Theorem 1 *Let $A \in M_{mn}(F)$, where $F = \mathbb{R}$ or $\mathbb{C}$. Then the following quantities are all equal to the same number, called the* rank *of A and denoted r or $r(A)$:*

1. $\dim C(A)$
2. $\dim R(A)$
3. *The number of pivot columns of A*
4. *The number of nonzero rows in any echelon form of A.*

In particular, $r(A) \leq m$ and $r(A) \leq n$ if A is $m \times n$. $r(A) = m$ if and only if A has a pivot in every row, and $r(A) = n$ if and only if A has a pivot in every column.

Theorem 2 *Let $A \in M_{mn}(F)$, where $F = \mathbb{R}$ or $\mathbb{C}$. Then a basis for $C(A)$ consists of the pivot columns of A, and a basis for $R(A)$ consists of the nonzero rows of any echelon form of A.*

Theorem 3 (Rank-Nullty Theorem) *Let $A \in M_{mn}(F)$, where $F = \mathbb{R}$ or $\mathbb{C}$. Then:*

1. $\dim C(A) + \dim N(A^H) = m$ *and*
2. $\dim R(A) + \dim N(A) = n$.

In other words, if A has rank $r = r(A)$, then the null space of A has dimension $n - r$ and the left nullspace of A has dimension $m - r$.

Theorem 4 (Fundamental Theorem of Linear Algebra) *Let $A \in M_{mn}(F)$, where $F = \mathbb{R}$ or $\mathbb{C}$. Then (with respect to the Hermitian inner product):*

$$[C(A)]^\perp = N(A^H) \text{ and } [R(A)]^\perp = N(A).$$

Theorem 5 (Summary of Theory of Systems of Linear Equations) *Let $A \in M_{mn}(F)$ (where $F = \mathbb{R}$ or $\mathbb{C}$) have rank r. Then:*

1. *The system $A\mathbf{x} = \mathbf{0}$ always has at least one solution (namely $\mathbf{x} = \mathbf{0}$). There are two possible situations, (a) or (b):*
   (a) $A\mathbf{x} = \mathbf{0}$ *has more than one solution. This is equivalent to all of the following:*
   - $N(A) \neq \{\mathbf{0}\}$
   - $\dim N(A) \geq 1$
   - $r < n$ *(i.e. there is at least one free column)* .

   In this case, for any $\mathbf{b} \in F^n$:
   - $A\mathbf{x} = \mathbf{b}$ *has no solution if and only if $\mathbf{b} \notin C(A)$;*
   - $A\mathbf{x} = \mathbf{b}$ *has infinitely many solutions if and only if $\mathbf{b} \in C(A)$ (in this case the solution set of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x}_p + N(A)$ where $\mathbf{x}_p$ is any particular solution of the system) (this case is assured if $r = m < n$);*
   - $A\mathbf{x} = \mathbf{b}$ *never has exactly one solution.*

   (b) $A\mathbf{x} = \mathbf{0}$ *has exactly one solution (only $\mathbf{x} = \mathbf{0}$). This is equivalent to all of the following:*
   - $N(A) = \{\mathbf{0}\}$
   - $\dim N(A) = 0$
   - $r = n$ *(i.e. all the columns of A are pivot columns).*

   In this case, for any $\mathbf{b} \in F^n$:
   - $A\mathbf{x} = \mathbf{b}$ *has no solution if and only if $\mathbf{b} \notin C(A)$;*
   - $A\mathbf{x} = \mathbf{b}$ *has exactly one solution if and only if $\mathbf{b} \in C(A)$ (this is assured if $r = m = n$; see below);*
   - $A\mathbf{x} = \mathbf{b}$ *never has infinitely many solutions.*

   Notice that if $m < n$ (that is, there are fewer variables than equations), case (b) above is impossible because $r \leq m$ (so r cannot be equal to n).
2. *In the special case where $r = m = n$ (i.e. A is square and has* full rank*), then $C(A) = R(A) = F^n$ and $N(A) = N(A^H) = \{\mathbf{0}\}$. Furthermore, A is invertible and for every $\mathbf{b} \in F^n$, the system $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.*
3. *The system $A\mathbf{x} = \mathbf{b}$ has no solution if and only if $\mathbf{b} \notin C(A)$ if and only if an echelon form of the augmented matrix $(A \,|\, \mathbf{b})$ contains a* false row *of the form $(0\ 0\ \cdots\ 0\ z)$ where $z \neq 0$.*