Definition: Given a field F, a *vector space* over F is a set V together with two operations:
- *addition:* $+ : V \times V \to V$ (i.e. $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} + \mathbf{v}$)
- *scalar multiplication*, $F \times V \to V$ (i.e. $(c, \mathbf{v}) \mapsto c\mathbf{v}$)

such that the following rules (called the "Vector Space Laws") are satisfied:

1. *Addition is closed:* For all $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} \in V$.
2. *Scalar multiplication is closed:* For all $c \in F$ and all $\mathbf{v} \in V$, $c\mathbf{v} \in V$.
3. *Addition is commutative:* For all $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
4. *Addition is associative:* For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
5. *Additive identity element:* There exists an element of V called $\mathbf{0}$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
6. *Additive inverses exist:* For all $\mathbf{v} \in V$, there exists an element $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
7. *Distributivity I:* For all $c \in F$ and all $\mathbf{u}, \mathbf{v} \in V$, $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. *Distributivity II:* For all $c, d \in F$ and all $\mathbf{v} \in V$, $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$.
9. *Scalar multiplication is associative:* For all $c, d \in F$ and all $\mathbf{v} \in V$, $(cd)\mathbf{v} = c(d\mathbf{v})$.
10. *Identity element for scalar multiplication:* $1\mathbf{v} = \mathbf{v}$ where 1 refers to the multiplicative identity element of F.

Language: Given a vector space V, the field F over which it lies is called the *underlying field* of V; elements of the underlying field are called *scalars*; elements of the vector space are called *vectors*.

Notation: Vectors are usually referred to by boldface letters (like $\mathbf{v}$) when typed, and as letters with arrows over them (like $\vec{v}$) when hand-written. However, sometimes we get lazy and just refer to a vector with a letter (like v). The zero scalar is denoted 0; the zero vector is denoted $\mathbf{0}$ or $\vec{0}$.

Theorem: $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ is a vector space over $\mathbb{R}$, where the addition and scalar multiplication are defined coordinate-wise, i.e.

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \quad \text{and} \quad c(x_1, y_1) = (cx_1, cy_1).$$

Proof: Let $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be arbitrary elements of $\mathbb{R}^2$, and let $c, d \in \mathbb{R}$. By definition of $\mathbb{R}^2$, we have $\mathbf{u} = (u_1, u_2)$; $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$. We check the vector space laws one by one:

1. *Addition is closed:* This is obvious by the definition of vector addition.
2. *Scalar multiplication is closed:* This is obvious by the definition of scalar multiplication.
3. *Addition is commutative:* We need to check $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$:

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= (u_1, u_2) + (v_1, v_2) \\
&= (u_1 + v_1, u_2 + v_2) \quad \text{(by defn of + in } \mathbb{R}^2\text{)} \\
&= (v_1 + u_1, v_2 + u_2) \quad \text{(since + is commutative in } \mathbb{R}\text{)} \\
&= (v_1, v_2) + (u_1, u_2) \quad \text{(by defn of + in } \mathbb{R}^2\text{)} \\
&= \mathbf{v} + \mathbf{u}.
\end{aligned}$$

4. *Addition is associative:*

$$\begin{aligned}
\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (u_1, u_2) + ((v_1, v_2) + (w_1, w_2)) \\
&= (u_1, u_2) + (v_1 + w_1, v_2 + w_2) \quad \text{(by defn of + in } \mathbb{R}^2\text{)} \\
&= (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2)) \quad \text{(by defn of + in } \mathbb{R}^2\text{)} \\
&= ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2) \quad \text{(since + is associative in } \mathbb{R}\text{)} \\
&= (u_1 + v_1, u_2 + v_2) + (w_1, w_2) \quad \text{(by defn of + in } \mathbb{R}^2\text{)} \\
&= ((u_1, u_2) + (v_1, v_2)) + (w_1, w_2) \quad \text{(by defn of + in } \mathbb{R}^2\text{)} \\
&= (\mathbf{u} + \mathbf{v}) + \mathbf{w}.
\end{aligned}$$

5. *Additive identity element:* Let $\mathbf{0} = (0, 0)$. Then

$$\begin{aligned}
\mathbf{v} + \mathbf{0} &= (v_1, v_2) + (0, 0) \\
&= (v_1 + 0, v_2 + 0) \quad \text{(by defn of + in } \mathbb{R}^2\text{)} \\
&= (v_1, v_2) \quad \text{(since 0 is additive identity in } \mathbb{R}\text{)} \\
&= \mathbf{v}.
\end{aligned}$$

6. *Additive inverses exist:* Let $-\mathbf{v} = (-v_1, -v_2)$. Then

$$\begin{aligned}
\mathbf{v} + (-\mathbf{v}) &= (v_1, v_2) + (-v_1, -v_2) \\
&= (v_1 + (-v_1), v_2 + (-v_2)) \quad \text{(by defn of + in } \mathbb{R}^2\text{)} \\
&= (0, 0) \quad \text{(by properties of additive inverses in } \mathbb{R}\text{)} \\
&= \mathbf{0} \quad \text{(by defn of 0).}
\end{aligned}$$

7. *Distributivity I:*

$$\begin{aligned}
c(\mathbf{u} + \mathbf{v}) &= c((u_1, u_2) + (v_1, v_2)) \\
&= c(u_1 + v_1, u_2 + v_2) \quad \text{(by defn of + in } \mathbb{R}^2\text{)} \\
&= (c(u_1 + v_1), c(u_2 + v_2)) \quad \text{(by defn of scalar multiplication)} \\
&= (cu_1 + cv_1, cu_2 + cv_2) \quad \text{(by distributivity of } \mathbb{R}\text{)} \\
&= (cu_1, cu_2) + (cv_1, cv_2) \quad \text{(by defn of + in } \mathbb{R}^2\text{)} \\
&= c(u_1, u_2) + c(v_1, v_2) \quad \text{(by defn of scalar multiplication)} \\
&= c\mathbf{u} + c\mathbf{v}.
\end{aligned}$$

8. *Distributivity II:* For all $c, d \in F$ and all $\mathbf{v} \in V$, $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$.

$$\begin{aligned}
(c + d)\mathbf{v}) &= (c + d)(v_1, v_2) \\
&= ((c + d)v_1, (c + d)v_2) \quad \text{(by defn of scalar multiplication)} \\
&= (cv_1 + dv_1, cv_2 + dv_2) \quad \text{(by distributivity of } \mathbb{R}\text{)} \\
&= (cv_1, cv_2) + (dv_1, dv_2) \quad \text{(by defn of + in } \mathbb{R}^2\text{)} \\
&= c(v_1, v_2) + d(v_1, v_2) \quad \text{(by defn of scalar multiplication)} \\
&= c\mathbf{v} + d\mathbf{v}.
\end{aligned}$$

9. *Scalar multiplication is associative:* For all $c, d \in F$ and all $\mathbf{v} \in V$, $(cd)\mathbf{v} = c(d\mathbf{v})$.

$$\begin{aligned}
(cd)\mathbf{v} &= (cd)(v_1, v_2) \\
&= ((cd)v_1, (cd)v_2) \quad \text{(by defn of scalar multiplication)} \\
&= (c(dv_1), c(dv_2)) \quad \text{(by associativity of } \cdot \text{ in } \mathbb{R}\text{)} \\
&= c(dv_1, dv_2) \quad \text{(by defn of scalar multiplication)} \\
&= c(d(v_1, v_2)) \quad \text{(by defn of scalar multiplication)} \\
&= c(d\mathbf{v}).
\end{aligned}$$

10. *Identity element for scalar multiplication:*

$$\begin{aligned}
1\mathbf{v} &= 1(v_1, v_2) \\
&= (1v_1, 1v_2) \quad \text{(by defn of scalar multiplication)} \\
&= (v_1, v_2) \quad \text{(since 1 is mult. identity in } \mathbb{R}\text{)} \\
&= \mathbf{v}.
\end{aligned}$$

Since all the laws hold, $\mathbb{R}^2$ is indeed a vector space over $\mathbb{R}$. $\square$

1. *"Traditional" vectors:* Given any field F and any $n \in \mathbf{N}$, $F^n = \{(x_1, ..., x_n) : x_j \in F \,\forall\, j\}$ is a vector space over F, where the addition and scalar multiplication are defined coordinate-wise. (The proof of this is the same as the proof that $\mathbb{R}^2$ is a vector space, essentially.)

2. *Fields:* Given any field F, F is a vector space over itself (where the addition and scalar multiplication are the field operations). In particular, $F = F^1$.

3. *Zero vector spaces:* Given any field F, $\{\mathbf{0}\}$ is a vector space over F. In particular, $F^0 = \{\mathbf{0}\}$.

4. $\mathbb{C}$ is a vector space over $\mathbb{R}$.

5. $\mathbb{C}$ is also a vector space over $\mathbb{C}$. But, the vector spaces $\mathbb{C}$ (over $\mathbb{R}$) and $\mathbb{C}$ (over $\mathbb{C}$) are two different vector spaces.

6. *Matrix spaces:* Given any field F, the set of $m \times n$ matrices (this means m rows and n columns) with elements in F, denoted $M_{mn}(F)$, is a vector space over F where the addition and scalar multiplication are defined entry-wise. (Notation: the set of square $n \times n$ matrices with entries in F is denoted $M_n(F)$ rather than $M_{nn}(F)$.)

7. *Sequence spaces:* In these examples, the addition and scalar multiplication are defined term-by-term, i.e.

   $$(x_1, x_2, x_3, ...) + (y_1, y_2, y_3, ...) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, ...) \quad \text{and} \quad c(x_1, x_2, ...) = (cx_1, cx_2, ...).$$

   (a) Given any field F, the set $F^{\mathbb{N}}$ of infinite sequences of elements of F is a vector space over F (where the addition and scalar multiplication are done term-by-term).
   (b) The set F^{∞} of infinite sequences where all but finitely many elements of the sequence are 0 also forms a vector space over F, with the same operations. Here, some care needs to be taken to ensure that addition is closed.
   (c) The set of convergent sequences of real numbers forms a vector space over $\mathbb{R}$.

8. *Function spaces:* For all these spaces of functions, the addition is described by $(f + g)(x) = f(x) + g(x)$ and the scalar multiplication is $(cf)(x) = c \cdot f(x)$

   (a) *Polynomials:* Given any field F, the set $F[x]$ of polynomials whose coefficients are in F is a vector space over F, where the addition and scalar multiplication come from usual addition and scalar multiplication of functions.
   (b) *Continuous functions:* The set $C(\mathbb{R}, \mathbb{R})$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ is a vector space over $\mathbb{R}$.
   (c) *Differentiable functions:* The set of differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ is a vector space over $\mathbb{R}$ is a vector space over $\mathbb{R}$.
   (d) *Analytic functions:* The set of analytic functions from $\mathbb{R}$ to $\mathbb{R}$ is a vector space over $\mathbb{R}$ is a vector space over $\mathbb{R}$ (a function is *analytic* if it can be written as a power series which converges everywhere).
   (e) *L^2 functions:* The set of "measurable" functions from $\mathbb{R}$ to $\mathbb{R}$ satsfying $\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty$ is a vector space over $\mathbb{R}$ (loosely speaking, a function is *measurable* if for every a and b in $\mathbb{R}$, $\int_a^b f(x) \, dx$ exists... all continuous and piecewise-continuous functions are measurable).