Suppose you are given a matrix (it can be an augmented matrix $(A\,\mathbf{b})$, or just a coefficient matrix A, or any other matrix) and you want to put it into echelon form. Here's a formulaic way which always works (this is not the only method, however).

Part 1: Transforming a matrix into row-echelon form: "Downward reduction"

1. Place an imaginary cursor at the upper-left entry of the matrix.
2. If the cursor entry and all entries below the cursor entry are zero, move the cursor one column to the right. Repeat this step if necessary.
3. The cursor should now be in a position such that there is a nonzero entry either in the cursor position or directly below the cursor. If necessary, switch the cursor row with a row *beneath* the cursor row to put a nonzero entry in the cursor position.
4. Add multiples of the cursor row to each of the rows beneath the cursor row to create zeros in all entries beneath the cursor.
5. Move the cursor one row down and one column to the right. If the cursor is not on the bottom row, return to step 2. If you reach the last nonzero row, you are done and the matrix is in row-echelon form.

Note: These steps are called *downward reduction* because the cursor is always moving down (from the upper-left corner downward and to the right).

Part 2: Transforming a row-echelon form into a reduced row-echelon form: "Upward reduction"

1. If necessary, carry out downward reduction as in Part 1 to place the matrix in row-echelon form.
2. Multiply each pivot row by the reciprocal of its pivot, so that all the pivots become 1.
3. Place the cursor on the right-most pivot position.
4. If necessary, add multiples of the cursor row to every row above the cursor row so that all entries above the cursor become zero.
5. Once all entries above the cursor are zero, move the cursor up one row and to the left until you reach a pivot position.
6. If the cursor is on the first row, you are done. The matrix is in rref form. Otherwise, return to step 4.

Note: These steps are called *upward reduction* because the cursor is always moving up (from the bottom pivot row up and to the left).

An example: Let $A = \begin{pmatrix} 0 & 6 & 4 & -12 \\ 3 & 3 & 0 & 9 \\ 2 & 0 & -3 & 10 \end{pmatrix}$. Let's put A into rref form (throughout these operations, the cursor position is underlined).

Part 1: Downward reduction

Matrix	Current Step	Thought Process	Subsequent Row Operation
$\begin{pmatrix} \underline{0} & 6 & 4 & -12 \\ 3 & 3 & 0 & 9 \\ 2 & 0 & -3 & 10 \end{pmatrix}$	1,2	Cursor column contains nonzero entries below cursor. Proceed to step 3.	
$\begin{pmatrix} \underline{0} & 6 & 4 & -12 \\ 3 & 3 & 0 & 9 \\ 2 & 0 & -3 & 10 \end{pmatrix}$	3	I need a nonzero entry at the cursor position.	Swap rows 1 and 2
$\begin{pmatrix} \underline{3} & 3 & 0 & 9 \\ 0 & 6 & 4 & -12 \\ 2 & 0 & -3 & 10 \end{pmatrix}$	4	I need to make all entries at below the cursor zero.	Add $-\left(\frac{2}{3}\right)$ times row 1 to row 2
$\begin{pmatrix} \underline{3} & 3 & 0 & 9 \\ 0 & 6 & 4 & -12 \\ 0 & -2 & -3 & 4 \end{pmatrix}$	5	Move the cursor down and right. Return to step 2.	
$\begin{pmatrix} 3 & 3 & 0 & 9 \\ 0 & \underline{6} & 4 & -12 \\ 0 & -2 & -3 & 4 \end{pmatrix}$	2	Cursor column contains nonzero entries below cursor; cursor entry is nonzero. Proceed to step 4.	
$\begin{pmatrix} 3 & 3 & 0 & 9 \\ 0 & \underline{6} & 4 & -12 \\ 0 & -2 & -3 & 4 \end{pmatrix}$	4	I need to make all entries at below the cursor zero.	Add $\frac{1}{3}$ times row 2 to row 3
$\begin{pmatrix} 3 & 3 & 0 & 9 \\ 0 & \underline{6} & 4 & -12 \\ 0 & 0 & -\frac{5}{3} & 0 \end{pmatrix}$	5	Move the cursor down and right. It'll be on the bottom row, so I'm done.	

Notice that this matrix is in row-echelon form. What you would typically write down on your sheet of paper is something like this:

$$\begin{pmatrix} 0 & 6 & 4 & -12 \\ 3 & 3 & 0 & 9 \\ 2 & 0 & -3 & 10 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 3 & 3 & 0 & 9 \\ 0 & 6 & 4 & -12 \\ 2 & 0 & -3 & 10 \end{pmatrix}$$

$$\xrightarrow{-\left(\frac{2}{3}\right)R_1 + R_2} \begin{pmatrix} 3 & 3 & 0 & 9 \\ 0 & 6 & 4 & -12 \\ 0 & -2 & -3 & 4 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{3}R_2 + R_3} \begin{pmatrix} 3 & 3 & 0 & 9 \\ 0 & 6 & 4 & -12 \\ 0 & 0 & -\frac{5}{3} & 0 \end{pmatrix}.$$

Part 2: Upward reduction

Matrix	Current Step	Thought Process	Subsequent Row Operation
$\begin{pmatrix} 3 & 3 & 0 & 9 \\ 0 & 6 & 4 & -12 \\ 0 & 0 & -\frac{5}{3} & 0 \end{pmatrix}$	1,2	Matrix is in ref form. Need to make pivots 1.	Multiply row 1 by $\frac{1}{3}$; multiply row 2 by $\frac{1}{6}$; multiply row 3 by $-\left(\frac{3}{5}\right)$.
$\begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & \frac{2}{3} & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	3	Place cursor on right-most pivot.	
$\begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & \frac{2}{3} & -2 \\ 0 & 0 & \underline{1} & 0 \end{pmatrix}$	4	I need to make all entries at above the cursor zero.	Add $-\left(\frac{2}{3}\right)$ times row 3 to row 2
$\begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & \underline{1} & 0 \end{pmatrix}$	5	Move the cursor up and left. Return to step 4.	
$\begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & \underline{1} & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	4	I need to make all entries at above the cursor zero.	Add -1 times row 2 to row 1
$\begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & \underline{1} & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	5	Move the cursor up and left. It'll be on the first row, so I'm done.	

Notice that this matrix is in its rref form. What you would typically write down on your sheet of paper for the upward steps is something like this:

$$\begin{pmatrix} 3 & 3 & 0 & 9 \\ 0 & 6 & 4 & -12 \\ 0 & 0 & -\frac{5}{3} & 0 \end{pmatrix} \xrightarrow{\frac{1}{3}\cdot R_1, \frac{1}{6}\cdot R_2, -\frac{3}{5}\cdot R_3} \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & \frac{2}{3} & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{-\left(\frac{2}{3}\right)R_3+R_2} \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{-1\cdot R_2+R_1} \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$