Here are some important facts describing the algebra of linear transformations. In particular, the first two facts below show that $L(V_1, V_2)$, the set of linear transformations from vector space V_1 to vector space V_2 (both vector spaces must be over the same field F), is itself a vector space over F. The additive identity element of $L(V_1, V_2)$ is the zero transformation (described below).

Sums of linear transformations are linear If $T : V_1 \to V_2$ and $S : V_1 \to V_2$ are known to be linear, then $T + S : V_1 \to V_2$, defined by setting $(T + S)(\mathbf{x}) = T(\mathbf{x}) + S(\mathbf{x})$, is linear.

Scalar multiples of linear transformations are linear If $T : V_1 \to V_2$ is linear, so is $rT : V_1 \to V_2$, defined by setting $(rT)(\mathbf{x}) = rT(\mathbf{x})$.

Compositions of linear transformations are linear If $T : V_1 \to V_2$ and $S : V_2 \to V_3$ are known to be linear, then $ST : V_1 \to V_3$, defined by setting $(ST)(\mathbf{x}) = S(T(\mathbf{x}))$, is linear.

Inverses of invertible linear transformations are linear If $T : V_1 \to V_2$ is known to be a bijective linear transformation, then $T^{-1} : V_2 \to V_1$, defined by setting $T^{-1}(\mathbf{x})$ to be the $\mathbf{y} \in V_1$ such that $T(\mathbf{y}) = \mathbf{x}$, is linear.

The following are typical examples of linear transformations $T : V_1 \to V_2$ (henceforth, in Math 28S you may assume any function in any of these classes is linear without proof):

The zero transformation defined by setting $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in V_1$.

Scalar multiplication $T(\mathbf{x}) = k\mathbf{x}$ is a linear transformation from V_1 to V_1 for any $k \in F$.

Matrix multiplication Every matrix $A \in M_{mn}(F)$ defines a linear transformation $T : F^n \to F^m$ by $T(\mathbf{x}) = A\mathbf{x}$.

Inner product with a fixed second vector If V_1 is an inner product space, then for every fixed $\mathbf{v} \in V_1$, the formula $T(\mathbf{x}) =< \mathbf{x}, \mathbf{v} >$ defines a linear transformation from V_1 to F.

Coordinate mappings Suppose $\dim(V_1) = n < \infty$ and $\mathcal{B}$ is any basis of V_1. Then the function $T(\mathbf{x}) = [\mathbf{x}]_{\mathbf{B}}$, the $\mathcal{B}-$*coordinate mapping*, is a linear transformation from V_1 to F^n.

Evaluations Let V_1 be some vector space of functions from X to F. Then, for any $x_0 \in X$, the formula $T(f) = f(x_0)$ defines a linear transformation $V_1 \to F$ called an *evaluation map*.

Differentiation $T : C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R})$ defined by $T(f) = f'$.

Definite integration $T : C([a, b], \mathbb{R}) \to \mathbb{R}$ defined by $T(f) = \int_a^b f(x)\,dx$.

Projection onto a vector If V_1 is an inner product space, then for any fixed $\mathbf{v} \in V_1$, the formula $T(\mathbf{x}) = \text{proj}_{\mathbf{v}}(\mathbf{x})$ is linear.

Projection onto a finite-dimensional subspace If V_1 is an inner product space, then for any finite-dimensional subspace $W \subseteq V_1$, the formula $T(\mathbf{x}) = \mathbf{x}^W$ (a.k.a. $\text{proj}_W(\mathbf{x})$) is linear.

Transposes $T(A) = A^T$ defines a linear transformation from $M_{mn}(F)$ to $M_{nm}(F)$.

Traces $T(A) = tr(A)$ defines a linear transformation from $M_{mn}(F)$ to F.