Definition: A *field* is a set F (we require F to have at least two elements) together with two binary operations:

- *addition*, denoted $+$, and
- *multiplication*, denoted $\cdot$,

such that the following rules (called the "Field Laws") are satisfied:

1. *Addition is closed:* For all $x, y \in F$, $x + y \in F$.
2. *Multiplication is closed:* For all $x, y \in F$, $xy \in F$.
3. *Addition is commutative:* For all $x, y \in F$, $x + y = y + x$.
4. *Multiplication is commutative:* For all $x, y \in F$, $xy = yx$.
5. *Addition is associative:* For all $x, y, z \in F$, $x + (y + z) = (x + y) + z$.
6. *Multiplication is associative:* For all $x, y, z \in F$, $x(yz) = (xy)z$.
7. *Additive identity element:* There exists an element of F called 0 such that $x + 0 = x$ for all $x \in F$.
8. *Multiplicative identity element:* There exists an element of F called 1 such that $1x = x$ for all $x \in F$.
9. *Additive inverses exist:* For all $x \in F$, there exists an element $-x \in F$ such that $x + (-x) = 0$.
10. *Reciprocals exist:* For all $x \neq 0 \in F$, there exists an element $x^{-1} \in F$ such that $x(x^{-1}) = 1$.
11. *Distributivity:* For all $x, y, z \in F$, $x(y + z) = xy + xz$.

From the Field Laws, one can deduce many other nice properties of fields. They are all properties that mimic the arithmetic properties of the real numbers that you are familar with. Among the properties that hold for any field F:

- The identity elements for addition and multiplication are unique.
- Additive and multiplicative inverses are unique.
- $0 \neq 1$.
- $-0 = 0$.
- For all $x \in F$, $0x = 0$.
- For every $x \in F$, $-x = -1 \cdot x$.
- For every $x, y, z \in F$, $(x + y)z = xz + yz$.
- If $x + y = z + y$, then $x = z$. (This is called the Cancellation Law.)
- If $x + y = x$ for any $y \in F$, then $x = 0$.

The most common fields are the "Big Three": $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}$. There are other (more exotic, also perhaps more interesting) fields which I will mention in class, but we won't mess with these much. For the most part, when we say "let F be a field", you should think of F as being the real numbers or complex numbers. In actuality, F is some abstract set with some notion of addition and multiplication that behave in a way that mimics addition and multiplication in $\mathbb{R}$.

Things that are not fields: $\mathbb{Z}$ is the prototype (because reciprocals do not exist; for example, the reciprocal of 2 is not an integer).