**Definition 1** *A matrix $Q \in M_n(\mathbb{R})$ is called* orthogonal *if $Q^{-1} = Q^T$. The set of $n \times n$ orthogonal matrices is denoted $O_n$.*

**Theorem 1 (Equivalent characterizations of orthogonal matrices)** *The following are equivalent:*

1. *$Q$ is orthogonal;*
2. *$Q^T$ is orthogonal;*
3. *$Q$ is invertible, and $Q^{-1}$ is orthogonal;*
4. *the columns of $Q$ form an orthonormal basis of $\mathbb{R}^n$ (w.r.t. the Euclidean inner product);*
5. *the rows of $Q$ form an orthonormal basis of $\mathbb{R}^n$ (w.r.t. the Euclidean inner product);*
6. *$Q$ "preserves dot product", i.e. $< Q\mathbf{x}, Q\mathbf{y} >=< \mathbf{x}, \mathbf{y} >$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ ($<, >$ is the Euclidean inner product);*
7. *$Q$ "preserves angles", i.e. for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the angle (computed w.r.t. Euclidean inner product) between $\mathbf{x}$ and $\mathbf{y}$ is the same as the angle between $Q\mathbf{x}$ and $Q\mathbf{y}$;*
8. *$Q$ "preserves norms", i.e. for every $\mathbf{x} \in \mathbb{R}^n$, $||Q\mathbf{x}|| = ||\mathbf{x}||$ (where $|| \cdot ||$ is the Euclidean norm);*
9. *$Q$ "preserves distances", i.e. for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the distance (computed w.r.t. Euclidean inner product) between $\mathbf{x}$ and $\mathbf{y}$ is the same as the distance between $Q\mathbf{x}$ and $Q\mathbf{y}$;*

To get a handle on what orthogonal matrices are, let's describe all the $2 \times 2$ orthogonal matrices, using property 4 of Theorem 1:

**Theorem 2 (Classification of $2 \times 2$ orthogonal matrices)** *$Q \in O_2$ if and only if $Q$ has one of the following two forms:*

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

Suppose $Q$ is of the first type, i.e. $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Then for every $\mathbf{x} \in \mathbb{R}^2$, $Q\mathbf{x}$ is the vector $\mathbf{x}$, rotated counterclockwise by the angle $\theta$. In other words, the first class of $2 \times 2$ orthogonal matrices are **rotations** by $\theta$.

Suppose $Q$ is of the second type, i.e. $Q = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$. Then, for every $\mathbf{x} \in \mathbb{R}^2$, $Q\mathbf{x}$ is the vector $\mathbf{x}$ reflected through a line which sits at angle $\theta/2$ to the positive $x-$axis. In other words, the first class of $2 \times 2$ orthogonal matrices are **reflections**.

In high-school Euclidean geometry you learn about various notions of *congruence* of geometric objects, i.e. two triangles are congruent if they have the same side lengths and same angle measures, two line segments are congruent if they have the same length, two circles are congruent if they have the same radius, etc. There is really only one notion of congruence that encompasses all these, however:

**Definition 2** *Two subsets $A$ and $B$ of $\mathbb{R}^2$ are* congruent *if there is an orthogonal matrix $Q \in O_2$ and a vector $\mathbf{v} \in \mathbb{R}^2$ such that $QA + \mathbf{v} = B$.*

(Here, $QA$ is a set... $QA = \{Q\mathbf{x} : \mathbf{x} \in A\}$). In other words, this definition says $A$ and $B$ are congruent objects if you can get from $A$ to $B$ by rotating or reflecting $A$ and then translating it (by $\mathbf{v}$) to get $B$ (and gives an algebraic characterization of this). This idea generalizes to higher dimensions:

**Definition 3** *Two subsets $A$ and $B$ of $\mathbb{R}^n$ are* congruent *if there is an orthogonal matrix $Q \in O_n$ and a vector $\mathbf{v} \in \mathbb{R}^n$ such that $QA + \mathbf{v} = B$.*

One caveat: higher-dimensional orthogonal matrices are not all rotations and reflections; classifying them is the kind of thing you would do in an upper-level geometry course.

*Remark:* Here we are using the standard inner product to describe all the geometric objects. If you start with a different inner product, then we can define a class of matrices which are *orthogonal* relative to that inner product by saying $Q$ is "orthogonal relative to $<,>$" if $Q$ the columns of $Q$ form an orthonormal (relative to $<,>$) basis of $\mathbb{R}^n$. Such matrices will preserve the inner product you start with, and will preserve the associated (weird) notions of angle, distance and norm. Then you can call two subsets of $\mathbb{R}^n$ "congruent" (relative to $<,>$) if there is an orthogonal matrix (relative to $<,>$) $Q$ and a vector $\mathbf{v}$ such that $B = QA + \mathbf{v}$.

The complex analogues of orthogonal matrices are called unitary matrices:

**Definition 4** *A matrix $U \in M_n(\mathbb{C})$ is called* unitary *if $U^{-1} = U^H$. The set of $n \times n$ orthogonal matrices is denoted $U_n$.*

**Theorem 3 (Equivalent characterizations of unitary matrices)** *The following are equivalent:*

1. *$U$ is unitary;*
2. *$U^T$ is unitary;*
3. *$U^H$ is unitary;*
4. *$\overline{U}$ is unitary;*
5. *$U$ is invertible and $U^{-1}$ is unitary;*
6. *the columns of $U$ form an orthonormal basis of $\mathbb{C}^n$ (w.r.t. the Hermitian inner product);*
7. *the rows of $U$ form an orthonormal basis of $\mathbb{C}^n$ (w.r.t. the Hermitian inner product);*
8. *$U$ "preserves dot product", i.e. $< U\mathbf{x}, U\mathbf{y} >=< \mathbf{x}, \mathbf{y} >$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ ($<,>$ is the Hermitian inner product);*
9. *$U$ "preserves norms", i.e. for every $\mathbf{x} \in \mathbb{C}^n$, $||U\mathbf{x}|| = ||\mathbf{x}||$ (where $|| \cdot ||$ is the norm associated to Hermitian inner product);*
10. *$U$ "preserves distances", i.e. for every $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, the distance (computed w.r.t. Euclidean inner product) between $\mathbf{x}$ and $\mathbf{y}$ is the same as the distance between $U\mathbf{x}$ and $U\mathbf{y}$;*