# Old MATH 320 Final Exams

David M. McClendon

Department of Mathematics
Ferris State University

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# Contents

*Chapter 1*

# General information about these exams

These are the final exams I have given between 2018 and 2024 in Calculus 3 courses. To help give you some guidance on what questions are appropriate, each question on each exam is followed by a section number in parenthesis (like "(3.2)"). That means that question can be solved using material from that section (or from earlier sections) in the 2024 version of my *Vector Calculus Lecture Notes*.

## 1.1 Spring 2024 Final Exam

1. Throughout this problem, let $\mathbf{w} = (1, 2, -5, 4)$, $\mathbf{x} = (3, 0, 5, 1)$, $\mathbf{y} = (1, -7, 4)$ and $\mathbf{z} = (2, 0, -3)$.

   a) (2.3) Of the following two expressions, circle the one that is defined, and the compute it:
   $$\mathbf{w} \cdot \mathbf{x} \qquad\qquad \mathbf{x} \cdot \mathbf{y}$$

   b) (2.2) Of the following two expressions, circle the one that is defined, and the compute it:
   $$2\mathbf{w} - 3\mathbf{z} \qquad\qquad 2\mathbf{y} - 3\mathbf{z}$$

   c) (2.6) Of the following two expressions, circle the one that is defined, and the compute it:
   $$\mathbf{w} \times \mathbf{x} \qquad\qquad \mathbf{y} \times \mathbf{z}$$

2. Throughout this problem, let $A = \begin{pmatrix} 3 & -4 \\ -4 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 \\ 1 & -3 \\ 5 & 2 \end{pmatrix}$.

   a) (2.4) Of the following two expressions, circle the one that is defined, and the compute it:
   $$AB \qquad\qquad BA$$

   b) (2.5) Of the following two expressions, circle the one that is defined, and the compute it:
   $$\det A \qquad\qquad \det B$$

   c) (3.1) The function $\mathbf{f}(\mathbf{x}) = B\mathbf{x}$ defines a function from what domain to what codomain?

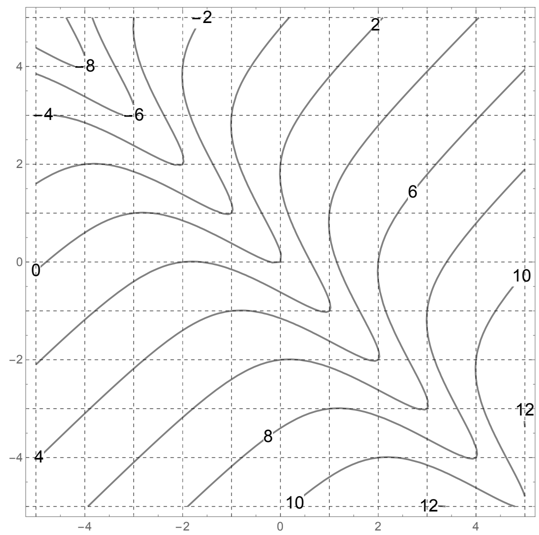
   d) (6.1) Is the matrix $A$ positive definite, negative definite or neither?

3. (3.5) Compute each limit, or explain (with justification) why the limit does not exist:

   a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x - y}{3x + y}$

   b) $\displaystyle\lim_{\mathbf{x}\to\mathbf{0}} \frac{xyz}{x^2 + y^2 + z^2}$

4. A contour plot for an unknown function $f : \mathbb{R}^2 \to \mathbb{R}$ is shown here:




Use this contour plot to answer these questions:

a) (3.2) Estimate $f(4, 2)$.

b) (4.2) Estimate $f_y(1, -3)$.

c) (4.2) Is $f_{xx}(2, -1)$ positive, negative or zero?

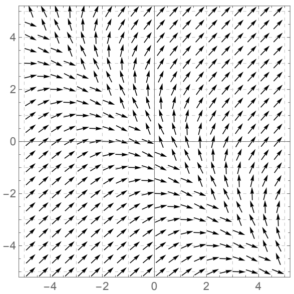
d) (4.2) Is $f_{yx}(1, 1)$ positive, negative or zero?

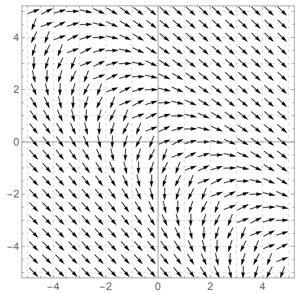
e) (3.2) Estimate a number $x$ so that $f(x, 3) = 5$.

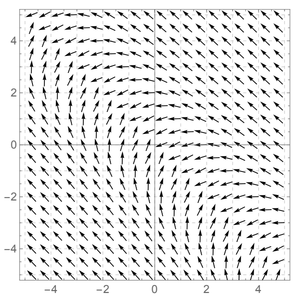
f) (3.2) What is the maximum value of $f$ on the region $[0, 2] \times [0, 2]$?

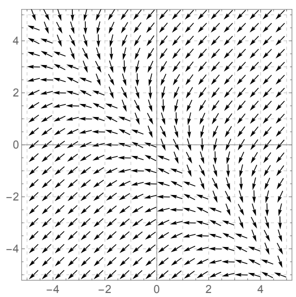
g) (3.2) What is the minimum value of $f$, subject to the constraint $y = x - 2$?

h) (4.5) Which of the pictures below is a picture of $\nabla f$?

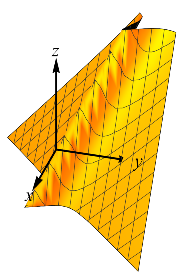
A. 

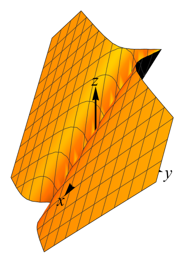
C. 

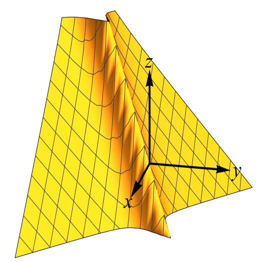
B. 

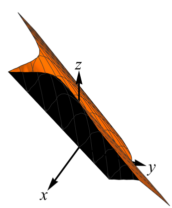
D. 

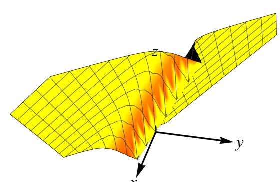
i) (3.2) Which of the pictures below is a graph of $f$?

A. 

B. 

C. 

D. 

E. 

5. (4.3) Compute the linearization of $f(x, y) = \ln(x + y^2)$ at the point $(1, 0)$, and use that linearization to estimate $f(.9, .2)$.

6. Throughout this problem, let $g(x, y) = 4x^2y - 3xy^2 + 2x + 7$.

   a) (4.5) Compute the gradient of $g$.

   b) (4.2) Compute $\dfrac{\partial^2 g}{\partial x \partial y}$.

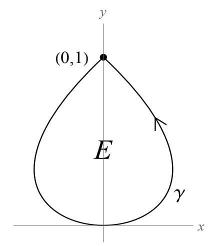
c) (4.3) Write an equation of the plane tangent to $g$ at the point $(1, -1, 2)$.

7. (6.1) Find all critical points of the function $f(x, y) = x^3 - y^3 - 12xy$. Classify each critical point as a local maximum, local minimum or saddle.

8. (6.3) Find the maximum value of $f(x, y, z) = 3x + 6y + 6z$, subject to the constraint $2x^2 + y^2 + 4z^2 = 8800$.

9. A figure skater is skating so that her position (measured in meters) at time $t$ (measured in seconds) is $\mathbf{x}(t) = (t - t^3, t^2)$.

   For $-1 \le t \le 1$, she skates the path $\gamma$ shown at right.




   a) (5.1) Compute the skater's velocity at time $\dfrac{1}{2}$.

   b) (5.1) Compute the skater's acceleration at time $\dfrac{1}{2}$.

   c) (5.4) Compute the curvature of the skater's path at time $\dfrac{1}{2}$.

   d) (8.5) Compute the area of the region $E$ enclosed by the skater's path from $t = -1$ to $t = 1$.

10. Compute each double integral:

   a) (7.3) $\displaystyle\iint_E (2x + 6x^2 y)\, dA$, where $E \subseteq \mathbb{R}^2$ is the rectangle $[0, 5] \times [0, 3]$.

   b) (7.5) $\displaystyle\iint_E 8x\, dA$, where $E \subseteq \mathbb{R}^2$ is the set of points in the first quadrant that lie inside the circle of radius $3$ centered at the origin.

   c) (7.3) $\displaystyle\iint_E e^{y^2}\, dA$, where $E \subseteq \mathbb{R}^2$ is the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 1)$.

11. (7.2) In this problem, suppose $f$ and $g$ are functions from $\mathbb{R}^2$ to $\mathbb{R}$ so that

$$\int_0^3 \int_0^3 f(x, y)\, dy\, dx = 10; \qquad \int_0^3 \int_3^4 f(x, y)\, dy\, dx = 8;$$

$$\int_0^3 \int_0^3 g(x, y)\, dy\, dx = 7; \qquad \int_0^3 \int_0^4 g(x, y)\, dy\, dx = 12.$$

   Use this information to compute each quantity:

   a) $\displaystyle\int_0^3 \int_0^3 [2f(x, y) - g(x, y)]\, dy\, dx$

b) $\int_0^3 \int_0^4 f(x, y)\, dy\, dx$

c) $\int_0^3 \int_3^4 g(x, y)\, dy\, dx$

d) $\int_0^3 \int_0^3 [f(x) + 2]\, dy\, dx$

12. (7.5) Compute $\iiint_E z\, dV$, where $E \subseteq \mathbb{R}^3$ is the set of points lying inside the sphere $x^2 + y^2 + z^2 = 1$, above the $xy$-plane, and inside the cone $z^2 = x^2 + y^2$.

13. Choose one of these two questions:

a) (8.4) Compute $\int_\gamma y^2\, ds$, where $\gamma$ is the top half of the circle of radius $2$ centered at the origin, parametrized counterclockwise.

b) (8.6) Compute $\int_\gamma \mathbf{f} \cdot d\mathbf{s}$, where $\mathbf{f}(x, y) = (10xy^3, 15x^2y^2 + 4)$ and $\gamma$ is parametrized by $\mathbf{x}(t) = (e^{t^2-t} + t, e^{t^3-t})$ for $0 \le t \le 1$.

## Solutions

1. a) $\boxed{\mathbf{w}\cdot\mathbf{x}}$ is defined and equal to $1(3)+2(0)-5(5)+4(1)=\boxed{-18}$.

   b) $\boxed{2\mathbf{y}-3\mathbf{z}}$ is defined and equal to $2(1,-7,4)-3(2,0,-3)=(2,-14,8)-(6,0,-9)=\boxed{(-4,-14,17)}$.

   c) $\boxed{\mathbf{y}\times\mathbf{z}}$ is defined and equal to $(-7(-3)-4(0),4(2)-(-3)1,1(0)-(-7)2)=\boxed{(21,11,14)}$.

2. a) $\boxed{BA}$ is defined and equal to $\boxed{\begin{pmatrix} -8 & 14 \\ 15 & -25 \\ 7 & -6 \end{pmatrix}}$.

   b) $\boxed{\det A}$ is defined and equal to $3(7)-(-4)^2=\boxed{5}$.

   c) $\boxed{\mathbf{f}:\mathbb{R}^2\to\mathbb{R}^3}$ since $B$ is $3\times 2$.

   d) Since $A$ is a $2\times 2$ symmetric matrix with positive trace and positive determinant, $A$ is $\boxed{\text{positive definite}}$.

3. a) Along the path $y=0$, we have $\displaystyle\lim_{(x,0)\to(0,0)}\frac{3x-0}{3x+0}=\lim_{x\to 0}1=1$, but along the path $x=0$ we have $\displaystyle\lim_{(0,y)\to(0,0)}\frac{0-y}{0+y}=\lim_{y\to 0}-1=-1$. Since the limits along different paths approaching $\mathbf{0}$ are unequal, $\displaystyle\lim_{(x,y)\to(0,0)}\frac{3x-y}{3x+y}$ $\boxed{\text{DNE}}$.

   b) Use spherical coordinates:

$$\begin{aligned}\lim_{\mathbf{x}\to\mathbf{0}}\frac{xyz}{x^2+y^2+z^2}&=\lim_{\rho\to 0}\frac{(\rho\sin\varphi\cos\theta)(\rho\sin\varphi\sin\theta)(\rho\cos\varphi)}{\rho^2}\\&=\lim_{\rho\to 0}\rho(\sin^2\varphi\cos\theta\sin\theta\cos\varphi=\boxed{0}\end{aligned}$$

   irrrespective of the values of $\phi$ and/or $\theta$.

4. a) $f(4,2)\approx\boxed{7}$.

   b) $f_y(1,-3)\approx\boxed{-2.5}$ since $f$ decreases by about $2.5$ per unit of increase of $y$ near $(1,-3)$.

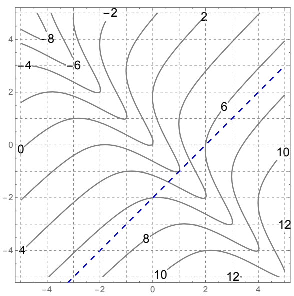
   c) As $x$ changes at $(2,-1)$, $f_x$ is decreasing from about $2.5$ to about $1.5$, so $f_{xx}(2,-1)$ is $\boxed{\text{negative}}$.

   d) As $x$ changes at $(1,1)$, $f_y$ decreases from about $-.5$ to about $-1$, so $f_{yx}(1,1)$ is $\boxed{\text{negative}}$.

   e) $f(x,3)=5$ when $\boxed{x\approx 3}$.

f) The maximum value of $f$ on the region $[0,2] \times [0,2]$ occurs at the lower right-hand corner of the square $[0,2] \times [0,2]$; this maximum value is $\boxed{6}$.

g) The line $y = x - 2$ is the dashed line shown on the picture below; the smallest value of $f$ achieved on this line is $\boxed{2}$ (at the point $(1,-1)$).




h) $\nabla f$ points in the direction of greatest increase of $f$, which is generally southeast. Thus $\nabla f$ must be picture $\boxed{\textbf{C}}$.

i) The highest values of $f$ occur when $x$ is positive and $y$ is negative; the only graph for which this is true is $\boxed{\textbf{D}}$.

5. $f : \mathbb{R}^2 \to \mathbb{R}$ so the total derivative of $f$ is the $1 \times 2$ matrix

$$Df(x,y) = \begin{pmatrix} f_x & f_y \end{pmatrix} = \begin{pmatrix} \frac{1}{x+y^2} & \frac{2y}{x+y^2} \end{pmatrix}$$
$$\Rightarrow Df(1,0) = \begin{pmatrix} \frac{1}{1+0^2} & \frac{2(0)}{1+0^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Thus the linearization of $f$ at $(1,0)$ is

$$\begin{aligned} L(x,y) &= f(1,0) + Df(1,0)(x-1, y-0) \\ &= \ln(1+0^2) + \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x-1 \\ y \end{pmatrix} \\ &= 0 + x - 1 = \boxed{x-1}. \end{aligned}$$

Plugging in $(x,y) = (.9,.2)$, we get $f(.9,.2) \approx L(.9,.2) = .9 - 1 = \boxed{-.1}$.

6. a) $\nabla g = (g_x, g_y) = \boxed{(8xy - 3y^2 + 2, 4x^2 - 6xy)}$.

b) $\dfrac{\partial^2 g}{\partial x \partial y} = \dfrac{\partial}{\partial x}(g_y) = \dfrac{\partial}{\partial x}(4x^2 - 6xy) = \boxed{8x - 6y}$.

c) The tangent plane has normal vector $\mathbf{n} = (g_x(1,-1,2), g_y(1,-1,2), -1)$ so using the answer to part (a), we see that $\mathbf{n} = (8(1)(-1) - 3(-1)^2 + 2, 4(1^2) - 6(1)(-1), -1) = (-9, 10, -1)$. This makes the normal equation of the plane

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) &= 0 \\ (-9, 10, -1) \cdot (x-1, y+1, z-2) &= 0 \\ -9(x-1) + 10(y+1) - (z-2) &= 0. \end{aligned}$$

This rearranges into $\boxed{-9x + 10y - z = -21}$.

7. To find the CPs, set the gradient equal to $\mathbf{0}$ and solve for $x$ and $y$. First, $\nabla f = (f_x, f_y) = (3x^2 - 12y, -3y^2 - 12x)$. Setting $f_x = 0$ gives $3x^2 - 12y = 0$, i.e. $\frac{1}{4}x^2 = y$. Substitute this into the second equation to get $-3\left(\frac{1}{4}x^2\right)^2 - 12x = 0$, i.e. $-\frac{3}{16}x^4 - 12x = 0$, which factors as $-3x\left(\frac{x^3}{16} + 4\right) = 0$. From $-3x = 0$, we get $x = 0$ (and therefore $y = \frac{1}{4}0^2 = 0$) and from $\frac{x^3}{16} + 4 = 0$, we get $x^3 = -64$, i.e. $x = -4$ (which goes with $y = \frac{1}{4}(4^2) = 4$). Thus the two critical points of $f$ are $(0, 0)$ and $(-4, 4)$. To classify these, use the Hessian:

$$Hf(x, y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6x & -12 \\ -12 & 6y \end{pmatrix}$$

so

$$Hf(0, 0) = \begin{pmatrix} 0 & -12 \\ -12 & 0 \end{pmatrix}$$

which has negative determinant, making $\boxed{(0, 0) \text{ a saddle}}$ and

$$Hf(-4, 4) = \begin{pmatrix} -24 & -12 \\ -12 & -24 \end{pmatrix}$$

which has negative trace and positive determinant, making $\boxed{(-4, 4) \text{ a local maximum}}$.

8. Use Lagrange's method. Write $g(x, y, z) = 2x^2 + y^2 + 4z^2$ so that we have

$$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} 3 &= \lambda(4x) \\ 6 &= \lambda(2y) \\ 6 &= \lambda(8z) \end{cases} \Rightarrow \lambda = \frac{3}{4x} = \frac{3}{y} = \frac{3}{4z}.$$

From this, we see $x = z$ and $y = 4x$. Substituting into the constraint, we get $2x^2 + (4x)^2 + 4x^2 = 8800$, i.e. $22x^2 = 8800$, i.e. $x^2 = 400$ so $x = \pm 20$. This gives two candidate points $(20, 80, 20)$ and $(-20, -80, -20)$. Test these candidate points in the utility to find the maximum value:

$$f(20, 80, 20) = 3(20) + 6(80) + 6(20) = 60 + 480 + 120 = 660$$
$$f(-20, -80, -20) = 3(-20) + 6(-80) + 6(-20) = -60 - 480 - 120 = -660$$

Thus the maximum value is $\boxed{660}$.

9. a) The skater's velocity is $\mathbf{x}'\left(\frac{1}{2}\right) = (1-3t^2, 2t)|_{t=1/2} = \left(1-\frac{3}{4}, \frac{2}{2}\right) = \boxed{\left(\frac{1}{4}, 1\right)}$.

b) The skater's acceleration is $\mathbf{x}''\left(\frac{1}{2}\right) = (-6t, 2)|_{t=1/2} = \boxed{(-3, 2)}$.

c) Treat the path as though it is in $\mathbb{R}^3$ by setting $z = 0$. Then, the curvature of the skater's path at time $\frac{1}{2}$ is

$$\begin{aligned}
\frac{||\mathbf{x}'(\frac{1}{2}) \times \mathbf{x}''(\frac{1}{2})||}{||\mathbf{x}'(\frac{1}{2})||^3} &= \frac{||\left(\frac{1}{4}, 1, 0\right) \times (-3, 2, 0)||}{||(\left(\frac{1}{4}, 1, 0\right)||^3} \\
&= \frac{||\left(0, 0, \frac{7}{2}\right)||}{\left(\frac{17}{16}\right)^{3/2}} \\
&= \frac{\frac{7}{2}}{\frac{17^{3/2}}{64}} = \boxed{224 \cdot 17^{-3/2}}.
\end{aligned}$$

d) Use the area formula coming from Green's Theorem:

$$\begin{aligned}
Area(E) &= \frac{1}{2} \oint_{\partial E} (x\,dy - y\,dx) \\
&= \frac{1}{2} \int_{-1}^{1} \left[(t-t^3)(2t\,dt) - t^2(1-3t^2)\,dt\right] \\
&= \frac{1}{2} \int_{-1}^{1} \left[2t^2 - 2t^4 - t^2 + 3t^4\right] dt \\
&= \frac{1}{2} \int_{-1}^{1} \left(t^2 + t^4\right) dt \\
&= \frac{1}{2} \left[\frac{t^3}{3} + \frac{t^5}{5}\right]_{-1}^{1} = \frac{1}{2}\left[\frac{1}{3} + \frac{1}{5}\right] - \frac{1}{2}\left[-\frac{1}{3} - \frac{1}{5}\right] = \boxed{\frac{8}{15}}.
\end{aligned}$$

10. a) This is

$$\begin{aligned}
\iint_E (2x + 6x^2y)\,dA &= \int_0^5 \int_0^3 (2x + 6x^2y)\,dy\,dx \\
&= \int_0^5 \left[2xy + 3x^2y^2\right]_0^3 dx \\
&= \int_0^5 \left[6x + 27x^2\right] dx \\
&= \left[3x^2 + 9x^3\right]_0^5 = 3(5^2) + 9(5^3) = 75 + 1125 = \boxed{1200}.
\end{aligned}$$

b) Using polar coordinates, the region $E$ can be described with the inequalities $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq r \leq 3$. Thus

$$\begin{aligned}\iint_E 8x\,dA &= \int_0^{\pi/2}\int_0^3 8r\cos\theta r\,dr\,d\theta \\ &= \int_0^{\pi/2}\int_0^3 8r^2\cos\theta\,dr\,d\theta \\ &= \int_0^{\pi/2}\left[\frac{8}{3}r^3\cos\theta\right]_0^3 d\theta \\ &= \int_0^{\pi/2} 72\cos\theta\,d\theta = 72\sin\theta\big|_0^{\pi/2} = 72 - 0 = \boxed{72}.\end{aligned}$$

c) The triangle $E$ can be described with the inequalities $0 \leq y \leq 1$, $0 \leq x \leq y$, so this is

$$\begin{aligned}\iint_E e^{y^2}\,dA &= \int_0^1\int_0^y e^{y^2}\,dx\,dy \\ &= \int_0^1 \left[e^{y^2}x\right]_0^y dy = \int_0^1 e^{y^2}y\,dy.\end{aligned}$$

This integral is done with the $u$-sub $u = y^2$, $\frac{1}{2}du = y\,dy$ to get

$$\int_0^1 \frac{1}{2}e^u\,du = \frac{1}{2}e^u\bigg|_0^1 = \boxed{\frac{1}{2}e - \frac{1}{2}}.$$

**NOTE:** This integral is not doable if you try to do the integration in the other order ($dy\,dx$).

11. a) By linearity,

$$\begin{aligned}\int_0^3\int_0^3 \left[2f(x,y) - g(x,y)\right]\,dy\,dx &= 2\int_0^3\int_0^3 f(x,y)\,dy\,dx - \int_0^3\int_0^3 g(x,y)\,dy\,dx \\ &= 2(10) - 7 = \boxed{13}.\end{aligned}$$

b) By additivity,

$$\begin{aligned}\int_0^3\int_0^4 f(x,y)\,dy\,dx &= \int_0^3\int_0^3 f(x,y)\,dy\,dx + \int_0^3\int_3^4 f(x,y)\,dy\,dx \\ &= 10 + 8 = \boxed{18}.\end{aligned}$$

c) By additivity,

$$\begin{aligned}\int_0^3\int_0^4 g(x,y)\,dy\,dx &= \int_0^3\int_0^3 g(x,y)\,dy\,dx + \int_0^3\int_3^4 g(x,y)\,dy\,dx \\ 12 &= 7 + \int_0^3\int_3^4 g(x,y)\,dy\,dx \\ \boxed{5} &= \int_0^3\int_3^4 g(x,y)\,dy\,dx.\end{aligned}$$

d) By linearity,

$$\begin{aligned}\int_0^3\int_0^3 [f(x)+2]\ dy\,dx &= \int_0^3\int_0^3 f(x,y)\,dy\,dx + \int_0^3\int_0^3 2dy\,dx\\ &= 10 + 2(Area([0,3]\times[0,3]))\\ &= 10+2(3)(3) = 10+18 = \boxed{28}.\end{aligned}$$

12. Using spherical coordinates, the set $E$ can be described with the inequalities $0 \le \theta \le 2\pi$, $0 \le \varphi \le \frac{\pi}{4}$ (from the cone) and $0 \le r \le 1$ (from the sphere). So the integral is

$$\begin{aligned}\iiint_E z\,dV &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 (\rho\cos\varphi)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta\\ &= 2\pi\int_0^{\pi/4}\int_0^1 \rho^3\cos\varphi\sin\varphi\,d\rho\,d\varphi\\ &= 2\pi\int_0^{\pi/4}\left[\frac{1}{4}\rho^r\cos\varphi\sin\varphi\right]_0^1 d\varphi\\ &= 2\pi\int_0^{\pi/4}\frac{1}{4}\cos\varphi\sin\varphi\,d\varphi\end{aligned}$$

For this last integral, use the $u$-sub $u=\sin\varphi$, $du = \cos\varphi\,du$ to get

$$2\pi\int_0^{\sqrt{2}/2}\frac{1}{4}u\,du = \left[\frac{\pi}{4}u^2\right]_0^{\sqrt{2}/2} = \frac{\pi}{4}\left(\frac{\sqrt{2}}{2}\right)^2 = \boxed{\frac{\pi}{8}}.$$

13. a) Parameterize $\gamma$ by $\mathbf{x}(t) = (2\cos t, 2\sin t)$ for $0 \le t \le \pi$. Then $\mathbf{x}'(t) = (-2\sin t, 2\cos t)$ so the speed is $||\mathbf{x}'(t)|| = \sqrt{4\sin^2 t + 4\cos^2 t} = \sqrt{4} = 2$. That means

$$\begin{aligned}\int_\gamma f\,ds &= \int_0^\pi f(\mathbf{x}(t))\,||\mathbf{x}'(t)||\,dt\\ &= \int_0^\pi f(2\cos t, 2\sin t)2\,dt\\ &= \int_0^\pi 8\sin^2 t\,dt\\ &= \int_0^\pi 8\left(\frac{1-\cos 2t}{2}\right)\,dt\\ &= \int_0^\pi (4-4\cos 2t)\,dt\\ &= [4t - 2\sin 2t]_0^\pi = \boxed{4\pi}.\end{aligned}$$

b) We first show $\mathbf{f}$ is conservative by finding a function $f : \mathbb{R}^2 \to \mathbb{R}$ so that $\mathbf{f} = \nabla f$:

$$f_x = 10xy^3 \Rightarrow f = \int 10xy^3\,dx = 5x^2y^3 + A(y)$$
$$f_y = 15x^2y^2 + 4 \Rightarrow f = \int (15x^2y^2 + 4)\,dy = 5x^2y^3 + 4y + B(x).$$

To reconcile these, set $A(y) = 4y$ and $B(x) = 0$ so that $f(x, y) = 5x^2y^3+4y$ is a potential function for $\mathbf{f}$. Then, by the Fundamental Theorem of Line Integrals,

$$\begin{aligned}\int_\gamma \mathbf{f} \cdot d\mathbf{s} &= f(\mathbf{x}(1)) - f(\mathbf{x}(0)) \\ &= f(e^0 + 1, e^0) - f(e^0 + 0, e^0) \\ &= f(2, 1) - f(1, 1) \\ &= 5(2^2)(1^3) + 4(1) - [5(1^2)(1^3) + 4(1)] = \boxed{15}.\end{aligned}$$

## 1.2 Fall 2021 Final Exam

1. Throughout this problem, let $\mathbf{v} = (-3, 1, 2)$, $\mathbf{w} = (-1, 5, 0)$ and $\mathbf{x} = (4, 1, -1)$.

   a) (2.3) Compute the distance between $\mathbf{v}$ and $\mathbf{w}$.

   b) (2.3) Is the angle between $\mathbf{v}$ and $\mathbf{w}$ acute, obtuse or right? Explain.

   c) (2.4) If $A = \begin{pmatrix} 1 & 0 & -4 \\ 2 & 1 & 0 \end{pmatrix}$, compute $A\mathbf{v}$.

   d) (2.7) Write parametric equations for the line passing through $\mathbf{w}$ and $\mathbf{x}$.

   e) (2.7) Write a normal equation of the plane containing $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{x}$.

2. For each given limit, compute the value of the limit, or explain why the limit does not exist.

   a) (3.5) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y}$.

   b) (3.5) $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{x^2+z^2}{x^2+y^2+z^2}$.

3. Suppose $f(x, y) = 4x^2y^2 - 3xy^3$.

   a) (4.5) Compute the gradient of $f$.

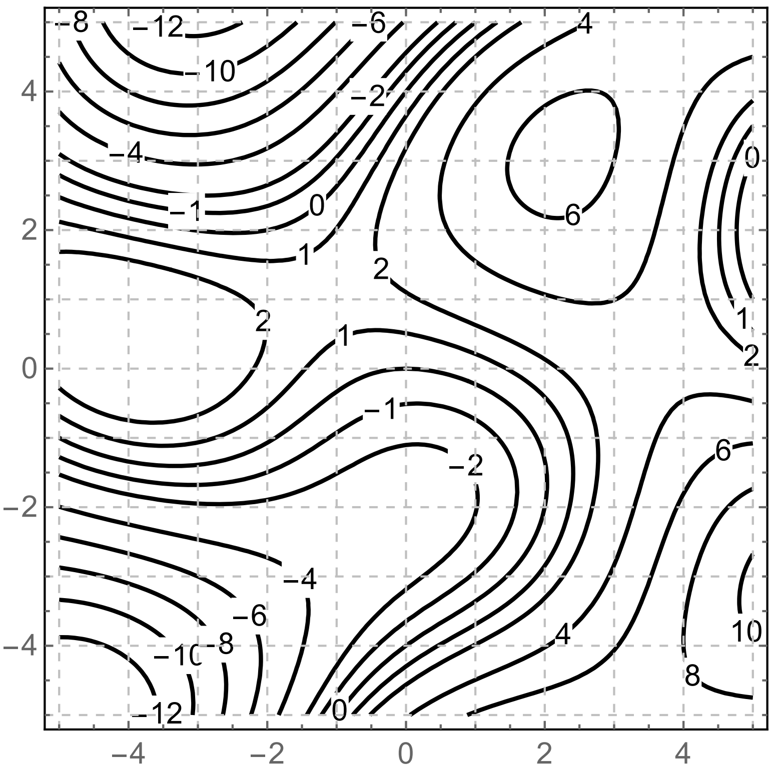
   b) (4.2) Compute $f_x(1, 2)$.

   c) (4.2) Compute $\frac{\partial^3 f}{\partial y^2 \partial x}$.

   d) (4.3) Write the equation of the plane tangent to the graph of $f$ at the point $(2, -1, 22)$.

   e) (8.4) Compute $\int_\gamma f\, ds$, where $\gamma$ is the line segment beginning at $(0, 0)$ and ending at $(2, 1)$.

4. A contour plot for an unknown function $f : \mathbb{R}^2 \to \mathbb{R}$ is given below:




Use this contour plot to answer the following questions.

a) (3.2) Estimate $f(4, -1)$.

b) (4.5) In which compass direction does $\nabla f(5, 2)$ point?

c) (4.2) Estimate $\frac{\partial f}{\partial y}(-2, 2)$.

d) (4.5) Is $D_{\mathbf{u}} f(2, -1)$ positive, negative or zero, if $\mathbf{u}$ is in the direction $(1, 1)$?

e) (3.2) Find the minimum value of $f(2, y)$, for $-5 \le y \le 5$.

f) (6.1) Estimate the coordinates of a local maximum of $f$.

g) (6.1) Estimate the coordinates of a saddle of $f$.

5. (4.3) Compute the linearization of $f(x, y, z) = x^2 \sin(yz)$ at the point $(2, 3, 0)$, and use that linearization to estimate $f(1.9, 3.3, .2)$.

6. Suppose that a particle is moving in $\mathbb{R}^3$ so that its position at time $t$ is $\left(t^2, t, \frac{2}{3}t^3\right)$.

a) (5.1) Compute the velocity of the particle at time $0$.

b) (5.2) Compute the tangential component of the acceleration of the particle at time $0$.

c) (5.2) What does the sign of your answer to part (b) tell you about the motion of the particle at time $0$?

d) (5.4) Compute the curvature of the path the particle travels at time $0$.

e) (5.2) Compute the distance travelled by the particle from time $0$ to time $2$.

7. (6.1) Find all critical points of the function $f(x,y) = 2x^3 + 6xy^2 - 9x^2 + 9y^2$. Classify, with appropriate reasoning, each critical point as a local maximum, local minimum or saddle.

8. (6.3) Compute the absolute maximum value of the function $f(x,y) = xy$, subject to the constraint $x^2 + 4y^2 = 8$.

9. a) (7.3) Compute $\iint_D \cos(x+y)\,dA$, where $D$ is the square $\left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$.

   b) (7.3) Compute $\iint_E 6y^2\,dA$, where $E$ is the triangle with vertices $(0,0)$, $(4,0)$ and $(2,2)$.

10. (7.3) Compute each iterated integral:

(a) $\displaystyle\int_0^1 \int_x^1 e^{y^2}\,dy\,dx$ (b) $\displaystyle\int_0^1 \int_0^y \int_{xy}^x 12xz\,dz\,dx\,dy$

11. (7.5) Compute the volume of the set of points in $\mathbb{R}^3$ lying above the parallelogram in $\mathbb{R}^2$ with vertices $(1,-1)$, $(-1,1)$, $(2,0)$ and $(0,2)$, and lying below the graph of $z = x^2$.

12. (7.5) Compute

$$\iiint_E xz\,dV,$$

where $E$ is the set of points in $\mathbb{R}^3$ satisfying $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x^2 + y^2 + z^2 \leq 1$.

## Solutions

1. a) $dist(\mathbf{v}, \mathbf{w}) = ||\mathbf{v} - \mathbf{w}|| = ||(-2, -4, 2)|| = \sqrt{2^2 + (-4)^2 + 2^2} = \boxed{\sqrt{24}}$.

   b) $\mathbf{v} \cdot \mathbf{w} = (-3)(-1) + 1(5) + 2(0) = 8 > 0$, so the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\boxed{\textbf{acute}}$.

   c) By regular matrix multiplication, $A\mathbf{v} = (1(-3) + 0(1) + 2(-4), 2(-3) + 1(1) + 2(0)) = \boxed{(-11, -5)}$.

   d) A direction vector for the line is $\mathbf{x} - \mathbf{w} = (5, -4, -1)$; the line then has parametric equations

$$\boxed{\begin{cases} x &= -1 + 5t \\ y &= 5 - 4t \\ z &= -t \end{cases}}.$$

   e) The plane contains vectors $\mathbf{w} - \mathbf{v} = (2, 4, -2)$ and $\mathbf{x} - \mathbf{w} = (5, -4, -1)$; a normal vector to the plane is therefore $\mathbf{n} = (2, 4, -2) \times (5, -4, -1) = (-12, -8, -28)$. Any nonzero multiple of this is also a normal vector, so I will use $\mathbf{n} = (3, 2, 7)$. Thus the plane has normal equation

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{x} - \mathbf{v}) &= 0 \\ \text{i.e. } (3, 2, 7)(x + 3, y - 1, z - 2) &= 0 \\ \text{i.e. } 3(x + 3) + 2(y - 1) + 7(z - 2) &= 0 \\ \text{i.e. } \boxed{3x + 2y + 7z = 7}&. \end{aligned}$$

2. a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{(x-y)(x+y)}{x+y} = \lim_{(x,y)\to(0,0)} (x - y) = 0 - 0 = \boxed{0}$.

   b) Along the $y$-axis, we have $\displaystyle\lim_{(0,y,0)\to(0,0)} \frac{x^2+z^2}{x^2+y^2+z^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$, but along the $z$-axis, we have $\displaystyle\lim_{(0,0,z)\to(0,0)} \frac{x^2+z^2}{x^2+y^2+z^2} = \lim_{z\to 0} \frac{z^2}{z^2} = 1$. Therefore the limit $\boxed{\textbf{does not exist}}$.
   (This limit could also be done with spherical coordinates.)

3. a) $\nabla f(x, y) = (f_x, f_y) = \boxed{(8xy^2 - 3y^3, 8x^2y - 9xy^2)}$.

   b) $f_x(1, 2) = (8xy^2 - 3y^3)|_{(1,2)} = 32 - 24 = \boxed{8}$.

   c) $\frac{\partial^3 f}{\partial y^2 \partial x} = f_{xyy} = (8xy^2 - 3y^3)_{yy} = (16xy - 9y^2)_y = \boxed{16x - 18y}$.

   d) Observe $f_x(2, -1) = 16 - (-3) = 19$ and $f_y(2, -1) = -32 - 18 = -50$, so the tangent plane has equation

$$\begin{aligned} z &= f(2, -1) + f_x(2, -1)(x - 2) + f_x(2, -1)(y + 1) \\ z &= 22 + 19(x - 2) - 50(y + 1) \\ \boxed{z = 19x - 50y - 66}&. \end{aligned}$$

e) $\gamma$ is parametrized by $\mathbf{x}(t) = (2t, t)$ for $0 \le t \le 1$, so $\mathbf{x}'(t) = (2,1)$ and $||\mathbf{x}'(t)|| = \sqrt{2^2+1^2} = \sqrt{5}$. Thus the line integral becomes

$$\begin{aligned}\int_\gamma f\,ds &= \int_0^1 f(2t,t)\sqrt{5}\,dt \\ &= \int_0^1 \left[4(2t)^2t^2 - 3(2t)t^3\right]\sqrt{5}\,dt \\ &= \sqrt{5}\int_0^1 10t^4\,dt = 2\sqrt{5}t^5\Big|_0^1 = \boxed{2\sqrt{5}}.\end{aligned}$$

4. a) $f(4,-1) \approx \boxed{5}$.

   b) $\nabla f(5,2)$ points toward the greatest increase in the value of $f$, which is $\boxed{\textbf{west}}$.

   c) $\frac{\partial f}{\partial y}(-2,2) \approx f(-2,3) - f(-2,2) = -3 - 0 = \boxed{-3}$.

   d) Is $D_{\mathbf{u}}f(2,-1)$ is $\boxed{\textbf{negative}}$ since $f$ decreases in the direction $(-1,-1)$ from the point $(2,-1)$.

   e) The minimum value of $f(2,y)$ for $-5 \le y \le 5$ is $\boxed{0}$, when $x \approx -1.5$.

   f) $f$ has alocal maximum at about $\boxed{(2.2, 3.1)}$.

   g) $f$ has two saddles in the viewing window: one at about $\boxed{(-1.2, 1)}$ and another at about $\boxed{(3.5, .25)}$.

5. The total derivative of $f$ is

$$Df(x,y,z) = \begin{pmatrix} f_x & f_y & f_z \end{pmatrix} = \begin{pmatrix} 2x\sin(yz) & x^2z\cos(yz) & x^2y\cos(yz) \end{pmatrix}.$$

   At the point $(2,3,0)$, this is $Df(2,3,0) = \begin{pmatrix} 0 & 0 & 12 \end{pmatrix}$. So the linearization of $f$ at $(2,3,0)$ is

$$\begin{aligned}L(x,y,z) &= f(2,3,0) + Df(2,3,0)(x-2, y-3, z-0) \\ &= 0 + \begin{pmatrix} 0 & 0 & 12 \end{pmatrix}(x-2, y-3, z-0) = \boxed{12z}.\end{aligned}$$

   That means

$$f(1.9, 3.3, .2) \approx L(1.9, 3.3, .2) = 12(.2) = \boxed{2.4}.$$

6. Suppose that a particle is moving in $\mathbb{R}^3$ so that its position at time $t$ is $\left(t^2, t, \frac{2}{3}t^3\right)$.

   a) $\mathbf{v}(0) = \mathbf{x}'(0) = (2t, 1, 2t^2)|_{t=0} = \boxed{(0,1,0)}$.

   b) First, $\mathbf{a}(0) = \mathbf{x}''(0) = (2, 0, 4t)|_{t=0} = (2,0,0)$. Therefore, $a_T(0) = \frac{\mathbf{a}(0)\cdot\mathbf{v}(0)}{||\mathbf{v}(0)||} = \frac{0}{1} = \boxed{0}$.

c) Since $a_T(0) = 0$, at time $0$ the object is neither speeding up nor slowing down at that instant.

d) $\kappa(0) = \frac{||\mathbf{v}(0)\times\mathbf{a}(0)||}{||\mathbf{v}(0)||^3} = \frac{||(0,0,-2)||}{1^3} = \boxed{2}$.

e) The arc length is

$$\begin{aligned}\int_0^2 ||\mathbf{x}'(t)||\,dt &= \int_0^2 \sqrt{(2t)^2 + 1^2 + (2t^2)^2}\,dt \\ &= \int_0^2 \sqrt{4t^2 + 1 + 4t^4}\,dt \\ &= \int_0^2 \sqrt{(2t^2+1)^2}\,dt \\ &= \int_0^2 (2t^2+1)\,dt \\ &= \frac{2}{3}t^3 + t\Big|_0^2 = \boxed{\frac{22}{3}}.\end{aligned}$$

7. The gradient of $f$ is $\nabla f(x,y) = (f_x, f_y) = (6x^2 + 6y^2 - 18x, 12xy + 18y)$. Set the gradient equal to $(0,0)$ to produce the system

$$\begin{cases} 6x^2 + 6y^2 - 18x = 0 \\ 12xy + 18y = 0 \Rightarrow 6y(2x+3) = 0 \Rightarrow y = 0 \text{ or } x = -\frac{3}{2}. \end{cases}$$

If $y = 0$, then the first equation gives $6x^2 - 18x = 0$, i.e. $x = 0$ or $x = 3$, giving the critical points $(0,0)$ and $(3,0)$. If $x = -\frac{3}{2}$, the first equation gives $y^2 = -36$, which has no solution. Thus there are two critical points: $(0,0)$ and $(3,0)$. We test these using the Hessian:

$$Hf(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x - 18 & 12y \\ 12y & 12x + 18 \end{pmatrix}.$$

We have $\det Hf(0,0) = \det \begin{pmatrix} -18 & 0 \\ 0 & 18 \end{pmatrix} < 0$, so $\boxed{(0,0) \text{ is a saddle}}$. Finally, we see that $Hf(3,0) = \begin{pmatrix} 18 & 0 \\ 0 & 54 \end{pmatrix}$ has positive determinant and trace, so $Hf(3,0) > 0$, so $\boxed{(3,0) \text{ is a local minimum}}$.

8. Use Lagrange's method: let $g(x,y) = x^2 + 4y^2$ and start with $\nabla f = \lambda \nabla g$ to get

$$\begin{cases} y &= \lambda(2x) \\ x &= \lambda(8y) \end{cases}$$

Plugging the first equation into the second, we get $x = 16\lambda^2 x$, so $x = 0$ or $16\lambda^2 = 1$ so $\lambda = \pm\frac{1}{4}$. If $x = 0$, then from the first equation $y = 0$, but $(0,0)$

isn't on the constraint, so we can discard that point. That leaves $\lambda = \pm\frac{1}{4}$: from the first equation above, that means $y = \left(\pm\frac{1}{4}\right)(2x) = \pm\frac{1}{2}x$. Plugging into the constraint gives $x^2 + 4\left(\pm\frac{1}{2}x\right)^2 = 8$, i.e. $2x^2 = 8$, i.e. $x = \pm 2$. since $y = \pm\frac{1}{2}x$, that gives four critical points $(\pm 2, \pm 1)$; plug these into the utility $f(x,y) = xy$ to see that the maximum value is $\boxed{2}$.

9. a) By Fubini's theorem, this is

$$\begin{aligned}\iint_D \cos(x+y)\,dA &= \int_0^{\pi/2}\int_0^{\pi/2} \cos(x+y)\,dy\,dx \\ &= \int_0^{\pi/2} \sin(x+y)\big|_0^{\pi/2}\,dx \\ &= \int_0^{\pi/2} \left[\sin(x+\frac{\pi}{2}) - \sin x\right]dx \\ &= \left[-\cos(x+\frac{\pi}{2}) + \cos x\right]_0^{\pi/2} \\ &= [(1+0)-(0+1)] = \boxed{0}.\end{aligned}$$

b) $E$ is horizontally simple with $0 \le y \le 2$, $y \le x \le 4-y$, so Fubini's theorem gives

$$\begin{aligned}\iint 6y^2\,dA &= \int_0^2\int_y^{4-y} 6y^2\,dx\,dy \\ &= \int_0^2 \left[6y^2x\right]_y^{4-y}\,dy \\ &= \int_0^2 \left[24y^2 - 12y^3\right]dx \\ &= \left[8y^3 - 3y^4\right]_0^2 = 64 - 48 = \boxed{16}.\end{aligned}$$

10. a) This is a double integral over a triangle with vertices $(0,0)$, $(0,1)$ and $(1,1)$, and by reversing the order of integration we get

$$\int_0^1\int_x^1 e^{y^2}\,dy\,dx = \int_0^1\int_0^y e^{y^2}\,dx\,dy = \int_0^1 ye^{y^2}\,dy.$$

Now use the $u$-sub $u = y^2$, $du = 2y\,dy$ to rewrite this integral as

$$\int_0^1 \frac{1}{2}e^u\,du = \frac{1}{2}e^u\Big|_0^1 = \boxed{\frac{1}{2}(e-1)}.$$

b) Compute this directly:

$$\begin{aligned}
\int_0^1 \int_0^y \int_{xy}^x 12xz\,dz\,dx\,dy &= \int_0^1 \int_0^y \left[6xz^2\right]_{xy}^x\,dx\,dy \\
&= \int_0^1 \int_0^y \left[6x^3 - 6x^3y^2\right]\,dx\,dy \\
&= \int_0^1 \left[\frac{3}{2}x^4 - \frac{3}{2}x^4y^2\right]_0^y\,dy \\
&= \int_0^1 \left[\frac{3}{2}y^4 - \frac{3}{2}y^6\right]\,dy \\
&= \left[\frac{3}{10}y^5 - \frac{3}{14}y^7\right]_0^1 \\
&= \frac{3}{10} - \frac{3}{14} = \boxed{\frac{3}{35}}.
\end{aligned}$$

11. The four sides of the parallelogram $E$ have equations $x+y=0$, $x+y=2$, $y-x=-2$ and $y-x=-2$, so we use the change of variables $(x,y) \overset{\phi}{\mapsto} (u,v)$ where $u = x+y$ and $v = y-x$. Thus

$$J(\phi) = \det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \det\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2,$$

so the volume is

$$V = \iint_E x^2\,dA = \int_0^2 \int_{-2}^2 x^2 \frac{1}{|J(\phi)|}\,dv\,du = \int_0^2 \int_{-2}^2 \frac{1}{2}x^2\,dv\,du.$$

Now we back-solve for $x$ in terms of $u$ and $v$; add the equations $u = x+y$ and $v = y-x$ to get $2y = u+v$, i.e. $y = \frac{1}{2}(u+v)$. Thus $x = u-y = u - \frac{1}{2}(u+v) = \frac{1}{2}(v-u)$, so the integral becomes

$$\begin{aligned}
\int_0^2 \int_{-2}^2 \frac{1}{2}\left[\frac{1}{2}(v-u)\right]^2\,dv\,du &= \frac{1}{8}\int_0^2 \int_{-2}^2 (v-u)^2\,dv\,du \\
&= \frac{1}{8}\int_0^2 \left[\frac{1}{3}(v-u)^3\right]_{-2}^2\,du \\
&= \frac{1}{24}\int_0^2 \left[(2-u)^3 - (-2-u)^3\right]\,du \\
&= \frac{1}{24}\int_0^2 \left[(2-u)^3 + (2+u)^3\right]\,du \\
&= \frac{1}{24}\left[-\frac{1}{4}(2-u)^4 + \frac{1}{4}(-2-u)^4\right]_0^2 \\
&= \frac{1}{96}\left[(0+4^4) - (-2^4+2^4)\right]_2^{10} = \frac{4^4}{96} = \boxed{\frac{8}{3}}.
\end{aligned}$$

12. In spherical coordinates, this region is $0 \le \rho \le 1$, $0 \le \theta \le \frac{\pi}{2}$, and $0 \le \varphi \le \frac{\pi}{2}$. So the integral becomes

$$\begin{aligned}
\iiint_E xz\,dV &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\sin\varphi\cos\theta)(\rho\cos\varphi)\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi \\
&= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^4\sin^2\varphi\cos\varphi\cos\theta\,d\rho\,d\theta\,d\varphi \\
&= \int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{1}{5}\rho^5\sin^2\varphi\cos\varphi\cos\theta\right]_0^1 d\theta\,d\varphi \\
&= \frac{1}{5}\int_0^{\pi/2}\int_0^{\pi/2}\sin^2\varphi\cos\varphi\cos\theta\,d\theta\,d\varphi \\
&= \frac{1}{5}\int_0^{\pi/2}\left[\sin^2\varphi\cos\varphi\sin\theta\right]_0^{\pi/2} d\varphi \\
&= \frac{1}{5}\int_0^{\pi/2}\sin^2\varphi\cos\varphi\,d\varphi.
\end{aligned}$$

Now use the $u$-sub $u = \sin\varphi$, $du = \cos\varphi\,d\varphi$ to rewrite this integral as

$$\frac{1}{5}\int_0^1 u^2\,du = \frac{1}{15}u^3\Big|_0^1 = \boxed{\frac{1}{15}}.$$

## 1.3 Spring 2021 Final Exam

1. Fill in the blanks in these sentences with sets so that the sentence is true.

   a) (4.1) Suppose $\mathbf{f}$ is such that for each $\mathbf{x}$, $D\mathbf{f}(\mathbf{x})$ is a $4 \times 2$ matrix. In this situation, $\mathbf{f}$ must be a function from ________ to ________ .

   b) (4.5) Suppose $\mathbf{f}$ is such that $\nabla\mathbf{f}(3, 1, -5)$ exists. In this setting, $\mathbf{f}$ must be a function from _______ to _______, and $\nabla\mathbf{f}(3, 1, -5)$ is an element of _______ .

   c) (8.2) Suppose $\mathbf{f}$ is such that div $\mathbf{f}(3, -2)$ exists. In this setting, $\mathbf{f}$ must be a function from ________ to _______, and div $\mathbf{f}(3, -2)$ is an element of _______ .

   d) (6.1) Suppose $\mathbf{f}$ is such that $H\mathbf{f}(4, 8)$ exists. In this situation, $\mathbf{f}$ must be a function from ________ to ________, and $H\mathbf{f}(4, 8)$ is an element of _______ .

   e) (4.5) Suppose $\mathbf{f}$ is such that $D_{\mathbf{u}}\mathbf{f}(-1, -4, 0)$ exists. In this situation, $\mathbf{f}$ must be a function from ________ to ________, $\mathbf{u}$ must be an element of ________, and $D_{\mathbf{u}}\mathbf{f}(-1, -4, 0)$ is an element of ________ .

2. (8.5) Green's Theorem says that under suitable hypotheses, some equation equating two types of integrals is true. Write that equation here:

3. (4.1) To say that a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is **differentiable** at $\mathbf{x}$ means that there exists some matrix $D\mathbf{f}(x)$ such that some limit exists and is equal to $0$. Write that limit here:

4. (3.5) Explain why the limit $\lim\limits_{\mathbf{x} \to \mathbf{0}} \frac{x-y+z}{x+y+z}$ does not exist.

5. Let $\mathbf{v} = (1, 3, 0)$ and $\mathbf{w} = (-2, -1, 2)$.

   a) (2.3) Compute $(\mathbf{v} + 2\mathbf{w}) \cdot \mathbf{w}$.

   b) (2.6) Find a nonzero vector in $\mathbb{R}^3$ which is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

   c) (2.3) Compute the measure of the angle between $\mathbf{v}$ and $\mathbf{w}$.

   d) (2.3) Compute the distance between $\mathbf{v}$ and $\mathbf{w}$.

6. Throughout this problem, let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x, y) = 2xy^2 + x^3 - 3y^4$.

   a) (4.2) Compute all second-order partial derivatives of $f$.

   b) (4.2) Compute the slope of the line tangent to the graph of $f$ which is parallel to the $y$-axis, that passes through the point $(1, -2)$.

   c) (4.5) Find the direction in which the value of $f$ is decreasing most rapidly, at the point $(2, 1)$.

d) (4.5) Compute the rate of change of $f$ in the direction $(-3,4)$ at the point $(2,1)$.

7. Suppose $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ are differentiable functions satisfying

$$\mathbf{f}(1,5) = (2,3); \quad \mathbf{g}(1,5) = (4,-1);$$

$$D\mathbf{f}(1,5) = \begin{pmatrix} 1 & -3 \\ 3 & 4 \end{pmatrix}; \quad D\mathbf{g}(1,5) = \begin{pmatrix} 0 & 2 \\ -7 & 2 \end{pmatrix}; \quad D\mathbf{g}(2,3) = \begin{pmatrix} -1 & 3 \\ 5 & 0 \end{pmatrix}.$$

In each part of this problem, you are given a quantity.

- If the given information in this problem is sufficient to compute the quantity, compute it.
- If the given information cannot be used to compute the quantity, write "not enough information".

a) (4.1) $D(\mathbf{f} + 2\mathbf{g})(1,5)$

b) (4.4) $D(\mathbf{f} \circ \mathbf{g})(1,5)$

c) (4.4) $D(\mathbf{g} \circ \mathbf{f})(1,5)$

8. (6.1) Find all the critical points of the function $f(x,y) = 4xy - x^4 - y^4 + 12$. Classify each critical point as a local maximum, local minimum or saddle.

9. (6.3) The profit of a company is given by $P(x,y,z) = 4x + 8y + 6z$, where $x$, $y$ and $z$ are units of three different products the company manufactures. Find the maximum profit of the company, given that $x^2 + 4y^2 + 2z^2 = 800$.

10. An object is moving in $\mathbb{R}^3$ so that its position at time $t$ is $\mathbf{x}(t) = (3\cos 2t, 4\sin 2t, \frac{1}{\pi}t)$.

a) (5.1) Compute the velocity of the object at time $t = \frac{\pi}{3}$.

b) (5.1) Compute the speed of the object at time $t = \frac{\pi}{3}$.

c) (5.1) Compute the acceleration of the object at time $t = \frac{\pi}{3}$.

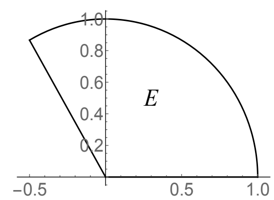
d) (4.3) Find parametric equations of the line which is tangent to the path the object travels at $t = \frac{\pi}{3}$.

11. In each part of this problem, you are given a set $E \subseteq \mathbb{R}^2$. For each given $E$, compute

$$\iint_E 8x \, dA.$$

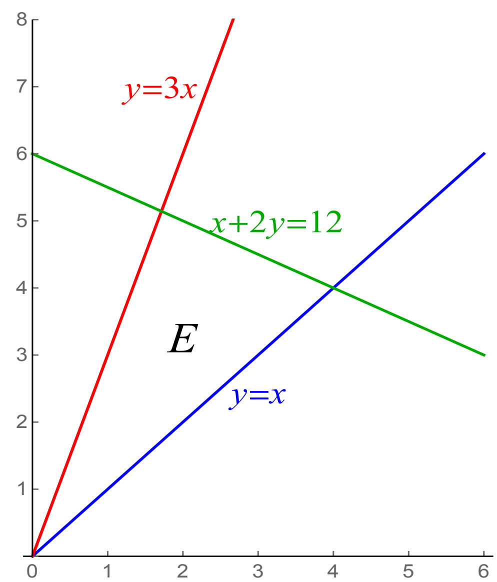
a) (7.3) $E = [0,1] \times [0,4]$

b) (7.5) $E$ is the one-third of a circle pictured here:




c) (7.3) $E = \{(x, y) : y \geq 0, y^2 \leq x \leq y + 2\}$

12. (7.5) Compute the volume of the set of points in $\mathbb{R}^3$ lying below the graph of $z = \frac{y^2(x+2y)^2}{x^5}$ and above the triangular region $E$ bounded by the red, blue and green lines shown below:

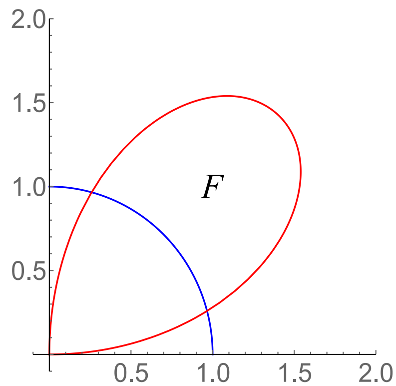



13. (7.5) Compute the volume of the set of points in $\mathbb{R}^3$ lying above the $xy$-plane, inside the sphere $x^2 + y^2 + z^2 = 16$, and inside the cone $z^2 = x^2 + y^2$.

14. (8.4) Compute the line integral $\int_\gamma \mathbf{f} \cdot d\mathbf{s}$, where $\gamma$ is the line segment beginning at $(2, -1, 3)$ and ending at $(4, 0, 1)$, and $\mathbf{f}(x, y, z) = (3z, x + y, 2x + z)$.

15. **(Bonus)** (7.5) Compute the area of the region $F$ of points lying in the first quadrant, outside the circle $x^2 + y^2 = 1$ but inside the graph of the polar

function $r = 2\sin 2\theta$. This region is shown below:



## Solutions

1. a) Suppose $\mathbf{f}$ is such that for each $\mathbf{x}$, $D\mathbf{f}(\mathbf{x})$ is a $4\times 2$ matrix. In this situation, $\mathbf{f}$ must be a function from $\boxed{\mathbb{R}^2}$ to $\boxed{\mathbb{R}^4}$.

   b) Suppose $\mathbf{f}$ is such that $\nabla\mathbf{f}(3,1,-5)$ exists. In this setting, $\mathbf{f}$ must be a function from $\boxed{\mathbb{R}^3}$ to $\boxed{\mathbb{R}}$, and $\nabla\mathbf{f}(3,1,-5)$ is an element of $\boxed{\mathbb{R}^3}$.

   c) Suppose $\mathbf{f}$ is such that div $\mathbf{f}(3,-2)$ exists. In this setting, $\mathbf{f}$ must be a function from $\boxed{\mathbb{R}^2}$ to $\boxed{\mathbb{R}^2}$, and div $\mathbf{f}(3,-2)$ is an element of $\boxed{\mathbb{R}}$.

   d) Suppose $\mathbf{f}$ is such that $H\mathbf{f}(4,8)$ exists. In this situation, $\mathbf{f}$ must be a function from $\boxed{\mathbb{R}^2}$ to $\boxed{\mathbb{R}}$, and $H\mathbf{f}(4,8)$ is an element of $\boxed{M_2(\mathbb{R})}$.

   e) Suppose $\mathbf{f}$ is such that $D_\mathbf{u}\mathbf{f}(-1,-4,0)$ exists. In this situation, $\mathbf{f}$ must be a
   function from $\boxed{\mathbb{R}^3}$ to $\boxed{\mathbb{R}}$, $\mathbf{u}$ must be an element of $\boxed{\mathbb{R}^3}$, and $D_\mathbf{u}\mathbf{f}(-1,-4,0)$ is an element of $\boxed{\mathbb{R}}$.

2. The formula of Green's Theorem is $\boxed{\oint_{\partial E} \mathbf{f}\cdot d\mathbf{s} = \iint_E (N_x - M_y)\ dA}$.

   (This is under the assumption that $\mathbf{f} = (M,N)$, that $E$ is compact with a piecewise $C^1$ boundary and that $\partial E$ has been oriented so that as you move along $\partial E$, $E$ is on the left.)

3. To say that a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is **differentiable** at $\mathbf{x}$ means that there exists some matrix $D\mathbf{f}(x)$ such that

$$\boxed{\lim_{\mathbf{h}\to\mathbf{0}} \frac{||\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{x})\mathbf{h}||}{||\mathbf{h}||} = 0.}$$

4. Along the $z$-axis, we have

$$\lim_{(0,0,z)\to(0,0,0)} \frac{x-y+z}{x+y+z} = \lim_{(0,0,z)\to(0,0,0)} \frac{0-0+z}{0+0+z} = 1,$$

   and along the $y$-axis, we have

$$\lim_{(0,y,0)\to(0,0,0)} \frac{x-y+z}{x+y+z} = \lim_{(0,0,z)\to(0,0,0)} \frac{0-y+0}{0+y+0} = -1.$$

   Since limits along different paths, are unequal, the limit does not exist.

5. a) Compute $(\mathbf{v}+2\mathbf{w})\cdot\mathbf{w} = (-3,1,4)\cdot(-2,-1,2) = (-3)(-2)+1(-1)+4(2) = \boxed{13}$.

   b) $\mathbf{v}\times\mathbf{w} = (3(2)-0(-1), 0(-2)-1(2), 1(-1)-3(-2)) = \boxed{(6,-2,5)}$.

c) First, $\mathbf{v} \cdot \mathbf{w} = 1(-2) + 3(-1) + 0(2) = -5$. Next, $||\mathbf{v}|| = \sqrt{1^2 + 3^2 + 0^2} = \sqrt{10}$ and $||\mathbf{w}|| = \sqrt{(-2)^2 + 1^2 + 2^2} = 3$. So from the angle formula for dot product, we have

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= ||\mathbf{v}||\,||\mathbf{w}||\,\cos\theta \\ -5 &= \sqrt{10}\,(3)\cos\theta \\ \frac{-5}{3\sqrt{10}} &= \cos\theta \\ \boxed{\arccos\left(\frac{-5}{3\sqrt{10}}\right)} &= \theta. \end{aligned}$$

d) This is $||\mathbf{v} - \mathbf{w}|| = ||(3, 4, -2)|| = \sqrt{3^2 + 4^2 + (-2)^2} = \sqrt{9 + 16 + 4} = \boxed{\sqrt{29}}$.

6. (6 pts each) Throughout this problem, let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x, y) = 2xy^2 + x^3 - 3y^4$.

a) First, the first-order partial derivatives are $f_x(x, y) = 2y^2 + 3x^2$ and $f_y(x, y) = 4xy - 12y^3$. Differentiate again to get

$$\boxed{f_{xx}(x, y) = 6x} \quad \boxed{f_{xy}(x, y) = f_{yx}(x, y) = 4y} \quad \boxed{f_{yy}(x, y) = 4x - 36y^2}$$

b) This is $f_y(1, -2) = 4(1)(-2) - 12(-2)^3 = -8 + 96 = \boxed{88}$.

c) This is $-\nabla f(2, 1) = -\left(f_x(2, 1), f_y(2, 1)\right) = -\left(2 \cdot 1^2 + 3 \cdot 2^2, 4 \cdot 2 \cdot 1 - 12 \cdot 1^3\right) = \boxed{(-14, 4)}$.

d) First, a unit vector in the direction $(-3, 4)$ is $\mathbf{u} = \frac{1}{||(-3,4)||}(-3, 4) = \left(\frac{-3}{5}, \frac{4}{5}\right)$. The question asks for a directional derivative:

$$D_{\mathbf{u}} f(-3, 4) = \nabla f(2, 1) \cdot \mathbf{u} = (14, -4) \cdot \left(\frac{-3}{5}, \frac{4}{5}\right) = \frac{-42}{5} - \frac{16}{5} = \boxed{\frac{-58}{5}}.$$

7. a) This follows from the Sum and Constant Multiple Rules:

$$\begin{aligned} D(\mathbf{f} + 2\mathbf{g})(1, 5) &= D\mathbf{f}(1, 5) + 2D\mathbf{g}(1, 5) \\ &= \begin{pmatrix} 1 & -3 \\ 3 & 4 \end{pmatrix} + 2\begin{pmatrix} 0 & 2 \\ -7 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 1 \\ -11 & 8 \end{pmatrix}}. \end{aligned}$$

b) Since we don't know what $\mathbf{g}(1, 5)$ is, $D(\mathbf{f} \circ \mathbf{g})(1, 5)$ cannot be computed. $\boxed{\text{Not enough information}}$.

c) This can be computed using the Chain Rule:

$$\begin{aligned} D(\mathbf{g}\circ\mathbf{f})(1,5) &= D\mathbf{g}(\mathbf{f}(1,5))D\mathbf{f}(1,5) \\ &= D\mathbf{g}(2,3)D\mathbf{f}(1,5) \\ &= \begin{pmatrix} -1 & 3 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 3 & 4 \end{pmatrix} = \boxed{\begin{pmatrix} 8 & 15 \\ 5 & -15 \end{pmatrix}}. \end{aligned}$$

8. First, find the critical points by setting the gradient equal to $\mathbf{0}$:

$$\nabla f(x,y) = (f_x(x,y), f_y(x,y) = (4y-4x^3, 4x-4y^3).$$

Setting each coordinate equal to $0$, we see from the first equation that $y = x^3$ and from the second equation that $x = y^3$. Substituting the first equation into the second gives $x = (x^3)^3$, i.e. $x = x^9$, i.e. $x^9 - x = x(x^8-1) = 0$, so $x = 0$, $x = 1$ or $x = -1$. From $y = x^3$, we get respective $y$-values $0$, $1$ and $-1$. This gives three critical points: $(0,0)$, $(1,1)$ and $(-1,-1)$, which we test by plugging them into the Hessian:

$$Hf(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} -12x^2 & 4 \\ 4 & -12y^2 \end{pmatrix}.$$

Testing the critical points, we get

| CP | $Hf$ | $\det Hf$ | $tr\ Hf$ | classification |
|---|---|---|---|---|
| $(0,0)$ | $\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$ | $-16$ | N/A | saddle |
| $(1,1)$ | $\begin{pmatrix} -12 & 4 \\ 4 & -12 \end{pmatrix}$ | $128$ | $-24$ | local max |
| $(-1,-1)$ | $\begin{pmatrix} -12 & 4 \\ 4 & -12 \end{pmatrix}$ | $128$ | $-24$ | local max |

9. We use Lagrange's method. Let $g(x,y,z) = x^2+4y^2+2z^2$; we set $\nabla P = \lambda\nabla g$ to get the system of equations

$$\begin{cases} 4 &= \lambda 2x \\ 8 &= \lambda 8y \\ 6 &= \lambda 4z \end{cases}$$

These equations lead to $x = \frac{2}{\lambda}$, $y = \frac{1}{\lambda}$ and $z = \frac{3}{2\lambda}$. Plugging into the constraint gives

$$800 = x^2+4y^2+2z^2 = \left(\frac{2}{\lambda}\right)^2 + 4\left(\frac{1}{\lambda}\right)^2 + 2\left(\frac{3}{2\lambda}\right)^2 = \frac{25}{2\lambda^2}$$

so $\lambda^2 = \frac{25}{1600} = \frac{1}{64}$ and $\lambda = \pm\frac{1}{8}$. Since $x, y$ and $z$ have to be nonnegative, we can drop $\lambda = -\frac{1}{8}$. $\lambda = \frac{1}{8}$ leads to $x = 16$, $y = 8$ and $z = 12$. This is the location of the maximum, and the maximum profit is $P(16, 8, 12) = 4(16)+8(8)+6(12) = \boxed{200}$.

10. a) $\mathbf{v}(t) = \mathbf{x}'(t) = (-6\sin 2t, 8\cos 2t, \frac{1}{\pi})$. $\mathbf{v}\left(\frac{\pi}{3}\right) = \boxed{\left(-3\sqrt{3}, -4, \frac{1}{\pi}\right)}$.

b) $||\mathbf{v}\left(\frac{\pi}{3}\right)|| = \sqrt{(-3\sqrt{3})^2 + (-4)^2 + \left(\frac{1}{\pi}\right)^2} = \boxed{\sqrt{43 + \frac{1}{\pi^2}}}$.

c) $\mathbf{a}(t) = \mathbf{x}''(t) = (-12\cos 2t, -16\sin 2t, 0)$ so $\mathbf{a}\left(\frac{\pi}{3}\right) = \boxed{\left(6, -8\sqrt{3}, 0\right)}$.

d) The line passes through $\mathbf{x}\left(\frac{\pi}{3}\right) = \left(\frac{-3}{2}, 2\sqrt{3}, \frac{1}{3}\right)$ and has direction vector $\mathbf{x}'\left(\frac{\pi}{3}\right) = \left(-3\sqrt{3}, -4, \frac{1}{\pi}\right)$ (computed in part (a)), so its parametric equations are

$$\boxed{\begin{cases} x &= \frac{-3}{2} - 3\sqrt{3}\,t \\ y &= 2\sqrt{3} - 4t \\ z &= \frac{1}{3} + \frac{1}{\pi}t \end{cases}}.$$

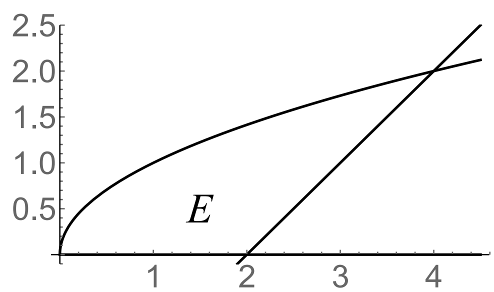
11. a) Compute directly with Fubini's Theorem:

$$\iint_E 8x\,dA = \int_0^1 \int_0^4 8x\,dy\,dx = \int_0^1 8xy|_0^4\,dx = \int_0^1 32x\,dx = 16x^2\Big|_0^1 = \boxed{16}.$$

b) Change to polar coordinates, since $E = \left\{(r, \theta) : 0 \le \theta \le \frac{2\pi}{3}, 0 \le r \le 1\right\}$:

$$\begin{aligned}\iint_E 8x\,dA &= \int_0^{2\pi/3} \int_0^1 8r\cos\theta\, r\,dr\,d\theta \\ &= \int_0^{2\pi/3} \int_0^1 8r^2\cos\theta\,dr\,d\theta \\ &= \int_0^{2\pi/3} \frac{8}{3}r^3\cos\theta\Big|_0^1\,d\theta \\ &= \int_0^{2\pi/3} \frac{8}{3}\cos\theta\,d\theta = \frac{8}{3}\sin\theta\Big|_0^{2\pi/3} = \boxed{\frac{4}{3}\sqrt{3}}.\end{aligned}$$

c) (10 pts) Sketch a picture of $E$:



Either from the picture, or by doing some algebra (setting $y^2 = y+2$ and solving for $y$), we find that the upper-right corner of $E$ is $(4,2)$. So you can compute the integral directly with Fubini's Theorem:

$$\begin{aligned}\iint_E 8x\,dA &= \int_0^2 \int_{y^2}^{y+2} 8x\,dx\,dy \\ &= \int_0^2 4x^2\Big|_{y^2}^{y+2}\,dy \\ &= \int_0^2 \left[4(y+2)^2 - 4y^4\right]\,dy \\ &= \left[\frac{4}{3}(y+2)^3 - \frac{4}{5}y^5\right]_0^2 = \left[\frac{4}{3}4^3 - \frac{4}{5}(32)\right] - \frac{4}{3}(2^3) = \boxed{\frac{736}{15}}.\end{aligned}$$

12. The volume is given by $\iint_E \frac{y^2(x+2y)^2}{x^5}\,dA$. To compute this integral, use the change of variable $u = y/x$ and $v = x+2y$ so that if $\varphi(x,y) = (u,v)$, then $\varphi(E) = \{(u,v) : 1 \le u \le 3, 0 \le v \le 12\}$. Then the Jacobian of $\varphi$ is

$$J(\varphi) = \det D\varphi = \det\begin{pmatrix} \frac{-y}{x^2} & \frac{1}{x} \\ 1 & 2 \end{pmatrix} = \frac{-2y}{x^2} - \frac{1}{x} = \frac{-(2y+x)}{x^2} = \frac{-v}{x^2}.$$

Back-solving for $x$ and $y$ in terms of $u$ and $v$, we get $u = y/x$ so $y = xu$. Plugging in the equation for $v$ gives $v = x + 2xu = x(1+2u)$, so $x = \frac{v}{1+2u}$ and finally, $y = xu = \frac{uv}{1+2u}$. Now the integral can be computed:

$$\begin{aligned}\iint_E \frac{y^2(x+2y)^2}{x^5}\,dA &= \iint_{\varphi(E)} \frac{y^2(x+2y)^2}{x^5}\frac{1}{|J(\varphi)|}\,dA \\ &= \int_0^{12}\int_1^3 \frac{\left(\frac{uv}{1+2u}\right)^2 v^2}{\left(\frac{v}{1+2u}\right)^5} \cdot \frac{\left(\frac{v}{1+2u}\right)^2}{v}\,du\,dv \\ &= \int_0^{12}\int_1^3 (1+2u)u^2\,du\,dv \\ &= \int_0^{12}\int_1^3 (u^2+2u^3)\,du\,dv \\ &= \int_0^{12}\left[\frac{1}{3}u^3 + \frac{1}{2}u^4\right]_1^3\,dv \\ &= \int_0^{12}\left(\left[9+\frac{81}{2}\right] - \left[\frac{1}{3}+\frac{1}{2}\right]\right)\,dv \\ &= \int_0^{12}\frac{146}{3}\,dv = \frac{146}{3}(12) = \boxed{584}.\end{aligned}$$

13. This solid, in spherical coordinates, is $0 \le \theta \le 2\pi$, $0 \le \rho \le 4$ and $0 \le \varphi \le \frac{\pi}{4}$.

So the volume is

$$\begin{aligned}\int_0^{2\pi}\int_0^4\int_0^{\pi/4} \rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta &= \int_0^{2\pi}\int_0^4 -\rho^2\cos\varphi\Big|_0^{\pi/4} \, d\rho \, d\theta \\ &= \int_0^{2\pi}\int_0^4 \rho^2\left(1-\frac{\sqrt{2}}{2}\right) d\rho \, d\theta \\ &= \int_0^{2\pi} \frac{1}{3}\rho^3\left(1-\frac{\sqrt{2}}{2}\right)\Bigg|_0^4 \, d\theta \\ &= \int_0^{2\pi} \frac{64}{3}\left(1-\frac{\sqrt{2}}{2}\right) d\theta \\ &= 2\pi\cdot\frac{64}{3}\left(1-\frac{\sqrt{2}}{2}\right) = \boxed{\frac{128\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right)}.\end{aligned}$$

14. Since $\gamma$ is a line segment, $\gamma$ is parametrized by

$$\mathbf{x}(t) = (2,-1,3) + t((4,0,1)-(2,-1,3)) = (2+2t, -1+t, 3-2t)$$

for $0 \le t \le 1$ and $\mathbf{x}'(t) = (2,1,-2)$ so $d\mathbf{s} = (2,1,-2)\,dt$. Thus, the line integral is

$$\begin{aligned}\int_\gamma \mathbf{f}\cdot d\mathbf{s} &= \int_0^1 (3z, x+y, 2x+z)\cdot(2,-1,2)\,dt \\ &= \int_0^1 (3(3-2t), (2+2t)+(-1+t), 2(2+2t)+3-2t)\cdot(2,1,-2)\,dt \\ &= \int_0^1 (9-6t, 1+3t, 7+2t)\cdot(2,1,-2)\,dt \\ &= \int_0^1 (18-12t+1+3t-14-4t)\,dt \\ &= \int_0^1 (5-13t)\,dt = \left[5t+\frac{13}{2}t^2\right]_0^1 = 5-\frac{13}{2} = \boxed{\frac{-3}{2}}.\end{aligned}$$

15. Start by finding the intersection points of the curves. In polar coordinates, the circle is $r=1$, so the curves intersect when $1 = 2\sin 2\theta$, i.e. $\frac{1}{2} = \sin 2\theta$, i.e. $2\theta \in \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$, i.e. $\theta = \frac{\pi}{12}$ and $\theta = \frac{5\pi}{12}$. So $F$, in polar coordinates, is

$$F = \left\{(r,\theta) : \frac{\pi}{12} \le \theta \le \frac{5\pi}{12}, 1 \le r \le 2\sin 2\theta\right\}.$$

Therefore the area of $F$ is

$$
\begin{aligned}
\iint_F 1\,dA &= \int_{\pi/12}^{5\pi/12}\int_1^{2\sin 2\theta} r\,dr\,d\theta \\
&= \int_{\pi/12}^{5\pi/12} \frac{1}{2}r^2\Big|_1^{2\sin 2\theta}\,d\theta \\
&= \int_{\pi/12}^{5\pi/12}\left[2\sin^2(2\theta)-\frac{1}{2}\right]d\theta \\
&= \int_{\pi/12}^{5\pi/12}\left[2\left(\frac{1-\cos 2(2\theta)}{2}\right)-\frac{1}{2}\right]d\theta \\
&= \int_{\pi/12}^{5\pi/12}\left[\frac{1}{2}-\cos 4\theta\right]d\theta \\
&= \left[\frac{\theta}{2}-\frac{1}{4}\sin 4\theta\right]_{\pi/12}^{5\pi/12} \\
&= \left[\frac{5\pi}{24}-\frac{1}{4}\sin\left(\frac{5\pi}{3}\right)\right]-\left[\frac{\pi}{24}-\frac{1}{4}\sin\left(\frac{\pi}{3}\right)\right] \\
&= \boxed{\frac{\pi}{6}+\frac{\sqrt{3}}{4}}.
\end{aligned}
$$

## 1.4 Fall 2020 Final Exam

1. Throughout this problem, suppose $\mathbf{f} : \mathbb{R} \to \mathbb{R}^3$, $g : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{h} : \mathbb{R}^3 \to \mathbb{R}^3$ are $C^\infty$ functions. Assume $\mathbf{u}$ is a unit vector in $\mathbb{R}^3$.

   In each part of this problem, you are given an expression. Determine if that expression is a **scalar**, a **vector** in $\mathbb{R}^3$, a **matrix** (in which case you should give its size), or **nonsense**.

   a) (3.1) $\mathbf{f}(1)$
   b) (4.2) $g_x(1,2,3)$
   c) (4.2) $\mathbf{h}_x(1,2,3)$
   d) (4.1) $D\mathbf{f}(1)$
   e) (4.1) $D\mathbf{f}(1,2,3)$
   f) (4.1) $D\mathbf{h}(1,2,3)$
   g) (4.5) $\nabla\mathbf{f}(1)$
   h) (4.5) $\nabla g(1,2,3)$
   i) (6.1) $H\mathbf{f}(1)$
   j) (6.1) $Hg(1,2,3)$
   k) (7.5) $J(\mathbf{f})$
   l) (7.5) $J(\mathbf{h})$
   m) (4.5) $D_\mathbf{u}\mathbf{f}(1)$
   n) (4.5) $D_\mathbf{u}g(1,2,3)$

2. Let $\mathbf{v} = (1,3,-7)$ and $\mathbf{w} = (2,5,2)$.

   a) (2.7) Write parametric equations for the line that passes through the point $(0,-6,11)$ and has direction vector $\mathbf{v}$.
   b) (2.3) Compute the dot product of $\mathbf{v}$ and $\mathbf{w}$.
   c) (2.3) Based on your answer to the previous question, is the angle between $\mathbf{v}$ and $\mathbf{w}$ acute, right, or obtuse? Explain your reasoning.
   d) (2.6) Find a nonzero vector in $\mathbb{R}^3$ which is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.
   e) (2.7) Write the normal equation of the plane that contains the point $(-8,-2,3)$ and contains lines whose direction vectors are $\mathbf{v}$ and $\mathbf{w}$.

3. Let $A = \begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix}$ and let $B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

   a) (2.5) Compute $\det A$.
   b) (2.4) Compute $B^T A$.
   c) (6.1) Is $A$ positive definite, negative definite, or neither? Explain.

4. Compute each limit (or explain why the limit does not exist):

   a) (3.5) $\lim\limits_{\mathbf{x}\to\mathbf{0}} \frac{x+y}{x}$
   b) (3.5) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^2}{x}$

5. Throughout this problem, let $f : \mathbb{R}^3 \to \mathbb{R}$ be $f(x,y,z) = x^2y - 2xz^3 + 4y^3z^2$.

   a) (4.2) Compute all first-order partial derivatives of $f$.
   b) (4.2) Compute $f_{yyz}$.

c) (4.5) Find the direction in which the value of $f$ is increasing most rapidly, at the point $(3, 1, 1)$.

d) (4.5) Write the normal equation of the plane tangent to the level surface $f(x, y, z) = 19$ at the point $(3, 1, 1)$.

e) (4.5) Use your answer to part (d) to estimate the value of $y$ so that $f(3.1, y, .8) = 19$.

f) (4.4) If $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^3$ is a differentiable function such that $\mathbf{g}(-2, 7) = (3, 1, 1)$ and $D\mathbf{g}(-2, 7) = \begin{pmatrix} 3 & -2 \\ 1 & 0 \\ 3 & -1 \end{pmatrix}$, compute $D(f \circ \mathbf{g})(-2, 7)$.

6. (6.3) Find the absolute maximum and absolute minimum values of $f(x, y) = x^2 - 2x - y^2$ over the region of points $(x, y)$ satisfying $x^2 + 4y^2 \leq 4$.

7. Suppose an object is moving in $\mathbb{R}^3$ so that its velocity at time $t$ is given by

$$\mathbf{v}(t) = \left(\frac{t^2}{2} - 3, t - t^2, 2t + 1\right).$$

a) (5.1) Compute the displacement of the object from time $0$ to time $1$.

b) (5.1) Compute the acceleration of the object at time $2$.

c) (5.2) Compute the tangential component of the object's acceleration at time $2$.

d) (5.4) Compute the normal component of the object's acceleration at time $2$.

8. In each part of this problem, you are given a set $E \subseteq \mathbb{R}^2$. For each given $E$, compute

$$\iint_E 10x^2 y \, dA.$$

a) (7.3) $E = [0, 3] \times [0, 2]$

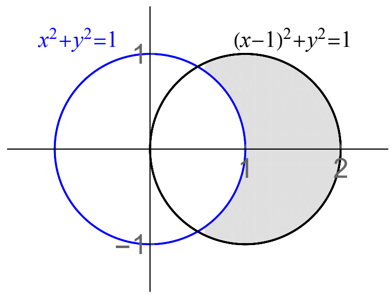
b) (7.3) $E = \{(x, y) : 0 \leq x \leq y \leq 2\}$

9. (7.5) Compute

$$\iint_E 12(y - x)^2 \, dA$$

where $E$ is the parallelogram with vertices $(1, 0)$, $(0, 2)$, $(6, 5)$ and $(5, 7)$.

10. (7.6) Compute the area of the region of points lying inside the circle $(x-1)^2 + y^2 = 1$ but outside the circle $x^2 + y^2 = 1$. This region is shaded in the picture below:




11. Let $S \subseteq \mathbb{R}^3$ be the solid consisting of points $(x, y, z)$ lying above the set $\{(x, y) : x^2 + y^2 \leq 4\}$ and below the function $z = x^2 + y^2$.

    a) (7.6) Compute the volume of $S$.

    b) (7.5) Compute $\iiint\limits_S z^2 \, dV$.

## Solutions

1. a) $\mathbf{f}(1)$ is a **vector** in $\mathbb{R}^3$.

   b) $g_x(1,2,3)$ is a **scalar**.

   c) $\mathbf{h}_x(1,2,3)$ is **nonsense**, since the range of $\mathbf{h}$ isn't $\mathbb{R}$.

   d) $D\mathbf{f}(1)$ is a $3 \times 1$ matrix, which is really a **vector** in $\mathbb{R}^3$.

   e) $D\mathbf{f}(1,2,3)$ is **nonsense**, since the inputs of $\mathbf{f}$ belong to $\mathbb{R}$, not $\mathbb{R}^3$.

   f) $D\mathbf{h}(1,2,3)$ is a $3 \times 3$ **matrix**.

   g) $\nabla\mathbf{f}(1)$ is **nonsense**, since the outputs of $\mathbf{f}$ do not belong to $\mathbb{R}$.

   h) $\nabla g(1,2,3)$ is a **vector** in $\mathbb{R}^3$.

   i) $H\mathbf{f}(1)$ is **nonsense**, since the outputs of $\mathbf{f}$ do not belong to $\mathbb{R}$.

   j) $Hg(1,2,3)$ is a $3 \times 3$ **matrix**.

   k) $J(\mathbf{f})$ is **nonsense**, since the domain and range of $\mathbf{f}$ aren't the same vector space.

   l) $J(\mathbf{h})$ is a $3 \times 3$ **matrix**.

   m) $D_{\mathbf{u}}\mathbf{f}(1)$ is **nonsense**, since the outputs of $\mathbf{f}$ do not belong to $\mathbb{R}$.

   n) $D_{\mathbf{u}}g(1,2,3)$ is a **scalar**.

2. a) Let $\mathbf{p} = (0, -6, 11)$; the parametric equations of the line are

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \Leftrightarrow \boxed{\begin{cases} x = 0 + 1t \\ y = -6 + 3t \\ z = 11 - 7t \end{cases}}.$$

   b) $\mathbf{v} \cdot \mathbf{w} = 1(2) + 3(5) - 7(2) = 2 + 15 - 14 = \boxed{3}$.

   c) Since $\mathbf{v} \cdot \mathbf{w} > 0$, the angle between $\mathbf{v}$ and $\mathbf{w}$ is **acute**.

   d) $\mathbf{v} \times \mathbf{w} = (3(2) - (-7)5, -7(2) - 1(2), 1(5) - 3(2)) = (41, -16, -1)$.

   e) A normal vector to the plane is $\mathbf{n} = (41, -16, -1)$; since the plane contains $\mathbf{p} = (-8, -2, 3)$, the normal equation of the plane is

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) &= 0 \\ (41, -16, -1) \cdot (x + 8, y + 2, z - 3) &= 0 \\ 41(x + 8) - 16(y + 2) - (z - 3) &= 0 \\ \boxed{41x - 16y - z = -299}&. \end{aligned}$$

3. a) $\det A = 3(5) - 2(2) = \boxed{11}$.

b) $B^T A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix} = \boxed{\begin{pmatrix} 7 & 12 \\ 17 & 26 \end{pmatrix}}$.

c) From (a), $\det A = 11$. Note $tr(A) = 1 + 4 = 5$. Since $A$ is a symmetric $2 \times 2$ matrix with positive trace and positive determinant, $A$ is **positive definite**.

4. a) Along the $x$-axis, we have $\lim\limits_{(x,0)\to(0,0)} \frac{x+y}{x} = \lim\limits_{x\to 0} \frac{x+0}{x} = 1$. But along the line $y = x$, we have $\lim\limits_{(x,x)\to(0,0)} \frac{x+y}{x} = \lim\limits_{x\to 0} \frac{x+x}{x} = 2$. Since we have two different limits along two different paths approaching $\mathbf{0}$, $\lim\limits_{\mathbf{x}\to\mathbf{0}} \frac{x+y}{x}$ **DNE**.

b) Change to polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x} = \lim_{r\to 0} \frac{r^2}{r\cos\theta} = \lim_{r\to 0} r\sec\theta = \boxed{0}.$$

5. a) $f_x(x,y,z) = \boxed{2xy - 2z^3}$; $f_y(x,y,z) = \boxed{x^2 + 12y^2z^2}$; $f_z(x,y,z) = \boxed{-6xz^2 + 8y^3z}$.

b) $f_y(x,y,z) = x^2 + 12y^2z^2$ from (a). Differentiate again to get $f_{yy}(x,y,z) = 24yz^2$ and one more time to get $f_{yyz}(x,y,z) = \boxed{48yz}$.

c) The direction in which the value of $f$ is increasing most rapidly at the point $(3,1,1)$ is $\nabla f(3,1,1) = (f_x(3,1,1), f_y(3,1,1), f_z(3,1,1)) = (6-2, 9+12, -18+8) = \boxed{(4,21,-10)}$.

d) The normal vector to this tangent plane is $\nabla f(3,1,1)$, which was computed in (c) as $(4,21,-10)$. So the normal equation of the plane is

$$\begin{aligned} \nabla f(3,1,1) \cdot (\mathbf{x} - (3,1,1)) &= 0 \\ (4,21,-10) \cdot (x-3, y-1, z-1) &= 0 \\ 4(x-3) + 21(y-1) - 10(z-1) &= 0 \\ \boxed{4x + 21y - 10z = 23}. \end{aligned}$$

e) Plug in $x = 3.1$ and $z = .8$ to the answer to (d) to get

$$\begin{aligned} 4(3.1) + 21y - 10(.8) &= 23 \\ 12.4 + 21y - 8 &= 23 \\ 21y &= 18.6 \\ \boxed{y = \frac{18.6}{21} = \frac{31}{35}}. \end{aligned}$$

f) First, $Df(3,1,1) = [\nabla f(3,1,1)]^T = \begin{pmatrix} 4 & 21 & -10 \end{pmatrix}$. Then, by applying the Chain Rule, we get

$$\begin{aligned} D(f\circ \mathbf{g})(-2,7) &= Df(\mathbf{g}(-2,7))D\mathbf{g}(-2,7) \\ &= Df(3,1,1)\begin{pmatrix} 3 & -2 \\ 1 & 0 \\ 3 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 21 & -10 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 1 & 0 \\ 3 & -1 \end{pmatrix} = \boxed{\begin{pmatrix} 3 & 2 \end{pmatrix}}. \end{aligned}$$

6. Start by finding the critical points of $f$ in the desired region:

$$\nabla f(x,y) = (2x-2, -2y) = (0,0) \Rightarrow (x,y) = (1,0) \text{ CP}.$$

Next, optimize $f$ along the boundary $x^2+4y^2=4$ by setting $g(x,y) = x^2+4y^2$ and using Lagrange's method:

$$\nabla f = \lambda\, \nabla g \Rightarrow \begin{cases} 2x-2 &= \lambda(2x) \\ -2y &= \lambda(8y) \Rightarrow y = 0 \text{ or } \lambda = -\frac{1}{4}. \end{cases}$$

If $y=0$, then from the constraint $x^2+4y^2=4$ we have $x^2=4$, i.e. $x=\pm 2$, leading to the two critical points $(2,0)$ and $(-2,0)$. On the other hand, if $\lambda = -\frac{1}{4}$, then from the first equation we get $2x-2 = \frac{-1}{2}x$, leading to $x=\frac{4}{5}$. Plugging this into the constraint gives $\left(\frac{4}{5}\right)^2 + 4y^2 = 4$, i.e. $y = \pm\frac{\sqrt{21}}{5}$, generating the boundary critical points $\left(\frac{4}{5}, \frac{\sqrt{21}}{5}\right)$ and $\left(\frac{4}{5}, -\frac{\sqrt{21}}{5}\right)$. Test all these points in the utility $f$:

| | Point | Value of $f$ |
|---|---|---|
| CP | $(1,0)$ | $1^2 - 2(1) - 0 = -1$ |
| BCP | $(2,0)$ | $2^2 - 2(2) - 0 = 0$ |
| BCP | $(-2,0)$ | $(-2)^2 - 2(-2) - 0 = 8$ |
| BCP | $\left(\frac{4}{5}, \frac{\sqrt{21}}{5}\right)$ | $\left(\frac{4}{5}\right)^2 - 2\left(\frac{4}{5}\right) - \left(\frac{\sqrt{21}}{5}\right)^2 = \frac{16}{25} - \frac{8}{5} - \frac{21}{25} = \frac{-9}{5}$ |
| BCP | $\left(\frac{4}{5}, -\frac{\sqrt{21}}{5}\right)$ | $\left(\frac{4}{5}\right)^2 - 2\left(\frac{4}{5}\right) - \left(-\frac{\sqrt{21}}{5}\right)^2 = \frac{16}{25} - \frac{8}{5} - \frac{21}{25} = \frac{-9}{5}$ |

So the absolute maximum value is $\boxed{8}$ and the absolute minimum value is $\boxed{-\frac{9}{5}}$.

7. a) The displacement is

$$\begin{aligned}\int_0^1 \mathbf{v}(t)\,dt &= \int_0^1 \left(\frac{t^2}{2}-3, t-t^2, 2t+1\right)\,dt\\ &= \left(\frac{1}{6}t^3-3t, \frac{1}{2}t^2-\frac{1}{3}t^3, t^2+t\right)\Big|_0^1\\ &= \left(\frac{1}{6}-3, \frac{1}{2}-\frac{1}{3}, 1+1\right)\\ &= \boxed{\left(-\frac{17}{6}, \frac{1}{6}, 2\right)}.\end{aligned}$$

b) $\mathbf{a}(2) = \mathbf{v}'(2) = (t, 1-2t, 2)\,|_{t=2} = \boxed{(2,-3,2)}$.

c) At time 2, the velocity is $\mathbf{v}(2) = (-1,-2,5)$ and the acceleration is $\mathbf{a}(2) = (2,-3,2)$. Thus

$$a_T(2) = \frac{\mathbf{v}(2)\cdot\mathbf{a}(2)}{||\mathbf{v}(2)||} = \frac{(-1,-2,5)\cdot(2,-3,2)}{||(-1,-2,5)||} = \frac{-2+6+10}{\sqrt{1+4+25}} = \boxed{\frac{14}{\sqrt{30}}}.$$

d) By the Pythagorean Theorem for acceleration,

$$\begin{aligned}[a_T(2)]^2 + [a_N(2)]^2 &= ||\mathbf{a}(2)||^2\\ \left(\frac{14}{\sqrt{30}}\right)^2 + [a_N(2)]^2 &= ||(2,-3,2)||^2\\ \frac{196}{30} + [a_N(2)]^2 &= 17\\ \frac{98}{15} + [a_N(2)]^2 &= 17\\ a_N(2) &= \sqrt{17-\frac{98}{15}} = \boxed{\sqrt{\frac{157}{15}}}.\end{aligned}$$

8. a) For $E = [0,3]\times[0,2]$, we have

$$\iint_E 10x^2y\,dA = \int_0^3\int_0^2 10x^2y\,dy\,dx = \int_0^3 \left[5x^2y^2\right]_0^2 dx = \int_0^3 20x^2\,dx = \frac{20}{3}x^3\Big|_0^3 = \boxed{180}.$$

b) For $E = \{(x,y) : 0 \le x \le y \le 2\}$, we have

$$\iint_E 10x^2y\,dA = \int_0^2\int_0^y 10x^2y\,dx\,dy = \int_0^2 \left[\frac{10}{3}x^3y\right]_0^y dy = \int_0^2 \frac{10}{3}y^4\,dy = \frac{2}{3}y^5\Big|_0^2 = \boxed{\frac{64}{3}}.$$

9. The parallelogram $E$ is bounded by the lines $y+2x=2$, $y+2x=17$, $x-y=1$ and $x-y=-2$. So we set $u=y+2x$ and $v=x-y$ and let $(u,v)=\varphi(x,y)$. Thus $\varphi(E)=\{(u,v):2\le u\le 17, -2\le v\le 1$ and

$$J(\varphi)=\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}=\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}=-3,$$

so since $y-x=-(x-y)-v$, we have

$$\begin{aligned}\iint_E 12(y-x)^2\,dA &= \iint_{\varphi(E)} 12(-v)^2\frac{1}{|J(\varphi)|}\,dA\\ &= \int_2^{17}\int_{-2}^{1}\frac{12v^2}{|-3|}\,dv\,du\\ &= \int_2^{17}\int_{-2}^{1}4v^2\,dv\,du\\ &= \int_2^{17}\left[\frac{4}{3}v^3\right]_{-2}^{1}\,du\\ &= \int_2^{17}12\,du = 12(17-2)=12(15)=\boxed{180}.\end{aligned}$$

10. In polar coordinates, the equation of the left-hand circle is $r=1$ and the equation of the right-hand circle is $r=2\cos\theta$. These circles intersect when $1=2\cos\theta$, i.e. $\theta=\pm\frac{\pi}{3}$. So the shaded region, in polar coordinates, is

$$E=\{(r,\theta):0\le\theta\le\frac{\pi}{3}, 1\le r\le 2\cos\theta\}.$$

So the area of $E$ is

$$\begin{aligned}\iint_E 1\,dA &= \int_{-\pi/3}^{\pi/3}\int_1^{2\cos\theta} r\,dr\,d\theta\\ &= \int_{-\pi/3}^{\pi/3}\left[\frac{r^2}{2}\right]_1^{2\cos\theta}\,d\theta\\ &= \int_{-\pi/3}^{\pi/3}\left[2\cos^2\theta-\frac{1}{2}\right]\,d\theta\\ &= \int_{-\pi/3}^{\pi/3}\left[(1-\cos 2\theta)-\frac{1}{2}\right]\,d\theta\\ &= \int_{-\pi/3}^{\pi/3}\left[\cos 2\theta+\frac{1}{2}\right]\,d\theta\\ &= \left[\frac{1}{2}\sin 2\theta+\frac{1}{2}\theta\right]_{-\pi/3}^{\pi/3}\\ &= \left[\frac{1}{2}\sin\left(\frac{2\pi}{3}\right)+\frac{\pi}{6}\right]-\left[\frac{1}{2}\sin\left(\frac{-2\pi}{3}\right)-\frac{\pi}{6}\right]=\boxed{\frac{\sqrt{3}}{2}+\frac{\pi}{3}}.\end{aligned}$$

11. a) In cylindrical coordinates, $S$ is the set of points $(r, \theta, z)$ satisfying $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$, and $0 \leq z \leq x^2 + y^2 = r^2$. Therefore

$$\iiint_S 1\, dV = \int_0^2 \int_0^{2\pi} \int_0^{r^2} r\, dz\, d\theta\, dr = \int_0^2 \int_0^{2\pi} r^3\, d\theta\, dr = \int_0^2 2\pi r^3\, dr = \frac{1}{2}\pi r^4 \bigg|_0^2 = \boxed{8\pi}.$$

b) Using the same setup as part (a),

$$\iiint_S z^2\, dV = \int_0^2 \int_0^{2\pi} \int_0^{r^2} z^2 r\, dz\, d\theta\, dr = \int_0^2 \int_0^{2\pi} \frac{1}{3} r^7\, d\theta\, dr = \int_0^2 \frac{2}{3}\pi r^7\, dr$$
$$= \frac{1}{12}\pi r^8 \bigg|_0^2 = \boxed{\frac{64}{3}\pi}.$$

## 1.5 Spring 2018 Final Exam

1. (2.3) Throughout this problem, let $\mathbf{v} = (3, 8)$ and $\mathbf{w} = (-5, 2)$.

   a) Find the norm of $\mathbf{v} - \mathbf{w}$.
   b) Find the projection of $\mathbf{w}$ onto $\mathbf{v}$.
   c) Find the cosine of the angle $\theta$ between $\mathbf{v}$ and $\mathbf{w}$.

2. (2.7) In this problem, consider the two lines $l_1$ and $l_2$, where $l_1$ has symmetric equations
$$\frac{x-11}{-3} = \frac{y-2}{-1} = \frac{z+11}{5}$$
and $l_2$ is parameterized by $\mathbf{x}(t) = (4+t, -5-2t, -2t)$.

   a) Show that lines $l_1$ and $l_2$ intersect in a point (by computing that point of intersection).
   b) Find the normal equation of the plane containing lines $l_1$ and $l_2$.

3. (3.5) Compute the following limits (or explain why they do not exist):

   a) $\lim_{\mathbf{x} \to \mathbf{0}} \frac{y-x}{y+x}$
   b) $\lim_{(x,y)\to(0,0)} \frac{y^2-x^2}{y+x}$
   c) $\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 y}{x^2+y^2+z^2}$

4. a) Compute the total derivative of $\mathbf{f}(x, y) = (xe^{xy}, ye^{2x-y})$.
   b) Find all second-order partial derivatives of $f(x, y) = 20x^2 - 10x^2y^2 + 30y^4$.

5. a) (4.2) Compute the directional derivative of $f(x, y, z) = x^2z - 3yz^2$ in the direction $(1, 2, -2)$ at the point $(3, 0, 5)$.
   b) (4.2) Compute div $\mathbf{f}$ where $\mathbf{f}(x, y) = (\sin(2x - y), \cos(2x + y))$.

6. Let $\mathbf{x}(t) = \left(2t^2 + 3, t, \frac{4}{3}\sqrt{2}t^{3/2} + 1\right)$ represent the position of an object at time $t$.

   a) (5.2) Find the tangential and normal components of the object's acceleration at time $t = 2$.
   b) (5.2) At time $t = 2$, is the object speeding up or slowing down? Justify your answer.

7. (6.2 or 6.3) Find the absolute maximum value of the function $f(x, y) = x^2y^4$ on the region $\{(x, y) : x^2 + y^2 \leq 36\}$.

8. Consider the surface $z = 6\sin x \cos y + 8$.

   a) (4.3) Find the equation of the plane which is tangent to this surface at $(\pi, 0, 5)$.

   b) (7.3) Find the volume of the solid consisting of points in $\mathbb{R}^3$ lying above the rectangle $[0, \frac{\pi}{2}] \times [0, \frac{\pi}{3}]$ in the $xy$-plane, but below this surface.

9. (7.5) Compute the double integral

$$\iint_E (xy - x^2)\, dA$$

   where $E$ is the parallelogram with vertices $(2, 0)$, $(6, 4)$, $(4, 8)$ and $(0, 4)$.

10. Let $E$ be a circle of radius $R$.

    a) (7.5) Show that $E$ has area $\pi R^2$, by computing a double integral with polar coordinates.

    b) (8.5) Show that $E$ has area $\pi R^2$, by computing an appropriate line integral and using Green's Theorem.

11. a) (8.4) Compute

$$\int_\gamma (xy + yz)\, ds$$

    where $\gamma$ is the line segment from $(0, 1, 2)$ to $(5, 3, 3)$.

    b) (8.4) Compute

$$\int_\gamma \mathbf{f} \cdot d\mathbf{s}$$

    where $\mathbf{f}(x, y, z) = (2xy^2 z, 2x^2 yz, x^2 y^2)$ and $\gamma$ is parameterized by

$$\mathbf{x}(t) = \left(te^{\sin \pi t}, t^4\sqrt{\tan \pi t + 1}, t^{2018}\right).$$

    for $0 \leq t \leq 1$.

12. (7.6) Choose one of (a) or (b):

    a) Compute

$$\iiint_E y\, dV$$

    where $E$ is the set of points $(x, y, z)$ in the first octant lying below the plane $2x + 4y + z = 12$.

    b) Compute the volume of the set of points $(x, y, z)$ inside the cylinder $x^2 + y^2 = 1$ lying above the $xy$-plane but below the sphere of radius 2 centered at the origin.

## Solutions

1. a) $||\mathbf{v}-\mathbf{w}|| = ||(3,8)-(-5,2)|| = ||(8,6)|| = \sqrt{8^2+6^2} = \sqrt{100} = 10.$

   b) $\pi_{\mathbf{v}}(\mathbf{w}) = \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w} = \frac{1}{73}(3,8) = \left(\frac{3}{73}, \frac{8}{73}\right).$

   c) $\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\,||\mathbf{w}||} = \frac{1}{\sqrt{3^2+8^2}\sqrt{(-5)^2+2^2}} = \frac{1}{\sqrt{73}\sqrt{29}}.$

2. a) $l_1$ passes through $(11,2,-11)$ and has direction vector $(-3,-1,5)$ so we can write the parametric equations of $l_1$ as $\mathbf{y}(s) = (-3s+11, -s+2, 5s-11)$. Now we set the coordinates of $\mathbf{x}(t)$ equal to the coordinates of $\mathbf{y}(s)$:

$$\begin{cases} -3s+11 &= 4+t \\ -s+2 &= -5-2t \\ 5s-11 &= -2t \end{cases}$$

   Subtracting the third equation from the first gives $-6s+13=-5$, i.e. $s=3$; therefore $t=-2$. These values of $s$ and $t$ work in all three equations and produce the intersection point $\mathbf{x}(-2) = \mathbf{y}(3) = (2,-1,4)$.

   b) To get the normal vector to the plane, take the cross product of the direction vectors of the two lines:

$$\mathbf{n} = (-3,-1,5)\times(1,-2,-2) = (12,-1,7)$$

   Then the equation of the plane is

$$\begin{aligned} \mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0) &= 0 \\ \Rightarrow (12,-1,7)\cdot((x,y,z)-(11,2,-11)) &= 0 \\ \Rightarrow (12,-1,7)\cdot(x-11, y-2, z+11) &= 0 \\ \Rightarrow 12(x-11)-(y-2)+7(z+11) &= 0 \\ \Rightarrow 12x-y+7z &= 53 \end{aligned}$$

3. a) $\lim_{\mathbf{x}\to\mathbf{0}} \frac{y-x}{y+x}$ DNE (along the $x$-axis, the limit is $\lim_{(x,0)\to(0,0)} \frac{0-x}{0+x} = -1$, but along the $y$-axis, the limit is $\lim_{(0,y)\to(0,0)} \frac{y-0}{y+0} = 1$.)

   b) $\lim_{(x,y)\to(0,0)} \frac{y^2-x^2}{y+x} = \lim_{(x,y)\to(0,0)} \frac{(y-x)(y+x)}{y+x} = \lim_{(x,y)\to(0,0)} y-x = 0.$

   c) Change to polar coordinates to get

$$\lim_{\rho\to 0} \frac{(\rho^2\sin^2\varphi\cos^2\theta)(\rho\sin\varphi\sin\theta)}{\rho^2} = \lim_{\rho\to 0} \rho(\sin^3\varphi\cos^2\theta\sin\theta) = 0,$$

   no matter what $\varphi$ and $\theta$ are.

4. a) This is a direct computation:

$$Df(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} e^{xy} + xye^{xy} & x^2 e^{xy} \\ 2ye^{2x-y} & e^{2x-y} - ye^{2x-y} \end{pmatrix}.$$

b) First, $f_x(x,y) = 40x - 20xy^2$ and $f_y(x,y) = -20x^2y + 120y^3$. That means

$$\begin{aligned} f_{xx}(x,y) &= 40 - 20y^2 \\ f_{yy}(x,y) &= -20x^2 + 360y^2 \\ f_{xy}(x,y) &= f_{yx}(x,y) = -40xy \end{aligned}$$

5. a) First, find a unit vector in the direction $(1,2,-2)$:

$$\mathbf{u} = \frac{(1,2,-2)}{||(1,2,-2)||} = \frac{(1,2,-2)}{\sqrt{1+4+4}} = \left(\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}\right).$$

Next, the gradient of $f$ is $\nabla f = (2xz, -3z^2, x^2 - 6yz)$ so $\nabla f(3,0,5) = (2(3)5, -3(5^2), 3^2 - 6(0)5^2) = (30, -75, 9)$. Therefore the directional derivative is

$$D_{\mathbf{u}} f(3,0,5) = \nabla f(3,0,5) \cdot \mathbf{u} = (30,-75,9) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}\right) = 10 - 50 - 6 = -46.$$

b) div $\mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = 2\cos(2x - y) - \sin(2x + y)$.

6. a) At time $t \geq 0$, the velocity is $\mathbf{x}'(t) = (4t, 1, \sqrt{8t})$, and the speed is

$$s(t) = ||\mathbf{x}'(t)|| = \sqrt{(4t)^2 + 1 + 8t} = \sqrt{16t^2 + 8t + 1} = \sqrt{(4t+1)^2} = 4t + 1.$$

Therefore $a_T = \left.\frac{ds}{dt}\right|_{t=2} = 4$.

Now for the normal component. At time $t$,

$$\mathbf{a}(t) = \mathbf{x}''(t) = \left(4, 0, \sqrt{\frac{2}{t}}\right)$$

so at time $t = 2$, the acceleration is $\mathbf{a}(2) = (4, 0, 1)$. The normal component of the acceleration is

$$a_N = \sqrt{||\mathbf{a}(2)||^2 - a_T^2} = \sqrt{17 - 16} = 1.$$

b) The object is speeding up when $t = 2$. $a_T = \frac{ds}{dt}$, the rate of change of the speed with respect to time. Since $a_T = 4 > 0$, the speed is increasing.

7. First, find the critical points of $f$: the gradient is $\nabla f = (2xy^4, 4x^2y^3)$; setting this equal to $\mathbf{0}$ we get $x = 0$ and/or $y = 0$, in which case $f(x,y) = 0$.

   Second, we have to study the behavior of $f$ along the boundary of the constraint $x^2 + y^2 = 36$: let $g(x,y) = x^2 + y^2$ and use Lagrange multipliers to maximize $f$ subject to $g(x,y) = 16$: $\nabla f = (2x, 2y)$ so we have

$$\nabla f = \lambda \nabla g \;\Rightarrow\; \begin{cases} 2xy^4 = 2\lambda x \\ 4x^2y^3 = 2\lambda y \end{cases}$$

   From the first equation, $y^4 = \lambda$ and from the second equation, $2x^2y^2 = \lambda$. Thus $2x^2y^2 = y^4$, i.e. $2x^2 = y^2$. Substituting into the constraint, we get $x^2 + 2x^2 = 36$, i.e. $x^2 = 12$ and $y^2 = 2x^2 = 24$. Irrespective of whether $x$ and/or $y$ are positive or negative, for these values of $x$ and $y$ we get

$$f(x,y) = x^2y^4 = 12(24)^2 = 6912$$

   which, since it is greater than zero, is the maximum value of $f$ given the constraint.

8. a) The tangent plane has equation

$$\begin{aligned} z &= f_x(\pi,0)(x-\pi) + f_y(\pi,0)(y-0) + 5 \\ z &= (6\cos\pi\cos 0)(x-\pi) + (-6\sin\pi\sin 0)(y-0) + 5 \\ z &= -6(x-\pi) + 5 \end{aligned}$$

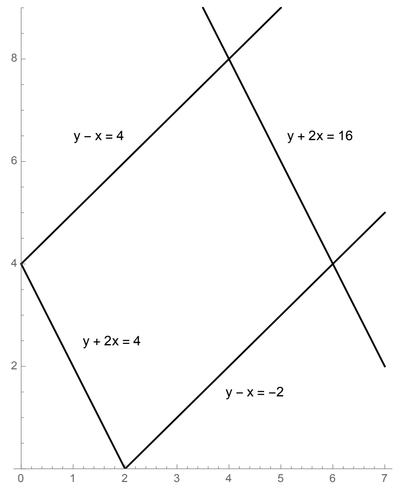
   and the normal equation of this plane is $6x + z = 6\pi + 5$.

   b) This volume is

$$\begin{aligned} \int_0^{\pi/2}\int_0^{\pi/3} (6\sin x\cos y + 8)\; dy\,dx &= \int_0^{\pi/2} \left[6\sin x\sin y + 8y\right]_0^{\pi/3}\; dx \\ &= \int_0^{\pi/2} \left(3\sqrt{3}\sin x + \frac{8\pi}{3}\right)\, dx \\ &= \left[-3\sqrt{3}\cos x + \frac{8\pi}{3}x\right]_0^{\pi/2} \\ &= \frac{4\pi^2}{3} + 3\sqrt{3}. \end{aligned}$$

9. First, sketch the parallelogram $E$ and write equations for the lines comprising

the four sides:




These lines suggest the change of variables $u = y - x$, $v = y + 2x$. Computing the Jacobian we have

$$J = \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \det \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} = -3$$

and notice also that $u - v = -3x$ so $x = \frac{-1}{3}(u - v)$. Therefore the integral becomes

$$\begin{aligned}
\iint_E (xy - x^2)\, dA &= \int_4^{16} \int_{-2}^{4} (xy - x^2) \frac{1}{|-3|}\, du\, dv \\
&= \frac{1}{3} \int_4^{16} \int_{-2}^{4} x(y - x)\, du\, dv \\
&= \frac{1}{3} \int_4^{16} \int_{-2}^{4} \frac{-1}{3}(u - v) u\, du\, dv \\
&= \frac{-1}{9} \int_4^{16} \int_{-2}^{4} (u^2 - uv)\, du\, dv \\
&= \frac{-1}{9} \int_4^{16} \left[ \frac{1}{3}u^3 - \frac{1}{2}u^2 v \right]_{-2}^{4} dv \\
&= \frac{-1}{9} \int_4^{16} [24 - 6v]\; dv \\
&= \frac{-1}{9} \left[ 24v - 3v^2 \right]_4^{16} \\
&= \frac{-1}{9}(-384 - 48) = 48.
\end{aligned}$$

10. a) In polar coordinates, $E = \{(r, \theta) : 0 \leq r \leq R, 0 \leq \theta \leq 2\pi\}$ so the area of $E$ is

$$\iint_E dA = \int_0^{2\pi} \int_0^R r\, dr\, d\theta = \int_0^{2\pi} \left[ \frac{1}{2} r^2 \right]_0^R d\theta = \int_0^{2\pi} \frac{1}{2} R^2\, d\theta = 2\pi \left( \frac{1}{2} R^2 \right) = \pi R^2.$$

b) Parameterize $\partial E$ by $\mathbf{x}(t) = (R\cos t, R\sin t)$ for $0 \le t \le 2\pi$. By Green's Theorem,

$$\begin{aligned}\iint_E dA &= \frac{1}{2}\oint_{\partial E} x\,dy - y\,dx \\ &= \frac{1}{2}\int_0^{2\pi} (R\cos t)(R\cos t\,dt) - (R\sin t)(-R\sin t\,dt) \\ &= \frac{1}{2}\int_0^{2\pi} (R^2\cos^2 t + R^2\sin^2 t)\,dt \\ &= \frac{1}{2}\oint_0^{2\pi} R^2\,dt \\ &= \frac{1}{2}(2\pi R^2) = \pi R^2.\end{aligned}$$

11. a) $\gamma$ is parameterized by $\mathbf{x}(t) = (5t, 2t+1, t+2)$ for $0 \le t \le 1$; we have

$$ds = ||\mathbf{x}'(t)||\,dt = \sqrt{5^2+2^2+1}\,dt = \sqrt{30}\,dt$$

and consequently

$$\begin{aligned}\int_\gamma (xy+yz)\,ds &= \int_0^1 (5t(2t+1) + (2t+1)(t+2))\,\sqrt{30}\,dt \\ &= \sqrt{30}\int_0^1 (12t^2+10t+2)\,dt \\ &= \sqrt{30}\left[4t^3+5t^2+2t\right]_0^1 \\ &= 11\sqrt{30}.\end{aligned}$$

b) Write $\mathbf{f} = (M, N, P)$. First,

$$\begin{aligned}\text{curl}\,\mathbf{f} &= (P_y - N_z, M_z - P_x, N_x - M_y) \\ &= (2x^2y - 2x^2y, 2xy^2 - 2xy^2, 4xyz - 4xyz) \\ &= \mathbf{0}\end{aligned}$$

so $\mathbf{f}$ is conservative. Next, find a potential function for $\mathbf{f}$ by integrating the components of $\mathbf{f}$:

$$\begin{aligned}f(x,y,z) &= \int M\,dx = \int 2xy^2z\,dx = x^2y^2z + A(y,z) \\ f(x,y,z) &= \int N\,dy = \int 2x^2yz\,dy = x^2y^2z + B(x,z) \\ f(x,y,z) &= \int P\,dz = \int x^2y^2\,dz = x^2y^2z + C(x,y)\end{aligned}$$

We see that by setting $A = B = C = 0$, the function $f(x,y,z) = x^2y^2z$ is a potential for $\mathbf{f}$. Now by the Fundamental Theorem of Line Integrals,

$$\int_\gamma \mathbf{f}\cdot d\mathbf{s} = f(\mathbf{x}(1)) - f(\mathbf{x}(0)) = f(1,1,1) - f(0,0,0) = 1 - 0 = 1.$$

12. a) $E$ can also be thought of as the set

$$\{(x,y,z) : 0 \leq y \leq 3, 0 \leq x \leq 6-2y, 0 \leq z \leq 12-2x-4y\}$$

so by Fubini's theorem, the triple integral is

$$\begin{aligned}
\iiint_E y\,dV &= \int_0^3 \int_0^{6-2y} \int_0^{12-2x-4y} y\,dz\,dx\,dy \\
&= \int_0^3 \int_0^{6-2y} [zy]_0^{12-2x-4y}\,dx\,dy \\
&= \int_0^3 \int_0^{6-2y} [y(12-2x-4y)]\,dx\,dy \\
&= \int_0^3 \int_0^{6-2y} \left(12y-2xy-4y^2\right)\,dx\,dy \\
&= \int_0^3 \left[12xy - x^2y - 4xy^2\right]_0^{6-2y}\,dx \\
&= \int_0^3 \left[12y(6-2y) - (6-2y)^2y - 4(6-2y)y^2\right]\,dy \\
&= \int_0^3 \left[36y - 24y^2 + 4y^3\right]\,dy \\
&= \left[18y^2 - 8y^3 + y^4\right]_0^3 \\
&= 18(9) - 8(27) + 81 = 27.
\end{aligned}$$

b) Let $E$ be the base of the figure (in the $xy$ plane) and use cylindrical coordinates, since

$$E = \{(r,\theta) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}.$$

The sphere of radius 2 centered at the origin is $x^2+y^2+z^2=4$, and the top half is $z = \sqrt{4-x^2-y^2} = \sqrt{4-r^2}$. Therefore we want the double integral

$$\begin{aligned}
\iint_E \sqrt{1-x^2-y^2}\,dA &= \int_0^{2\pi} \int_0^1 \sqrt{4-r^2}\,r\,dr\,d\theta \\
&= 2\pi \int_0^1 \sqrt{4-r^2}\,r\,dr \\
&= 2\pi \left[\frac{-1}{3}(4-r^2)^{3/2}\right]_0^1 \\
&= 2\pi \left[\frac{-1}{3}(3\sqrt{3}) + \frac{1}{3}(4^{3/2})\right] \\
&= \frac{16\pi}{3} - 2\pi\sqrt{3}.
\end{aligned}$$