Name:

Directions: This exam has nine questions; please complete all nine questions on your own time and return this exam in class on Monday, October 17. I expect your solutions to be highly legible and polished; this means you should work the problems out on scratch paper and edit them before transferring them to the exam pages. Your solutions will be graded mostly for correctness, but also for clarity and precise use of mathematical notation.

Any result from calculus, any homework problem we have done (refer to it by the problem number), and any theorem we have proven in class may be assumed without proof (if the theorem has a name, refer to it; otherwise just say "by theorem from class"). You may use your notes; you may not use books or other references. You may use a calculator for addition, subtraction, multiplication and division (but not anything more complicated). You are not to discuss the exam with others (this includes classmates, tutors, other professors, etc.)

Last, the underlying field for any vector space is the "obvious one" unless otherwise indicated. If you have any questions, feel free to drop by my office or send me an email.

Grading:

Problem	Points Possible	Points Earned
1	15	
2	10	
3	15	
4	10	
5	30	
6	10	
7	10	
8	10	
9	20	
Total	130	

1. Let $V = \mathbb{C}^2$, endowed with the Hermitian inner product. Let $\mathbf{v} = (1-i, 2+3i)$ and let $\mathbf{w} = (3i, 4+i)$.

   (a) (5 pts) Find a vector of norm 3 in the same direction as $\mathbf{v}$.

   (b) (5 pts) Find the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$.

   (c) (5 pts) Find the distance between $\mathbf{v}$ and $\mathbf{w}$.

2. (10 pts) Find the normal equation of the plane in $\mathbb{R}^3$ (with the usual inner product) whose parametric equations are

$$\begin{cases} x_1 = 2 - 3s + 4t \\ x_2 = s + t \\ x_3 = -1 - s \end{cases}.$$

3. Let $V = \mathbb{R}^3$ with the usual inner product; let $\mathbf{x}_1 = (1, -2, -2)$, $\mathbf{x}_2 = (0, 3, 3)$ and $\mathbf{x}_3 = (3, 2, 4)$. (You may assume without proof that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a linearly independent set, hence a basis of V.)

   (a) (10 pts) Apply the Gram-Schmidt process to find the associated orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of V.

   (b) (5 pts) Let $\mathbf{y} = (1, 1, 0)$; find the projection of $\mathbf{y}$ onto the subspace $W = Span(\mathbf{x}_1, \mathbf{x}_2)$.

4. For each of the given lists of vectors in the given vector spaces V, determine whether the set $\mathcal{S}$ is linearly independent or linearly dependent. Justify your answer:

   (a) (5 pts) $V = \mathbb{R}^4$; $\mathcal{S} = \{(1, 2, -1, 5), (2, -1, 6, -9), (0, 4, -1, 3), (-1, 3, 6, 8), (0, 2, -1, 6)\}$

   (b) (5 pts) $V = C(\mathbb{R}, \mathbb{R})$; $\mathcal{S} = \{\sin x, \cos x\}$.

5. (Each item 5 pts) For each of the following vector spaces V and subsets $W \subseteq V$:
   - Determine whether or not W is a subspace of V.
   - If W is a subspace, give its dimension.
   - If W has a basis, give a basis of W.

   Answers should be completely justified (i.e. proven).

   (a) $V = F^3$; $W = \{(x_1, x_2, x_3) : x_1x_2x_3 = 0\}$.

   (b) $V = \mathbb{C}^3$ (over $\mathbb{C}$); $W = \{(5r - 3s + 8t, 3s, 6r + 6t) : r, s, t \in \mathbb{C}\}$.

(c) $V = F^5$; $W = \{(x_1, x_2, x_3, x_4, x_5) : x_1 = x_3 \text{ and } x_4 = -x_5\}$.

(d) $V = M_2(\mathbb{R})$; $W = \{A \in M_2(\mathbb{R}) : A = A^T\}$ where A^T is the *transpose* of A, i.e. if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then A^T is the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ (i.e. the b and c have switched places).

(e) $V = \mathbb{P}_4$; W is the set of polynomials $f \in V$ satisfying $f(x) = 2f(-x)$.

(f) $V = \mathbb{P}_2$ endowed with the inner product $< f, g >= f(0)g(0) + f(1)g(1) + f(-1)g(-1)$; $W = \big(Span(x^2)\big)^{\perp}$.

6. (10 pts) Prove or disprove: let V be an inner product space and $\mathbf{v} \in V$ be a nonzero vector. Then for any $\mathbf{x} \in V$ and any nonzero scalar r,

$$\text{proj}_{\mathbf{v}}\mathbf{x} = \text{proj}_{r\mathbf{v}}\mathbf{x}.$$

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7. (10 pts) Let V be an inner product space and let $\mathbf{v}, \mathbf{w} \in V$. Prove that if $\mathbf{v} \perp \mathbf{w}$ if and only if $||\mathbf{v}|| \leq ||\mathbf{v} + a\mathbf{w}||$ for all scalars a. **Please write only on the left-hand side of this vertical line:**

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8. (10 pts) Let V be a vector space over a field F where $\dim V = 2$. Suppose $\mathcal{B} = \{\mathbf{v}, \mathbf{w}\}$ is a basis of V. Let $a, b, c, d \in F$ and consider the set of two vectors $\mathcal{B}' = \{a\mathbf{v} + b\mathbf{w}, c\mathbf{v} + d\mathbf{w}\}$. Formulate and prove a theorem which says

$\mathcal{B}'$ is a basis if and only if ______________________.

In other words, you should fill in the blank with some (relatively simple) algebraic condition on the a, b, c and d. **Please write only on the left-hand side of this vertical line:**

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9. Let W_1 and W_2 be finite-dimensional subspaces of a vector space V. We have proven in class that both $W_1 + W_2$ and $W_1 \bigcap W_2$ are also subspaces of V (these facts were proven in homework problems 5.2 and 5.5 (e)).

   (a) (10 pts) Prove the following formula which relates the dimensions of all these subspaces:

$$\dim (W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim \left(W_1 \bigcap W_2\right).$$

   (b) (5 pts) Let $m \geq 2$. Prove that if $W_1, W_2, ..., W_m$ are all finite-dimensional subspaces of a vector space V, then

$$\dim(W_1 + ... + W_m) \leq \dim(W_1) + \dim(W_2) + ... + \dim(W_m).$$

   (c) (5 pts) Suppose W_1 is a $6-$dimensional subspace and W_2 is a $5-$dimensional subspace of a $9-$dimensional vector space V. In light of the formula of part (a) of this question (and other facts we have proven in this course), what are all of the possible dimensions of $W_1 \bigcap W_2$? Explain.

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1. (a) First, find the norm of $\mathbf{v}$:

$$< \mathbf{v}, \mathbf{v} >= (1-i)\overline{(1-i)} + (2+3i)\overline{(2+3i)} = (1-i)(1+i) + (2+3i)(2-3i) = 15;$$

therefore $||\mathbf{v}|| = \sqrt{< \mathbf{v}, \mathbf{v} >} = \sqrt{15}$. So a normalized version of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{1}{\sqrt{15}}(1-i, 2+3i)$$

and since $||\mathbf{u}|| = 1$, a vector in the same direction as $\mathbf{v}$ of norm 3 is

$$3\mathbf{u} = \frac{3}{\sqrt{15}}(1-i, 2+3i) = \frac{\sqrt{15}}{5}(1-i, 2+3i) = \left(\frac{\sqrt{15}}{5} - i\frac{\sqrt{15}}{5}, \frac{2\sqrt{15}}{5} + i\frac{3\sqrt{15}}{5}\right).$$

(b) This was a bad question, since angles are only defined in real vector spaces. Throw it out.

(c) We have $\mathbf{v} - \mathbf{w} = (1-4i, -2+2i)$; therefore the distance between $\mathbf{v}$ and $\mathbf{w}$ is

$$||\mathbf{v} - \mathbf{w}|| = \sqrt{(1-4i)\overline{(1-4i)} + (-2+2i)\overline{(-2+2i)}} = \sqrt{1+16+4+4} = 5.$$

2. Let $\mathcal{P}$ be the plane with the given parametric equations. We see that since $\mathcal{P}$ has equation

$$\mathbf{x} = (2,0,-1) + s(-3,1,-1) + t(4,1,0),$$

the vector $(2,0,-1)$ lies in $\mathcal{P}$. To find a normal vector to $\mathcal{P}$, take the cross product of the two direction vectors of $\mathcal{P}$:

$$\mathbf{n} = (-3,1,-1) \times (4,1,0) = (1(0) - (-1)1, -1(4) - (-3)(0), -3(1) - 1(4)) = (1,-4,-7).$$

Since $(1,4,-7)\cdot(2,0,-1) = 9$, the normal equation is $< \mathbf{n}, \mathbf{x} >= 3$, i.e. $< (1,-4,-7), (x,y,z) >= 9$, i.e. $x - 4y - 7z = 9$.

3. (a) Begin by converting $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ into an orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$:

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{x}_1 = (1,-2,-2); \\
\mathbf{w}_2 &= \mathbf{x}_2 - \text{proj}_{\mathbf{w}_1}\mathbf{x}_2 \\
&= (0,3,3) - \frac{< (0,3,3), (1,-2,-2) >}{< (1,-2,-2), (1,-2,-2) >}(1,-2,-2) \\
&= (0,3,3) - \frac{-12}{9}(1,-2,-2) = \left(\frac{4}{3}, \frac{1}{3}, \frac{1}{3}\right)
\end{aligned}$$

At this point, rescale $\mathbf{w}_2$ by multiplying it by 3 to clear the denominators, i.e. set $\mathbf{w}_2 = (4,1,1)$. Then

$$\begin{aligned}
\mathbf{w}_3 &= \mathbf{x}_3 - \text{proj}_{\mathbf{w}_1}\mathbf{x}_3 - \text{proj}_{\mathbf{w}_2}\mathbf{x}_3 \\
&= (3,2,4) - \frac{< (3,2,4), (1,-2,-2) >}{< (1,-2,-2), (1,-2,-2) >}(1,-2,-2) - \\
&\quad \frac{< (3,2,4), (4,1,1) >}{< (4,1,1), (4,1,1) >}(4,1,1) \\
&= (3,2,4) - \frac{-9}{9}(1,-2,-2) - \frac{18}{18}(4,1,1) \\
&= (3,2,4) + (1,-2,-2) - (4,1,1) = (0,-1,1).
\end{aligned}$$

Now to find the orthonormal basis, normalize the vectors obtained above:

$$\begin{aligned}
\mathbf{v}_1 &= \frac{\mathbf{w}_1}{||\mathbf{w}_1||} = \frac{1}{3}(1,-2,-2) = \left(\frac{1}{3}, \frac{-2}{3}, \frac{-2}{3}\right); \\
\mathbf{v}_2 &= \frac{\mathbf{w}_2}{||\mathbf{w}_2||} = \frac{\sqrt{2}}{6}(4,1,1) = \left(\frac{2\sqrt{2}}{3}, \frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}\right); \\
\mathbf{v}_3 &= \frac{\mathbf{w}_3}{||\mathbf{w}_3||} = \frac{\sqrt{2}}{2}(0,-1,1) = \left(0, \frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).
\end{aligned}$$

(b) From the Gram-Schmidt procedure, $W = Span(\mathbf{x}_1, \mathbf{x}_2) = Span(\mathbf{v}_1, \mathbf{v}_2)$; since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthornormal basis, we have

$$\begin{aligned}
\text{proj}_W(1,1,0) &= \text{proj}_{\mathbf{v}_1}(1,1,0) + \text{proj}_{\mathbf{v}_2}(1,1,0) \\
&= <(1,1,0), \mathbf{v}_1> \mathbf{v}_1 + <(1,1,0), \mathbf{v}_2> \mathbf{v}_2 \\
&= \frac{-1}{3}\left(\frac{1}{3}, \frac{-2}{3}, \frac{-2}{3}\right) + \frac{5\sqrt{2}}{6}\left(\frac{2\sqrt{2}}{3}, \frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}\right) \\
&= \left(\frac{-1}{9}, \frac{2}{9}, \frac{2}{9}\right) + \left(\frac{10}{9}, \frac{5}{18}, \frac{5}{18}\right) \\
&= \left(1, \frac{1}{2}, \frac{1}{2}\right).
\end{aligned}$$

4. (a) This is a set of five vectors in a four-dimensional space; hence the set is linearly dependent (by the Exchange Lemma).

(b) Suppose $a \sin x + b \cos x = 0$; plug in $x = 0$ to obtain $b = 0$ and plug in $x = \pi/2$ to obtain $a = 0$. Therefore by definition $\mathcal{S}$ is linearly independent.

5. (a) Observe $(1,0,0) \in W$ and $(0,1,1) \in W$ but $(1,0,0) + (0,1,1) = (1,1,1) \notin W$. Therefore W is not closed under addition and is therefore not a subspace of V.

(b) Observe that W is the set of vectors of the form $r(5,3,6) + s(-3,3,0) + t(8,0,6)$ hence $W = Span((5,3,6), (-3,3,0), (8,0,6))$ and since any span is a subspace, W is a subspace of V. Observe that $(8,0,6) = (5,3,6) - (-3,3,0)$ so $(8,0,6)$ can be dropped from the list of three vectors without changing the span, i.e. $W = Span((5,3,6), (-3,3,0))$. As these two vectors are non-parallel, they are linearly independent, hence $\mathcal{B} = \{(5,3,6), (-3,3,0)\}$ forms a basis of W and $\dim W = 2$.

(c) By substitution, a vector belongs to W if and only if it is of the form $(x_3, x_2, x_3, -x_5, x_5)$ for scalars x_2, x_3 and x_5. Restated, a vector belongs to W if and only if it is of the form

$$x_3(1,0,1,0,0) + x_2(0,1,0,0,0) + x_5(0,0,0,-1,1)$$

for scalars x_3, x_2, x_5, i.e. $W = Span((1,0,1,0,0), (0,1,0,0,0), (0,0,0,-1,1))$ so W is a subspace of V since it is a span. The first two vectors are nonparallel, and the last vector is not in the span of the first two, so the three vectors which span W are linearly independent and hence form a basis of W. We have $\dim W = 3$ and a basis of W is

$$\mathcal{B} = \{(1,0,1,0,0), (0,1,0,0,0), (0,0,0,-1,1)\}.$$

(d) We see that a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in W if and only if $b = c$, i.e. if and only if

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus
$$A = Span\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$$
and A is therefore a subspace (since it is a span). Now the first two matrices are clearly not parallel, and the third matrix is not in the span of the first two (because it has nonzero lower-right entry, and the span of the first two matrices contains only vectors with zero in the lower-right entry), so these matrices are linearly independent. Therefore $\dim W = 3$ and a basis is
$$\mathcal{B} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}.$$

(e) Suppose $f(x) = a + bx + cx^2 + dx^3 + ex^4 \in W$. Then
$$2f(-x) = 2a - 2bx + 2cx^2 - 2dx^3 + 2ex^4 = a + bx + cx^2 + dx^3 + ex^4 = f(x)$$
and therefore we have $a = 2a, -2b = b, 2c = c, -2d = d, 2e = e$. Therefore $a = b = c = d = e = 0$ and therefore $W = \{0\}$ which is a subspace of dimension zero. W has no basis.

(f) W is a subspace since it is the orthogonal complement of another subspace (namely $U = Span(x^2)$). Since U is spanned by one nonzero function, $\dim U = 1$ and by a homework problem, $\dim W = \dim V - \dim U = 3 - 1 = 2$. So to find a basis we need only find two nonparallel vectors in W, i.e. two nonparallel functions f satisfying $< f, x^2 >= 0$. Writing this equation out, we see $< f, x^2 >= 0$ if and only if $f(0)\cdot 0 + f(1)\cdot 1 + f(-1)\cdot 1 = 0$, i.e. $f(1) = -f(-1)$. Two nonparallel functions in V satisfying this are $f(x) = x$ and $f(x) = x^2 - 1$; hence a basis of W is $\mathcal{B} = \{x, x^2 - 1\}$.

6. This is true; here is a proof:
$$\text{proj}_{r\mathbf{v}}\mathbf{x} = \frac{< \mathbf{x}, r\mathbf{v} >}{< r\mathbf{v}, r\mathbf{v} >} r\mathbf{v} = \frac{\overline{r} < \mathbf{x}, \mathbf{v} >}{r\overline{r} < \mathbf{v}, \mathbf{v} >} r\mathbf{v} = \frac{< \mathbf{x}, \mathbf{v} >}{< \mathbf{v}, \mathbf{v} >}\mathbf{v} = \text{proj}_{\mathbf{v}}\mathbf{x}.$$

7. ($\Rightarrow$) Assume $\mathbf{v} \perp \mathbf{w}$. Then
$$\begin{aligned} ||\mathbf{v} + a\mathbf{w}||^2 - ||\mathbf{v}||^2 &= < \mathbf{v} + a\mathbf{w}, \mathbf{v} + a\mathbf{w} > - < \mathbf{v}, \mathbf{v} > \\ &= < \mathbf{v}, \mathbf{v} > + a < \mathbf{w}, \mathbf{v} > + \overline{a} < \mathbf{v}, \mathbf{w} > + a\overline{a} < \mathbf{w}, \mathbf{w} > - < \mathbf{v}, \mathbf{v} > \\ &= a \cdot 0 + \overline{a} \cdot 0 + |a|^2 ||\mathbf{w}||^2 \geq 0. \end{aligned}$$

($\Leftarrow$) As a special case, suppose $\mathbf{w} = \mathbf{0}$. Then $\mathbf{v} \perp \mathbf{w}$ since every vector is orthogonal to the zero vector (with respect to any inner product).

Henceforth, we assume $\mathbf{w} \neq \mathbf{0}$. Let $a = \frac{-<\mathbf{v},\mathbf{w}>}{||\mathbf{w}||^2}$; then $\overline{a} = \frac{-\overline{<\mathbf{v},\mathbf{w}>}}{||\mathbf{w}||^2} = \frac{-<\mathbf{w},\mathbf{v}>}{||\mathbf{w}||^2}$ since the denominator of a is real. So we have

$$\begin{aligned}
&||\mathbf{v}+a\mathbf{w}||^2 - ||\mathbf{v}||^2\\
&=<\mathbf{v}+a\mathbf{w},\mathbf{v}+a\mathbf{w}> - <\mathbf{v},\mathbf{v}>\\
&=<\mathbf{v},\mathbf{v}> + a<\mathbf{w},\mathbf{v}> + \overline{a}<\mathbf{v},\mathbf{w}> + a\overline{a}<\mathbf{w},\mathbf{w}> - <\mathbf{v},\mathbf{v}>\\
&= a<\mathbf{w},\mathbf{v}> + \overline{a<\mathbf{w},\mathbf{v}>} + |a|^2||\mathbf{w}||^2\\
&= a<\mathbf{w},\mathbf{v}> + \overline{a}<\mathbf{v},\mathbf{w}> + a\overline{a}||\mathbf{w}||^2\\
&= \frac{-<\mathbf{v},\mathbf{w}>}{||\mathbf{w}||^2}<\mathbf{w},\mathbf{v}> + \frac{-<\mathbf{w},\mathbf{v}>}{||\mathbf{w}||^2}<\mathbf{v},\mathbf{w}> + \left(\frac{-<\mathbf{v},\mathbf{w}>}{||\mathbf{w}||^2}\right)\left(\frac{-<\mathbf{w},\mathbf{v}>}{||\mathbf{w}||^2}\right)||\mathbf{w}||^2\\
&= -2\frac{<\mathbf{v},\mathbf{w}><\mathbf{w},\mathbf{v}>}{||\mathbf{w}||^2} + \frac{<\mathbf{v},\mathbf{w}><\mathbf{w},\mathbf{v}>}{||\mathbf{w}||^2}\\
&= -\frac{<\mathbf{v},\mathbf{w}><\mathbf{w},\mathbf{v}>}{||\mathbf{w}||^2}\\
&= -\frac{|<\mathbf{v},\mathbf{w}>|^2}{||\mathbf{w}||^2}
\end{aligned}$$

By hypothesis, this expression is ≥ 0 (since the expression is assumed nonnegative for all a). But since other than the $-$ sign, everything that appears in the expression is a norm or absolute value, the expression is also ≤ 0. Therefore it is must be equal to zero; therefore the numerator is zero; therefore $<\mathbf{v},\mathbf{w}>=0$ so $\mathbf{v}\perp\mathbf{w}$.

8. *Claim:* $\mathcal{B}'$ is a basis if and only if $\underline{ad-bc\neq 0.}$

   Proof: Since $\dim V=2$ and there are two vectors in $\mathcal{B}'$, we see $\mathcal{B}'$ is a basis if and only if the vectors in $\mathcal{B}'$ are linearly independent. So let's figure out when the vectors $a\mathbf{v}+b\mathbf{w}$ and $c\mathbf{v}+d\mathbf{w}$ are linearly independent:

   Special cases: If $a=c=0$, the both the vectors in $\mathcal{B}'$ are multiples of $\mathbf{w}$, hence parallel, hence do not form a basis. Similarly, if $b=d=0$, $\mathcal{B}'$ consists of two multiples of $\mathbf{v}$, hence not a basis.

   General case: Assume we are not in one of the special cases above, i.e. assume a or c is nonzero and assume either b or d is nonzero.

   Suppose there are scalars r and s such that $r(a\mathbf{v}+b\mathbf{w})+s(c\mathbf{v}+d\mathbf{w})=\mathbf{0}$. Then, by combining terms we see that
$$(ra+sc)\mathbf{v}+(rb+sd)\mathbf{w}=\mathbf{0}$$
   and so since $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, we have
$$\begin{cases} ra+sc=0\\ rb+sd=0\end{cases}.$$

   Now we solve this system for (r,s) in terms of the other variables. If $a\neq 0$, solve the first equation for r (to get $r=\frac{sc}{a}$), substitute this into the second equation to get
$$s\left(\frac{cb}{a}-d\right)=0.$$

   If $\frac{cb}{a}-d\neq 0$, then the system above has only one solution, namely $r=s=0$, and the vectors in $\mathcal{B}'$ are l. ind., hence form a basis. But if $\frac{cb}{a}-d=0$, then we can choose $s=1, r=\frac{c}{a}$ and this gives a nonzero solution of the above system. Hence the vectors in $\mathcal{B}'$ are l. dep., hence are not a basis.

To summarize: we have shown that $\mathcal{B}'$ is a basis if and only if $\frac{cb}{a} - d \neq 0$, which is true if and only if $cb - ad \neq 0$, which is true if and only if $ad - bc \neq 0$. Notice that the last two criteria include the special cases described above.

9. (a) Let $\dim(W_1 \bigcap W_2) = m$; let $\mathcal{B} = \{\mathbf{x}_1, ..., \mathbf{x}_m\}$ be a basis of $W_1 \bigcap W_2$.
Since $W_1 \bigcap W_2$ is a subspace of both W_1 and W_2, by the Basis Extension Theorem we can extend $\mathcal{B}$ to bases

$$\mathcal{B}_1 = \{\mathbf{x}_1, ..., \mathbf{x}_m, \mathbf{y}_1, ..., \mathbf{y}_n\}$$

and

$$\mathcal{B}_2 = \{\mathbf{x}_1, ..., \mathbf{x}_m, \mathbf{z}_1, ..., \mathbf{z}_p\}$$

of W_1 and W_2, respectively. In particular, we have $\dim W_1 = m+n$ and $\dim W_2 = m+p$.
Claim: $\mathcal{B}_1 \bigcup \mathcal{B}_2 = \{\mathbf{x}_1, ..., \mathbf{x}_m, \mathbf{y}_1, ..., \mathbf{y}_n, \mathbf{x}_1, ..., \mathbf{x}_m, \mathbf{z}_1, ..., \mathbf{z}_p\}$ is a basis of $W_1 + W_2$.
Proof of the result, assuming the claim: If the claim is true, then we have $\dim(W_1+W_2) = m+n+p$ and therefore the formula

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim\left(W_1 \bigcap W_2\right).$$

becomes $m+n+p = (m+n)+(m+p)-m$ which is obviously true. So if we prove the claim, we are done.
Proof of the claim: First, we show that $\mathcal{B}_1 \bigcup \mathcal{B}_2$ spans $W_1 + W_2$. Let $\mathbf{v} \in W_1 + W_2$; then $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$. Since $\mathcal{B}_1$ is a basis of W_1 and $\mathcal{B}_2$ is a basis of W_2, we have

$$\mathbf{w}_1 = \sum_{j=1}^{m} a_j \mathbf{x}_j + \sum_{j=1}^{n} b_j \mathbf{y}_j \text{ and } \mathbf{w}_1 = \sum_{j=1}^{m} c_j \mathbf{x}_j + \sum_{j=1}^{p} d_j \mathbf{z}_j;$$

therefore

$$\mathbf{v} = \sum_{j=1}^{m} (a_j + c_j)\mathbf{x}_j + \sum_{j=1}^{n} b_j \mathbf{y}_j + \sum_{j=1}^{p} d_j \mathbf{z}_j$$

so since every $\mathbf{v} \in W_1 + W_2$ is a linear combination of vectors in $\mathcal{B}_1 \bigcup \mathcal{B}_2$, we have shown $\mathcal{B}_1 \bigcup \mathcal{B}_2$ spans $W_1 + W_2$.
Next, we show $\mathcal{B}_1 \bigcup \mathcal{B}_2$ is linearly independent. Suppose there are scalars a_j, b_j, c_j such that

$$\sum_{j=1}^{m} a_j \mathbf{x}_j + \sum_{j=1}^{n} b_j \mathbf{y}_j + \sum_{j=1}^{p} c_j \mathbf{z}_j = \mathbf{0}. \tag{1}$$

Rewrite this as

$$\sum_{j=1}^{m} a_j \mathbf{x}_j + \sum_{j=1}^{n} b_j \mathbf{y}_j = -\sum_{j=1}^{p} c_j \mathbf{z}_j. \tag{2}$$

Let $\mathbf{v} = -\sum_{j=1}^{p} c_j \mathbf{z}_j$ (i.e. the right-hand side of Equation (2)... of course $\mathbf{v}$ is also the left-hand side). Now, $\mathbf{v} \in W_1$ (since on the left-hand side it is a linear combination of vectors in $\mathcal{B}_1$), but $\mathbf{v}$ is also in W_2 (since on the right-hand side is a linear combination of some of the vectors in $\mathcal{B}_2$). Therefore $\mathbf{v} \in W_1 \bigcap W_2$ and since $\mathcal{B}$ is a basis of $W_1 \bigcap W_2$, we can write

$$\mathbf{v} = \sum_{j=1}^{m} d_j \mathbf{x}_j$$

and therefore Equation (1) reduces to

$$\sum_{j=1}^{m}(d_j\mathbf{x}_j + \sum_{j=1}^{n} b_j\mathbf{y}_j = \mathbf{0}$$

for some choice of scalars d_j. But since $\mathcal{B}_1$ is a basis, it is a linearly independent set, hence for all j $d_j = 0$, and more importantly for all j, $b_j = 0$. Now equation (1) reduces to

$$\sum_{j=1}^{m} a_j\mathbf{x}_j + \sum_{j=1}^{p} c_j\mathbf{z}_j = \mathbf{0}$$

and since $\mathcal{B}_2$ is a basis, it is a linearly independent set, hence all the a_j and c_j are zero as well. Therefore $\mathcal{B}_1 \bigcup \mathcal{B}_2$ is linearly independent, hence a basis of $W_1 + W_2$.

(b) We proceed by induction on m. If $m = 2$, the statement is $\dim(W_1 + W_2) \leq \dim(W_1) + \dim(W_2)$ which holds by applying part (a): the equation $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \bigcap W_2)$ yields $\dim(W_1+W_2) \leq \dim(W_1)+\dim(W_2)$ because $\dim(W_1 \bigcap W_2) \geq 0$.

Now for the induction step: assume the statement is true when $m = k$, i.e.

$$\dim(W_1 + ... + W_k) \leq \dim(W_1) + \dim(W_2) + ... + \dim(W_k)$$

for any subspaces $W_1, ..., W_k$. Now consider subspaces $W_1, ..., W_{k+1}$;

$$\begin{aligned}\dim(W_1 + ... + W_{k+1}) &= \dim((W_1 + ... + W_k) + W_{k+1}) \\ &\leq \dim(W_1 + ... + W_k) + \dim W_{k+1} \quad \text{(by the } m = 2 \text{ case)} \\ &\leq \dim W_1 + ... + \dim W_k + \dim W_{k+1} \quad \text{(by the IH).}\end{aligned}$$

By induction, we are done.

(c) From (a) we have

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim\left(W_1 \bigcap W_2\right).$$

which, since $\dim V = 9$ and $W_1 + W_2 \subseteq V$, becomes

$$9 \geq 5 + 6 - \dim\left(W_1 \bigcap W_2\right).$$

Therefore $\dim(W_1 \bigcap W_2) \geq 2$. But since $W_1 \bigcap W_2 \subseteq W_2$, we have $\dim(W_1 \bigcap W_2) \leq 5$. Thus the dimension of $W_1 \bigcap W_2$ is 2,3,4 or 5.