In this document we will prove:

Theorem 1 (Spectral Theorem (real version)) *Let $A \in M_n(\mathbb{R})$. The following are equivalent:*

1. *A is symmetric.*
2. *There exists an orthonormal basis of $\mathbb{R}^n$, consisting of eigenvectors of A.*
3. *A is "orthogonally diagonalizable", i.e. there exists an orthogonal matrix Q such that $Q^{-1}AQ = Q^TAQ$ is diagonal.*
4. *There exists a diagonal matrix Λ and an orthogonal matrix Q such that $A = Q\Lambda Q^{-1} = Q\Lambda Q^T$.*

It is clear that statements 3 and 4 are equivalent; the Q in statement 3 is the Q^T in statement 4 and vice versa. The equivalence of statements 2 and 3 is similar to the proof given in class for the complex version of the spectral theorem (substitute the word "unitary" for "orthogonal" and take Hermitian rather than transpose and it is exactly the same). So we are left to show the equivalence of statement 1 to the others. To prove this, we first need a couple of lemmas:

Lemma 1 *Suppose $A \in M_n(\mathbb{R})$ is symmetric. Suppose α and β are real constants such that $\alpha^2 < 4\beta$. Then the matrix $A^2 + \alpha A + \beta I$ is invertible.*

Proof: Let $\mathbf{v} \neq \mathbf{0}$. Then

$$\begin{aligned} < (A^2 + \alpha A + \beta I)\mathbf{v}, \mathbf{v} > &= < A^2\mathbf{v}, \mathbf{v} > + \alpha < A\mathbf{v}, \mathbf{v} > + \beta < \mathbf{v}, \mathbf{v} > \\ &= < A\mathbf{v}, A^H\mathbf{v} > + \alpha < A\mathbf{v}, \mathbf{v} > + \beta ||\mathbf{v}||^2 \quad \text{(dual relations)} \\ &= < A\mathbf{v}, A\mathbf{v} > + \alpha < A\mathbf{v}, \mathbf{v} > + \beta ||\mathbf{v}||^2 \quad \text{(since } A \text{ is symmetric)} \\ &= ||A\mathbf{v}||^2 + \alpha < A\mathbf{v}, \mathbf{v} > + \beta ||\mathbf{v}||^2 \\ &> ||A\mathbf{v}||^2 + \alpha < A\mathbf{v}, \mathbf{v} > + \frac{1}{4}\alpha^2 ||\mathbf{v}||^2 \quad \text{(since } \alpha^2 < 4\beta). \end{aligned}$$

Treat this last expression as a function of α, and minimize it using calculus techniques: its minimum value occurs when $\alpha = \frac{-2<A\mathbf{v},\mathbf{v}>}{||\mathbf{v}||^2}$ and its minimum value is

$$||A\mathbf{v}||^2 - 2\frac{< A\mathbf{v}, \mathbf{v} >^2}{||\mathbf{v}||^2} + \frac{< A\mathbf{v}, \mathbf{v} >^2}{||\mathbf{v}||^2} = ||A\mathbf{v}||^2 - \frac{< A\mathbf{v}, \mathbf{v} >^2}{||\mathbf{v}||^2}.$$

This last expression is guaranteed to be ≥ 0 by the Cauchy-Schwarz inequality. Therefore, $< (A^2 + \alpha A + \beta I)\mathbf{v}, \mathbf{v} >$ is nonzero, so $(A^2 + \alpha A + \beta I)\mathbf{v} \neq \mathbf{0}$. Consequently $N(A^2 + \alpha A + \beta I) = \{\mathbf{0}\}$ so $A^2 + \alpha A + \beta I$ is invertible. $\Box$

Lemma 2 *Suppose $A \in M_n(\mathbb{R})$ is symmetric. Then A has a real eigenvalue.*

Proof: Let $\mathbf{v} \neq \mathbf{0}$ be a vector in $\mathbb{R}^n$. Then the list of $n+1$ vectors

$$\{\mathbf{v}, A\mathbf{v}, A^2\mathbf{v}, ..., A^n\mathbf{v}\}$$

is linearly dependent by the Exchange Lemma, so there exist constants $c_0, ..., c_n \in \mathbb{R}$, not all zero, such that

$$\sum_{j=0}^{n} c_j A^j \mathbf{v} = \mathbf{0}.$$

Define $p(x) = c_0 + c_1x + c_2x^2 + ... + c_nx^n \in \mathbb{P}_n$. Factor the polynomial $p(x)$, first by factoring out the constant c_n and then by factoring it into linear factors and quadratic factors with no real root. We obtain

$$p(x) = c_n \left[(x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_r)(x^2 + \alpha_1 x + \beta_1)(x^2 + \alpha_2 x + \beta_2) \cdots (x^2 + \alpha_s x + \beta_s)\right].$$

In this factorization we are assuming the quadratic factors $x^2 + \alpha_j x + \beta_j$ cannot be factored into two linear terms over the reals, i.e. that these quadratic factors have no real roots. But a quadratic of this type has no real roots if and only if $\alpha_j^2 < 4\beta_j$ by the quadratic formula. Therefore, we have

$$\begin{aligned}\mathbf{0} &= \sum_{j=0}^{n} c_j A^j \mathbf{v} \\ &= c_n \left[(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_r I)(A^2+\alpha_1 A+\beta_1 I)(A^2+\alpha_2 A+\beta_2 I)\cdots(A^2+\alpha_s A+\beta_s I)\right]\mathbf{v}\end{aligned}$$

Notice that all the matrices $(A^2 + \alpha_j A + \beta_j I)$ are invertible by Lemma 1. Thus, it must be the case that

$$\mathbf{0} = c_n \left[(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_r I)\right]\mathbf{v}$$

which means $N(A - \lambda_j I) \neq \{\mathbf{0}\}$ for some j (keep in mind $\mathbf{v} \neq \mathbf{0}$). Therefore λ_j is an eigenvalue of A. $\square$

Proof of the spectral theorem: $(4 \Rightarrow 1)$ Suppose $A = Q\Lambda Q^T$ for some orthogonal matrix Q and some diagonal matrix Λ (since Λ is diagonal, we have $\Lambda^T = \Lambda$). Then $A^T = (Q\Lambda Q^T)^T = (Q^T)^T\Lambda^T Q^T = Q\Lambda^T Q^T = Q\Lambda Q^T = A$ so A is symmetric.

$(1 \Rightarrow 2)$ We prove this by induction on n. When $n = 1$ the statement is obvious. Now we assume the statement is true when $n = k$, and suppose A is $(k+1)\times(k+1)$. By Lemma 2, A has an eigenvalue λ with eigenvector $\mathbf{x}$; without loss of generality we can assume $||\mathbf{x}|| = 1$ (otherwise normalize $\mathbf{x}$). Let $W = Span(\mathbf{x})$.

Claim: $\mathbf{v} \in W^\perp$ implies $A\mathbf{v} \in W^\perp$.

Proof of claim: A vector is in $W^\perp$ if and only if it is orthogonal to $\mathbf{x}$, since $\mathbf{x}$ spans W. Now suppose $\mathbf{v} \in W^\perp$, i.e. $<\mathbf{v}, \mathbf{x}> = 0$. Then $<A\mathbf{v}, \mathbf{x}> = <\mathbf{v}, A^T\mathbf{x}> = <\mathbf{v}, A\mathbf{x}> = <\mathbf{v}, \lambda\mathbf{x}> = \lambda <\mathbf{v}, \mathbf{x}> = 0$ so $A\mathbf{v} \in W^\perp$ as well; this proves the claim.

From the claim, if we define a linear transformation $T : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ by $T(\mathbf{x}) = A(\mathbf{x})$, we see $T(W^\perp) \subseteq W^\perp$. By the induction hypothesis, there is an orthonormal basis of $W^\perp$, say $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ of eigenvectors of A. It is not hard to see that $\{\mathbf{x}, \mathbf{v}_1, ..., \mathbf{v}_k\}$ is therefore an orthonormal basis of $\mathbb{R}^{k+1}$ of eigenvectors of A. This proves the theorem.