1. Consider these two matrices in $M_n(\mathbb{C})$:

$$A = \begin{pmatrix} 3i & 0 & 1+i \\ 0 & 2-i & 2i \\ 2 & 0 & -3 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 1-i \\ 0 & 1+i & 2i \\ -3i & 4-i & -1 \end{pmatrix}$$

   (a) Compute AB.
   (b) Find the determinant of A.

2. Find all solutions of the following system of linear equations:

$$\begin{cases} 2x + 3y + z = 3 \\ 3x - 2y + 2z = 11 \\ 3x + 11y + 4z = -2 \end{cases}.$$

3. In this problem, you are to assume that the following matrices A and B are row equivalent:

$$A = \begin{pmatrix} 1 & 4 & 0 & -2 & -1 & 2 & 0 \\ -1 & -4 & 2 & -4 & 1 & 0 & -4 \\ -2 & -8 & 1 & -2 & -4 & -3 & 1 \\ 1 & 4 & -1 & 1 & -1 & 0 & -1 \\ -2 & -8 & 1 & 2 & 4 & -3 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 4 & 0 & -2 & -1 & 2 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

   (a) Find a basis for the null space of A.
   (b) Find a basis for the row space of A.
   (c) True or false (brief explanation required):
      i. There is a vector $\mathbf{b} \in \mathbb{R}^5$ such that the system $A\mathbf{x} = \mathbf{b}$ has no solution.
      ii. There is a vector $\mathbf{b} \in \mathbb{R}^5$ such that the system $A\mathbf{x} = \mathbf{b}$ has exactly one solution.
      iii. The seventh column of A is in the span of the first six columns of A.

4. Determine whether each of the following transformations is linear. If the transformation is linear, give a basis for its kernel and its image.

   (a) $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by $T(M) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} M - M \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.
   (b) $T : M_3(\mathbb{R}) \to \mathbb{R}$ defined by $T(A) = \det A$.

5. Let V be the subspace of $C^\infty(\mathbb{R}, \mathbb{R})$ spanned by $\mathcal{B} = \{e^x \cos 2x, e^x \sin 2x\}$ (you may assume without proof that $\mathcal{B}$ is a basis of V).

   (a) Let $T : V \to V$ be the linear transformation defined by $T(f) = f''$. Find the matrix of T relative to $\mathcal{B}$.
   (b) Let $S : V \to \mathbb{R}^3$ be the linear transformation defined by $S(f) = (f(0), f'(0), f''(0))$. Find the matrix of S relative to $\mathcal{B}$ and the standard basis of $\mathbb{R}^3$.
   (c) Let $\mathcal{C} = \{(1, 0, -1), (2, 0, -1), (2, -1, 7)\}$ (this is a basis of $\mathbb{R}^3$). Find the matrix of S relative to $\mathcal{B}$ and $\mathcal{C}$.

6. Diagonalize the matrix $A = \begin{pmatrix} 0 & i \\ 1 & 1-i \end{pmatrix}$ and find A^{2011} (simplify your answer).

7. Let $A \in M_{mn}(\mathbb{C})$ be a matrix with linearly independent columns.
   (a) Prove that $A^H A$ is invertible.
   Hint: Suppose $A^H A\mathbf{x} = \mathbf{0}$. To what subspaces must $A\mathbf{x}$ belong?
   (b) Let $P = A(A^H A)^{-1} A^H$. Show that for any vector $\mathbf{x} \in \mathbb{C}^m$,
   $$P\mathbf{x} \in C(A) \quad \text{and} \quad \mathbf{x} - P\mathbf{x} \in [C(A)]^{\perp}.$$
   (c) A square matrix P which satisfies $P = A(A^H A)^{-1} A^H$ for some matrix A with linearly independent columns is called a *projection matrix*. In light of the facts proven in part (b), why is it appropriate to call P a "projection matrix"? Onto what space does P project vectors?
   (d) Prove that every projection matrix is Hermitian.
   (e) Prove that for any projection matrix P, $P^2 = P$. (In fact, a matrix P is a projection matrix if and only if it is Hermitian and satisfies $P^2 = P$, but you aren't being asked to prove that.)
   (f) Let A be a square matrix with linearly independent columns. Find P. Explain why this answer "makes sense" (given our understanding of projections, etc.).
   (g) Let P be a projection matrix. What are the only possible eigenvalues of P? Prove your answer.
   (h) Use the ideas developed in this problem to find the projection of $(1, 1, 1, 1, 1)$ onto the subspace of $\mathbb{R}^5$ spanned by $(2, 0, 4, 0, 1)$ and $(0, 1, 0, 0, -3)$. (You will not receive credit for performing Gram-Schmidt on the basis and computing inner products; you must use the ideas of this problem.)

8. Prove that if $T : \mathbb{R}^3 \to \mathbb{R}^3$ has a standard matrix which is skew-symmetric, then there is a vector $\mathbf{v} \in \mathbb{R}^3$ such that $T(\mathbf{x}) = \mathbf{x} \times \mathbf{v}$ for all $\mathbf{x} \in \mathbb{R}^3$.

   Hint: The entries of the standard matrix of T should have something to do with the components of $\mathbf{v}$.

9. Prove that every complex matrix $M \in M_n(\mathbb{C})$ can be written as $M = A + iB$ where A and B are both Hermitian matrices.

10. Prove that given any unitary matrix U, there is a skew-Hermitian matrix A such that $e^A = U$. (A matrix $B \in M_n(\mathbb{C})$ is *skew-Hermitian* if $B^H = -B$.)

11. Let $A \in M_n(F)$. Prove or disprove: A and A^T have the same eigenvalues.

12. (a) Let $A \in M_n(\mathbb{C})$ be an upper triangular matrix. Prove that e^A is upper triangular, and find the diagonal entries of e^A in terms of the diagonal entries of A.
   Hint: Show that if A is upper triangular, then so are A^2, A^3, A^4, etc. Then use the definition of the matrix exponential.
   (b) Prove that for any $A \in M_n(\mathbb{C})$, $\det(\exp A) = e^{trace(A)}$.
   Hint: Prove the result first for upper triangular matrices, then extend the result to arbitrary matrices.

1. (a) By direct computation, $AB = \begin{pmatrix} 3-3i & 5+3i & 2+2i \\ 6 & 5+9i & 2+2i \\ 9i & -12+3i & 5-2i \end{pmatrix}$.
   (b) $\det A = (3i)(2-i)(-3) - (2)(2-i)(1+i) = (18i-9) - (6+2i) = -15-20i$.

2. Write the augmented matrix and perform row reductions:

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 3 \\ 3 & -2 & 2 & 11 \\ 3 & 11 & 4 & -2 \end{array}\right) \xrightarrow{-\left(\frac{3}{2}\right)R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 2 & 3 & 1 & 3 \\ 0 & \frac{-13}{2} & \frac{1}{2} & \frac{13}{2} \\ 3 & 11 & 4 & -2 \end{array}\right)$$

$$\xrightarrow{-\left(\frac{3}{2}\right)R_1+R_3\to R_3} \left(\begin{array}{ccc|c} 2 & 3 & 1 & 3 \\ 0 & \frac{-13}{2} & \frac{1}{2} & \frac{13}{2} \\ 0 & \frac{13}{2} & \frac{5}{2} & \frac{-13}{2} \end{array}\right)$$

$$\xrightarrow{R_2+R_3\to R_3} \left(\begin{array}{ccc|c} 2 & 3 & 1 & 3 \\ 0 & \frac{-13}{2} & \frac{1}{2} & \frac{13}{2} \\ 0 & 0 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{R_2\times 2} \left(\begin{array}{ccc|c} 2 & 3 & 1 & 3 \\ 0 & -13 & 1 & 13 \\ 0 & 0 & 3 & 0 \end{array}\right)$$

We see that there are no free columns, so this system has one solution. From the last row, we see $3x_3 = 0$, i.e. $x_3 = 0$. Substituting this into the equation coming from the second row yields $-13x_2 = 13$, i.e. $x_2 = -1$. Plugging this into the equation coming from the first row gives $2x_1 + 3(-1) = 3$, i.e. $x_1 = 3$. Therefore the only solution is $(3, -1, 0)$.

3. (a) Consider the equation $A\mathbf{x} = \mathbf{0}$; write $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$. The free columns of B (hence the free columns of A) are the second and fifth columns. So we need to solve for x_1, x_3 and x_4 in terms of x_2 and x_5. (Of course, from the bottom two rows, we see that if $A\mathbf{x} = \mathbf{0}$ then $x_6 = x_7 = 0$.) From the third row we obtain $x_4 + 2x_5 = 0$ so $x_4 = -2x_5$. From the second row, we see $x_3 - 3x_4 = 0$ so $x_3 = 3x_4 = -6x_5$. From the first row, we see $x_1 + 4x_2 - 2x_4 - x_5 = 0$ implies $x_1 = -4x_2 + 2x_4 + x_5 = -4x_2 - 3x_5$. Therefore, if $\mathbf{x} \in N(A)$, then $A\mathbf{x} = \mathbf{0}$ so

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} -4x_2 - 3x_5 \\ x_2 \\ -6x_5 \\ -2x_5 \\ x_5 \\ 0 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -3 \\ 0 \\ -6 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

   Therefore a basis for $N(A)$ is $\{(-4, 1, 0, 0, 0, 0, 0), (-3, 0, -6, -2, 1, 0, 0)\}$.

   (b) A basis for the row space consists of the five rows of B, since B is an echelon form of A and all the rows of B are linearly independent (since they all contain a pivot). (The five rows of A also work, but only since the matrix A has full row rank.)

   (c) i. Since A has five rows, $C(A) \subseteq \mathbb{R}^5$. But because A has five linearly independent columns, $\dim C(A) = 5$ and therefore $C(A) = \mathbb{R}^5$. So every $\mathbf{b} \in \mathbb{R}^5$ belongs to the column space of A, so this statement is **false**.

   ii. From part (a), $N(A) \neq \{\mathbf{0}\}$. Therefore, since if $\mathbf{x}$ solves the system $A\mathbf{x} = \mathbf{b}$, so does $\mathbf{x} + \mathbf{n}$ for any $\mathbf{n} \in N(A)$, this statement is also **false**.

   iii. Since the seventh column of A is a pivot column, it does not depend on the previous columns. Thus this statement is **false**.

4. (a) Let A denote the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Thus $T(M) = AM - MA$. Then

$$T(rM) = A(rM) - (rM)A = r(AM) - r(MA) = r(AM - MA) = rT(M)$$

and

$$\begin{aligned} T(M+N) &= A(M+N) - (M+N)A \\ &= AM + AN - MA - NA \\ &= (AM - MA) + (AN - NA) \\ &= T(M) + T(N) \end{aligned}$$

so T is linear. To describe the image and kernel of T, let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$T(M) = \begin{pmatrix} a+c & b+d \\ a+c & b+d \end{pmatrix} - \begin{pmatrix} a+b & a+b \\ c+d & c+d \end{pmatrix} = \begin{pmatrix} c-b & d-a \\ a-d & b-c \end{pmatrix}$$

Image: from above, every matrix in the image of T is of the form

$$(c-b)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + (d-a)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

therefore the image of T is $2-$dimensional and has basis $\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}$.

Kernel: from above, $M \in \ker(T)$ if and only if $a = d$ and $b = c$, i.e.

$$M \in \ker(T) \Leftrightarrow M = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \Leftrightarrow M = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore $\ker(T)$ is $2-$dimensional, and has basis $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$.

(b) Let I denote the 3×3 identity matrix. We see $T(2I) = 2^3 T(I) = 8$ but $2T(I) = 2(1) = 2$. Therefore $T(2I) \neq 2T(I)$ so T is not linear.

5. (a) By direct calculation, $T(e^x \cos 2x) = -3e^x \cos 2x - 4e^x \sin 2x$ and $T(e^x \sin 2x) = 4e^x \cos 2x - 3e^x \sin 2x$. Therefore the matrix of T relative to $\mathcal{B}$ is

$$\begin{pmatrix} -3 & 4 \\ -4 & -3 \end{pmatrix}.$$

(b) By direct calculation, $S(e^x \cos 2x) = (1, 1, -3)$ and $S(e^x \sin 2x) = (0, 2, 4)$. Therefore the matrix of S relative to $\mathcal{B}$ and the standard basis of $\mathbb{R}^3$ is

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ -3 & 4 \end{pmatrix}.$$

(c) Let $P = P_{\mathcal{S} \leftarrow \mathcal{C}}$ be the change of basis matrix from $\mathcal{C}$ to the standard basis $\mathcal{S}$; the columns of P are the elements of $\mathcal{C}$ so

$$P = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & -1 \\ -1 & -1 & 7 \end{pmatrix}.$$

Now the matrix B of S relative to $\mathcal{B}$ and $\mathcal{C}$ is therefore $B = P^{-1}A$ (where A is as in part (b)). We see that

$$B = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & -1 \\ -1 & -1 & 7 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} -11 & 40 \\ 7 & 22 \\ -1 & -2 \end{pmatrix}$$

6. First, find eigenvalues of A:

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & i \\ 1 & 1-i-\lambda \end{pmatrix} = -\lambda(1-i-\lambda) - i = \lambda^2 - \lambda + i\lambda - i = (\lambda - 1)(\lambda + i).$$

Therefore the eigenvalues of A are $\lambda = 1$ and $\lambda = -i$. Next, find corresponding eigenvectors $\mathbf{x} = (x, y)$:

- $\lambda = 1$: $A\mathbf{x} = \mathbf{x}$ becomes
$$\begin{cases} iy = x \\ x + y - iy = y \end{cases}$$
which yields $x = iy$, so one eigenvector for $\lambda = 1$ is $(i, 1)$.
- $\lambda = -i$: $A\mathbf{x} = -i\mathbf{x}$ becomes
$$\begin{cases} iy = -ix \\ x + y - iy = -iy \end{cases}$$
which yields $y = -x$, so one eigenvector for $\lambda = -i$ is $(1, -1)$.

Now we have $A = P\Lambda P^{-1}$ where

$$P = \begin{pmatrix} i & 1 \\ 1 & -1 \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}.$$

Therefore

$$\begin{aligned} A^{2011} = (P\Lambda P^{-1})^{2011} &= P\Lambda^{2011}P^{-1} \\ &= \begin{pmatrix} i & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}^{2011} \begin{pmatrix} i & 1 \\ 1 & -1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} i & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1^{2011} & 0 \\ 0 & (-i)^{2011} \end{pmatrix} \frac{1}{-1-i} \begin{pmatrix} -1 & -1 \\ -1 & i \end{pmatrix} \\ &= \frac{1}{-1-i} \begin{pmatrix} i & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -1 & i \end{pmatrix} \\ &= \frac{1}{-1-i} \begin{pmatrix} i & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -1 & i \end{pmatrix} \\ &= \frac{1}{-1-i} \begin{pmatrix} -2i & -1-i \\ -1+i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1+i & 1 \\ -i & 0 \end{pmatrix}. \end{aligned}$$

7. (a) First, notice that $A^H A \in M_n(\mathbb{C})$ (i.e. $A^H A$ is square). Suppose $\mathbf{x} \in \mathbb{C}^n$ is such that $A^H A\mathbf{x} = \mathbf{0}$. By definition, $A\mathbf{x} \in C(A)$ and $A\mathbf{x} \in N(A^H) = [C(A)]^\perp$. Therefore $A\mathbf{x} = \mathbf{0}$ so $\mathbf{x} \in N(A)$. But since A has linearly independent columns, $N(A) = \{\mathbf{0}\}$ so $\mathbf{x} = \mathbf{0}$. We have proven that $N(A^H A) = \{\mathbf{0}\}$, so $A^H A$ is invertible.

(b) For the first statement, let $\mathbf{z} = (A^H A)^{-1} A^H \mathbf{x}$. Notice $P\mathbf{x} = A\left[(A^H A)^{-1} A^H \mathbf{x}\right] = A\mathbf{z} \in C(A)$. For the second statement, first recall that $[C(A)]^\perp = N(A^H)$ by the Fundamental Theorem of Linear Algebra. So it is sufficient to show $A^H(\mathbf{x} - P\mathbf{x}) = \mathbf{0}$. This follows from a direct calculation:

$$\begin{aligned} A^H\left[\mathbf{x} - P\mathbf{x}\right] &= A^H\mathbf{x} - A^H\left[A(A^H A)^{-1} A^H \mathbf{x}\right] \\ &= A^H\mathbf{x} - (A^H A)(A^H A)^{-1} A^H \mathbf{x} \\ &= A^H\mathbf{x} - A^H\mathbf{x} = \mathbf{0}. \end{aligned}$$

(c) By part (b), $P\mathbf{x}$ is the projection of $\mathbf{x}$ onto the column space of A.

(d) Let $P = A(A^H A)^{-1} A^H$; then

$$\begin{aligned} P^H &= \left[A(A^H A)^{-1} A^H\right]^H \\ &= (A^H)^H[(A^H A)^{-1}]^H A^H \\ &= A[(A^H A)^H]^{-1} A^H \\ &= A(A^H A)^{-1} A^H = P \end{aligned}$$

so P is Hermitian.

(e) Let $P = A(A^H A)^{-1} A^H$; then

$$\begin{aligned} P^2 &= \left[A(A^H A)^{-1} A^H\right]\left[A(A^H A)^{-1} A^H\right] \\ &= A(A^H A)^{-1}(A^H A)(A^H A)^{-1} A^H \\ &= A(A^H A)^{-1} A^H = P. \end{aligned}$$

(f) If A is a square matrix with linearly independent columns, then A and A^H are invertible and $(A^H A)^{-1} = A^{-1}[A^H]^{-1}$. So $P = A(A^H A)^{-1} A^H = AA^{-1}[A^H]^{-1} A^H = II = I$. This makes sense, because if A is square and has linearly independent columns, then $C(A) = \mathbb{C}^n$. Thus P is the projection onto all of $\mathbb{C}^n$, which is the identity map.

(g) Suppose λ is an eigenvalue of P. Then there is $\mathbf{x} \neq \mathbf{0}$ such that $P\mathbf{x} = \lambda\mathbf{x}$. Then $P^2\mathbf{x} = P(P\mathbf{x}) = P(\lambda\mathbf{x}) = \lambda P\mathbf{x} = \lambda^2\mathbf{x}$. But by part (e), $P^2 = P$ so $P^2\mathbf{x} = P\mathbf{x} = \lambda\mathbf{x}$. Putting this together, we see $\lambda^2\mathbf{x} = \lambda\mathbf{x}$ so $(\lambda^2 - \lambda)\mathbf{x} = \mathbf{0}$. Since $\mathbf{x}$ is an eigenvector, $\mathbf{x} \neq \mathbf{0}$ so $\lambda^2 - \lambda = 0$. Therefore λ must be 0 or 1.
(In fact, every vector in $C(A)$ is an eigenvector of P corresponding to $\lambda = 1$ and every vector in $[C(A)]^\perp = N(A^H)$ is an eigenvector of P corresponding to $\lambda = 0$.)

(h) Let V be the subspace of $\mathbb{R}^5$ spanned by $(2, 0, 4, 0, 1)$ and $(0, 1, 0, 0, -3)$; we seek the projection of $\mathbf{x} = (1, 1, 1, 1, 1)$ onto V. Note that $V = C(A)$ if we let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 4 & 0 \\ 0 & 0 \\ 1 & -3 \end{pmatrix}.$$

By part (b), the projection of $\mathbf{x}$ onto V is given by $P\mathbf{x}$ where $P = A(A^H A)^{-1} A^H$. So what is left to do is compute this P. First,

$$A^H A = \begin{pmatrix} 2 & 0 & 4 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 4 & 0 \\ 0 & 0 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 21 & -3 \\ -3 & 10 \end{pmatrix}.$$

Next,

$$(A^H A)^{-1} = \frac{1}{201}\begin{pmatrix} 10 & 3 \\ 3 & 21 \end{pmatrix}.$$

Therefore

$$\begin{aligned} P &= A(A^H A)^{-1} A^H \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 4 & 0 \\ 0 & 0 \\ 1 & -3 \end{pmatrix} \frac{1}{201}\begin{pmatrix} 10 & 3 \\ 3 & 21 \end{pmatrix}\begin{pmatrix} 2 & 0 & 4 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 \end{pmatrix} \\ &= \frac{1}{201}\begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 4 & 0 \\ 0 & 0 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 20 & 3 & 40 & 0 & 1 \\ 6 & 21 & 12 & 0 & -60 \end{pmatrix} \\ &\frac{1}{201}\begin{pmatrix} 40 & 6 & 80 & 0 & 2 \\ 6 & 21 & 12 & 0 & -60 \\ 80 & 12 & 160 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & -60 & 4 & 0 & 181 \end{pmatrix}. \end{aligned}$$

Finally, the projection is

$$P\mathbf{x} = \frac{1}{201}\begin{pmatrix} 40 & 6 & 80 & 0 & 2 \\ 6 & 21 & 12 & 0 & -60 \\ 80 & 12 & 160 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & -60 & 4 & 0 & 181 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{201}\begin{pmatrix} 128 \\ -21 \\ 256 \\ 0 \\ 127 \end{pmatrix}.$$

8. Let the standard matrix of T be

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

(every 3×3 skew-symmetric real matrix is of this form). This means $T(\mathbf{e}_j)$ is the j^{th} column of this matrix. Now let $\mathbf{v} = (c, -b, a)$. Notice

$$\begin{aligned} \mathbf{e}_1 \times \mathbf{v} &= (0, -a, -b); \\ \mathbf{e}_2 \times \mathbf{v} &= (-a, 0, -c); \\ \mathbf{e}_3 \times \mathbf{v} &= (b, c, 0). \end{aligned}$$

In particular, $T(\mathbf{e}_j) = \mathbf{e}_j \times \mathbf{v}$ for $j = 1, 2, 3$. Now for an arbitrary $\mathbf{x} \in \mathbb{R}^3$, write $\mathbf{x} = \sum_{j=1}^3 c_j \mathbf{e}_j$. We see

$$T(\mathbf{x}) = T\left(\sum_{j=1}^3 c_j \mathbf{e}_j\right) = \sum_{j=1}^3 c_j T(\mathbf{e}_j) = \sum_{j=1}^3 c_j\left(\mathbf{e}_j \times \mathbf{v}\right) = \left(\sum_{j=1}^3 c_j \mathbf{e}_j\right) \times \mathbf{v} = \mathbf{x} \times \mathbf{v}.$$

9. Given M, let $A = \frac{1}{2}(M + M^H)$ and $B = \frac{1}{2i}(M - M^H)$. Notice that

$$A + iB = \frac{1}{2}(M + M^H) + i\frac{1}{2i}(M - M^H) = \frac{1}{2}(M + M^H) + \frac{1}{2}(M - M^H) = M.$$

Next, we show A is Hermitian:

$$A^H = \left[\frac{1}{2}(M + M^H)\right]^H = \frac{1}{2}(M + M^H)^H = \frac{1}{2}(M^H + M) = A$$

Last, we show B is Hermitian:

$$B^H = \left[\frac{1}{2i}(M - M^H)\right]^H = \overline{\left(\frac{1}{2i}\right)}(M - M^H)^H = \frac{1}{-2i}(M^H - M) = \frac{1}{2i}(M - M^H) = B.$$

10. Let U be $n \times n$. By the spectral theorem, there is a unitary matrix M and a diagonal matrix D such that $U = MDM^H$. In particular, the diagonal entries of D are the eigenvalues of U which by a homework problem must be complex numbers of modulus 1 since U is unitary. Writing these complex numbers in polar form, we see that

$$D = \begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_n} \end{pmatrix}.$$

Let

$$B = \begin{pmatrix} i\theta_1 & & & \\ & i\theta_2 & & \\ & & \ddots & \\ & & & i\theta_n \end{pmatrix}.$$

Clearly, B is skew-Hermitian and $e^B = D$. Now, set $A = MBM^H$. Applying the fact M is unitary (i.e. $M^H = M^{-1}$), we have

$$e^A = e^{MBM^H} = e^{MBM^{-1}} = Me^BM^{-1} = MDM^H = U$$

and

$$A^H = (MBM^H)^H = MB^HM^H = M(-B)M^H = -MBM^H = -A$$

so A has the desired properties.

11. Applying properties of determinants and matrix transposes, we see $\det(A - \lambda I) = \det[(A - \lambda I)^T] = \det[A^T - \lambda I^T] = \det(A^T - \lambda I)$ so A and A^T have the same characteristic polynomial. Since the eigenvalues of a matrix are the roots of its characteristic polynomial, it follows that A and A^T have the same eigenvalues.

12. (a) Let A be

$$\begin{pmatrix} \lambda_1 & & & (*) \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}.$$

Then

$$A^2 = \begin{pmatrix} \lambda_1 & & & (*) \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}\begin{pmatrix} \lambda_1 & & & (*) \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & & & (*) \\ & \lambda_2^2 & & \\ & & \ddots & \\ 0 & & & \lambda_n^2 \end{pmatrix}$$

and similarly (by an obvious induction proof which doesn't need to be done)

$$A^k = \begin{pmatrix} \lambda_1^k & & & (*) \\ & \lambda_2^k & & \\ & & \ddots & \\ 0 & & & \lambda_n^k \end{pmatrix}.$$

for $k = 0, 1, 2, 3, ...$ Finally,

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{\lambda_1{}^k}{k!} & & & (*) \\ & \sum_{k=0}^{\infty} \frac{\lambda_2{}^k}{k!} & & \\ & & \ddots & \\ 0 & & & \sum_{k=0}^{\infty} \frac{\lambda_n{}^k}{k!} \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & & & (*) \\ & e^{\lambda_2} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n} \end{pmatrix}.$$

(b) *Upper triangular case:* Suppose first that A is upper triangular with diagonal entries $\lambda_1, ..., \lambda_n$. Then by part (a), e^A is upper triangular with diagonal entries $e^{\lambda_1}, ..., e^{\lambda_n}$ and has determinant $e^{\lambda_1} \cdots e^{\lambda_n} = e^{\lambda_1 + ... + \lambda_n} = e^{trace(A)}$.

General case: Now let A be an arbitrary matrix. By a homework problem there is an invertible matrix P such that PAP^{-1} is upper triangular. From above, we know $\det(e^{PAP^{-1}}) = e^{trace(PAP^{-1})}$. But

$$\det(e^{PAP^{-1}}) = \det(Pe^A P^{-1}) = \det P \cdot \det e^A \cdot \det P^{-1} = \det e^A$$

and

$$e^{trace(PAP^{-1})} = e^{trace(AP^{-1}P)} = e^{trace(A)}$$

(the first equality follows from the fact that $trace(M_1 M_2) = trace(M_2 M_1)$); putting these equalities together proves the result.