Theorem 1 (Polar Decomposition) *Let $A \in M_n(\mathbb{C})$. Then there exists a unitary matrix $U \in U_n$ such that*

$$A = U\sqrt{A^H A}.$$

Proof: Let $A \in M_n(\mathbb{C})$. Then

$$||A\mathbf{v}||^2 =< A\mathbf{v}, A\mathbf{v} >=< A^H A\mathbf{v}, \mathbf{v} >=< \sqrt{A^H A}\mathbf{v}, \sqrt{A^H A}\mathbf{v} >= ||\sqrt{A^H A}\mathbf{v}||^2$$

so by taking square roots, we have $||A\mathbf{v}|| = ||\sqrt{A^H A}\mathbf{v}||$ for all $\mathbf{v} \in \mathbb{C}^n$.

Next, let $T : \mathbb{C}^n \to \mathbb{C}^n$ be defined by $T(\mathbf{x}) = A\mathbf{x}$ and let $R : \mathbb{C}^n \to \mathbb{C}^n$ be defined by $R(\mathbf{x}) = \sqrt{A^H A}\mathbf{x}$. Now, define $S : image(R) \to image(T)$ by setting $S(R(\mathbf{x})) = T(\mathbf{x})$. It's not obvious that this can be done (i.e. that S is "well-defined"; in particular, if there were two vectors $\mathbf{x}$ and $\mathbf{y}$ such that $R(\mathbf{x}) = R(\mathbf{y})$ but $T(\mathbf{x}) \neq T(\mathbf{y})$, then we could not define S at $R(\mathbf{x}) = R(\mathbf{y})$ so that it satisified $S(R(\mathbf{x})) = T(\mathbf{x})$ and $S(R(\mathbf{y})) = T(\mathbf{y})$. But, in fact, this can't happen:

Claim: If $R(\mathbf{x}) = R(\mathbf{y})$ then $T(\mathbf{x}) = T(\mathbf{y})$.

Proof of claim: Suppose $R(\mathbf{x}) = R(\mathbf{y})$. Then

$$0 = ||R(\mathbf{x}) - R(\mathbf{y})|| = ||R(\mathbf{x} - \mathbf{y})|| = ||\sqrt{A^H A}(\mathbf{x} - \mathbf{y})|| = ||A(\mathbf{x} - \mathbf{y})|| = ||A\mathbf{x} - A\mathbf{y}|| = ||T(\mathbf{x}) - T(\mathbf{y})||$$

so $T(\mathbf{x}) = T(\mathbf{y})$ (the fourth equality follows from the first paragraph of the proof of the theorem).