1 Change of basis matrices

Suppose V is a vector space over field F with dimension n. Then, we know that $V \cong F^n$ and every such isomorphism is a coordinate mapping. Stated another way, if we choose any basis $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ of V, the coordinate mapping $\phi_{\mathcal{B}} : V \to F^n$ defined by

$$\phi_{\mathcal{B}}(\mathbf{x}) = (s_1, ..., s_n) \Leftrightarrow \mathbf{x} = \sum_{j=1}^{n} s_j \mathbf{b}_j$$

is an isomorphism (invertible linear transformation).

Suppose we take two bases of V, say $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, ..., \mathbf{c}_n\}$. Then the transformation $T : F^n \to F^n$ defined by $T = \phi_{\mathcal{C}} \circ \phi_{\mathcal{B}}$ is a transformation which converts $\mathcal{B}-$coordinates to $\mathcal{C}-$coordinates, i.e.

$$T([\mathbf{x}]_{\mathcal{B}}) = [\mathbf{x}]_{\mathcal{C}}$$

for every $\mathbf{x} \in V$. Since T maps F^n to F^n, it has a standard $n \times n$ matrix which we call $P_{\mathcal{C}\leftarrow\mathcal{B}}$. Since both $\phi_{\mathcal{B}}$ and $\phi_{\mathcal{C}}$ are isomorphisms, so is the composition $T = \phi_{\mathcal{C}} \circ \phi_{\mathcal{B}}^{-1}$, so this standard matrix of T is invertible. This proves:

Theorem 1 *Let $\mathcal{B}$ and $\mathcal{C}$ be two bases of V, where V is a finite-dimensional vector space over F of dimension n. Then there is an invertible matrix $P_{\mathcal{C}\leftarrow\mathcal{B}} \in M_n(F)$ such that for every $\mathbf{x} \in V$,*

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}.$$

In other words, $P_{\mathcal{C}\leftarrow\mathcal{B}}$ "converts $\mathcal{B}-$coordinates to $\mathcal{C}-$coordinates".

Question: What are the entries of $P_{\mathcal{C}\leftarrow\mathcal{B}}$?

Answer: For each vector $\mathbf{b}_j \in \mathcal{B}$, let $\mathbf{r}_j = [\mathbf{b}_j]_{\mathcal{C}}$. In other words, if we let $\mathbf{r}_j = (r_{1j}, r_{2j}, ..., r_{nj})$, we see $\mathbf{b}_j = \sum_{k=1}^{n} r_{kj}\mathbf{c}_j$. Define

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \cdots & \mathbf{r}_n \end{pmatrix} = \begin{pmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{pmatrix}.$$

Now, let $\mathbf{x} \in V$.

$$\begin{aligned}
[\mathbf{x}]_{\mathcal{B}} = (s_1, ..., s_n) \Leftrightarrow \mathbf{x} = \sum_{j=1}^{n} s_j \mathbf{b}_j &= \sum_{j=1}^{n} s_j \left(\sum_{k=1}^{n} r_{kj}\mathbf{c}_j \right) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} s_j r_{kj} \mathbf{c}_j \\
&= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} s_j r_{kj} \right) \mathbf{c}_j.
\end{aligned}$$

Therefore $[\mathbf{x}]_{\mathcal{C}} = (d_1, ..., d_n)$ where $d_k = \sum_{j=1}^{n} s_j r_{kj} = \sum_{j=1}^{n} r_{kj} s_j$. Notice that the s_j come from the column vector $[\mathbf{x}]_{\mathcal{B}}$ and the r_{kj} come from the k^{th} row of $P_{\mathcal{C}\leftarrow\mathcal{B}}$. Therefore d_k is the k^{th} entry of $P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$; this means

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

for every $\mathbf{x} \in V$ as desired.

Proposition 2 *Let $\mathcal{B}$ and $\mathcal{C}$ be two bases of V, where V is a finite-dimensional vector space over F of dimension n. Then $P_{\mathcal{C}\leftarrow\mathcal{B}}^{-1} = P_{\mathcal{B}\leftarrow\mathcal{C}}$.*

Proposition 3 *Let $\mathcal{B}$ be a basis of V, where V is a finite-dimensional vector space over F of dimension n. Then $P_{\mathcal{B}\leftarrow\mathcal{B}} = I$.*

Proposition 4 *Let $V = F^n$, let $\mathcal{S}$ be the standard basis $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ and let $\mathcal{B}$ be any other basis of V. Then the columns of $P_{\mathcal{S}\leftarrow\mathcal{B}}$ are the basis vectors of $\mathcal{B}$, listed in order. In other words,*

$$P_{\mathcal{S}\leftarrow\mathcal{B}} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix}.$$

Proposition 5 *Let $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be three bases of V, where V is a finite-dimensional vector space over F of dimension n. Then $P_{\mathcal{D}\leftarrow\mathcal{B}} = P_{\mathcal{D}\leftarrow\mathcal{C}} P_{\mathcal{C}\leftarrow\mathcal{B}}$.*

Proposition 6 *(Special case of Proposition 4) Let $\mathcal{B}$ and $\mathcal{C}$ be two bases of F^n. Then Then $P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}\leftarrow\mathcal{S}} P_{\mathcal{S}\leftarrow\mathcal{B}}$.*

2 Matrices of linear transformations

Theorem 7 *Let V and W be vector spaces over the same field F; suppose $\dim(V) = n < \infty$ and $\dim(W) = m < \infty$. Let $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ be a basis of V and let $\mathcal{C} = \{\mathbf{c}_1, ..., \mathbf{c}_m\}$ be a basis of W. Let $T : V \to W$ be a linear transformation.*

Then, there is a matrix $A \in M_{mn}(F)$, called the matrix of T relative to $\mathcal{B}$ and $\mathcal{C}$, *such that for all $\mathbf{x} \in V$, $[T(\mathbf{x})]_{\mathcal{C}} = A[\mathbf{x}]_{\mathcal{B}}$. In other words, the following commutative diagram holds:*

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ {\scriptstyle \phi_{\mathcal{B}}}\downarrow & & \downarrow{\scriptstyle \phi_{\mathcal{C}}} \\ F^n & \xrightarrow[\mathbf{x}\mapsto A\mathbf{x}]{} & F^n \end{array}$$

Proof: Define A by setting

$$A = \begin{pmatrix} [T(\mathbf{b}_1)]_{\mathcal{C}} & [T(\mathbf{b}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{C}} \end{pmatrix}.$$

The condition $[T(\mathbf{x})]_{\mathcal{C}} = A[\mathbf{x}]_{\mathcal{B}}$ can be checked by hand.

Remark: If $T : V \to V$ (i.e. the domain and codomain are the same vector space), then we say A is the *matrix of T relative to $\mathcal{B}$* if A is as above, where we are choosing $\mathcal{B}$ to be the basis for both the domain and range; i.e. the following diagram should hold:

$$\begin{array}{ccc} V & \xrightarrow{T} & V \\ {\scriptstyle \phi_{\mathcal{B}}}\downarrow & & \downarrow{\scriptstyle \phi_{\mathcal{B}}} \\ F^n & \xrightarrow[\mathbf{x}\mapsto A\mathbf{x}]{} & F^n \end{array}$$

If $V = F^n$ and $\mathcal{B}$ is the standard basis, then the A in this situation is the standard matrix of T.

Theorem 8 *Let V and W be vector spaces over the same field F; suppose* $\dim(V) = n < \infty$ *and* $\dim(W) = m < \infty$. *Let $\mathcal{B}$ and $\mathcal{B}'$ be two bases of V and let $\mathcal{C}$ and $\mathcal{C}'$ be two bases of W. Let $T : V \to W$ be a linear transformation whose matrix relative to $\mathcal{B}$ and $\mathcal{C}$ is A and whose matrix relative to $\mathcal{B}'$ and $\mathcal{C}'$ is A'. Then*

$$A' = P_{\mathcal{C}' \leftarrow \mathcal{C}} A P_{B \leftarrow B'} = \left(P_{C \leftarrow C'}\right)^{-1} A P_{B \leftarrow B'} = Q^{-1} A P$$

if we set $P = P_{B \leftarrow B'}$ and $Q = P_{C \leftarrow C'}$.

Theorem 9 *Let V be a vector space over the same field F; suppose* $\dim(V) = n < \infty$ *and* $\dim(W) = m < \infty$. *Let $\mathcal{B}$ and $\mathcal{B}'$ be two bases of V. Let $T : V \to B$ be a linear transformation whose matrix relative to $\mathcal{B}$ is A (here we mean $\mathcal{B}$ is chosen as the basis for both the domain and range) and whose matrix relative to $\mathcal{B}'$ is A'. Then*

$$A' = P_{\mathcal{B}' \leftarrow \mathcal{B}} A P_{B \leftarrow B'} = \left(P_{B \leftarrow B'}\right)^{-1} A P_{B \leftarrow B'} = P^{-1} A P$$

if we set $P = P_{B \leftarrow B'}$.