Definition 1 *Given vector space V over field F, a set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ of vectors is called* linearly dependent *(*l. dep.*) if there exist scalars $c_1, ..., c_n$, with not all of the c_j zero, such that*

$$c_1\mathbf{v}_1 + ... + c_n\mathbf{v}_n = \mathbf{0}.$$

Definition 2 *Given vector space V over field F, a set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ of vectors is called* linearly independent *(*l. ind.*) if the set is not linearly dependent, i.e. if whenever we have scalars $c_1, ..., c_n$ such that*

$$c_1\mathbf{v}_1 + ... + c_n\mathbf{v}_n = \mathbf{0},$$

then it must be the case that $c_1 = c_2 = ... = c_n = 0$.

How to prove a set of vectors is linearly independent: There are many tricks we will develop and theorems you can use (see below), but one technique is to assume that there are scalars $c_1, ..., c_n$ such that $c_1\mathbf{v}_1 + ... + c_n\mathbf{v}_n = \mathbf{0}$. If you can prove all the c_j's must be zero, then the vectors are l. ind.

How to prove a set of vectors is linearly dependent: There are many tricks we will develop and theorems you can use (see below), but one technique is to find explicit scalars $c_1, ..., c_n$ such that at least one of the c_js is nonzero where $c_1\mathbf{v}_1 + ... + c_n\mathbf{v}_n = \mathbf{0}$.

Elementary theorems on linear independence: Learn them, know them and love them

Theorem 1 *Given vector space V over field F, a set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ of vectors is linearly dependent if and only if there exists a $k \leq n$ such that $\mathbf{v}_k \in Span(\mathbf{v}_1, ..., \mathbf{v}_{k-1})$ (we describe this by saying that* $\mathbf{v}_k$ depends on $\mathbf{v}_1, ..., \mathbf{v}_{k-1}$*).*

Theorem 2 *The zero vector is never part of a linearly independent set.*

Theorem 3 *A set $\{\mathbf{v}\}$ of one vector is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$.*

Theorem 4 *A set $\{\mathbf{v}, \mathbf{w}\}$ of two vector is linearly dependent if and only if $\mathbf{v} \,||\, \mathbf{w}$ are not parallel.*

Theorem 5 *Any subset of a linearly independent set of vectors is itself linearly independent.*

Theorem 6 *Any set of vectors containing a linearly dependent set is itself linearly dependent.*

Theorem 7 *Suppose $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is a linearly dependent set where $\mathbf{v}_k = \sum_{j=1}^{k-1} d_j\mathbf{v}_j$ (this must hold for at least one $k \leq n$ by Theorem 1). Then*

$$Span(\mathbf{v}_1, ..., \mathbf{v}_n) = Span(\mathbf{v}_1, ..., \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, ..., \mathbf{v}_n)$$

(in other words, a vector which depends on the previous vectors in the list can be removed from the list without changing the span).

Theorem 8 *Suppose $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ spans V where V is a vector space (or subspace). Then there is a linearly independent subset of $\mathcal{S}$ which still spans V.*

Theorem 9 (Exchange Lemma) *Suppose V is a vector space (or subspace) where $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ spans V and $\{\mathbf{w}_1, ..., \mathbf{w}_m\}$ is a linearly independent set. Then $m \leq n$. (In English, this says that any linearly independent set of vectors cannot have more vectors in it than any spanning set.)*