1. Complete each of the following to define the italicized term:
   (a) Given a vector space V over field F, two vectors $\mathbf{v}$ and $\mathbf{w}$ are *parallel* if...
   (b) Given a vector space V over field F, a collection of vectors $\mathbf{v}_1, ..., \mathbf{v}_n$ are called *linearly independent* if...
   (c) Given a finite-dimensional vector space V over field F, the *dimension* of V is...

2. Find parametric equations of the plane in $\mathbb{R}^3$ containing the point $(1, 1, 2)$ and the line l which has parametric equations

$$\begin{cases} x_1 = 2 - t \\ x_2 = 4 + t \\ x_3 = -1 + 3t \end{cases}.$$

3. Let $\mathbf{v} = (3 + i, 4)$ and $\mathbf{w} = (1 - i, i)$.
   (a) Calculate the Hermitian inner product of $\mathbf{v}$ and $\mathbf{w}$. Simplify your answer.
   (b) Find the distance (with respect to Hermitian innner product) between $\mathbf{v}$ and $\mathbf{w}$. Simplify your answer.

4. Let z be an arbitrary complex number.
   (a) Prove $\text{Re}(iz) = -\text{Im}(z)$.
   (b) Prove that z and its conjugate have the same modulus.

5. Suppose $\mathbf{v}$ and $\mathbf{w}$ are two linearly independent vectors in some vector space V over $\mathbb{R}$. Prove that $\{\mathbf{v} - \mathbf{w}, 5\mathbf{v} + 2\mathbf{w}\}$ also forms a linearly independent set.

6. In each of the following situations, you are given a vector space V over the field $\mathbb{R}$ and a subset $W \subseteq V$. For each situation: (i) determine whether or not W is a subspace of V; (ii) if W is a subspace of V, give its dimension; (iii) if W is a finite-dimensional subspace, give a basis of W. Justify your answers.
   (a) $V = \mathbb{R}^2$; $W = \{\mathbf{v} \in V :< \mathbf{v}, \mathbf{v} >= 1\}$
   (b) $V = \mathbb{R}^3$; $W = \{(x_1, x_2, x_3) \in V : x_1 + x_2 = x_3\}$
   (c) $V = C(\mathbb{R}, \mathbb{R})$; $W = V$
   (d) $V = \mathbb{R}^4$; $W = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right)$
   (e) $V = \mathbb{R}^3$; $W = \{(r + s - t, 3r - 3t, 5s) : r, s, t \in \mathbb{R}\}$.

7. Let V be a vector space (over $\mathbb{R}$ or $\mathbb{C}$) with norm $||\cdot||$ coming from some inner product $<, >$. Prove that for all $\mathbf{x}, \mathbf{y} \in V$,

$$||\mathbf{x} - \mathbf{y}|| \, ||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}||^2 + ||\mathbf{y}||^2.$$

1. (a) Given a vector space V over field F, two vectors $\mathbf{v}$ and $\mathbf{w}$ are *parallel* if **one is a scalar multiple of the other.**

   (b) Given a vector space V over field F, a collection of vectors $\mathbf{v}_1, ..., \mathbf{v}_n$ are called *linearly independent* if **whenever $c_1\mathbf{v}_1 + ... + c_n\mathbf{v}_n = \mathbf{0}$ for scalars $c_1, ..., c_n$, it must be the case that $c_1 = c_2 = ... = c_n = 0$.**

   (c) Given a finite-dimensional vector space V over field F, the *dimension* of V is **the number of elements in any basis of V.**

2. By setting $t = 0$ in the given parametric equations for l, we see that the point $(2, 4, -1)$ is in the plane. So two vectors in the plane are $\mathbf{v} = (1, 1, 2) - (2, 4, -1) = (-1, -3, 3)$ and the direction vector for the line, i.e. $\mathbf{w} = (-1, 1, 3)$; it is clear that $\mathbf{v}$ and $\mathbf{w}$ are not parallel. We take the point $\mathbf{p} = (1, 1, 2)$ in the plane and construct parametric equations for the plane coming from the vector equation $\mathbf{x} = \mathbf{p} + s\mathbf{v} + t\mathbf{w}$ which are

$$\begin{cases} x_1 = 1 - s - t \\ x_2 = 1 - 3s + t \\ x_3 = 2 + 3s + 3t \end{cases}.$$

   Of course these are not the only possible parametric equations for the plane.

3. Let $\mathbf{v} = (3 + i, 4)$ and $\mathbf{w} = (1 - i, i)$.

   (a) This is a direct calculation:

$$\begin{aligned} < (3+i, 4), (1-i, i) > &= (3+i)\overline{(1-i)} + 4\bar{i} \\ &= (3+i)(1+i) - 4i \\ &= 3 + i + 3i - 1 - 4i = 2. \end{aligned}$$

   (b) The distance between the vectors is

$$\begin{aligned} ||\mathbf{v} - \mathbf{w}|| &= ||(2 + 2i, 4 - i)|| \\ &= \sqrt{< (2+2i, 4-i), (2+2i, 4-i) >} \\ &= \sqrt{(2+2i)(2-2i) + (4-i)(4+i)} \\ &= \sqrt{2^2 + 2^2 + 4^2 + 1^2} \\ &= 5. \end{aligned}$$

4. (a) Let $z = a + ib$, then $-\mathrm{Im}(z) = -b$. Also we see

$$\mathrm{Re}(iz) = \mathrm{Re}(ai - b) = -b = -\mathrm{Im}(z).$$

   (b) Again let $z = a + ib$; we have

$$|\overline{z}| = |a - ib| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|.$$

   Alternate solution: $|z|^2 = z \cdot \overline{z} = \overline{z} \cdot z = \overline{z} \cdot \overline{\overline{z}} = |\overline{z}|^2$; take the square root of both sides.

5. Suppose there are two scalars c_1 and c_2 such that

$$c_1(\mathbf{v} - \mathbf{w}) + c_2(5\mathbf{v} + 2\mathbf{w}) = \mathbf{0}.$$

By combining like terms, we see that this implies

$$(c_1 + 5c_2)\mathbf{v} + (-c_1 + 2c_2)\mathbf{w} = \mathbf{0};$$

since $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, this means that

$$c_1 + 5c_2 = 0 \quad \text{and} \quad -c_1 + 2c_2 = 0.$$

Adding these two equations, we see that $7c_2 = 0$ so $c_2 = 0$ and therefore $c_1 = 0$ as well. This shows $\{\mathbf{v} - \mathbf{w}, 5\mathbf{v} + 2\mathbf{w}\}$ to be linearly independent.

6. (a) Observe $\mathbf{0} \notin W$ since $< \mathbf{0}, \mathbf{0} >= 0 \neq 1$ so W cannot be a subspace. Alternatively, let $\mathbf{w} = (1, 0)$ which is in W. But $2\mathbf{w} = (2, 0)$ is not in W because $< (2, 0), (2, 0) >= 4$. So W is not closed under scalar multiplication and hence is not a subspace.

   (b) Subtract x_3 from both sides of the equation to see that

$$W = (x_1, x_2, x_3) : x_1 + x_2 - x_3 = 0;$$

   by a homework problem we see that this is a plane which passes through the origin and hence is a subspace of dimension 2. So any two linearly independent (i.e. non-parallel) vectors in W form a basis; for example we can set

$$\mathcal{B} = \{(1, 0, 1), (0, 1, 1)\}.$$

   (c) Every vector space is a subspace of itself, so W is a subspace of V. It is infinite-dimensional (this was proven in class).

   (d) The span of any collection of vectors is always a subspace, so W is a subspace. Writing the given vectors which span W as $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5$, we see that $\mathbf{w}_3 = -\mathbf{w}_2$, $\mathbf{w}_4 = \mathbf{w}_1$ and $\mathbf{w}_5 = \frac{1}{2}\mathbf{w}_1 + \frac{1}{2}\mathbf{w}_2$ so we can delete $\mathbf{w}_3$, $\mathbf{w}_4$ and $\mathbf{w}_5$ from the set without changing the span, that is

$$W = \text{Span}((1, 1, 1, 1), (-1, 1, -1, 1)).$$

   The two vectors $(1, 1, 1, 1)$ and $(-1, 1, -1, 1)$ are obviously not parallel, so they are linearly independent and since they span W they form a basis; W therefore has dimension 2.

   (e) By separating like terms, we see that

$$W = \{r(1, 3, 0) + s(1, 0, 5) + t(-1, -3, 0) : r, s, t \in \mathbb{R}\},$$

   i.e. $W = \text{Span}((1, 3, 0), (1, 0, 5), (-1, -3, 0))$ and is therefore a subspace. The third of these vectors is (-1) times the first vector, so it can be removed from the set without changing the span, i.e.

$$W = \text{Span}((1, 3, 0), (1, 0, 5)).$$

   Clearly the vectors $(1, 3, 0)$ and $(1, 0, 5)$ are nonparallel, so they are linearly independent and therefore form a basis. In particular, $\dim(W) = 2$.

7. We start by expanding the square of the left-hand side:

$$\begin{aligned}
||\mathbf{x}-\mathbf{y}||^2\,||\mathbf{x}+\mathbf{y}||^2 &= <(\mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{y})><(\mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y})> \\
&= [<\mathbf{x},\mathbf{x}> - <\mathbf{y},\mathbf{x}> - <\mathbf{x}.\mathbf{y}> + <\mathbf{y},\mathbf{y}>]\cdot \\
&\quad\ [<\mathbf{x},\mathbf{x}> + <\mathbf{x},\mathbf{y}> + <\mathbf{y},\mathbf{x}> + <\mathbf{y},\mathbf{y}>] \\
&= \left[\left(||\mathbf{x}||^2+||\mathbf{y}||^2\right) - 2\,\mathrm{Re} <\mathbf{x},\mathbf{y}>\right]\cdot \\
&\quad\ \left[\left(||\mathbf{x}||^2+||\mathbf{y}||^2\right) + 2\,\mathrm{Re} <\mathbf{x},\mathbf{y}>\right] \\
&= \left(||\mathbf{x}||^2+||\mathbf{y}||^2\right)^2 - (2\,\mathrm{Re} <\mathbf{x},\mathbf{y}>)^2 \\
&\leq \left(||\mathbf{x}||^2+||\mathbf{y}||^2\right)^2
\end{aligned}$$

Take the square root of both sides of this inequality to obtain the result.