; then

$$\begin{aligned}\int_0^1 \int_x^1 \sqrt{y^2 + 1}\, dy\, dx &= \iint_E \sqrt{y^2 + 1}\, dA \\ &= \int_0^1 \int_0^y \sqrt{y^2 + 1}\, dx\, dy \\ &= \int_0^1 \left[x\sqrt{y^2 + 1}\right]_0^y \, dy \\ &= \int_0^1 y\sqrt{y^2 + 1}\, dy.\end{aligned}$$

Now use the substitution $u = y^2 + 1$. Thus $du = 2y\,dy$ so $\frac{1}{2}du = y\,dy$. As for the limits of integration, when $y = 0$, $u = 0^2 + 1 = 1$ and when $y = 1$, $u = 1^2 + 1 = 2$. So the integral becomes

$$\int_1^2 \frac{1}{2}\sqrt{u}\,du = \int_1^2 \frac{1}{2}u^{1/2}\,du = \frac{1}{3}u^{3/2}\Big|_1^2 = \frac{1}{3}\left(2^{3/2} - 1\right).$$

5. Change the integral to polar coordinates, since $E = \{(r,\theta) : 0 \le r \le 3, 0 \le \theta \le \frac{\pi}{4}\}$. Therefore

$$\begin{aligned}
\iint_E x\,dA &= \int_0^{\pi/4}\int_0^3 (r\cos\theta)r\,dr\,d\theta \\
&= \int_0^{\pi/4}\int_0^3 r^2\cos\theta\,dr\,d\theta \\
&= \int_0^{\pi/4}\left[\frac{1}{3}r^3\cos\theta\right]_0^3 d\theta \\
&= \int_0^{\pi/4} 9\cos\theta\,d\theta \\
&= 9\sin\theta\big|_0^{\pi/4} = 9\left(\frac{\sqrt{2}}{2}\right) - 0 = \frac{9\sqrt{2}}{2}.
\end{aligned}$$

6. By Green's Theorem, the area is

$$area(E) = \oint_{\partial E} -y\,dx = \int_{\gamma_1} -y\,dx + \int_{\gamma_2} -y\,dx$$

where γ_1 is the cycloid and γ_2 is the line segment across the bottom of E (running from $(0,0)$ to $(2\pi R, 0)$.

Now γ_2 is parameterized by $x(t) =$ something, $y(t) = 0$ so $dy = 0$. This means

$$\int_{\gamma_2} -y\,dx = \int_0^1 0\text{ (something) }dt = 0,$$

so all we really have to compute is $\int_{\gamma_1} -y\,dx$. To do this integral, use the given parameterization and first compute $dx = R(\cos t - 1)\,dt$. Therefore

$$\begin{aligned}
area(E) = \oint_{\partial E} -y\,dx &= \int_{\gamma_1} -y\,dx \\
&= \int_0^{2\pi} -R(1-\cos t)R(\cos t - 1)\,dt \\
&= R^2\int_0^{2\pi}(1-\cos t)^2\,dt \\
&= R^2\int_0^{2\pi}(1 - 2\cos t + \cos^2 t)\,dt.
\end{aligned}$$

Using the given integration fact, this becomes

$$R^2\left[t - 2\sin t + \frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = R^2\left[(2\pi - 0 + \pi + 0) - (0)\right] = 3\pi R^2.$$

7. We can do this with either a double integral in polar coordinates, or a triple integral in cylindrical coordinates.

   Double integral solution: let D be the disk of radius 2 centered at the origin; we want

$$\begin{aligned}
\iint_D (4-y)\,dA &= \int_0^2 \int_0^{2\pi} (4-y) r\,d\theta\,dr \\
&= \int_0^2 \int_0^{2\pi} (4 - r\sin\theta) r\,d\theta\,dr \\
&= \int_0^2 \int_0^{2\pi} (4r - r^2\sin\theta)\,d\theta\,dr \qquad (*) \\
&= \int_0^2 \left[4r\theta + r^2\cos\theta\right]_0^{2\pi} dr \\
&= \int_0^2 \left[(8\pi r + r^2) - (0 + r^2)\right] dr \\
&= \int_0^2 8\pi r\,dr = 4\pi r^2\Big|_0^2 = 16\pi.
\end{aligned}$$

   Triple integral solution: in cylindrical coordinates, the solid E whose volume we want is

$$\{(r,\theta,z) : 0 \le r \le 2, 0 \le \theta \le 2\pi, z \le 4 - y = 4 - r\sin\theta\}.$$

   Therefore its volume is

$$\begin{aligned}
vol(E) = \iiint_E 1\,dV &= \int_0^2 \int_0^{2\pi} \int_0^{4 - r\sin\theta} r\,dz\,d\theta\,dr \\
&= \int_0^2 \int_0^{2\pi} [rz]_0^{4-r\sin\theta}\,d\theta\,dr \\
&= \int_0^2 \int_0^{2\pi} (4r - r^2\sin\theta)\,d\theta\,dr \qquad (*)
\end{aligned}$$

   This is the same integral as in the double integral solution, and is evaluated the same way to get 16π.

   Non-calculus solution: if you slice through the solid horizontally along the plane $z = 4$, the portion of the solid above the plane can be flipped over and placed on the rest of the solid to obtain a cylinder with constant height 4. Thus the volume of the solid is the same as the volume of a cylinder with radius 2 and height 4, which is $V = \pi r^2 h = \pi 2^2(4) = 16\pi$.

3.10 Fall 2020 Exam 3

NOTE: This exam did not cover Chapter 8. In Fall 2020, that chapter was skipped due to disruptions to the course schedule related to the COVID-19 pandemic.

Also, problems 1 and 2 of this exam were open-note and unlimited time; the rest of the exam was closed-note with a time limit of 60 minutes.

1. (7.5) Compute
$$\iint\limits_E x^2\, dA$$
where E is the region shown bounded by the x-axis, the lines $x + y = 1$, $x + y = 3$, and $y = 4x$.

2. (7.6) Compute the volume of the solid consisting of the points $(x, y, z) \in \mathbb{R}^3$ lying above the xy-plane, outside the cylinder $x^2 + y^2 = 16$ but inside the sphere $x^2 + y^2 + z^2 = 36$.

3. (7.2) Suppose $g : \mathbb{R}^2 \to \mathbb{R}$ is a function such that
$$\int_0^2 \int_0^2 g(x,y)\, dy\, dx = 5; \qquad \int_0^2 \int_0^4 g(x,y)\, dy\, dx = 8;$$
$$\int_0^4 \int_0^2 g(x,y)\, dy\, dx = 3; \qquad \int_0^4 \int_0^4 g(x,y)\, dy\, dx = 10.$$
Use this given information to evaluate each quantity:
   a) $\int_0^2 \int_0^4 g(x,y)\, dx\, dy$
   b) $\int_2^4 \int_2^4 g(x,y)\, dy\, dx$
   c) $\int_0^4 \int_0^2 3g(x,y)\, dx\, dy$
   d) $\int_0^4 \int_2^2 g(x,y)\, dy\, dx$
   e) $\int_0^2 \int_0^2 (2 + 5g(x,y))\, dy\, dx$

4. (7.5) Compute
$$\iiint\limits_E (x^2 + y^2 + z^2)^2\, dV$$
where E is the sphere of radius 2 centered at the origin.

5. Let $f(x,y) = x + 2y$.
   a) (7.3) Compute
$$\iint\limits_E f(x,y)\, dA$$
   where E is the rectangle with vertices $(0,0)$, $(4,0)$, $(0,2)$ and $(4,2)$.

b) (7.3) Compute

$$\iint_D f(x,y)\,dA$$

where D is the triangle with vertices $(0,0)$, $(0,2)$ and $(2,2)$.

6. (7.4) Compute the iterated integral:

$$\int_0^4 \int_0^1 \int_0^2 xz\,dx\,dy\,dz$$

7. (7.5) Compute the iterated integral:

$$\int_0^4 \int_{\sqrt{x}}^2 \left(y^3+2\right)^{3/2}\,dy\,dx$$

Solutions

1. Let $(u,v) = \varphi(x,y)$ where $u = x+y$ and $v = y/x$. Thus the region E can be described as the set of (u,v) satisfying $1 \le u \le 3$ and $0 \le v \le 4$. Next,

$$J(\varphi) = \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \det \begin{pmatrix} 1 & 1 \\ \frac{-y}{x^2} & \frac{1}{x} \end{pmatrix} = \frac{1}{x} + \frac{y}{x^2} = \frac{x+y}{x^2} = \frac{u}{x^2}.$$

We next need to back-solve for x and y in terms of u and v. Since $v = y/x$, $y = vx$ so $u = x + vx = x(1+v)$. That means $x = \frac{u}{1+v}$ and $y = vx = \frac{uv}{1+v}$. So by the change of variable formula, the integral is

$$\begin{aligned}
\int_0^4 \int_1^3 x^2 \frac{1}{|J(\varphi)|}\,du\,dv &= \int_0^4 \int_1^3 x^2 \left(\frac{x^2}{u}\right)\,du\,dv \\
&= \int_0^4 \int_1^3 \frac{x^4}{u}\,du\,dv \\
&= \int_0^4 \int_1^3 \frac{\left(\frac{u}{1+v}\right)^4}{u}\,du\,dv \\
&= \int_0^4 \int_1^3 \frac{u^3}{(1+v)^4}\,du\,dv \\
&= \int_0^4 \left[\frac{u^4}{4}(1+v)^{-4}\right]_1^3\,dv \\
&= \int_0^4 20(1+v)^{-4}\,dv \\
&= \frac{-20}{3}(1+v)^{-3}\Big|_0^4 = \frac{-20}{3}\left[\frac{1}{125} - 1\right] = \frac{-20}{3} \cdot \frac{-124}{125} = \boxed{\frac{496}{75}}.
\end{aligned}$$

2. Let S denote the solid described in the problem. In cylindrical coordinates, this solid consists of the points (r, θ, z) with $0 \leq \theta \leq 2\pi$, $4 \leq r \leq 6$ and $0 \leq z \leq \sqrt{36 - r^2}$. So the volume can be evaluated as follows:

$$\begin{aligned}
V = \iiint_S 1\, dV &= \int_0^{2\pi} \int_4^6 \int_0^{\sqrt{36-r^2}} r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi} \int_4^6 rz\big|_0^{\sqrt{36-r^2}}\, dr\, d\theta \\
&= \int_0^{2\pi} \int_4^6 r\sqrt{36 - r^2}\, dr\, d\theta \\
&\quad \text{(Here, use the } u\text{-sub } u = 36 - r^2,\ du = -2r\, dr) \\
&= \int_0^{2\pi} \int_{20}^0 \frac{-1}{2}\sqrt{u}\, du\, d\theta \\
&= \int_0^{2\pi} \frac{-1}{3} u^{3/2}\Big|_{20}^0\, d\theta \\
&= \int_0^{2\pi} \frac{1}{3}(20)^{3/2}\, d\theta = \boxed{\frac{2\pi}{3}(20)^{3/2}}.
\end{aligned}$$

3. a) By Fubini's Theorem, the order of integration can be reversed over a rectangle like $[0, 4] \times [0, 2]$, so we get $\int_0^2 \int_0^4 g(x, y)\, dx\, dy = \int_0^4 \int_0^2 g(x, y)\, dy\, dx = \boxed{3}$.

   b) By additivity, we have

$$\begin{aligned}
10 = \int_0^4 \int_0^4 g(x, y)\, dy\, dx &= \int_0^2 \int_0^4 g(x, y)\, dy\, dx + \int_2^4 \int_0^4 g(x, y)\, dy\, dx \\
&= 8 + \int_2^4 \int_0^4 g(x, y)\, dy\, dx.
\end{aligned}$$

   Therefore $\int_2^4 \int_0^4 g(x, y)\, dy\, dx = 10 - 8 = 2$. Again using additivity, we have

$$\begin{aligned}
3 = \int_0^4 \int_0^2 g(x, y)\, dy\, dx &= \int_0^2 \int_0^2 g(x, y)\, dy\, dx + \int_2^4 \int_0^2 g(x, y)\, dy\, dx \\
&= 5 + \int_2^4 \int_0^2 g(x, y)\, dy\, dx
\end{aligned}$$

   so $\int_2^4 \int_0^2 g(x, y)\, dy\, dx = 3 - 5 = -2$. Using additivitiy a third time, we get

$$\begin{aligned}
\int_2^4 \int_0^4 g(x, y)\, dy\, dx &= \int_2^4 \int_0^2 g(x, y)\, dy\, dx + \int_2^4 \int_2^4 g(x, y)\, dy\, dx \\
2 &= -2 + \int_2^4 \int_2^4 g(x, y)\, dy\, dx
\end{aligned}$$

   so $\int_2^4 \int_2^4 g(x, y)\, dy\, dx = \boxed{4}$.

c) $\int_0^4 \int_0^2 3g(x,y)\,dx\,dy = 3\int_0^2 \int_0^4 g(x,y)\,dy\,dx = 3(8) = \boxed{24}$.

d) Since the upper and lower limits on the inside integral are the same, we obtain $\int_0^4 \int_2^2 g(x,y)\,dy\,dx = \boxed{0}$.

e) $\int_0^2 \int_0^2 (2+5g(x,y))\,dy\,dx = \int_0^2 \int_0^2 2\,dy\,dx + 5\int_0^2 \int_0^2 g(x,y)\,dy\,dx = 2(2)2 + 5(5) = 8 + 25 = \boxed{33}$.

4. In spherical coordinates, E is the set of points with $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$ and $0 \le \rho \le 2$. So this integral is

$$\begin{aligned}
\iiint_E (x^2+y^2+z^2)^2\,dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^2 (\rho^2)^2 \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^2 \rho^6 \sin\varphi\,d\rho\,d\varphi\,d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \frac{1}{7}\rho^7 \sin\varphi\Big|_0^2\,d\varphi\,d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \frac{1}{7}(2)^7 \sin\varphi\,d\varphi\,d\theta \\
&= \int_0^{2\pi} \frac{-1}{7}(2)^7 \cos\varphi\Big|_0^{\pi}\,d\theta \\
&= \int_0^{2\pi} \frac{-1}{7}(2)^7(-1-1)\,d\theta \\
&= \int_0^{2\pi} \frac{2^8}{7}\,d\theta \\
&= \boxed{\frac{2^9}{7}\pi}.
\end{aligned}$$

5. a) Notice E can be described by the inequalities $0 \le x \le 4$ and $0 \le y \le 2$. Therefore, by Fubini's Theorem, we have

$$\begin{aligned}
\iint_E (x+2y)\,dA &= \int_0^4 \int_0^2 (x+2y)\,dy\,dx \\
&= \int_0^4 \left[xy+y^2\right]_0^2\,dx = \int_0^4 \left[2x+4\right]\,dx = \left[x^2+4x\right]_0^4 = \boxed{32}.
\end{aligned}$$

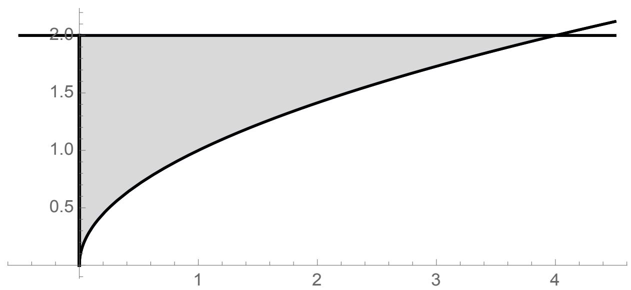
b) Notice D can be described by the inequalities $0 \le y \le 2$ and $0 \le x \le y$. Therefore, by Fubini's Theorem, we have

$$\begin{aligned}
\iint_D f(x,y)\,dA &= \int_0^2 \int_0^y (x+2y)\,dx\,dy = \int_0^2 \left[\frac{1}{2}x^2 + 2xy\right]_0^y\,dy \\
&= \int_0^2 \left[\frac{5}{2}y^2\right]\,dy = \left[\frac{5}{6}y^3\right]_0^2 = \boxed{\frac{20}{3}}.
\end{aligned}$$

6. Compute this directly:

$$\int_0^4 \int_0^1 \int_0^2 xz\,dx\,dy\,dz = \int_0^4 \int_0^1 \left[\frac{1}{2}x^2 z\right]_0^2 dy\,dz$$
$$= \int_0^4 \int_0^1 2z\,dy\,dz = \int_0^4 [2yz]_0^1\,dz = \int_0^4 2z\,dz = z^2|_0^4 = \boxed{16}.$$

7. You need to reverse the order of integration, because there's no way to come up with an antiderivative of $(y^3+2)^{3/2}$ with respect to y. A picture of the region over which you are integrating is the shaded region below (where the curve is $y = \sqrt{x}$):



This region can also be described as the set of (x, y) satisfying $0 \le y \le 2$, $0 \le x \le y^2$, so after reversing the order of the integrals we get

$$\int_0^4 \int_{\sqrt{x}}^2 \left(y^3+2\right)^{3/2} dy\,dx = \int_0^2 \int_0^{y^2} \left(y^3+2\right)^{3/2} dx\,dy$$
$$= \int_0^2 x\left(y^3+2\right)^{3/2}\Big]_0^{y^2} dy$$
$$= \int_0^2 y^2\left(y^3+2\right)^{3/2} dy$$

(Here, use the u-sub $u = y^3+2$, $du = 3y^2\,dy$)

$$= \int_2^{10} \frac{1}{3}u^{3/2}\,dy$$
$$= \frac{2}{15}u^{5/2}\Big|_2^{10} = \boxed{\frac{2}{15}\left(10^{5/2} - 2^{5/2}\right)}.$$

3.11 Spring 2021 Exam 3

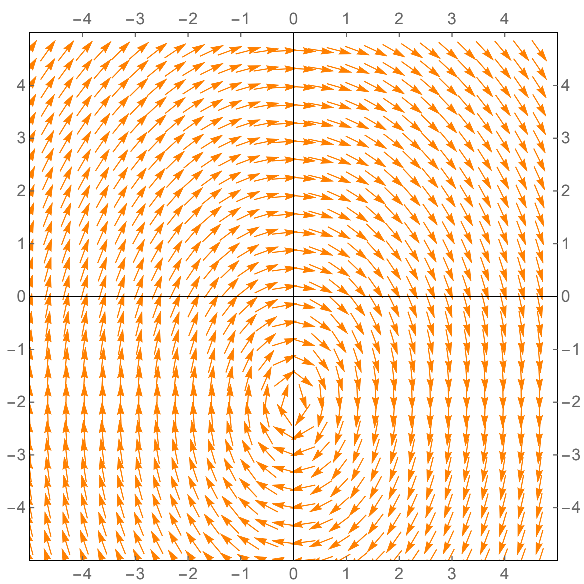
Note: Problems 1 and 2 of this exam were open-note and unlimited time; the rest of the exam was closed-note with a time limit of 60 minutes.

1. (7.5) Compute the volume of the solid consisting of the points $(x, y, z) \in \mathbb{R}^3$ lying above the xy-plane, lying outside the cone $z^2 = x^2 + y^2$, but inside the sphere $x^2 + y^2 + z^2 = 4$.

2. (7.3) Compute the exact value of this iterated integral:

$$\int_1^2 \int_{1/2}^{1/x} 2xe^{\left(y+\frac{1}{y}\right)}\, dy\, dx.$$

3. The picture of some unknown vector field $\mathbf{f}$ on $\mathbb{R}^2$ is shown here:



   a) (8.1) On the picture above, sketch the flow line to $\mathbf{f}$ passing through the point $(1, 0)$.

   b) Use the picture to answer the following questions :

      i. (8.2) Is div $\mathbf{f}(3, 2)$ positive, negative or zero?

      ii. (8.2) If you thought of $\mathbf{f}$ as a vector field on $\mathbb{R}^3$ by setting its z-coordinate equal to 0, would the z-coordinate of curl $\mathbf{f}(0, 0, 0)$ be positive, negative or zero?

      iii. (8.4) Let γ be the line segment with initial point $(0, 3)$ and terminal point $(3, 0)$. Is $\int_\gamma \mathbf{f} \cdot d\mathbf{s}$ positive, negative or zero?

4. (8.2) Compute the divergence of the vector field $\mathbf{f}(x, y) = (3e^{2x-y}, 4e^{x+5y})$.

5. (8.5) Compute the area of the region in $\mathbb{R}^2$ bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{25} = 1$.

6. (7.4) Compute $\int_0^2 \int_y^2 \int_0^1 (6y + 8xz)\ dz\,dx\,dy$.

7. (8.4) Compute $\int_\gamma f\,ds$, where $f(x, y) = 4xy$ and γ is the line segment starting at $(1, 1)$ and ending at $(5, -2)$.

8. (7.5) Compute $\int_0^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} x^2\,dy\,dx$.

9. (7.2) Compute the volume of the solid consisting of the points $(x, y, z) \in \mathbb{R}^3$ satisfying the inequalities $0 \leq z \leq 12x^2y$, $x^2 \leq y \leq x$, and $0 \leq x \leq 1$.

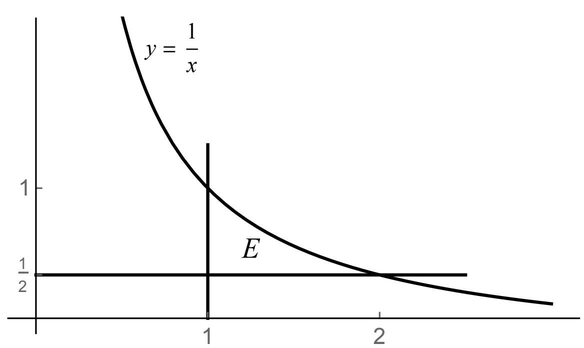
10. (7.3) Compute $\iint_E y^2\,dA$, where E is the triangle with vertices $(6, 0)$, $(6, 3)$ and $(0, 3)$.

Solutions

1. The solid S can be described in spherical coordinates as $0 \le \rho \le 2$ (since the sphere is $\rho = 2$), $0 \le \theta \le 2\pi$ and $\frac{\pi}{4} \le \varphi \le \frac{\pi}{2}$ (since the cone is $\varphi = \frac{\pi}{4}$ and the xy-plane is $\varphi = \frac{\pi}{2}$). So the volume is

$$\begin{aligned}\iiint_S 1\,dV &= \int_0^{2\pi}\int_0^2\int_{\pi/4}^{\pi/2} \rho^2 \sin\varphi\,d\varphi\,d\rho\,d\theta \\ &= \int_0^{2\pi}\int_0^2 -\rho^2\cos\varphi\Big|_{\pi/4}^{\pi/2}\,d\rho\,d\theta \\ &= \int_0^{2\pi}\int_0^2 -\rho^2\left(0 - \frac{\sqrt{2}}{2}\right)d\rho\,d\theta \\ &= \int_0^{2\pi} \frac{\sqrt{2}}{2}\cdot\frac{\rho^3}{3}\Big|_0^2\,d\theta \\ &= \int_0^{2\pi} \frac{4}{3}\sqrt{2}\,d\theta \\ &= 2\pi\cdot\frac{4}{3}\sqrt{2} = \boxed{\frac{8}{3}\pi\sqrt{2}}.\end{aligned}$$

2. This integral is $\iint_E 2xe^{\left(y+\frac{1}{y}\right)}\,dA$, where E is as shown here:



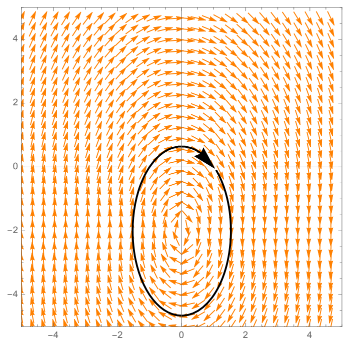
The curve $y = \frac{1}{x}$ can also be written as $x = \frac{1}{y}$. So by using Fubini's theorem to reverse the order of integration, we see that the given double integral is also

$$\begin{aligned}\int_{1/2}^1\int_1^{1/y} 2xe^{\left(y+\frac{1}{y}\right)}\,dx\,dy &= \int_{1/2}^1 \left[x^2e^{\left(y+\frac{1}{y}\right)}\right]_1^{1/y} dy \\ &= \int_{1/2}^1 \left(\frac{1}{y^2} - 1\right)e^{\left(y+\frac{1}{y}\right)}\,dy.\end{aligned}$$

Now perform the u-substitution $u = y + \frac{1}{y}$, $du = \left(1 - \frac{1}{y^2}\right)dy$ to get

$$\int_{5/2}^2 -e^u\,du = -e^u\big|_{5/2}^2 = \boxed{-e^2 + e^{5/2}}.$$

3. a) The flow line is an ellipse:



b) Use the picture to answer the following questions :

i. div $\mathbf{f}(3,2)$ is $\boxed{\text{zero}}$, since the arrows entering and leaving $(3,2)$ appear to have the same length.

ii. curl $\mathbf{f}(0,0,0)$ is $\boxed{\text{negative}}$, since the vector field rotates clockwise (or by the right-hand rule).

iii. Since γ predominantly goes in the same direction as $\mathbf{f}$, $\int_\gamma \mathbf{f} \cdot d\mathbf{s}$ is $\boxed{\text{positive}}$.

4. Write the coordinates of $\mathbf{f}$ as f_1 and f_2. Then

$$\text{div}\,\mathbf{f}(x,y) = (f_1)_x + (f_2)_y = \boxed{6e^{2x-y} + 20e^{x+5y}}.$$

5. Let E be the region whose area we want; the boundary ∂E is parametrized by $x = 3\cos t, y = 5\sin t$ for $0 \le t \le 2\pi$. This means $dx = -3\sin t\,dt$ and $dy = 5\cos t\,dt$. So by the area formula coming from Green's Theorem,

$$\begin{aligned}
area(E) = \iint_E 1\,dA &= \frac{1}{2}\oint_{\partial E}(x\,dy - y\,dx) \\
&= \frac{1}{2}\int_0^{2\pi}((3\cos t)dy - (5\sin t)\,dx) \\
&= \frac{1}{2}\int_0^{2\pi}(3\cos t(5\cos t) - 5\sin t(-3\sin t))\;dt \\
&= \frac{1}{2}\int_0^{2\pi}\left(15\cos^2 t + 15\sin^2 t\right)\;dt \\
&= \frac{1}{2}\int_0^{2\pi}15\,dt = \frac{1}{2}(15)2\pi = \boxed{15\pi}.
\end{aligned}$$

6. This is a direct computation:

$$\begin{aligned}
\int_0^2 \int_y^2 \int_0^1 (6y + 8xz)\ dz\,dx\,dy &= \int_0^2 \int_y^2 \left[6yz + 4xz^2\right]_0^1\ dx\,dy \\
&= \int_0^2 \int_y^2 (6y + 4x)\ dx\,dy \\
&= \int_0^2 \left[6yx + 2x^2\right]_y^2\ dy \\
&= \int_0^2 \left[(12y + 8) - (6y^2 + 2y^2)\right]\ dy \\
&= \int_0^2 (-8y^2 + 12y + 8)\,dy \\
&= \left[-\frac{8}{3}y^3 + 6y^2 + 8y\right]_0^2 \\
&= -\frac{64}{3} + 24 + 16 = \boxed{\frac{56}{3}}.
\end{aligned}$$

7. Parametrize γ by $\mathbf{x}(t) = (x(t), y(t)) = (1 + 4t, 1 - 3t)$ for $0 \leq t \leq 1$. Then, $||\mathbf{x}'(t)|| = ||\,(4, -3)\,|| = 5$. So the line integral is

$$\begin{aligned}
\int_\gamma f\,ds &= \int_0^1 f\,(1 + 4t, 1 - 3t)\ ||\mathbf{x}'(t)||\,dt \\
&= \int_0^1 4(1 + 4t)(1 - 3t)5\,dt \\
&= \int_0^1 20(1 + t - 12t^2)\,dt \\
&= 20t + 10t^2 - 80t^3\Big|_0^1 = 20 + 10 - 80 = \boxed{-50}.
\end{aligned}$$

8. Change to polar coordinates (this is the right-half of a circle of radius 5 centered at the origin):

$$\begin{aligned}
\int_0^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} x\,dy\,dx &= \int_{-\pi/2}^{\pi/2} \int_0^5 (r\cos\theta)r\,dr\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} \int_0^5 r^2\cos\theta\,dr\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} \frac{1}{3}r^3\cos\theta\Big|_0^5\ d\theta \\
&= \frac{125}{3}\int_{-\pi/2}^{\pi/2} \cos\theta\,d\theta \\
&= \frac{125}{3}\sin\theta\big|_{-\pi/2}^{\pi/2} = \frac{125}{3}(1 - (-1)) = \boxed{\frac{250}{3}}.
\end{aligned}$$

9. This is a direct computation, starting with either a triple integral in the first line of this solution, or the double integral in the third line:

$$\begin{aligned}\iiint_S 1\,dV &= \int_0^1 \int_{x^2}^{x} \int_0^{12x^2y} 1\,dz\,dy\,dx \\ &= \int_0^1 \int_{x^2}^{x} z\big|_0^{12x^2y}\,dy\,dx \\ &= \int_0^1 \int_{x^2}^{x} 12x^2y\,dy\,dx \\ &= \int_0^1 6x^2y^2\Big|_{x^2}^{x}\,dx \\ &= \int_0^1 (6x^4 - 6x^6)\,dx \\ &= \frac{6}{5}x^5 - \frac{6}{7}x^7\Big|_0^1 = \frac{6}{5} - \frac{6}{7} = \boxed{\frac{12}{35}}.\end{aligned}$$

10. The region is bounded by the vertical line $x = 6$, the horizontal line $y = 3$ and the diagonal line $x + 2y = 6$, i.e. $x = 6 - 2y$, i.e. $y = 6 - \frac{1}{2}x$. This integral can be done in either order, but I'll do it $dx\,dy$ to avoid using a slope of $-\frac{1}{2}$ in the line.

$$\begin{aligned}\int_0^3 \int_{6-2y}^{6} y^2\,dx\,dy &= \int_0^3 y^2x\Big|_{6-2y}^{6}\,dy \\ &= \int_0^3 \left[6y^2 - y^2(6-2y)\right]\,dy \\ &= \int_0^3 2y^3\,dy = \frac{1}{2}y^4\Big| 0^3 = \boxed{\frac{81}{2}}.\end{aligned}$$

3.12 Fall 2021 Exam 3

Note: Problems 1 and 2 of this exam were open-note and unlimited time; the rest of the exam was closed-note with a time limit of 75 minutes.

1. (7.6) Find the volume of the solid in $\mathbb{R}^3$ consisting of the points lying inside the cylinder $(x-1)^2 + y^2 = 1$, above the xy-plane, and under the graph of $f(x, y) = \sqrt{x^2 + y^2}$.

2. (8.5) Compute the area enclosed by the curve parametrized by
$$\mathbf{x}(t) = (t^2 - t^4, t^2 - t^8)$$
where $0 \le t \le 1$.

3. Throughout this problem, let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field
$$\mathbf{f}(x, y, z) = (2x - z, 3z, -2xy + 3z^3).$$
   a) (8.2) Compute the divergence of $\mathbf{f}$ at the point $(1, 2, 3)$.
   b) (8.2) What does the sign of your answer to part (a) tell you about behavior of the vector field at $(1, 2, 3)$?
   c) (8.2) Compute curl $\mathbf{f}$.
   d) (8.4) Rewrite the line integral $\int_\gamma \mathbf{f} \cdot d\mathbf{s}$ as a Riemann integral, where γ is the straight line segment starting at the origin and ending at $(-5, -2, 1)$. (The only variable allowed in your answer is t.)

4. Let $A = [0, 5] \times [0, 2]$ and let $B = [0, 5] \times [2, 4]$, and suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that $\iint_A f(x, y)\, dA = 7$ and $\iint_B f(x, y) = -2$.
   a) (7.2) Compute $\iint_{A \cup B} f(x, y)\, dA$.
   b) (7.2) Compute $\iint_{A \cap B} f(x, y)\, dA$.
   c) (7.2) Compute $\iint_A 2f(x, y)\, dA$.
   d) (7.2) Compute $\iint_A [f(x, y) + 3]\, dA$.
   e) (7.4) Compute the average value of f on A.

5. (7.5) Compute
$$\iiint_E y\, dV$$
where E is the triangular pyramid in $\mathbb{R}^3$ consisting of points (x, y, z) satisfying $x \ge 0$, $y \ge 0$, $z \ge 0$ and $2x + 2y + z \le 2$.

6. Throughout this problem, let $f(x, y) = x^2 + 3y^2$.

a) (7.3) Compute $\iint_E f(x,y)\,dA$, where E is the rectangle with vertices $(0,0)$, $(3,0)$, $(0,1)$ and $(3,1)$.

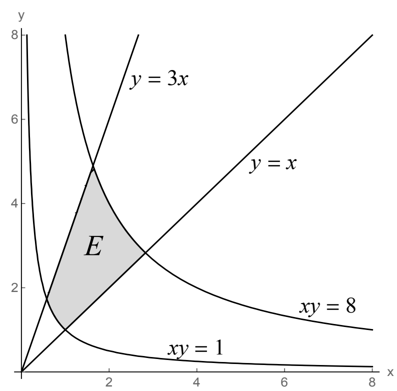
b) (8.4) Compute $\int_\gamma f\,ds$, where γ is the circle of radius 2 centered at the origin, oriented counterclockwise.

7. (7.3) Compute $\iint_E e^x\,dA$, where E is the triangle with vertices $(0,0)$, $(2,0)$ and $(4,2)$.

8. (7.5) Compute

$$\iint_E y^2\,dA$$

where E is the region pictured below:



Solutions

1. The base of the solid is a circle of radius 1 centered at $(1,0)$, which has polar equation $r = 2\cos\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Using this, the solid can be described in cylindrical coordinates by the inequalities $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $0 \le r \le 2\cos\theta$, $0 \le z \le r$. So the volume of the solid E is

$$\begin{aligned}\iiint_E 1\,dV &= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\int_0^r r\,dz\,dr\,d\theta \\ &= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} \frac{8}{3}\cos^3\theta\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} \frac{8}{3}\cos^2\theta\cos\theta\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} \frac{8}{3}(1-\sin^2\theta)\cos\theta\,d\theta.\end{aligned}$$

Now use the u-sub $u = \sin\theta$, $du = \cos\theta\, d\theta$ to get

$$\begin{aligned}\int_{-1}^{1} \frac{8}{3}(1-u^2)\,du &= \frac{8}{3}\left[u - \frac{1}{3}u^3\right]_{-1}^{1} \\ &= \frac{8}{3}\left[1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)\right] = \frac{8}{3}\left[\frac{4}{3}\right] = \boxed{\frac{32}{9}}.\end{aligned}$$

2. Use Green's Theorem with $\mathbf{f}(x,y) = (0,x)$ to get

$$area(E) = \oint_{\partial E} x\,dy.$$

Since the curve is parametrized by $x = t^2 - t^4$, $y = t^2 - t^8$, we have $dy = (2t - 8t^7)\,dt$ so we obtain

$$\begin{aligned}area(E) &= \int_0^1 (t^2 - t^4)(2t - 8t^7)\,dt \\ &= \int_0^1 (2t^3 - 2t^5 - 8t^9 + 8t^{11})\,dt \\ &= \left[\frac{1}{2}t^4 - \frac{1}{3}t^6 - \frac{4}{5}t^{10} + \frac{2}{3}t^{12}\right]_0^1 \\ &= \frac{1}{2} - \frac{1}{3} - \frac{4}{5} + \frac{2}{3} = \boxed{\frac{1}{30}}.\end{aligned}$$

3. a) div $\mathbf{f}(x,y,z) = (f_1)_x + (f_2)_y + (f_3)_z = 2 + 0 + 9z^2$, so div $\mathbf{f}(1,2,3) = 2 + 9(3^2) = \boxed{83}$.

   b) Since div $\mathbf{f}(1,2,3)$ is positive, the arrow in the picture of $\mathbf{f}$ ending at $(1,2,3)$ is shorter than the arrow in the picture of $\mathbf{f}$ starting at $(1,2,3)$, i.e. the vector field has more net flow out of $(1,2,3)$ than in, i.e. the vector field is "spreading out" at $(1,2,3)$.

   c) By the usual formula for curl,

$$\begin{aligned}\nabla \times \mathbf{f} &= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - z & 3z & -2xy + 3z^3 \end{pmatrix} \\ &= (-2x - 3, -1 - (-2y), 0 - 0) \\ &= \boxed{(-2x - 3, 2y - 1, 0)}.\end{aligned}$$

d) γ is parametrized by $\mathbf{x}(t) = (-5t, -2t, t)$ for $0 \leq t \leq$; we have $d\mathbf{s} = (-5, -2, 1)\,dt$ so the integral becomes

$$\begin{aligned}
\int_\gamma \mathbf{f} \cdot d\mathbf{s} &= \int_0^1 (2x - z, 3z, -2xy + 3z^3) \cdot (-5, -2, 1)\,dt \\
&= \int_0^1 (-10t - t, 3t, -2(-5t)(-2t) + 3t^3) \cdot (-5, -2, 1)\,dt \\
&= \int_0^1 (55t - 6t + 3t^3 - 20t^2)\,dt \\
&= \boxed{\int_0^1 (49t - 20t^2 + 3t^3)\,dt}.
\end{aligned}$$

4. a) $\iint_{A\cup B} f(x,y)\,dA = \iint_A f(x,y)\,dA + \iint_B f(x,y)\,dA = 7 + (-2) = \boxed{5}$.
   b) $\iint_{A\cap B} f(x,y)\,dA = \boxed{0}$ since $A \cap B$ has zero area.
   c) $\iint_A 2f(x,y)\,dA = 2\iint_A f(x,y)\,dA = 2(7) = \boxed{14}$.
   d) $\iint_A [f(x,y)+3]\,dA = \iint_A f(x,y)\,dA + \iint_A 3\,dA = 7+3\cdot area(A) = 7+3(5)2 = \boxed{37}$.
   e) The average value of f on A is $\frac{1}{area(A)} \iint_A f(x,y)\,dA = \frac{1}{10}(7) = \boxed{\frac{7}{10}}$.

5. The solid E can be described by the inequalities $0 \leq y \leq 1$, $0 \leq x \leq 1-y$, $0 \leq z \leq 2 - 2x - 2y$, so the integral is

$$\begin{aligned}
\iiint_E y\,dV &= \int_0^1 \int_0^{1-y} \int_0^{2-2x-2y} y\,dz\,dx\,dy \\
&= \int_0^1 \int_0^{1-y} (2y - 2xy - 2y^2)\,dx\,dy \\
&= \int_0^1 \left[2xy - x^2y - 2xy^2\right]_0^{1-y}\,dy \\
&= \int_0^1 \left[2(1-y)y - (1-y)^2y - 2(1-y)y^2\right]\,dy \\
&= \int_0^1 \left[y^3 - 2y^2 + y\right]\,dy \\
&= \left[\frac{1}{4}y^4 - \frac{2}{3}y^3 + \frac{1}{2}y^2\right]_0^1 = \frac{1}{4} - \frac{3}{2} + \frac{1}{2} = \boxed{\frac{1}{12}}.
\end{aligned}$$

6. a) Apply Fubini's Theorem:

$$\begin{aligned}\iint_E f(x,y)\,dA &= \int_0^3 \int_0^1 (x^2+3y^2)\,dy\,dx \\ &= \int_0^3 \left[x^2 y + y^3\right]_0^1 \,dx \\ &= \int_0^3 \left[x^2+1\right]\,dx \\ &= \left[\frac{1}{3}x^3 + x\right]_0^3 = \frac{1}{3}(27)+3-0 = \boxed{12}.\end{aligned}$$

b) Here, parametrize γ by $\mathbf{x}(t) = (2\cos t, 2\sin t)$ for $0 \le t \le 2\pi$. We have $||\mathbf{x}'(t)|| = ||(-2\sin t, 2\cos t)|| = \sqrt{4\sin^2 t + 4\cos^2 t} = \sqrt{4} = 2$, so the integral becomes

$$\begin{aligned}\int_\gamma f\,ds &= \int_0^{2\pi} (4\cos^2 t + 12\sin^2 t)2\,dt \\ &= \int_0^{2\pi} (8+16\sin^2 t)\,dt \\ &= \int_0^{2\pi} (8+8(1-\cos 2t)\,dt \\ &= \int_0^{2\pi} (16-8\cos 2t)\,dt \\ &= [16t - 4\sin 2t]_0^{2\pi} = \boxed{32\pi}.\end{aligned}$$

7. The triangle can be described by the inequalities $0 \le y \le 2$, $2y \le x \le y+2$, so the integral becomes

$$\begin{aligned}\iint_F f(x,y)\,dA &= \int_0^2 \int_{2y}^{y+2} e^x\,dx\,dy \\ &= \int_0^2 \left[e^{y+2} - e^{2y}\right]\,dy \\ &= \left[e^{y+2} - \frac{1}{2}e^{2y}\right]_0^2 \\ &= e^4 - \frac{1}{2}e^4 - \left[e^2 - \frac{1}{2}\right] = \boxed{\frac{1}{2}e^4 - e^2 + \frac{1}{2}}.\end{aligned}$$

8. Use the change of variable $(x,y) \overset{\varphi}{\mapsto} (u,v)$ given by $u = xy$, $v = \frac{y}{x}$, so that $\varphi(E) = \{(u,v) : 1 \le u \le 8, 1 \le v \le 3\}$. The Jacobian of φ is

$$J(\varphi) = \det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \det\begin{pmatrix} y & x \\ \frac{-y}{x^2} & \frac{1}{x} \end{pmatrix} = \frac{y}{x} + \frac{y}{x} = \frac{2y}{x} = 2v.$$

Next, solve for x and y in terms of u and v: $v = \frac{y}{x}$ gives $vx = y$, so $u = xvx$ so $x = \sqrt{\frac{u}{v}}$ and $y = vx = \sqrt{uv}$. So $y^2 = uv$ and the change of variable formula therefore gives

$$\begin{aligned}
\iint_E y^2\, dA &= \int_1^8 \int_1^3 uv \frac{1}{|2v|}\, dv\, du \\
&= \int_1^8 \int_1^3 \frac{1}{2} u\, dv\, du \quad \text{(since $v > 0$ we can disregard the $|\cdot|$)} \\
&= \int_1^8 \left[\frac{1}{2} uv\right]_1^3 du \\
&= \int_1^8 u\, du = \frac{1}{2}u^2\bigg|_1^8 = 32 - \frac{1}{2} = \boxed{\frac{63}{2}}.
\end{aligned}$$

3.13 Spring 2018 Final Exam

1. (2.3) Throughout this problem, let $\mathbf{v} = (3, 8)$ and $\mathbf{w} = (-5, 2)$.

   a) Find the norm of $\mathbf{v} - \mathbf{w}$.

   b) Find the projection of $\mathbf{w}$ onto $\mathbf{v}$.

   c) Find the cosine of the angle θ between $\mathbf{v}$ and $\mathbf{w}$.

2. (2.7) In this problem, consider the two lines l_1 and l_2, where l_1 has symmetric equations
$$\frac{x-11}{-3} = \frac{y-2}{-1} = \frac{z+11}{5}$$
and l_2 is parameterized by $\mathbf{x}(t) = (4+t, -5-2t, -2t)$.

   a) Show that lines l_1 and l_2 intersect in a point (by computing that point of intersection).

   b) Find the normal equation of the plane containing lines l_1 and l_2.

3. (3.5) Compute the following limits (or explain why they do not exist):

   a) $\lim\limits_{\mathbf{x}\to\mathbf{0}} \frac{y-x}{y+x}$

   b) $\lim\limits_{(x,y)\to(0,0)} \frac{y^2-x^2}{y+x}$

   c) $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{x^2y}{x^2+y^2+z^2}$

4. a) Compute the total derivative of $\mathbf{f}(x, y) = (xe^{xy}, ye^{2x-y})$.

   b) Find all second-order partial derivatives of $f(x, y) = 20x^2 - 10x^2y^2 + 30y^4$.

5. a) (4.2) Compute the directional derivative of $f(x, y, z) = x^2z - 3yz^2$ in the direction $(1, 2, -2)$ at the point $(3, 0, 5)$.

   b) (4.2) Compute div $\mathbf{f}$ where $\mathbf{f}(x, y) = (\sin(2x - y), \cos(2x + y))$.

6. Let $\mathbf{x}(t) = \left(2t^2 + 3, t, \frac{4}{3}\sqrt{2}t^{3/2} + 1\right)$ represent the position of an object at time t.

   a) (5.2) Find the tangential and normal components of the object's acceleration at time $t = 2$.

   b) (5.2) At time $t = 2$, is the object speeding up or slowing down? Justify your answer.

7. (6.2 or 6.3) Find the absolute maximum value of the function $f(x, y) = x^2y^4$ on the region $\{(x, y) : x^2 + y^2 \leq 36\}$.

8. Consider the surface $z = 6\sin x \cos y + 8$.

   a) (4.3) Find the equation of the plane which is tangent to this surface at $(\pi, 0, 5)$.

   b) (7.3) Find the volume of the solid consisting of points in $\mathbb{R}^3$ lying above the rectangle $[0, \frac{\pi}{2}] \times [0, \frac{\pi}{3}]$ in the xy-plane, but below this surface.

9. (7.5) Compute the double integral

$$\iint_E (xy - x^2)\, dA$$

   where E is the parallelogram with vertices $(2, 0)$, $(6, 4)$, $(4, 8)$ and $(0, 4)$.

10. Let E be a circle of radius R.

    a) (7.5) Show that E has area πR^2, by computing a double integral with polar coordinates.

    b) (8.5) Show that E has area πR^2, by computing an appropriate line integral and using Green's Theorem.

11. a) (8.4) Compute

$$\int_\gamma (xy + yz)\, ds$$

    where γ is the line segment from $(0, 1, 2)$ to $(5, 3, 3)$.

    b) (8.4) Compute

$$\int_\gamma \mathbf{f} \cdot d\mathbf{s}$$

    where $\mathbf{f}(x, y, z) = (2xy^2z, 2x^2yz, x^2y^2)$ and γ is parameterized by

$$\mathbf{x}(t) = \left(te^{\sin \pi t}, t^4\sqrt{\tan \pi t + 1}, t^{2018}\right).$$

    for $0 \le t \le 1$.

12. (7.6) Choose one of (a) or (b):

    a) Compute

$$\iiint_E y\, dV$$

    where E is the set of points (x, y, z) in the first octant lying below the plane $2x + 4y + z = 12$.

    b) Compute the volume of the set of points (x, y, z) inside the cylinder $x^2 + y^2 = 1$ lying above the xy-plane but below the sphere of radius 2 centered at the origin.

Solutions

1. a) $||\mathbf{v}-\mathbf{w}|| = ||(3,8)-(-5,2)|| = ||(8,6)|| = \sqrt{8^2+6^2} = \sqrt{100} = 10.$

   b) $\pi_{\mathbf{v}}(\mathbf{w}) = \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w} = \frac{1}{73}(3,8) = \left(\frac{3}{73}, \frac{8}{73}\right).$

   c) $\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\,||\mathbf{w}||} = \frac{1}{\sqrt{3^2+8^2}\sqrt{(-5)^2+2^2}} = \frac{1}{\sqrt{73}\sqrt{29}}.$

2. a) l_1 passes through $(11,2,-11)$ and has direction vector $(-3,-1,5)$ so we can write the parametric equations of l_1 as $\mathbf{y}(s) = (-3s+11, -s+2, 5s-11)$. Now we set the coordinates of $\mathbf{x}(t)$ equal to the coordinates of $\mathbf{y}(s)$:

$$\begin{cases} -3s+11 &= 4+t \\ -s+2 &= -5-2t \\ 5s-11 &= -2t \end{cases}$$

   Subtracting the third equation from the first gives $-6s+13 = -5$, i.e. $s = 3$; therefore $t = -2$. These values of s and t work in all three equations and produce the intersection point $\mathbf{x}(-2) = \mathbf{y}(3) = (2,-1,4)$.

   b) To get the normal vector to the plane, take the cross product of the direction vectors of the two lines:

$$\mathbf{n} = (-3,-1,5)\times(1,-2,-2) = (12,-1,7)$$

   Then the equation of the plane is

$$\begin{aligned} \mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0) &= 0 \\ \Rightarrow (12,-1,7)\cdot((x,y,z)-(11,2,-11)) &= 0 \\ \Rightarrow (12,-1,7)\cdot(x-11, y-2, z+11) &= 0 \\ \Rightarrow 12(x-11)-(y-2)+7(z+11) &= 0 \\ \Rightarrow 12x-y+7z &= 53 \end{aligned}$$

3. a) $\lim_{\mathbf{x}\to\mathbf{0}} \frac{y-x}{y+x}$ DNE (along the x-axis, the limit is $\lim_{(x,0)\to(0,0)} \frac{0-x}{0+x} = -1$, but along the y-axis, the limit is $\lim_{(0,y)\to(0,0)} \frac{y-0}{y+0} = 1$.)

   b) $\lim_{(x,y)\to(0,0)} \frac{y^2-x^2}{y+x} = \lim_{(x,y)\to(0,0)} \frac{(y-x)(y+x)}{y+x} = \lim_{(x,y)\to(0,0)} y-x = 0.$

   c) Change to polar coordinates to get

$$\lim_{\rho\to 0} \frac{(\rho^2\sin^2\varphi\cos^2\theta)(\rho\sin\varphi\sin\theta)}{\rho^2} = \lim_{\rho\to 0} \rho(\sin^3\varphi\cos^2\theta\sin\theta) = 0,$$

   no matter what φ and θ are.

4. a) This is a direct computation:

$$Df(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} e^{xy} + xye^{xy} & x^2e^{xy} \\ 2ye^{2x-y} & e^{2x-y} - ye^{2x-y} \end{pmatrix}.$$

b) First, $f_x(x,y) = 40x - 20xy^2$ and $f_y(x,y) = -20x^2y + 120y^3$. That means

$$\begin{aligned} f_{xx}(x,y) &= 40 - 20y^2 \\ f_{yy}(x,y) &= -20x^2 + 360y^2 \\ f_{xy}(x,y) &= f_{yx}(x,y) = -40xy \end{aligned}$$

5. a) First, find a unit vector in the direction $(1,2,-2)$:

$$\mathbf{u} = \frac{(1,2,-2)}{||(1,2,-2)||} = \frac{(1,2,-2)}{\sqrt{1+4+4}} = \left(\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}\right).$$

Next, the gradient of f is $\nabla f = (2xz, -3z^2, x^2 - 6yz)$ so $\nabla f(3,0,5) = (2(3)5, -3(5^2), 3^2 - 6(0)5^2) = (30, -75, 9)$. Therefore the directional derivative is

$$D_{\mathbf{u}}f(3,0,5) = \nabla f(3,0,5) \cdot \mathbf{u} = (30,-75,9) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}\right) = 10 - 50 - 6 = -46.$$

b) div $\mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = 2\cos(2x - y) - \sin(2x + y)$.

6. a) At time $t \geq 0$, the velocity is $\mathbf{x}'(t) = (4t, 1, \sqrt{8t})$, and the speed is

$$s(t) = ||\mathbf{x}'(t)|| = \sqrt{(4t)^2 + 1 + 8t} = \sqrt{16t^2 + 8t + 1} = \sqrt{(4t+1)^2} = 4t + 1.$$

Therefore $a_T = \left.\frac{ds}{dt}\right|_{t=2} = 4$.

Now for the normal component. At time t,

$$\mathbf{a}(t) = \mathbf{x}''(t) = \left(4, 0, \sqrt{\frac{2}{t}}\right)$$

so at time $t = 2$, the acceleration is $\mathbf{a}(2) = (4, 0, 1)$. The normal component of the acceleration is

$$a_N = \sqrt{||\mathbf{a}(2)||^2 - a_T^2} = \sqrt{17 - 16} = 1.$$

b) The object is speeding up when $t = 2$. $a_T = \frac{ds}{dt}$, the rate of change of the speed with respect to time. Since $a_T = 4 > 0$, the speed is increasing.

7. First, find the critical points of f: the gradient is $\nabla f = (2xy^4, 4x^2y^3)$; setting this equal to **0** we get $x = 0$ and/or $y = 0$, in which case $f(x, y) = 0$.

   Second, we have to study the behavior of f along the boundary of the constraint $x^2 + y^2 = 36$: let $g(x, y) = x^2 + y^2$ and use Lagrange multipliers to maximize f subject to $g(x, y) = 16$: $\nabla f = (2x, 2y)$ so we have

$$\nabla f = \lambda \nabla g \;\Rightarrow\; \begin{cases} 2xy^4 = 2\lambda x \\ 4x^2y^3 = 2\lambda y \end{cases}$$

   From the first equation, $y^4 = \lambda$ and from the second equation, $2x^2y^2 = \lambda$. Thus $2x^2y^2 = y^4$, i.e. $2x^2 = y^2$. Substituting into the constraint, we get $x^2 + 2x^2 = 36$, i.e. $x^2 = 12$ and $y^2 = 2x^2 = 24$. Irrespective of whether x and/or y are positive or negative, for these values of x and y we get

$$f(x, y) = x^2y^4 = 12(24)^2 = 6912$$

   which, since it is greater than zero, is the maximum value of f given the constraint.

8. a) The tangent plane has equation

$$\begin{aligned} z &= f_x(\pi, 0)(x - \pi) + f_y(\pi, 0)(y - 0) + 5 \\ z &= (6\cos\pi\cos 0)(x - \pi) + (-6\sin\pi\sin 0)(y - 0) + 5 \\ z &= -6(x - \pi) + 5 \end{aligned}$$

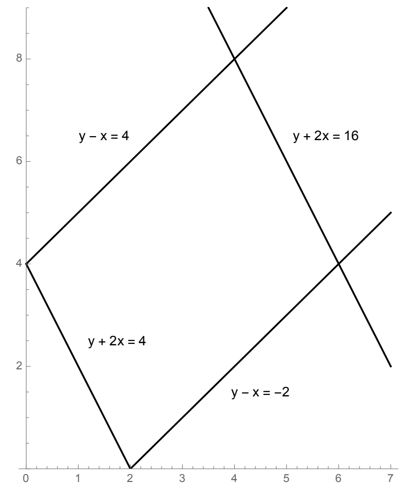
   and the normal equation of this plane is $6x + z = 6\pi + 5$.

   b) This volume is

$$\begin{aligned} \int_0^{\pi/2}\int_0^{\pi/3} (6\sin x\cos y + 8)\; dy\, dx &= \int_0^{\pi/2} [6\sin x\sin y + 8y]_0^{\pi/3}\; dx \\ &= \int_0^{\pi/2} \left(3\sqrt{3}\sin x + \frac{8\pi}{3}\right) dx \\ &= \left[-3\sqrt{3}\cos x + \frac{8\pi}{3}x\right]_0^{\pi/2} \\ &= \frac{4\pi^2}{3} + 3\sqrt{3}. \end{aligned}$$

9. First, sketch the parallelogram E and write equations for the lines comprising

the four sides:



These lines suggest the change of variables $u = y - x$, $v = y + 2x$. Computing the Jacobian we have

$$J = \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \det \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} = -3$$

and notice also that $u - v = -3x$ so $x = \frac{-1}{3}(u - v)$. Therefore the integral becomes

$$\begin{aligned} \iint_E (xy - x^2)\, dA &= \int_4^{16} \int_{-2}^{4} (xy - x^2) \frac{1}{|-3|}\, du\, dv \\ &= \frac{1}{3} \int_4^{16} \int_{-2}^{4} x(y - x)\, du\, dv \\ &= \frac{1}{3} \int_4^{16} \int_{-2}^{4} \frac{-1}{3}(u - v) u\, du\, dv \\ &= \frac{-1}{9} \int_4^{16} \int_{-2}^{4} (u^2 - uv)\, du\, dv \\ &= \frac{-1}{9} \int_4^{16} \left[\frac{1}{3}u^3 - \frac{1}{2}u^2 v \right]_{-2}^{4} dv \\ &= \frac{-1}{9} \int_4^{16} [24 - 6v]\, dv \\ &= \frac{-1}{9} \left[24v - 3v^2 \right]_4^{16} \\ &= \frac{-1}{9}(-384 - 48) = 48. \end{aligned}$$

10. a) In polar coordinates, $E = \{(r, \theta) : 0 \le r \le R, 0 \le \theta \le 2\pi\}$ so the area of E is

$$\iint_E dA = \int_0^{2\pi} \int_0^R r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{1}{2} r^2 \right]_0^R d\theta = \int_0^{2\pi} \frac{1}{2} R^2\, d\theta = 2\pi \left(\frac{1}{2} R^2 \right) = \pi R^2.$$

b) Parameterize ∂E by $\mathbf{x}(t) = (R\cos t, R\sin t)$ for $0 \le t \le 2\pi$. By Green's Theorem,

$$\begin{aligned}
\iint_E dA &= \frac{1}{2}\oint_{\partial E} x\,dy - y\,dx \\
&= \frac{1}{2}\int_0^{2\pi} (R\cos t)(R\cos t\,dt) - (R\sin t)(-R\sin t\,dt) \\
&= \frac{1}{2}\int_0^{2\pi} (R^2\cos^2 t + R^2\sin^2 t)\,dt \\
&= \frac{1}{2}\oint_0^{2\pi} R^2\,dt \\
&= \frac{1}{2}(2\pi R^2) = \pi R^2.
\end{aligned}$$

11. a) γ is parameterized by $\mathbf{x}(t) = (5t, 2t+1, t+2)$ for $0 \le t \le 1$; we have

$$ds = ||\mathbf{x}'(t)||\,dt = \sqrt{5^2 + 2^2 + 1}\,dt = \sqrt{30}\,dt$$

and consequently

$$\begin{aligned}
\int_\gamma (xy + yz)\,ds &= \int_0^1 (5t(2t+1) + (2t+1)(t+2))\ \sqrt{30}\,dt \\
&= \sqrt{30}\int_0^1 (12t^2 + 10t + 2)\,dt \\
&= \sqrt{30}\left[4t^3 + 5t^2 + 2t\right]_0^1 \\
&= 11\sqrt{30}.
\end{aligned}$$

b) Write $\mathbf{f} = (M, N, P)$. First,

$$\begin{aligned}
\text{curl } \mathbf{f} &= (P_y - N_z, M_z - P_x, N_x - M_y) \\
&= (2x^2y - 2x^2y, 2xy^2 - 2xy^2, 4xyz - 4xyz) \\
&= \mathbf{0}
\end{aligned}$$

so $\mathbf{f}$ is conservative. Next, find a potential function for $\mathbf{f}$ by integrating the components of $\mathbf{f}$:

$$\begin{aligned}
f(x,y,z) &= \int M\,dx = \int 2xy^2z\,dx = x^2y^2z + A(y,z) \\
f(x,y,z) &= \int N\,dy = \int 2x^2yz\,dy = x^2y^2z + B(x,z) \\
f(x,y,z) &= \int P\,dz = \int x^2y^2\,dz = x^2y^2z + C(x,y)
\end{aligned}$$

We see that by setting $A = B = C = 0$, the function $f(x, y, z) = x^2y^2z$ is a potential for $\mathbf{f}$. Now by the Fundamental Theorem of Line Integrals,

$$\int_\gamma \mathbf{f} \cdot d\mathbf{s} = f(\mathbf{x}(1)) - f(\mathbf{x}(0)) = f(1, 1, 1) - f(0, 0, 0) = 1 - 0 = 1.$$

12. a) E can also be thought of as the set

$$\{(x, y, z) : 0 \leq y \leq 3, 0 \leq x \leq 6 - 2y, 0 \leq z \leq 12 - 2x - 4y\}$$

so by Fubini's theorem, the triple integral is

$$\begin{aligned}
\iiint_E y\,dV &= \int_0^3 \int_0^{6-2y} \int_0^{12-2x-4y} y\,dz\,dx\,dy \\
&= \int_0^3 \int_0^{6-2y} [zy]_0^{12-2x-4y}\ dx\,dy \\
&= \int_0^3 \int_0^{6-2y} [y(12 - 2x - 4y)]\ dx\,dy \\
&= \int_0^3 \int_0^{6-2y} \left(12y - 2xy - 4y^2\right)\ dx\,dy \\
&= \int_0^3 \left[12xy - x^2y - 4xy^2\right]_0^{6-2y}\ dx \\
&= \int_0^3 \left[12y(6 - 2y) - (6 - 2y)^2y - 4(6 - 2y)y^2\right]\ dy \\
&= \int_0^3 \left[36y - 24y^2 + 4y^3\right]\ dy \\
&= \left[18y^2 - 8y^3 + y^4\right]_0^3 \\
&= 18(9) - 8(27) + 81 = 27.
\end{aligned}$$

b) Let E be the base of the figure (in the xy plane) and use cylindrical coordinates, since

$$E = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}.$$

The sphere of radius 2 centered at the origin is $x^2 + y^2 + z^2 = 4$, and the top half is $z = \sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}$. Therefore we want the double integral

$$\begin{aligned}
\iint_E \sqrt{1-x^2-y^2}\,dA &= \int_0^{2\pi}\int_0^1 \sqrt{4-r^2}\,r\,dr\,d\theta \\
&= 2\pi \int_0^1 \sqrt{4-r^2}\,r\,dr \\
&= 2\pi \left[\frac{-1}{3}(4-r^2)^{3/2}\right]_0^1 \\
&= 2\pi \left[\frac{-1}{3}(3\sqrt{3}) + \frac{1}{3}(4^{3/2})\right] \\
&= \frac{16\pi}{3} - 2\pi\sqrt{3}.
\end{aligned}$$

3.14 Fall 2020 Final Exam

1. Throughout this problem, suppose $\mathbf{f} : \mathbb{R} \to \mathbb{R}^3$, $g : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{h} : \mathbb{R}^3 \to \mathbb{R}^3$ are C^∞ functions. Assume $\mathbf{u}$ is a unit vector in $\mathbb{R}^3$.

   In each part of this problem, you are given an expression. Determine if that expression is a **scalar**, a **vector** in $\mathbb{R}^3$, a **matrix** (in which case you should give its size), or **nonsense**.

   a) (3.1) $\mathbf{f}(1)$
   b) (4.2) $g_x(1,2,3)$
   c) (4.2) $\mathbf{h}_x(1,2,3)$
   d) (4.1) $D\mathbf{f}(1)$
   e) (4.1) $D\mathbf{f}(1,2,3)$
   f) (4.1) $D\mathbf{h}(1,2,3)$
   g) (4.5) $\nabla\mathbf{f}(1)$
   h) (4.5) $\nabla g(1,2,3)$
   i) (6.1) $H\mathbf{f}(1)$
   j) (6.1) $Hg(1,2,3)$
   k) (7.5) $J(\mathbf{f})$
   l) (7.5) $J(\mathbf{h})$
   m) (4.5) $D_{\mathbf{u}}\mathbf{f}(1)$
   n) (4.5) $D_{\mathbf{u}}g(1,2,3)$

2. Let $\mathbf{v} = (1,3,-7)$ and $\mathbf{w} = (2,5,2)$.

   a) (2.7) Write parametric equations for the line that passes through the point $(0,-6,11)$ and has direction vector $\mathbf{v}$.
   b) (2.3) Compute the dot product of $\mathbf{v}$ and $\mathbf{w}$.
   c) (2.3) Based on your answer to the previous question, is the angle between $\mathbf{v}$ and $\mathbf{w}$ acute, right, or obtuse? Explain your reasoning.
   d) (2.6) Find a nonzero vector in $\mathbb{R}^3$ which is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.
   e) (2.7) Write the normal equation of the plane that contains the point $(-8,-2,3)$ and contains lines whose direction vectors are $\mathbf{v}$ and $\mathbf{w}$.

3. Let $A = \begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix}$ and let $B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

   a) (2.5) Compute $\det A$.
   b) (2.4) Compute $B^T A$.
   c) (6.1) Is A positive definite, negative definite, or neither? Explain.

4. Compute each limit (or explain why the limit does not exist):

   a) (3.5) $\lim\limits_{\mathbf{x}\to\mathbf{0}} \frac{x+y}{x}$
   b) (3.5) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^2}{x}$

5. Throughout this problem, let $f : \mathbb{R}^3 \to \mathbb{R}$ be $f(x,y,z) = x^2y - 2xz^3 + 4y^3z^2$.

   a) (4.2) Compute all first-order partial derivatives of f.
   b) (4.2) Compute f_{yyz}.

c) (4.5) Find the direction in which the value of f is increasing most rapidly, at the point $(3,1,1)$.

d) (4.5) Write the normal equation of the plane tangent to the level surface $f(x,y,z) = 19$ at the point $(3,1,1)$.

e) (4.5) Use your answer to part (d) to estimate the value of y so that $f(3.1, y, .8) = 19$.

f) (4.4) If $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^3$ is a differentiable function such that $\mathbf{g}(-2,7) = (3,1,1)$ and $D\mathbf{g}(-2,7) = \begin{pmatrix} 3 & -2 \\ 1 & 0 \\ 3 & -1 \end{pmatrix}$, compute $D(f \circ \mathbf{g})(-2,7)$.

6. (6.3) Find the absolute maximum and absolute minimum values of $f(x,y) = x^2 - 2x - y^2$ over the region of points (x,y) satisfying $x^2 + 4y^2 \leq 4$.

7. Suppose an object is moving in $\mathbb{R}^3$ so that its velocity at time t is given by

$$\mathbf{v}(t) = \left(\frac{t^2}{2} - 3, t - t^2, 2t + 1\right).$$

a) (5.1) Compute the displacement of the object from time 0 to time 1.

b) (5.1) Compute the acceleration of the object at time 2.

c) (5.2) Compute the tangential component of the object's acceleration at time 2.

d) (5.4) Compute the normal component of the object's acceleration at time 2.

8. In each part of this problem, you are given a set $E \subseteq \mathbb{R}^2$. For each given E, compute

$$\iint_E 10x^2 y \, dA.$$

a) (7.3) $E = [0,3] \times [0,2]$

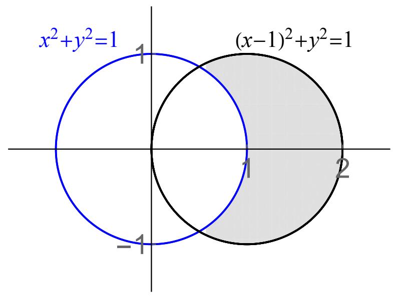
b) (7.3) $E = \{(x,y) : 0 \leq x \leq y \leq 2\}$

9. (7.5) Compute

$$\iint_E 12(y-x)^2 \, dA$$

where E is the parallelogram with vertices $(1,0)$, $(0,2)$, $(6,5)$ and $(5,7)$.

10. (7.6) Compute the area of the region of points lying inside the circle $(x-1)^2+y^2=1$ but outside the circle $x^2+y^2=1$. This region is shaded in the picture below:



11. Let $S \subseteq \mathbb{R}^3$ be the solid consisting of points (x,y,z) lying above the set $\{(x,y) : x^2+y^2 \leq 4\}$ and below the function $z = x^2+y^2$.

    a) (7.6) Compute the volume of S.

    b) (7.5) Compute $\iiint\limits_S z^2\,dV$.

Solutions

1. a) $\mathbf{f}(1)$ is a **vector** in $\mathbb{R}^3$.

   b) $g_x(1,2,3)$ is a **scalar**.

   c) $\mathbf{h}_x(1,2,3)$ is **nonsense**, since the range of $\mathbf{h}$ isn't $\mathbb{R}$.

   d) $D\mathbf{f}(1)$ is a 3×1 matrix, which is really a **vector** in $\mathbb{R}^3$.

   e) $D\mathbf{f}(1,2,3)$ is **nonsense**, since the inputs of $\mathbf{f}$ belong to $\mathbb{R}$, not $\mathbb{R}^3$.

   f) $D\mathbf{h}(1,2,3)$ is a 3×3 **matrix**.

   g) $\nabla\mathbf{f}(1)$ is **nonsense**, since the outputs of $\mathbf{f}$ do not belong to $\mathbb{R}$.

   h) $\nabla g(1,2,3)$ is a **vector** in $\mathbb{R}^3$.

   i) $H\mathbf{f}(1)$ is **nonsense**, since the outputs of $\mathbf{f}$ do not belong to $\mathbb{R}$.

   j) $Hg(1,2,3)$ is a 3×3 **matrix**.

   k) $J(\mathbf{f})$ is **nonsense**, since the domain and range of $\mathbf{f}$ aren't the same vector space.

   l) $J(\mathbf{h})$ is a 3×3 **matrix**.

   m) $D_{\mathbf{u}}\mathbf{f}(1)$ is **nonsense**, since the outputs of $\mathbf{f}$ do not belong to $\mathbb{R}$.

   n) $D_{\mathbf{u}}g(1,2,3)$ is a **scalar**.

2. a) Let $\mathbf{p} = (0, -6, 11)$; the parametric equations of the line are

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \Leftrightarrow \boxed{\begin{cases} x = 0 + 1t \\ y = -6 + 3t \\ z = 11 - 7t \end{cases}}.$$

   b) $\mathbf{v} \cdot \mathbf{w} = 1(2) + 3(5) - 7(2) = 2 + 15 - 14 = \boxed{3}$.

   c) Since $\mathbf{v} \cdot \mathbf{w} > 0$, the angle between $\mathbf{v}$ and $\mathbf{w}$ is **acute**.

   d) $\mathbf{v} \times \mathbf{w} = (3(2) - (-7)5, -7(2) - 1(2), 1(5) - 3(2)) = (41, -16, -1)$.

   e) A normal vector to the plane is $\mathbf{n} = (41, -16, -1)$; since the plane contains $\mathbf{p} = (-8, -2, 3)$, the normal equation of the plane is

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) &= 0 \\ (41, -16, -1) \cdot (x + 8, y + 2, z - 3) &= 0 \\ 41(x + 8) - 16(y + 2) - (z - 3) &= 0 \\ \boxed{41x - 16y - z = -299}&. \end{aligned}$$

3. a) $\det A = 3(5) - 2(2) = \boxed{11}$.

b) $B^T A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix} = \boxed{\begin{pmatrix} 7 & 12 \\ 17 & 26 \end{pmatrix}}$.

c) From (a), $\det A = 11$. Note $tr(A) = 1 + 4 = 5$. Since A is a symmetric 2×2 matrix with positive trace and positive determinant, A is **positive definite**.

4. a) Along the x-axis, we have $\lim\limits_{(x,0)\to(0,0)} \frac{x+y}{x} = \lim\limits_{x\to 0} \frac{x+0}{x} = 1$. But along the line $y = x$, we have $\lim\limits_{(x,x)\to(0,0)} \frac{x+y}{x} = \lim\limits_{x\to 0} \frac{x+x}{x} = 2$. Since we have two different limits along two different paths approaching $\mathbf{0}$, $\lim\limits_{\mathbf{x}\to\mathbf{0}} \frac{x+y}{x}$ **DNE**.

b) Change to polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x} = \lim_{r\to 0} \frac{r^2}{r\cos\theta} = \lim_{r\to 0} r\sec\theta = \boxed{0}.$$

5. a) $f_x(x,y,z) = \boxed{2xy - 2z^3}$; $f_y(x,y,z) = \boxed{x^2 + 12y^2z^2}$; $f_z(x,y,z) = \boxed{-6xz^2 + 8y^3z}$.

b) $f_y(x,y,z) = x^2 + 12y^2z^2$ from (a). Differentiate again to get $f_{yy}(x,y,z) = 24yz^2$ and one more time to get $f_{yyz}(x,y,z) = \boxed{48yz}$.

c) The direction in which the value of f is increasing most rapidly at the point $(3,1,1)$ is $\nabla f(3,1,1) = (f_x(3,1,1), f_y(3,1,1), f_z(3,1,1)) = (6-2, 9+12, -18+8) = \boxed{(4,21,-10)}$.

d) The normal vector to this tangent plane is $\nabla f(3,1,1)$, which was computed in (c) as $(4,21,-10)$. So the normal equation of the plane is

$$\begin{aligned} \nabla f(3,1,1)\cdot(\mathbf{x} - (3,1,1)) &= 0 \\ (4,21,-10)\cdot(x-3, y-1, z-1) &= 0 \\ 4(x-3) + 21(y-1) - 10(z-1) &= 0 \\ \boxed{4x + 21y - 10z = 23}&. \end{aligned}$$

e) Plug in $x = 3.1$ and $z = .8$ to the answer to (d) to get

$$\begin{aligned} 4(3.1) + 21y - 10(.8) &= 23 \\ 12.4 + 21y - 8 &= 23 \\ 21y &= 18.6 \\ \boxed{y = \frac{18.6}{21} = \frac{31}{35}}&. \end{aligned}$$

f) First, $Df(3,1,1) = [\nabla f(3,1,1)]^T = \begin{pmatrix} 4 & 21 & -10 \end{pmatrix}$. Then, by applying the Chain Rule, we get

$$\begin{aligned} D(f \circ \mathbf{g})(-2,7) &= Df(\mathbf{g}(-2,7))D\mathbf{g}(-2,7) \\ &= Df(3,1,1)\begin{pmatrix} 3 & -2 \\ 1 & 0 \\ 3 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 21 & -10 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 1 & 0 \\ 3 & -1 \end{pmatrix} = \boxed{\begin{pmatrix} 3 & 2 \end{pmatrix}}. \end{aligned}$$

6. Start by finding the critical points of f in the desired region:

$$\nabla f(x,y) = (2x-2, -2y) = (0,0) \Rightarrow (x,y) = (1,0) \text{ CP}.$$

Next, optimize f along the boundary $x^2+4y^2=4$ by setting $g(x,y)=x^2+4y^2$ and using Lagrange's method:

$$\nabla f = \lambda\, \nabla g \Rightarrow \begin{cases} 2x-2 &= \lambda(2x) \\ \quad -2y &= \lambda(8y) \Rightarrow y=0 \text{ or } \lambda = -\frac{1}{4}. \end{cases}$$

If $y=0$, then from the constraint $x^2+4y^2=4$ we have $x^2=4$, i.e. $x=\pm 2$, leading to the two critical points $(2,0)$ and $(-2,0)$. On the other hand, if $\lambda = -\frac{1}{4}$, then from the first equation we get $2x-2 = \frac{-1}{2}x$, leading to $x=\frac{4}{5}$. Plugging this into the constraint gives $\left(\frac{4}{5}\right)^2 + 4y^2 = 4$, i.e. $y = \pm\frac{\sqrt{21}}{5}$, generating the boundary critical points $\left(\frac{4}{5}, \frac{\sqrt{21}}{5}\right)$ and $\left(\frac{4}{5}, -\frac{\sqrt{21}}{5}\right)$. Test all these points in the utility f:

	Point	Value of f
CP	$(1,0)$	$1^2-2(1)-0=-1$
BCP	$(2,0)$	$2^2-2(2)-0=0$
BCP	$(-2,0)$	$(-2)^2-2(-2)-0=8$
BCP	$\left(\frac{4}{5}, \frac{\sqrt{21}}{5}\right)$	$\left(\frac{4}{5}\right)^2 - 2\left(\frac{4}{5}\right) - \left(\frac{\sqrt{21}}{5}\right)^2 = \frac{16}{25} - \frac{8}{5} - \frac{21}{25} = \frac{-9}{5}$
BCP	$\left(\frac{4}{5}, -\frac{\sqrt{21}}{5}\right)$	$\left(\frac{4}{5}\right)^2 - 2\left(\frac{4}{5}\right) - \left(-\frac{\sqrt{21}}{5}\right)^2 = \frac{16}{25} - \frac{8}{5} - \frac{21}{25} = \frac{-9}{5}$

So the absolute maximum value is $\boxed{8}$ and the absolute minimum value is $\boxed{-\frac{9}{5}}$.

7. a) The displacement is

$$\begin{aligned}\int_0^1 \mathbf{v}(t)\,dt &= \int_0^1 \left(\frac{t^2}{2}-3, t-t^2, 2t+1\right)\,dt\\ &= \left(\frac{1}{6}t^3-3t, \frac{1}{2}t^2-\frac{1}{3}t^3, t^2+t\right)\Big|_0^1\\ &= \left(\frac{1}{6}-3, \frac{1}{2}-\frac{1}{3}, 1+1\right)\\ &= \boxed{\left(-\frac{17}{6}, \frac{1}{6}, 2\right)}.\end{aligned}$$

b) $\mathbf{a}(2) = \mathbf{v}'(2) = (t, 1-2t, 2)\,|_{t=2} = \boxed{(2,-3,2)}$.

c) At time 2, the velocity is $\mathbf{v}(2) = (-1,-2,5)$ and the acceleration is $\mathbf{a}(2) = (2,-3,2)$. Thus

$$a_T(2) = \frac{\mathbf{v}(2)\cdot\mathbf{a}(2)}{||\mathbf{v}(2)||} = \frac{(-1,-2,5)\cdot(2,-3,2)}{||(-1,-2,5)||} = \frac{-2+6+10}{\sqrt{1+4+25}} = \boxed{\frac{14}{\sqrt{30}}}.$$

d) By the Pythagorean Theorem for acceleration,

$$\begin{aligned}[a_T(2)]^2 + [a_N(2)]^2 &= ||\mathbf{a}(2)||^2\\ \left(\frac{14}{\sqrt{30}}\right)^2 + [a_N(2)]^2 &= ||(2,-3,2)||^2\\ \frac{196}{30} + [a_N(2)]^2 &= 17\\ \frac{98}{15} + [a_N(2)]^2 &= 17\\ a_N(2) = \sqrt{17-\frac{98}{15}} &= \boxed{\sqrt{\frac{157}{15}}}.\end{aligned}$$

8. a) For $E = [0,3]\times[0,2]$, we have

$$\iint_E 10x^2y\,dA = \int_0^3\int_0^2 10x^2y\,dy\,dx = \int_0^3 \left[5x^2y^2\right]_0^2 dx = \int_0^3 20x^2\,dx = \frac{20}{3}x^3\Big|_0^3 = \boxed{180}.$$

b) For $E = \{(x,y) : 0 \le x \le y \le 2\}$, we have

$$\iint_E 10x^2y\,dA = \int_0^2\int_0^y 10x^2y\,dx\,dy = \int_0^2 \left[\frac{10}{3}x^3y\right]_0^y dy = \int_0^2 \frac{10}{3}y^4\,dy = \frac{2}{3}y^5\Big|_0^2 = \boxed{\frac{64}{3}}.$$

9. The parallelogram E is bounded by the lines $y+2x = 2$, $y+2x = 17$, $x-y = 1$ and $x - y = -2$. So we set $u = y + 2x$ and $v = x - y$ and let $(u, v) = \varphi(x, y)$. Thus $\varphi(E) = \{(u, v) : 2 \leq u \leq 17, -2 \leq v \leq 1$ and

$$J(\varphi) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} = -3,$$

so since $y - x = -(x - y) - v$, we have

$$\begin{aligned} \iint_E 12(y-x)^2\, dA &= \iint_{\varphi(E)} 12(-v)^2 \frac{1}{|J(\varphi)|}\, dA \\ &= \int_2^{17} \int_{-2}^{1} \frac{12v^2}{|-3|}\, dv\, du \\ &= \int_2^{17} \int_{-2}^{1} 4v^2\, dv\, du \\ &= \int_2^{17} \left[\frac{4}{3}v^3\right]_{-2}^{1} du \\ &= \int_2^{17} 12\, du = 12(17-2) = 12(15) = \boxed{180}. \end{aligned}$$

10. In polar coordinates, the equation of the left-hand circle is $r = 1$ and the equation of the right-hand circle is $r = 2\cos\theta$. These circles intersect when $1 = 2\cos\theta$, i.e. $\theta = \pm\frac{\pi}{3}$. So the shaded region, in polar coordinates, is

$$E = \{(r, \theta) : 0 \leq \theta \leq \frac{\pi}{3}, 1 \leq r \leq 2\cos\theta\}.$$

So the area of E is

$$\begin{aligned} \iint_E 1\, dA &= \int_{-\pi/3}^{\pi/3} \int_1^{2\cos\theta} r\, dr\, d\theta \\ &= \int_{-\pi/3}^{\pi/3} \left[\frac{r^2}{2}\right]_1^{2\cos\theta} d\theta \\ &= \int_{-\pi/3}^{\pi/3} \left[2\cos^2\theta - \frac{1}{2}\right] d\theta \\ &= \int_{-\pi/3}^{\pi/3} \left[(1-\cos 2\theta) - \frac{1}{2}\right] d\theta \\ &= \int_{-\pi/3}^{\pi/3} \left[\cos 2\theta + \frac{1}{2}\right] d\theta \\ &= \left[\frac{1}{2}\sin 2\theta + \frac{1}{2}\theta\right]_{-\pi/3}^{\pi/3} \\ &= \left[\frac{1}{2}\sin\left(\frac{2\pi}{3}\right) + \frac{\pi}{6}\right] - \left[\frac{1}{2}\sin\left(\frac{-2\pi}{3}\right) - \frac{\pi}{6}\right] = \boxed{\frac{\sqrt{3}}{2} + \frac{\pi}{3}}. \end{aligned}$$

11. a) In cylindrical coordinates, S is the set of points (r, θ, z) satisfying $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$, and $0 \leq z \leq x^2 + y^2 = r^2$. Therefore

$$\iiint_S 1\,dV = \int_0^2 \int_0^{2\pi} \int_0^{r^2} r\,dz\,d\theta\,dr = \int_0^2 \int_0^{2\pi} r^3\,d\theta\,dr = \int_0^2 2\pi r^3\,dr = \frac{1}{2}\pi r^4 \Big|_0^2 = \boxed{8\pi}.$$

b) Using the same setup as part (a),

$$\iiint_S z^2\,dV = \int_0^2 \int_0^{2\pi} \int_0^{r^2} z^2 r\,dz\,d\theta\,dr = \int_0^2 \int_0^{2\pi} \frac{1}{3} r^7\,d\theta\,dr = \int_0^2 \frac{2}{3}\pi r^7\,dr$$
$$= \frac{1}{12}\pi r^8 \Big|_0^2 = \boxed{\frac{64}{3}\pi}.$$

3.15 Spring 2021 Final Exam

1. Fill in the blanks in these sentences with sets so that the sentence is true.

   a) (4.1) Suppose $\mathbf{f}$ is such that for each $\mathbf{x}$, $D\mathbf{f}(\mathbf{x})$ is a 4×2 matrix. In this situation, $\mathbf{f}$ must be a function from ________ to ________ .

   b) (4.5) Suppose $\mathbf{f}$ is such that $\nabla\mathbf{f}(3, 1, -5)$ exists. In this setting, $\mathbf{f}$ must be a function from _______ to _______, and $\nabla\mathbf{f}(3, 1, -5)$ is an element of _______ .

   c) (8.2) Suppose $\mathbf{f}$ is such that div $\mathbf{f}(3, -2)$ exists. In this setting, $\mathbf{f}$ must be a function from ________ to _______, and div $\mathbf{f}(3, -2)$ is an element of _______ .

   d) (6.1) Suppose $\mathbf{f}$ is such that $H\mathbf{f}(4, 8)$ exists. In this situation, $\mathbf{f}$ must be a function from ________ to ________, and $H\mathbf{f}(4, 8)$ is an element of _______ .

   e) (4.5) Suppose $\mathbf{f}$ is such that $D_{\mathbf{u}}\mathbf{f}(-1, -4, 0)$ exists. In this situation, $\mathbf{f}$ must be a function from ________ to ________, $\mathbf{u}$ must be an element of ________, and $D_{\mathbf{u}}\mathbf{f}(-1, -4, 0)$ is an element of ________ .

2. (8.5) Green's Theorem says that under suitable hypotheses, some equation equating two types of integrals is true. Write that equation here:

3. (4.1) To say that a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is **differentiable** at $\mathbf{x}$ means that there exists some matrix $D\mathbf{f}(x)$ such that some limit exists and is equal to 0. Write that limit here:

4. (3.5) Explain why the limit $\lim\limits_{\mathbf{x}\to\mathbf{0}} \frac{x-y+z}{x+y+z}$ does not exist.

5. Let $\mathbf{v} = (1, 3, 0)$ and $\mathbf{w} = (-2, -1, 2)$.

   a) (2.3) Compute $(\mathbf{v} + 2\mathbf{w}) \cdot \mathbf{w}$.

   b) (2.6) Find a nonzero vector in $\mathbb{R}^3$ which is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

   c) (2.3) Compute the measure of the angle between $\mathbf{v}$ and $\mathbf{w}$.

   d) (2.3) Compute the distance between $\mathbf{v}$ and $\mathbf{w}$.

6. Throughout this problem, let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x, y) = 2xy^2 + x^3 - 3y^4$.

   a) (4.2) Compute all second-order partial derivatives of f.

   b) (4.2) Compute the slope of the line tangent to the graph of f which is parallel to the y-axis, that passes through the point $(1, -2)$.

   c) (4.5) Find the direction in which the value of f is decreasing most rapidly, at the point $(2, 1)$.

d) (4.5) Compute the rate of change of f in the direction $(-3,4)$ at the point $(2,1)$.

7. Suppose $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ are differentiable functions satisfying

$$\mathbf{f}(1,5) = (2,3); \quad \mathbf{g}(1,5) = (4,-1);$$

$$D\mathbf{f}(1,5) = \begin{pmatrix} 1 & -3 \\ 3 & 4 \end{pmatrix}; \quad D\mathbf{g}(1,5) = \begin{pmatrix} 0 & 2 \\ -7 & 2 \end{pmatrix}; \quad D\mathbf{g}(2,3) = \begin{pmatrix} -1 & 3 \\ 5 & 0 \end{pmatrix}.$$

In each part of this problem, you are given a quantity.

- If the given information in this problem is sufficient to compute the quantity, compute it.
- If the given information cannot be used to compute the quantity, write "not enough information".

a) (4.1) $D(\mathbf{f} + 2\mathbf{g})(1,5)$

b) (4.4) $D(\mathbf{f} \circ \mathbf{g})(1,5)$

c) (4.4) $D(\mathbf{g} \circ \mathbf{f})(1,5)$

8. (6.1) Find all the critical points of the function $f(x,y) = 4xy - x^4 - y^4 + 12$. Classify each critical point as a local maximum, local minimum or saddle.

9. (6.3) The profit of a company is given by $P(x,y,z) = 4x + 8y + 6z$, where x, y and z are units of three different products the company manufactures. Find the maximum profit of the company, given that $x^2 + 4y^2 + 2z^2 = 800$.

10. An object is moving in $\mathbb{R}^3$ so that its position at time t is $\mathbf{x}(t) = (3\cos 2t, 4\sin 2t, \frac{1}{\pi}t)$.

a) (5.1) Compute the velocity of the object at time $t = \frac{\pi}{3}$.

b) (5.1) Compute the speed of the object at time $t = \frac{\pi}{3}$.

c) (5.1) Compute the acceleration of the object at time $t = \frac{\pi}{3}$.

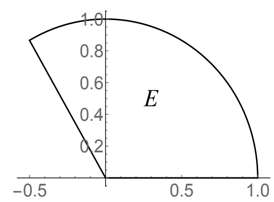
d) (4.3) Find parametric equations of the line which is tangent to the path the object travels at $t = \frac{\pi}{3}$.

11. In each part of this problem, you are given a set $E \subseteq \mathbb{R}^2$. For each given E, compute

$$\iint_E 8x \, dA.$$

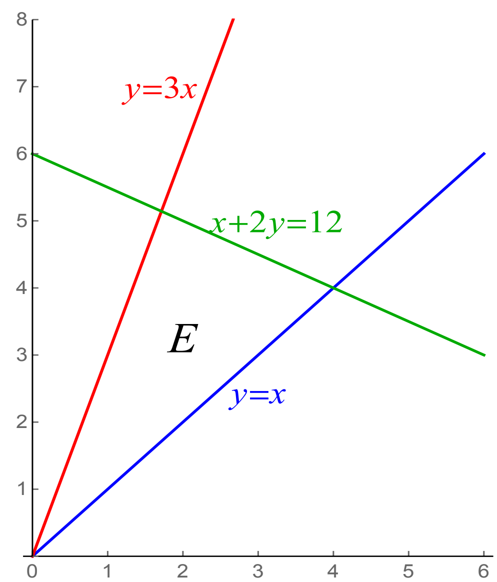
a) (7.3) $E = [0,1] \times [0,4]$

b) (7.5) E is the one-third of a circle pictured here:



c) (7.3) $E = \{(x, y) : y \geq 0, y^2 \leq x \leq y + 2\}$

12. (7.5) Compute the volume of the set of points in $\mathbb{R}^3$ lying below the graph of $z = \frac{y^2(x+2y)^2}{x^5}$ and above the triangular region E bounded by the red, blue and green lines shown below:

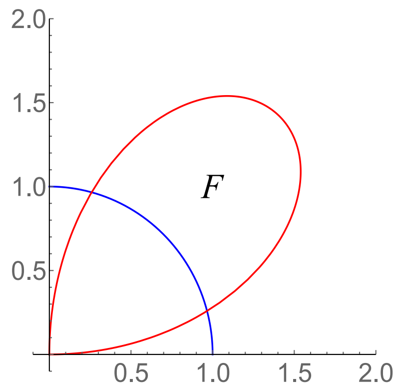


13. (7.5) Compute the volume of the set of points in $\mathbb{R}^3$ lying above the xy-plane, inside the sphere $x^2 + y^2 + z^2 = 16$, and inside the cone $z^2 = x^2 + y^2$.

14. (8.4) Compute the line integral $\int_\gamma \mathbf{f} \cdot d\mathbf{s}$, where γ is the line segment beginning at $(2, -1, 3)$ and ending at $(4, 0, 1)$, and $\mathbf{f}(x, y, z) = (3z, x + y, 2x + z)$.

15. **(Bonus)** (7.5) Compute the area of the region F of points lying in the first quadrant, outside the circle $x^2 + y^2 = 1$ but inside the graph of the polar

function $r = 2\sin 2\theta$. This region is shown below:



Solutions

1. a) Suppose $\mathbf{f}$ is such that for each $\mathbf{x}$, $D\mathbf{f}(\mathbf{x})$ is a 4×2 matrix. In this situation, $\mathbf{f}$ must be a function from $\boxed{\mathbb{R}^2}$ to $\boxed{\mathbb{R}^4}$.

   b) Suppose $\mathbf{f}$ is such that $\nabla\mathbf{f}(3,1,-5)$ exists. In this setting, $\mathbf{f}$ must be a function from $\boxed{\mathbb{R}^3}$ to $\boxed{\mathbb{R}}$, and $\nabla\mathbf{f}(3,1,-5)$ is an element of $\boxed{\mathbb{R}^3}$.

   c) Suppose $\mathbf{f}$ is such that div $\mathbf{f}(3,-2)$ exists. In this setting, $\mathbf{f}$ must be a function from $\boxed{\mathbb{R}^2}$ to $\boxed{\mathbb{R}^2}$, and div $\mathbf{f}(3,-2)$ is an element of $\boxed{\mathbb{R}}$.

   d) Suppose $\mathbf{f}$ is such that $H\mathbf{f}(4,8)$ exists. In this situation, $\mathbf{f}$ must be a function from $\boxed{\mathbb{R}^2}$ to $\boxed{\mathbb{R}}$, and $H\mathbf{f}(4,8)$ is an element of $\boxed{M_2(\mathbb{R})}$.

   e) Suppose $\mathbf{f}$ is such that $D_{\mathbf{u}}\mathbf{f}(-1,-4,0)$ exists. In this situation, $\mathbf{f}$ must be a
   a
   function from $\boxed{\mathbb{R}^3}$ to $\boxed{\mathbb{R}}$, $\mathbf{u}$ must be an element of $\boxed{\mathbb{R}^3}$, and $D_{\mathbf{u}}\mathbf{f}(-1,-4,0)$ is an element of $\boxed{\mathbb{R}}$.

2. The formula of Green's Theorem is $\boxed{\oint_{\partial E} \mathbf{f}\cdot d\mathbf{s} = \iint_E (N_x - M_y)\, dA}$.

   (This is under the assumption that $\mathbf{f} = (M,N)$, that E is compact with a piecewise C^1 boundary and that ∂E has been oriented so that as you move along ∂E, E is on the left.)

3. To say that a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is **differentiable** at $\mathbf{x}$ means that there exists some matrix $D\mathbf{f}(x)$ such that

$$\boxed{\lim_{\mathbf{h}\to\mathbf{0}} \frac{||\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{x})\mathbf{h}||}{||\mathbf{h}||} = 0.}$$

4. Along the z-axis, we have

$$\lim_{(0,0,z)\to(0,0,0)} \frac{x-y+z}{x+y+z} = \lim_{(0,0,z)\to(0,0,0)} \frac{0-0+z}{0+0+z} = 1,$$

   and along the y-axis, we have

$$\lim_{(0,y,0)\to(0,0,0)} \frac{x-y+z}{x+y+z} = \lim_{(0,0,z)\to(0,0,0)} \frac{0-y+0}{0+y+0} = -1.$$

   Since limits along different paths, are unequal, the limit does not exist.

5. a) Compute $(\mathbf{v}+2\mathbf{w})\cdot\mathbf{w} = (-3,1,4)\cdot(-2,-1,2) = (-3)(-2)+1(-1)+4(2) = \boxed{13}$.

   b) $\mathbf{v}\times\mathbf{w} = (3(2)-0(-1), 0(-2)-1(2), 1(-1)-3(-2)) = \boxed{(6,-2,5)}$.

c) First, $\mathbf{v} \cdot \mathbf{w} = 1(-2) + 3(-1) + 0(2) = -5$. Next, $||\mathbf{v}|| = \sqrt{1^2 + 3^2 + 0^2} = \sqrt{10}$ and $||\mathbf{w}|| = \sqrt{(-2)^2 + 1^2 + 2^2} = 3$. So from the angle formula for dot product, we have

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= ||\mathbf{v}||\,||\mathbf{w}|| \cos\theta \\ -5 &= \sqrt{10}\,(3)\cos\theta \\ \frac{-5}{3\sqrt{10}} &= \cos\theta \\ \boxed{\arccos\left(\frac{-5}{3\sqrt{10}}\right)} &= \theta. \end{aligned}$$

d) This is $||\mathbf{v} - \mathbf{w}|| = ||(3, 4, -2)|| = \sqrt{3^2 + 4^2 + (-2)^2} = \sqrt{9 + 16 + 4} = \boxed{\sqrt{29}}$.

6. (6 pts each) Throughout this problem, let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x, y) = 2xy^2 + x^3 - 3y^4$.

a) First, the first-order partial derivatives are $f_x(x, y) = 2y^2 + 3x^2$ and $f_y(x, y) = 4xy - 12y^3$. Differentiate again to get

$$\boxed{f_{xx}(x, y) = 6x} \quad \boxed{f_{xy}(x, y) = f_{yx}(x, y) = 4y} \quad \boxed{f_{yy}(x, y) = 4x - 36y^2}$$

b) This is $f_y(1, -2) = 4(1)(-2) - 12(-2)^3 = -8 + 96 = \boxed{88}$.

c) This is $-\nabla f(2, 1) = -\left(f_x(2, 1), f_y(2, 1)\right) = -\left(2 \cdot 1^2 + 3 \cdot 2^2, 4 \cdot 2 \cdot 1 - 12 \cdot 1^3\right) = \boxed{(-14, 4)}$.

d) First, a unit vector in the direction $(-3, 4)$ is $\mathbf{u} = \frac{1}{||(-3,4)||}(-3, 4) = \left(\frac{-3}{5}, \frac{4}{5}\right)$. The question asks for a directional derivative:

$$D_{\mathbf{u}}f(-3, 4) = \nabla f(2, 1) \cdot \mathbf{u} = (14, -4) \cdot \left(\frac{-3}{5}, \frac{4}{5}\right) = \frac{-42}{5} - \frac{16}{5} = \boxed{\frac{-58}{5}}.$$

7. a) This follows from the Sum and Constant Multiple Rules:

$$\begin{aligned} D(\mathbf{f} + 2\mathbf{g})(1, 5) &= D\mathbf{f}(1, 5) + 2D\mathbf{g}(1, 5) \\ &= \begin{pmatrix} 1 & -3 \\ 3 & 4 \end{pmatrix} + 2\begin{pmatrix} 0 & 2 \\ -7 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 1 \\ -11 & 8 \end{pmatrix}}. \end{aligned}$$

b) Since we don't know what $\mathbf{g}(1, 5)$ is, $D(\mathbf{f} \circ \mathbf{g})(1, 5)$ cannot be computed. $\boxed{\text{Not enough information}}$.

c) This can be computed using the Chain Rule:

$$\begin{aligned} D(\mathbf{g}\circ\mathbf{f})(1,5) &= D\mathbf{g}(\mathbf{f}(1,5))D\mathbf{f}(1,5) \\ &= D\mathbf{g}(2,3)D\mathbf{f}(1,5) \\ &= \begin{pmatrix} -1 & 3 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 3 & 4 \end{pmatrix} = \boxed{\begin{pmatrix} 8 & 15 \\ 5 & -15 \end{pmatrix}}. \end{aligned}$$

8. First, find the critical points by setting the gradient equal to $\mathbf{0}$:

$$\nabla f(x,y) = (f_x(x,y), f_y(x,y) = (4y - 4x^3, 4x - 4y^3).$$

Setting each coordinate equal to 0, we see from the first equation that $y = x^3$ and from the second equation that $x = y^3$. Substituting the first equation into the second gives $x = (x^3)^3$, i.e. $x = x^9$, i.e. $x^9 - x = x(x^8 - 1) = 0$, so $x = 0$, $x = 1$ or $x = -1$. From $y = x^3$, we get respective y-values 0, 1 and -1. This gives three critical points: $(0,0)$, $(1,1)$ and $(-1,-1)$, which we test by plugging them into the Hessian:

$$Hf(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} -12x^2 & 4 \\ 4 & -12y^2 \end{pmatrix}.$$

Testing the critical points, we get

CP	Hf	$\det Hf$	$tr\ Hf$	classification
$(0,0)$	$\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$	-16	N/A	saddle
$(1,1)$	$\begin{pmatrix} -12 & 4 \\ 4 & -12 \end{pmatrix}$	128	-24	local max
$(-1,-1)$	$\begin{pmatrix} -12 & 4 \\ 4 & -12 \end{pmatrix}$	128	-24	local max

9. We use Lagrange's method. Let $g(x,y,z) = x^2 + 4y^2 + 2z^2$; we set $\nabla P = \lambda \nabla g$ to get the system of equations

$$\begin{cases} 4 &= \lambda 2x \\ 8 &= \lambda 8y \\ 6 &= \lambda 4z \end{cases}$$

These equations lead to $x = \frac{2}{\lambda}$, $y = \frac{1}{\lambda}$ and $z = \frac{3}{2\lambda}$. Plugging into the constraint gives

$$800 = x^2 + 4y^2 + 2z^2 = \left(\frac{2}{\lambda}\right)^2 + 4\left(\frac{1}{\lambda}\right)^2 + 2\left(\frac{3}{2\lambda}\right)^2 = \frac{25}{2\lambda^2}$$

so $\lambda^2 = \frac{25}{1600} = \frac{1}{64}$ and $\lambda = \pm\frac{1}{8}$. Since x, y and z have to be nonnegative, we can drop $\lambda = -\frac{1}{8}$. $\lambda = \frac{1}{8}$ leads to $x = 16$, $y = 8$ and $z = 12$. This is the location of the maximum, and the maximum profit is $P(16, 8, 12) = 4(16)+8(8)+6(12) = \boxed{200}$.

10. a) $\mathbf{v}(t) = \mathbf{x}'(t) = (-6\sin 2t, 8\cos 2t, \frac{1}{\pi})$. $\mathbf{v}\left(\frac{\pi}{3}\right) = \boxed{\left(-3\sqrt{3}, -4, \frac{1}{\pi}\right)}$.

b) $||\mathbf{v}\left(\frac{\pi}{3}\right)|| = \sqrt{(-3\sqrt{3})^2 + (-4)^2 + \left(\frac{1}{\pi}\right)^2} = \boxed{\sqrt{43 + \frac{1}{\pi^2}}}$.

c) $\mathbf{a}(t) = \mathbf{x}''(t) = (-12\cos 2t, -16\sin 2t, 0)$ so $\mathbf{a}\left(\frac{\pi}{3}\right) = \boxed{\left(6, -8\sqrt{3}, 0\right)}$.

d) The line passes through $\mathbf{x}\left(\frac{\pi}{3}\right) = \left(\frac{-3}{2}, 2\sqrt{3}, \frac{1}{3}\right)$ and has direction vector $\mathbf{x}'\left(\frac{\pi}{3}\right) = \left(-3\sqrt{3}, -4, \frac{1}{\pi}\right)$ (computed in part (a)), so its parametric equations are

$$\boxed{\left\{\begin{array}{ll} x & = \frac{-3}{2} - 3\sqrt{3}\,t \\ y & = 2\sqrt{3} - 4t \\ z & = \frac{1}{3} + \frac{1}{\pi}t \end{array}\right.}.$$

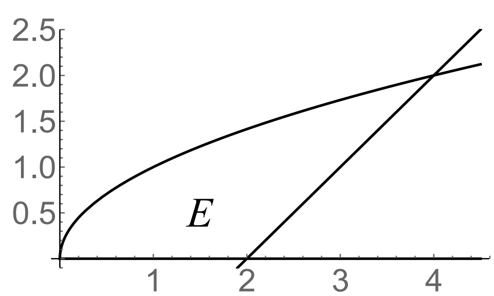
11. a) Compute directly with Fubini's Theorem:

$$\iint_E 8x\,dA = \int_0^1 \int_0^4 8x\,dy\,dx = \int_0^1 8xy|_0^4\,dx = \int_0^1 32x\,dx = 16x^2\Big|_0^1 = \boxed{16}.$$

b) Change to polar coordinates, since $E = \left\{(r,\theta) : 0 \le \theta \le \frac{2\pi}{3}, 0 \le r \le 1\right\}$:

$$\begin{aligned} \iint_E 8x\,dA &= \int_0^{2\pi/3} \int_0^1 8r\cos\theta\, r\,dr\,d\theta \\ &= \int_0^{2\pi/3} \int_0^1 8r^2\cos\theta\,dr\,d\theta \\ &= \int_0^{2\pi/3} \frac{8}{3}r^3\cos\theta\Big|_0^1\,d\theta \\ &= \int_0^{2\pi/3} \frac{8}{3}\cos\theta\,d\theta = \frac{8}{3}\sin\theta\Big|_0^{2\pi/3} = \boxed{\frac{4}{3}\sqrt{3}}. \end{aligned}$$

c) (10 pts) Sketch a picture of E:



Either from the picture, or by doing some algebra (setting $y^2 = y+2$ and solving for y), we find that the upper-right corner of E is $(4,2)$. So you can compute the integral directly with Fubini's Theorem:

$$\begin{aligned}
\iint_E 8x\,dA &= \int_0^2 \int_{y^2}^{y+2} 8x\,dx\,dy \\
&= \int_0^2 4x^2\Big|_{y^2}^{y+2}\,dy \\
&= \int_0^2 \left[4(y+2)^2 - 4y^4\right] dy \\
&= \left[\frac{4}{3}(y+2)^3 - \frac{4}{5}y^5\right]_0^2 = \left[\frac{4}{3}4^3 - \frac{4}{5}(32)\right] - \frac{4}{3}(2^3) = \boxed{\frac{736}{15}}.
\end{aligned}$$

12. The volume is given by $\iint_E \frac{y^2(x+2y)^2}{x^5}\,dA$. To compute this integral, use the change of variable $u = y/x$ and $v = x+2y$ so that if $\varphi(x,y) = (u,v)$, then $\varphi(E) = \{(u,v) : 1 \le u \le 3, 0 \le v \le 12\}$. Then the Jacobian of φ is

$$J(\varphi) = \det D\varphi = \det \begin{pmatrix} \frac{-y}{x^2} & \frac{1}{x} \\ 1 & 2 \end{pmatrix} = \frac{-2y}{x^2} - \frac{1}{x} = \frac{-(2y+x)}{x^2} = \frac{-v}{x^2}.$$

Back-solving for x and y in terms of u and v, we get $u = y/x$ so $y = xu$. Plugging in the equation for v gives $v = x + 2xu = x(1+2u)$, so $x = \frac{v}{1+2u}$ and finally, $y = xu = \frac{uv}{1+2u}$. Now the integral can be computed:

$$\begin{aligned}
\iint_E \frac{y^2(x+2y)^2}{x^5}\,dA &= \iint_{\varphi(E)} \frac{y^2(x+2y)^2}{x^5}\frac{1}{|J(\varphi)|}\,dA \\
&= \int_0^{12}\int_1^3 \frac{\left(\frac{uv}{1+2u}\right)^2 v^2}{\left(\frac{v}{1+2u}\right)^5} \cdot \frac{\left(\frac{v}{1+2u}\right)^2}{v}\,du\,dv \\
&= \int_0^{12}\int_1^3 (1+2u)u^2\,du\,dv \\
&= \int_0^{12}\int_1^3 (u^2+2u^3)\,du\,dv \\
&= \int_0^{12} \left[\frac{1}{3}u^3 + \frac{1}{2}u^4\right]_1^3 dv \\
&= \int_0^{12} \left(\left[9 + \frac{81}{2}\right] - \left[\frac{1}{3} + \frac{1}{2}\right]\right) dv \\
&= \int_0^{12} \frac{146}{3}\,dv = \frac{146}{3}(12) = \boxed{584}.
\end{aligned}$$

13. This solid, in spherical coordinates, is $0 \le \theta \le 2\pi$, $0 \le \rho \le 4$ and $0 \le \varphi \le \frac{\pi}{4}$.

So the volume is

$$\begin{aligned}\int_0^{2\pi}\int_0^4\int_0^{\pi/4}\rho^2\sin\varphi\,d\varphi\,d\rho\,d\theta &= \int_0^{2\pi}\int_0^4 -\rho^2\cos\varphi\Big|_0^{\pi/4}\,d\rho\,d\theta\\ &= \int_0^{2\pi}\int_0^4\rho^2\left(1-\frac{\sqrt{2}}{2}\right)d\rho\,d\theta\\ &= \int_0^{2\pi}\frac{1}{3}\rho^3\left(1-\frac{\sqrt{2}}{2}\right)\Big|_0^4\,d\theta\\ &= \int_0^{2\pi}\frac{64}{3}\left(1-\frac{\sqrt{2}}{2}\right)d\theta\\ &= 2\pi\cdot\frac{64}{3}\left(1-\frac{\sqrt{2}}{2}\right) = \boxed{\frac{128\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right)}.\end{aligned}$$

14. Since γ is a line segment, γ is parametrized by

$$\mathbf{x}(t) = (2,-1,3) + t((4,0,1)-(2,-1,3)) = (2+2t,-1+t,3-2t)$$

for $0\le t\le 1$ and $\mathbf{x}'(t) = (2,1,-2)$ so $d\mathbf{s} = (2,1,-2)\,dt$. Thus, the line integral is

$$\begin{aligned}\int_\gamma \mathbf{f}\cdot d\mathbf{s} &= \int_0^1 (3z,x+y,2x+z)\cdot(2,-1,2)\,dt\\ &= \int_0^1 (3(3-2t),(2+2t)+(-1+t),2(2+2t)+3-2t)\cdot(2,1,-2)\,dt\\ &= \int_0^1 (9-6t,1+3t,7+2t)\cdot(2,1,-2)\,dt\\ &= \int_0^1 (18-12t+1+3t-14-4t)\,dt\\ &= \int_0^1 (5-13t)\,dt = \left[5t+\frac{13}{2}t^2\right]_0^1 = 5-\frac{13}{2} = \boxed{\frac{-3}{2}}.\end{aligned}$$

15. Start by finding the intersection points of the curves. In polar coordinates, the circle is $r=1$, so the curves intersect when $1 = 2\sin 2\theta$, i.e. $\frac{1}{2} = \sin 2\theta$, i.e. $2\theta\in\left\{\frac{\pi}{6},\frac{5\pi}{6}\right\}$, i.e. $\theta = \frac{\pi}{12}$ and $\theta = \frac{5\pi}{12}$. So F, in polar coordinates, is

$$F = \left\{(r,\theta) : \frac{\pi}{12}\le\theta\le\frac{5\pi}{12}, 1\le r\le 2\sin 2\theta\right\}.$$

Therefore the area of F is

$$\begin{aligned}
\iint_F 1\,dA &= \int_{\pi/12}^{5\pi/12} \int_1^{2\sin 2\theta} r\,dr\,d\theta \\
&= \int_{\pi/12}^{5\pi/12} \frac{1}{2}r^2\Big|_1^{2\sin 2\theta}\,d\theta \\
&= \int_{\pi/12}^{5\pi/12} \left[2\sin^2(2\theta) - \frac{1}{2}\right] d\theta \\
&= \int_{\pi/12}^{5\pi/12} \left[2\left(\frac{1-\cos 2(2\theta)}{2}\right) - \frac{1}{2}\right] d\theta \\
&= \int_{\pi/12}^{5\pi/12} \left[\frac{1}{2} - \cos 4\theta\right] d\theta \\
&= \left[\frac{\theta}{2} - \frac{1}{4}\sin 4\theta\right]_{\pi/12}^{5\pi/12} \\
&= \left[\frac{5\pi}{24} - \frac{1}{4}\sin\left(\frac{5\pi}{3}\right)\right] - \left[\frac{\pi}{24} - \frac{1}{4}\sin\left(\frac{\pi}{3}\right)\right] \\
&= \boxed{\frac{\pi}{6} + \frac{\sqrt{3}}{4}}.
\end{aligned}$$

3.16 Fall 2021 Final Exam

1. Throughout this problem, let $\mathbf{v} = (-3, 1, 2)$, $\mathbf{w} = (-1, 5, 0)$ and $\mathbf{x} = (4, 1, -1)$.

   a) (2.3) Compute the distance between $\mathbf{v}$ and $\mathbf{w}$.

   b) (2.3) Is the angle between $\mathbf{v}$ and $\mathbf{w}$ acute, obtuse or right? Explain.

   c) (2.4) If $A = \begin{pmatrix} 1 & 0 & -4 \\ 2 & 1 & 0 \end{pmatrix}$, compute $A\mathbf{v}$.

   d) (2.7) Write parametric equations for the line passing through $\mathbf{w}$ and $\mathbf{x}$.

   e) (2.7) Write a normal equation of the plane containing $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{x}$.

2. For each given limit, compute the value of the limit, or explain why the limit does not exist.

   a) (3.5) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y}$.

   b) (3.5) $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{x^2+z^2}{x^2+y^2+z^2}$.

3. Suppose $f(x, y) = 4x^2y^2 - 3xy^3$.

   a) (4.5) Compute the gradient of f.

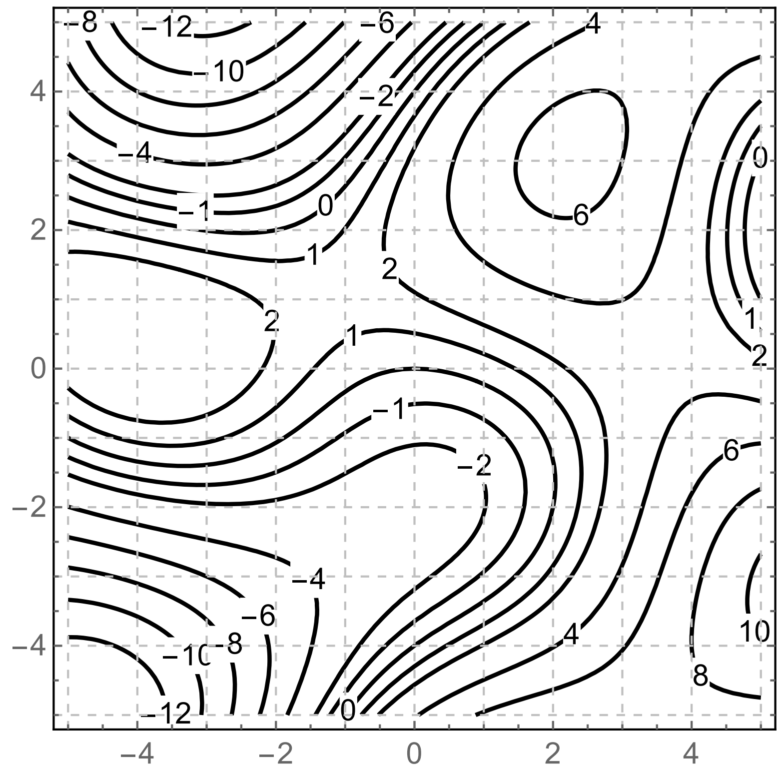
   b) (4.2) Compute $f_x(1, 2)$.

   c) (4.2) Compute $\frac{\partial^3 f}{\partial y^2 \partial x}$.

   d) (4.3) Write the equation of the plane tangent to the graph of f at the point $(2, -1, 22)$.

   e) (8.4) Compute $\int_\gamma f\, ds$, where γ is the line segment beginning at $(0, 0)$ and ending at $(2, 1)$.

4. A contour plot for an unknown function $f : \mathbb{R}^2 \to \mathbb{R}$ is given below:



Use this contour plot to answer the following questions.

a) (3.2) Estimate $f(4, -1)$.

b) (4.5) In which compass direction does $\nabla f(5, 2)$ point?

c) (4.2) Estimate $\frac{\partial f}{\partial y}(-2, 2)$.

d) (4.5) Is $D_{\mathbf{u}} f(2, -1)$ positive, negative or zero, if $\mathbf{u}$ is in the direction $(1, 1)$?

e) (3.2) Find the minimum value of $f(2, y)$, for $-5 \le y \le 5$.

f) (6.1) Estimate the coordinates of a local maximum of f.

g) (6.1) Estimate the coordinates of a saddle of f.

5. (4.3) Compute the linearization of $f(x, y, z) = x^2 \sin(yz)$ at the point $(2, 3, 0)$, and use that linearization to estimate $f(1.9, 3.3, .2)$.

6. Suppose that a particle is moving in $\mathbb{R}^3$ so that its position at time t is $\left(t^2, t, \frac{2}{3}t^3\right)$.

a) (5.1) Compute the velocity of the particle at time 0.

b) (5.2) Compute the tangential component of the acceleration of the particle at time 0.

c) (5.2) What does the sign of your answer to part (b) tell you about the motion of the particle at time 0?

d) (5.4) Compute the curvature of the path the particle travels at time 0.

e) (5.2) Compute the distance travelled by the particle from time 0 to time 2.

7. (6.1) Find all critical points of the function $f(x, y) = 2x^3 + 6xy^2 - 9x^2 + 9y^2$. Classify, with appropriate reasoning, each critical point as a local maximum, local minimum or saddle.

8. (6.3) Compute the absolute maximum value of the function $f(x, y) = xy$, subject to the constraint $x^2 + 4y^2 = 8$.

9. a) (7.3) Compute $\iint_D \cos(x + y)\, dA$, where D is the square $\left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$.

   b) (7.3) Compute $\iint_E 6y^2\, dA$, where E is the triangle with vertices $(0, 0)$, $(4, 0)$ and $(2, 2)$.

10. (7.3) Compute each iterated integral:

$$\text{(a)}\ \int_0^1 \int_x^1 e^{y^2}\, dy\, dx \qquad\qquad \text{(b)}\ \int_0^1 \int_0^y \int_{xy}^x 12xz\, dz\, dx\, dy$$

11. (7.5) Compute the volume of the set of points in $\mathbb{R}^3$ lying above the parallelogram in $\mathbb{R}^2$ with vertices $(1, -1)$, $(-1, 1)$, $(2, 0)$ and $(0, 2)$, and lying below the graph of $z = x^2$.

12. (7.5) Compute

$$\iiint_E xz\, dV,$$

where E is the set of points in $\mathbb{R}^3$ satisfying $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x^2 + y^2 + z^2 \leq 1$.

Solutions

1. a) $dist(\mathbf{v}, \mathbf{w}) = ||\mathbf{v} - \mathbf{w}|| = ||(-2, -4, 2)|| = \sqrt{2^2 + (-4)^2 + 2^2} = \boxed{\sqrt{24}}$.

   b) $\mathbf{v} \cdot \mathbf{w} = (-3)(-1) + 1(5) + 2(0) = 8 > 0$, so the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\boxed{\textbf{acute}}$.

   c) By regular matrix multiplication, $A\mathbf{v} = (1(-3) + 0(1) + 2(-4), 2(-3) + 1(1) + 2(0)) = \boxed{(-11, -5)}$.

   d) A direction vector for the line is $\mathbf{x} - \mathbf{w} = (5, -4, -1)$; the line then has parametric equations

$$\boxed{\left\{ \begin{array}{ll} x & = -1 + 5t \\ y & = 5 - 4t \\ z & = -t \end{array} \right.}.$$

   e) The plane contains vectors $\mathbf{w} - \mathbf{v} = (2, 4, -2)$ and $\mathbf{x} - \mathbf{w} = (5, -4, -1)$; a normal vector to the plane is therefore $\mathbf{n} = (2, 4, -2) \times (5, -4, -1) = (-12, -8, -28)$. Any nonzero multiple of this is also a normal vector, so I will use $\mathbf{n} = (3, 2, 7)$. Thus the plane has normal equation

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{x} - \mathbf{v}) &= 0 \\ \text{i.e. } (3, 2, 7)(x + 3, y - 1, z - 2) &= 0 \\ \text{i.e. } 3(x + 3) + 2(y - 1) + 7(z - 2) &= 0 \\ \text{i.e. } \boxed{3x + 2y + 7z = 7}&. \end{aligned}$$

2. a) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim\limits_{(x,y)\to(0,0)} \frac{(x-y)(x+y)}{x+y} = \lim\limits_{(x,y)\to(0,0)} (x - y) = 0 - 0 = \boxed{0}$.

   b) Along the y-axis, we have $\lim\limits_{(0,y,0)\to(0,0)} \frac{x^2+z^2}{x^2+y^2+z^2} = \lim\limits_{y\to 0} \frac{0}{y^2} = 0$, but along the z-axis, we have $\lim\limits_{(0,0,z)\to(0,0)} \frac{x^2+z^2}{x^2+y^2+z^2} = \lim\limits_{z\to 0} \frac{z^2}{z^2} = 1$. Therefore the limit $\boxed{\textbf{does not exist}}$.
   (This limit could also be done with spherical coordinates.)

3. a) $\nabla f(x, y) = (f_x, f_y) = \boxed{(8xy^2 - 3y^3, 8x^2y - 9xy^2)}$.

   b) $f_x(1, 2) = (8xy^2 - 3y^3)|_{(1,2)} = 32 - 24 = \boxed{8}$.

   c) $\frac{\partial^3 f}{\partial y^2 \partial x} = f_{xyy} = (8xy^2 - 3y^3)_{yy} = (16xy - 9y^2)_y = \boxed{16x - 18y}$.

   d) Observe $f_x(2, -1) = 16 - (-3) = 19$ and $f_y(2, -1) = -32 - 18 = -50$, so the tangent plane has equation

$$\begin{aligned} z &= f(2, -1) + f_x(2, -1)(x - 2) + f_x(2, -1)(y + 1) \\ z &= 22 + 19(x - 2) - 50(y + 1) \\ &\boxed{z = 19x - 50y - 66}. \end{aligned}$$

e) γ is parametrized by $\mathbf{x}(t) = (2t, t)$ for $0 \le t \le 1$, so $\mathbf{x}'(t) = (2, 1)$ and $||\mathbf{x}'(t)|| = \sqrt{2^2 + 1^2} = \sqrt{5}$. Thus the line integral becomes

$$\begin{aligned} \int_\gamma f\, ds &= \int_0^1 f(2t, t)\sqrt{5}\, dt \\ &= \int_0^1 \left[4(2t)^2t^2 - 3(2t)t^3\right]\sqrt{5}\, dt \\ &= \sqrt{5}\int_0^1 10t^4\, dt = 2\sqrt{5}t^5\Big|_0^1 = \boxed{2\sqrt{5}}. \end{aligned}$$

4. a) $f(4, -1) \approx \boxed{5}$.

   b) $\nabla f(5, 2)$ points toward the greatest increase in the value of f, which is $\boxed{\textbf{west}}$.

   c) $\frac{\partial f}{\partial y}(-2, 2) \approx f(-2, 3) - f(-2, 2) = -3 - 0 = \boxed{-3}$.

   d) Is $D_{\mathbf{u}}f(2, -1)$ is $\boxed{\textbf{negative}}$ since f decreases in the direction $(-1, -1)$ from the point $(2, -1)$.

   e) The minimum value of $f(2, y)$ for $-5 \le y \le 5$ is $\boxed{0}$, when $x \approx -1.5$.

   f) f has alocal maximum at about $\boxed{(2.2, 3.1)}$.

   g) f has two saddles in the viewing window: one at about $\boxed{(-1.2, 1)}$ and another at about $\boxed{(3.5, .25)}$.

5. The total derivative of f is

$$Df(x, y, z) = \begin{pmatrix} f_x & f_y & f_z \end{pmatrix} = \begin{pmatrix} 2x\sin(yz) & x^2z\cos(yz) & x^2y\cos(yz) \end{pmatrix}.$$

At the point $(2, 3, 0)$, this is $Df(2, 3, 0) = \begin{pmatrix} 0 & 0 & 12 \end{pmatrix}$. So the linearization of f at $(2, 3, 0)$ is

$$\begin{aligned} L(x, y, z) &= f(2, 3, 0) + Df(2, 3, 0)(x - 2, y - 3, z - 0) \\ &= 0 + \begin{pmatrix} 0 & 0 & 12 \end{pmatrix}(x - 2, y - 3, z - 0) = \boxed{12z}. \end{aligned}$$

That means

$$f(1.9, 3.3, .2) \approx L(1.9, 3.3, .2) = 12(.2) = \boxed{2.4}.$$

6. Suppose that a particle is moving in $\mathbb{R}^3$ so that its position at time t is $\left(t^2, t, \frac{2}{3}t^3\right)$.

   a) $\mathbf{v}(0) = \mathbf{x}'(0) = (2t, 1, 2t^2)|_{t=0} = \boxed{(0, 1, 0)}$.

   b) First, $\mathbf{a}(0) = \mathbf{x}''(0) = (2, 0, 4t)|_{t=0} = (2, 0, 0)$. Therefore, $a_T(0) = \frac{\mathbf{a}(0)\cdot\mathbf{v}(0)}{||\mathbf{v}(0)||} = \frac{0}{1} = \boxed{0}$.

c) Since $a_T(0) = 0$, at time 0 the object is neither speeding up nor slowing down at that instant.

d) $\kappa(0) = \frac{||\mathbf{v}(0)\times\mathbf{a}(0)||}{||\mathbf{v}(0)||^3} = \frac{||(0,0,-2)||}{1^3} = \boxed{2}$.

e) The arc length is

$$\begin{aligned}\int_0^2 ||\mathbf{x}'(t)||\,dt &= \int_0^2 \sqrt{(2t)^2+1^2+(2t^2)^2}\,dt\\ &= \int_0^2 \sqrt{4t^2+1+4t^4}\,dt\\ &= \int_0^2 \sqrt{(2t^2+1)^2}\,dt\\ &= \int_0^2 (2t^2+1)\,dt\\ &= \frac{2}{3}t^3+t\Big|_0^2 = \boxed{\frac{22}{3}}.\end{aligned}$$

7. The gradient of f is $\nabla f(x,y) = (f_x, f_y) = (6x^2+6y^2-18x, 12xy+18y)$. Set the gradient equal to $(0,0)$ to produce the system

$$\begin{cases} 6x^2+6y^2-18x = 0\\ 12xy+18y = 0 \Rightarrow 6y(2x+3) = 0 \Rightarrow y = 0 \text{ or } x = -\frac{3}{2}.\end{cases}$$

If $y = 0$, then the first equation gives $6x^2 - 18x = 0$, i.e. $x = 0$ or $x = 3$, giving the critical points $(0,0)$ and $(3,0)$. If $x = -\frac{3}{2}$, the first equation gives $y^2 = -36$, which has no solution. Thus there are two critical points: $(0,0)$ and $(3,0)$. We test these using the Hessian:

$$Hf(x,y) = \begin{pmatrix} f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{pmatrix} = \begin{pmatrix} 12x-18 & 12y\\ 12y & 12x+18\end{pmatrix}.$$

We have $\det Hf(0,0) = \det\begin{pmatrix} -18 & 0\\ 0 & 18\end{pmatrix} < 0$, so $\boxed{(0,0) \text{ is a saddle}}$. Finally, we see that $Hf(3,0) = \begin{pmatrix} 18 & 0\\ 0 & 54\end{pmatrix}$ has positive determinant and trace, so $Hf(3,0) > 0$, so $\boxed{(3,0) \text{ is a local minimum}}$.

8. Use Lagrange's method: let $g(x,y) = x^2+4y^2$ and start with $\nabla f = \lambda\,\nabla g$ to get

$$\begin{cases} y &= \lambda(2x)\\ x &= \lambda(8y)\end{cases}$$

Plugging the first equation into the second, we get $x = 16\lambda^2 x$, so $x = 0$ or $16\lambda^2 = 1$ so $\lambda = \pm\frac{1}{4}$. If $x = 0$, then from the first equation $y = 0$, but $(0,0)$

isn't on the constraint, so we can discard that point. That leaves $\lambda = \pm\frac{1}{4}$: from the first equation above, that means $y = \left(\pm\frac{1}{4}\right)(2x) = \pm\frac{1}{2}x$. Plugging into the constraint gives $x^2 + 4\left(\pm\frac{1}{2}x\right)^2 = 8$, i.e. $2x^2 = 8$, i.e. $x = \pm 2$. since $y = \pm\frac{1}{2}x$, that gives four critical points $(\pm 2, \pm 1)$; plug these into the utility $f(x, y) = xy$ to see that the maximum value is $\boxed{2}$.

9. a) By Fubini's theorem, this is

$$\begin{aligned}\iint_D \cos(x+y)\,dA &= \int_0^{\pi/2}\int_0^{\pi/2} \cos(x+y)\,dy\,dx \\ &= \int_0^{\pi/2} \sin(x+y)\big|_0^{\pi/2}\,dx \\ &= \int_0^{\pi/2} \left[\sin(x+\frac{\pi}{2}) - \sin x\right] dx \\ &= \left[-\cos(x+\frac{\pi}{2}) + \cos x\right]_0^{\pi/2} \\ &= [(1+0)-(0+1)] = \boxed{0}.\end{aligned}$$

b) E is horizontally simple with $0 \le y \le 2$, $y \le x \le 4-y$, so Fubini's theorem gives

$$\begin{aligned}\iint 6y^2\,dA &= \int_0^2\int_y^{4-y} 6y^2\,dx\,dy \\ &= \int_0^2 \left[6y^2x\right]_y^{4-y} dy \\ &= \int_0^2 \left[24y^2 - 12y^3\right] dx \\ &= \left[8y^3 - 3y^4\right]_0^2 = 64 - 48 = \boxed{16}.\end{aligned}$$

10. a) This is a double integral over a triangle with vertices $(0,0)$, $(0,1)$ and $(1,1)$, and by reversing the order of integration we get

$$\int_0^1\int_x^1 e^{y^2}\,dy\,dx = \int_0^1\int_0^y e^{y^2}\,dx\,dy = \int_0^1 ye^{y^2}\,dy.$$

Now use the u-sub $u = y^2$, $du = 2y\,dy$ to rewrite this integral as

$$\int_0^1 \frac{1}{2}e^u\,du = \frac{1}{2}e^u\Big|_0^1 = \boxed{\frac{1}{2}(e-1)}.$$

b) Compute this directly:

$$\begin{aligned}
\int_0^1 \int_0^y \int_{xy}^x 12xz\,dz\,dx\,dy &= \int_0^1 \int_0^y \left[6xz^2\right]_{xy}^x \,dx\,dy \\
&= \int_0^1 \int_0^y \left[6x^3 - 6x^3y^2\right] \,dx\,dy \\
&= \int_0^1 \left[\frac{3}{2}x^4 - \frac{3}{2}x^4y^2\right]_0^y \,dy \\
&= \int_0^1 \left[\frac{3}{2}y^4 - \frac{3}{2}y^6\right] \,dy \\
&= \left[\frac{3}{10}y^5 - \frac{3}{14}y^7\right]_0^1 \\
&= \frac{3}{10} - \frac{3}{14} = \boxed{\frac{3}{35}}.
\end{aligned}$$

11. The four sides of the parallelogram E have equations $x + y = 0$, $x + y = 2$, $y - x = -2$ and $y - x = -2$, so we use the change of variables $(x, y) \overset{\phi}{\mapsto} (u, v)$ where $u = x + y$ and $v = y - x$. Thus

$$J(\phi) = \det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \det\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2,$$

so the volume is

$$V = \iint_E x^2\,dA = \int_0^2 \int_{-2}^2 x^2 \frac{1}{|J(\phi)|}\,dv\,du = \int_0^2 \int_{-2}^2 \frac{1}{2}x^2\,dv\,du.$$

Now we back-solve for x in terms of u and v; add the equations $u = x + y$ and $v = y - x$ to get $2y = u + v$, i.e. $y = \frac{1}{2}(u + v)$. Thus $x = u - y = u - \frac{1}{2}(u + v) = \frac{1}{2}(v - u)$, so the integral becomes

$$\begin{aligned}
\int_0^2 \int_{-2}^2 \frac{1}{2}\left[\frac{1}{2}(v-u)\right]^2 \,dv\,du &= \frac{1}{8}\int_0^2 \int_{-2}^2 (v-u)^2\,dv\,du \\
&= \frac{1}{8}\int_0^2 \left[\frac{1}{3}(v-u)^3\right]_{-2}^2 \,du \\
&= \frac{1}{24}\int_0^2 \left[(2-u)^3 - (-2-u)^3\right] \,du \\
&= \frac{1}{24}\int_0^2 \left[(2-u)^3 + (2+u)^3\right] \,du \\
&= \frac{1}{24}\left[-\frac{1}{4}(2-u)^4 + \frac{1}{4}(-2-u)^4\right]_0^2 \\
&= \frac{1}{96}\left[(0 + 4^4) - (-2^4 + 2^4)\right]_2^{10} = \frac{4^4}{96} = \boxed{\frac{8}{3}}.
\end{aligned}$$

12. In spherical coordinates, this region is $0 \leq \rho \leq 1$, $0 \leq \theta \leq \frac{\pi}{2}$, and $0 \leq \varphi \leq \frac{\pi}{2}$. So the integral becomes

$$\begin{aligned}
\iiint_E xz\,dV &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\sin\varphi\cos\theta)(\rho\cos\varphi)\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi \\
&= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^4\sin^2\varphi\cos\varphi\cos\theta\,d\rho\,d\theta\,d\varphi \\
&= \int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{1}{5}\rho^5\sin^2\varphi\cos\varphi\cos\theta\right]_0^1 d\theta\,d\varphi \\
&= \frac{1}{5}\int_0^{\pi/2}\int_0^{\pi/2}\sin^2\varphi\cos\varphi\cos\theta\,d\theta\,d\varphi \\
&= \frac{1}{5}\int_0^{\pi/2}\left[\sin^2\varphi\cos\varphi\sin\theta\right]_0^{\pi/2} d\varphi \\
&= \frac{1}{5}\int_0^{\pi/2}\sin^2\varphi\cos\varphi\,d\varphi.
\end{aligned}$$

Now use the u-sub $u = \sin\varphi$, $du = \cos\varphi\,d\varphi$ to rewrite this integral as

$$\frac{1}{5}\int_0^1 u^2\,du = \frac{1}{15}u^3\Big|_0^1 = \boxed{\frac{1}{15}}.$$