1. Give a precise statement of each of the following theorems, including all appropriate hypotheses:
   (a) Cauchy-Schwarz Inequality
   (b) Orthogonal Decomposition Theorem
   (c) Rank-Nullty Theorem

2. Consider the basis $\mathcal{B} = \{(2,-5),(1,-3)\}$ of $\mathbb{R}^2$. Suppose that the coordinates of a vector $\mathbf{x}$ in $\mathbb{R}^2$ with respect to $\mathcal{B}$ are $[\mathbf{x}]_{\mathcal{B}} = (x,y)$. In terms of $x$ and $y$, find the coordinates of $\mathbf{x}$ with respect to the standard basis, and with respect to $\mathcal{B}' = \{(-1,2),(3,-7)\}$.

3. Consider the system $A\mathbf{x} = \mathbf{b}$ whose equations are

$$(*) \quad \left\{ \begin{array}{rcrcrcrcrcr} 2x_1 & - & 2x_2 & + & 2x_3 & + & 6x_4 & - & 2x_5 & = & 4 \\ & & 3x_2 & + & 9x_3 & + & 6x_4 & + & 4x_5 & = & 11 \\ -2x_1 & & & - & 8x_3 & - & 10x_4 & + & x_5 & = & -2 \end{array} \right. ;$$

the augmented matrix $(A\,\mathbf{b})$ corresponding to this system has reduced row-echelon form

$$\begin{pmatrix} 1 & 0 & 4 & 5 & 0 & \frac{19}{3} \\ 0 & 1 & 3 & 2 & 0 & \frac{19}{3} \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

   (a) Describe the solution set of the system.
   (b) Give a basis for the column space of $A$.
   (c) Let $W \subseteq \mathbb{R}^5$ be the subspace defined by

$$W = \text{Span}\left((2,-2,2,6,-2),(0,3,9,6,4),(-2,0,-8,-10,1)\right).$$

   Give a basis for $W^\perp$.
   (d) Let $\mathbf{a}_j$ represent the $j^{th}$ column of $A$. Are the following sets of vectors linearly independent or linearly dependent?
      i. $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, $\mathbf{a}_5$
      ii. $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_5$
      iii. $\mathbf{a}_1$, $\mathbf{a}_3$, $\mathbf{a}_5$
      iv. $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_5$, $\mathbf{b}$
      v. $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{b}$

4. Consider the matrix
$$A = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix}.$$
Find the inverse of $A$, and use your answer to find the solution set of the system $A\mathbf{x} = (0, 1, 1)$.

5. For each of the following linear transformations $T : V_1 \to V_2$: (i) give the standard matrix of the linear transformation; (ii) determine whether or not $T$ is injective; and (iii) determine whether or not $T$ is surjective.
   (a) $V_1 = V_2 = \mathbb{P}_3$; $T(f) = f'' - f$.
   (b) $V_1 = \mathbb{R}^2$; $V_2 = \mathbb{R}^4$; $T$ is the linear transformation satisfying $T(1, 0) = (1, 2, -1, 0)$ and $T(1, 1) = (2, 6, -3, -2)$.

6. (a) Let $A$ be a $6 \times 3$ real matrix of rank 3, and let $B$ be a $3 \times 3$ matrix of rank 3. Prove that the matrix $AB$ has rank 3.
   (b) Let $V$ be a vector space with inner product $<, >$; let $\mathbf{w}$ be a nonzero vector in $V$. Prove that for any $\mathbf{x} \in V$,
$$\text{proj}_{\mathbf{w}} (\text{proj}_{\mathbf{w}} \mathbf{x}) = \text{proj}_{\mathbf{w}} \mathbf{x}.$$

1. (a) **Cauchy-Schwarz Inequality:** Let $V$ be a vector space with inner product $<,>$ and associated norm $||\cdot||$. Then for all $\mathbf{v}, \mathbf{w} \in V$, $|<\mathbf{v}, \mathbf{w}>| \leq ||\mathbf{v}||\,||\mathbf{w}||$.

   (b) **Orthogonal Decomposition Theorem:** Let $V$ be a vector space with inner product $<,>$ and let $W$ be a subspace of $V$ with $\dim W < \infty$. Then for any vector $\mathbf{v} \in V$, we can write $\mathbf{v} = \mathbf{v}^W + \mathbf{v}^\perp$ where $\mathbf{v}^W \in W$ and $\mathbf{v}^\perp \in W^\perp$.

   (c) **Rank-Nullty Theorem:** Let $F = \mathbb{R}$ or $\mathbb{C}$. Then for any $A \in M_{m\times n}(F)$, $\dim R(A) + \dim N(A) = n$ and $\dim C(A) + \dim N(A^H) = m$.

2. We see from the given information that $\mathbf{x} = x(2,-5) + y(1,-3) = (2x + y, -5x - 3y)$. So the coordinates of $\mathbf{x}$ with respect to the standard basis are $\mathbf{x} = [\mathbf{x}]_{\{\mathbf{e}_1,\mathbf{e}_2\}} = (2x+y, -5x-3y)$.

   The coordinates $[\mathbf{x}]_{\mathcal{B}'}$ are given by the vector $P_{\mathcal{B}'}^{-1}(2x+y, -5x-3y)$ where

$$P_{\mathcal{B}'} = \begin{pmatrix} -1 & 3 \\ 2 & -7 \end{pmatrix}; \text{ therefore } = P_{\mathcal{B}'}^{-1} \begin{pmatrix} -7 & -3 \\ -2 & -1 \end{pmatrix}$$

   and $[\mathbf{x}]_{\mathcal{B}'} = P_{\mathcal{B}'}^{-1}(2x+y, -5x-3y) = (-7(2x+y) - 3(-5x-3y), -2(2x+y) - (-5x-3y)) = (x+2y, x+y)$.

3. (a) The reduced row-echelon form translates to the system of equations

$$\begin{cases} x_1 \quad\quad +4x_3 + 5x_4 \quad = \frac{19}{3} \\ \quad x_2 + 3x_3 + 2x_4 \quad = \frac{19}{3} \\ \quad\quad\quad\quad\quad\quad x_5 = -2 \end{cases};$$

   solving for the pivot variables $x_1$ and $x_2$ in terms of the free variables $x_3$ and $x_4$, we obtain the solution set

$$\left\{\left(\frac{19}{3} - 4x_3 - 5x_4, \frac{19}{3} - 3x_3 - 2x_4, x_3, x_4, -2\right) : x_3, x_4 \in \mathbb{R}\right\}$$

$$\left\{\left(\frac{19}{3}, \frac{19}{3}, 0, 0, -2\right) + x_3(-4,-3,1,0,0) + x_4(-5,-2,0,1,0) : x_3, x_4 \in \mathbb{R}\right\}$$

   (b) A basis for $C(A)$ is the pivot columns of $A$, i.e. the first, second and fifth columns of $A$:

$$\{(2,0,-2), (-2,3,0), (-2,4,1)\}$$

   (c) Observe that $W = R(A)$; by the fundamental theorem of linear algebra, $W^\perp = N(A)$. A basis for $N(A)$ is given by looking at the solution to part (a); a basis is

$$\{(-4,-3,1,0,0), (-5,-2,0,1,0)\}$$

(d) i. $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$ are linearly dependent because this is a list of five vectors in the three-dimensional vector space $\mathbb{R}^3$.

ii. $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5$ are linearly independent because they are the pivot columns of $A$.

iii. $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5$ are linearly independent because the corresponding columns of the reduced row-echelon form of $A$ are linearly independent.

iv. $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5, \mathbf{b}$ are linearly dependent because $\mathbf{b} \in C(A) = \mathrm{Span}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5)$. We know $\mathbf{b} \in C(A)$ since there is at least one solution of the system $A\mathbf{x} = \mathbf{b}$.

v. $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}$ are linearly independent because the last column of the rref form of the system is not a linear combination of the first two columns in the rref form; hence $\mathbf{b}$ is not a linear combination of the first two columns of $A$.

4. Use the Gauss-Jordan method:

$$(A \,|\, I) \;=\!=\; \left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{1\cdot R_1 + R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-1\cdot R_2 + R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{1/2\cdot R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & .5 & -.5 & .5 \end{array}\right)$$

$$\xrightarrow{-1\cdot R_3 + R_2,\, -2\cdot R_3 + R_1} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -.5 & 1.5 & -.5 \\ 0 & 0 & 1 & .5 & -.5 & .5 \end{array}\right)$$

$$\xrightarrow{2\cdot R_2 + R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & -2 \\ 0 & 1 & 0 & -.5 & 1.5 & -.5 \\ 0 & 0 & 1 & .5 & -.5 & .5 \end{array}\right) = (I \,|\, A^{-1}).$$

So $A^{-1} = \begin{pmatrix} -1 & 4 & -2 \\ -.5 & 1.5 & -.5 \\ .5 & -.5 & .5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 & 8 & -4 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix}$. Now, the solution set to $A\mathbf{x} = (0,1,1)$ is $\mathbf{x} = A^{-1}(0,1,1) = (2,1,0)$.

5. (a) The standard basis of $\mathbb{P}_3$ is $\{1, x, x^2, x^3\}$; we see

$$T(1) = -1, T(x) = -x, T(x^2) = 2 - x^2, T(x^3) = 6x - x^3$$

so the standard matrix of $T$ is

$$A = \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 6 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This matrix is a square matrix which is already in row-echelon form, and has as many pivots as rows. So $A$ is invertible,and therefore $T$ is invertible, hence bijective (both injective ans surjective).

(b) First, $T(0,1) = T(1,1) - T(1,0) = (1,4,-2,-2)$. So the standard matrix of $T$ is

$$A = (T(\mathbf{e}_1)\; T(\mathbf{e}_2)) = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ -1 & -2 \\ 0 & -2 \end{pmatrix}.$$

$T$ cannot be surjective because $2 = \dim V_1 < \dim V_2 = 4$. Since the columns of $A$ are not parallel, they are linearly independent so $\dim C(A) = 2$. By rank-nullty, $\dim N(A) = 2 - 2 = 0$ so $N(A) = \ker(T) = \{\mathbf{0}\}$ and $T$ is therefore injective.

6. (a) Consider the linear transformations $T_1 : F^3 \to F^3$ and $T_2 : F^3 \to F^6$ defined by $T_1(\mathbf{x}) = B\mathbf{x}$ and $T_2(\mathbf{x}) = A\mathbf{x}$. Since $A$ and $B$ have full column rank, both $T_1$ and $T_2$ are injective (by a theorem in class). Since $T_1$ and $T_2$ are both injective, the by another theorem in class, the composition $T_2T_1$ is injective, so the rank of the standard matrix of $T_2T_1$ (which is $AB$) must be equal to the number of columns of that matrix, which is 3.

(b) Start with the left-hand side and write it out using the formula which gives projection onto $\mathbf{w}$:

$$\begin{aligned}
\mathrm{proj}_{\mathbf{w}}\left(\mathrm{proj}_{\mathbf{w}}\mathbf{x}\right) &= \frac{\langle \mathrm{proj}_{\mathbf{w}}\mathbf{x}, \mathbf{w}\rangle}{<\mathbf{w},\mathbf{w}>}\mathbf{w} \\
&= \frac{\left\langle \frac{<\mathbf{x},\mathbf{w}>}{<\mathbf{w},\mathbf{w}>}\mathbf{w}, \mathbf{w}\right\rangle}{<\mathbf{w},\mathbf{w}>}\mathbf{w} \\
&= \frac{\frac{<\mathbf{x},\mathbf{w}>}{<\mathbf{w},\mathbf{w}>} <\mathbf{w},\mathbf{w}>}{<\mathbf{w},\mathbf{w}>}\mathbf{w} \\
&= \frac{<\mathbf{x},\mathbf{w}>}{<\mathbf{w},\mathbf{w}>}\mathbf{w} \\
&= \mathrm{proj}_{\mathbf{w}}\mathbf{x}.
\end{aligned}$$