Recall that an *inner product* on a vector space V is a formula $<,>$ satisfying:

Conjugate symmetry $<\mathbf{v},\mathbf{w}>=\overline{<\mathbf{w},\mathbf{v}>}$ for all $\mathbf{v},\mathbf{w}\in V$.
Linearity in first coordinate $<r\mathbf{v},\mathbf{w}>=r<\mathbf{v},\mathbf{w}>$ and $<\mathbf{v}_1+\mathbf{v}_2,\mathbf{w}>=<\mathbf{v}_1,\mathbf{w}>+<\mathbf{v}_2,\mathbf{w}>$ for all $\mathbf{v},\mathbf{v}_1,\mathbf{v}_2,\mathbf{w}\in V$ and all $r\in F$.
Positivity $<\mathbf{v},\mathbf{v}>\geq 0$ for all $\mathbf{v}\in V$, and
Definiteness $<\mathbf{v},\mathbf{v}>=0$ only when $\mathbf{v}=\mathbf{0}$.

On the face of things, there could be lots of formulas $<,>$ satisfying these properties. In fact, all the possible inner products on a finite-dimensional vector space can be classified. Problems 33 and 34 of the section of the homework on matrix theory give a complete classification of inner products on V when $V=\mathbb{R}^n$ or $\mathbb{C}^n$. Here are the relevant results:

Definition 1 *Let $F=\mathbb{R}$ or $\mathbb{C}$. A matrix A is called* positive definite *if*

- *for every $\mathbf{v}\in\mathbb{C}^n$, the product $\mathbf{v}^HA\mathbf{v}\geq 0$ (in particular, this means $\mathbf{v}^HA\mathbf{v}$ is real for every $\mathbf{v}$), and*
- *if $\mathbf{v}\in\mathbb{C}^n$ is such that $\mathbf{v}^HA\mathbf{v}=0$, then $\mathbf{v}=\mathbf{0}$.*

Lemma 1 (Dual relations) *Let $F=\mathbb{R}$ or $\mathbb{C}$ and let $<,>$ represent the standard (dot or Hermitian) inner product. Let $A\in M_{mn}(F)$ and $B\in M_{nm}(F)$ be given. Then for all $\mathbf{x}\in F^n$ and $\mathbf{y}\in F^m$, we have*

$$<A\mathbf{x},\mathbf{y}>=<\mathbf{x},A^H\mathbf{y}> \quad \text{and} \quad <\mathbf{x},B\mathbf{y}>=<B^H\mathbf{x},\mathbf{y}>.$$

Theorem 1 (Classification of inner products on $\mathbb{R}^n$)

1. *Suppose $A\in M_n(\mathbb{R})$ is symmetric and positive definite. Then the formula $<\mathbf{x},\mathbf{y}>=\mathbf{y}^TA\mathbf{x}$ defines an inner product on $\mathbb{R}^n$.*
2. *Suppose $<,>$ is some inner product on $\mathbb{R}^n$. Then there exists a symmetric and positive definite matrix A, called the* matrix associated to the inner product, *such that for every $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$,*

$$<\mathbf{x},\mathbf{y}>=\mathbf{y}^TA\mathbf{x}.$$

If you know the inner product $<,>$, the associated matrix A is computed by finding inner products of the standard basis vectors. In particular, if we denote the entries of A by a_{ij}, then $a_{ij}=<\mathbf{e}_j,\mathbf{e}_i>$ for all i,j.

This theorem says that there is a correspondence between inner products on $\mathbb{R}^n$ and symmetric, positive definite matrices A. The usual Euclidean inner product corresponds to $A=I$. This correspondence is useful because if you are given some inner product $<,>$ on $\mathbb{R}^n$, then you can always take the associated matrix A and write the inner product as $\mathbf{y}^TA\mathbf{x}$, then use matrix theory if you have to manipulate it.

The theorem also says that every inner product on $\mathbb{R}^n$ is related to the usual dot product; in particular, for a given inner product with associated matrix A, then for all $\mathbf{x}$ and $\mathbf{y}$, we have $<\mathbf{x},\mathbf{y}>=A\mathbf{x}\cdot\mathbf{y}=\mathbf{x}\cdot A\mathbf{y}$ where $\cdot$ represents Euclidean dot product. (The last equality follows from the dual relations and the fact that A is symmetric.)

Theorem 2 (Classification of inner products on $\mathbb{C}^n$)

1. *Suppose $A\in M_n(\mathbb{C})$ is Hermitian and positive definite. Then the formula $<\mathbf{x},\mathbf{y}>=\mathbf{y}^HA\mathbf{x}$ defines an inner product on $\mathbb{C}^n$.*
2. *Suppose $<,>$ is some inner product on $\mathbb{R}^n$. Then there exists a Hermitian and positive definite matrix A, called the* matrix associated to the inner product, *such that for every $\mathbf{x},\mathbf{y}\in\mathbb{C}^n$,*

$$<\mathbf{x},\mathbf{y}>=\mathbf{y}^HA\mathbf{x}.$$

As in the real case, if you know the inner product $<,>$, the associated matrix A is computed by finding inner products of the standard basis vectors. In particular, if we denote the entries of A by a_{ij}, then $a_{ij}=<\mathbf{e}_j,\mathbf{e}_i>$ for all i,j.

This theorem says that there is a correspondence between inner products on $\mathbb{C}^n$ and Hermitian, positive definite matrices A. The usual Hermitian inner product corresponds to $A = I$. This correspondence is useful because if you are given some inner product $<,>$ on $\mathbb{C}^n$, then you can always take the associated matrix A and write the inner product as $\mathbf{y}^H A\mathbf{x}$, then use matrix theory if you have to manipulate it.

The theorem also says that every inner product on $\mathbb{C}^n$ is related to the usual Hermitian inner product; in particular, for a given inner product with associated matrix A, then for all $\mathbf{x}$ and $\mathbf{y}$, we have $< \mathbf{x}, \mathbf{y} >= A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A^H\mathbf{y} = A\mathbf{y}$ where $\cdot$ represents Hermitian inner product. (The last equalities follows from the dual relations and the fact that A is a Hermitian matrix.)