1. Classify the following statements as true or false.

   (a) There is a system of linear equations in three variables that has exactly three solutions.

   (b) If the set $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ is linearly independent and $\mathbf{v}_{p+1}$ is not an element of $\text{Span}(\mathbf{v}_1, ..., \mathbf{v}_p)$, then the set $\{\mathbf{v}_1, ..., \mathbf{v}_p, \mathbf{v}_{p+1}\}$ is linearly independent.

   (c) The standard matrix of a linear transformation $T : \mathbb{R}^p \to \mathbb{R}^q$ has size $p \times q$.

   (d) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 2$ is a linear transformation.

   (e) The set $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 0\}$ is a subspace of $\mathbb{R}^2$.

   (f) A subset H of a vector space V is a subspace of V if H is closed under addition and scalar multiplication.

   (g) The vector spaces $\mathbb{P}_2$ and $\mathbb{R}^2$ are isomorphic.

   (h) If B is an echelon form of a matrix A, then the set of pivot columns of B is a basis for the column space of A.

   (i) The set $\{p(t) \in \mathbb{P}_2 : p(1) = 0\}$ is a subspace of $\mathbb{P}_2$.

   (j) Every linear system of equations with a free variable has infinitely many solutions.

2. (a) Prove that for any complex numbers z_1 and z_2, $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$.

   (b) Prove that for any complex numbers $z_1, z_2, ..., z_n$, $\overline{z_1 z_2 \ldots z_n} = \overline{z_1} \cdot \ldots \cdot \overline{z_n}$.

3. Let $V = \mathbb{C}^2$, endowed with the Hermitian inner product $\langle \, , \rangle$ and associated norm $|| \cdot ||$.

   (a) Suppose $\mathbf{v}, \mathbf{w}$ and $\mathbf{x}$ are vectors in V such that $\mathbf{v}$ and $\mathbf{w}$ have norm 2, $\mathbf{x}$ has norm 3, and the vectors have the following inner products:

   $$\langle \mathbf{v}, \mathbf{w} \rangle = 2 - i \quad \langle \mathbf{v}, \mathbf{x} \rangle = 1 + 3i \quad \langle \mathbf{w}, \mathbf{x} \rangle = 2i.$$

      i. Evaluate $\langle \mathbf{w}, 2\mathbf{v} + i\mathbf{x} \rangle$.

      ii. Find the distance between $\mathbf{v}$ and $\mathbf{w}$.

   (b) Prove that for any $\mathbf{v}, \mathbf{w} \in V$ and any matrix $A \in M_{2\times2}(\mathbb{C})$,

   $$\langle A\mathbf{v}, \mathbf{w} \rangle = \left\langle \mathbf{v}, A^H \mathbf{w} \right\rangle.$$

4. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose standard matrix is

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -1 & 1 & 0 \\ 0 & -2 & 4 \end{pmatrix}.$$

Let V be the set of 3×3 real matrices B such that if $S : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation with standard matrix B, then $T \circ S = 0$.

   (a) Describe all matrices B which belong to V.

   (b) Prove that V is a subspace of $M_{3\times3}(\mathbb{R})$.

(c) Find a basis for V.

5. Consider the linear system $A\mathbf{x} = \mathbf{b}$ where the augmented matrix $(A\,|\,\mathbf{b})$ of this system is
$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{array}\right).$$
   (a) Find the solution set of the system $A\mathbf{x} = \mathbf{b}$.
   (b) Is there a vector $\mathbf{c} \in \mathbb{R}^4$ for which the system $A\mathbf{x} = \mathbf{c}$ has no solution? Why or why not?
   (c) Is there a vector $\mathbf{c} \in \mathbb{R}^4$ for which the system $A\mathbf{x} = \mathbf{c}$ has exactly one solution? Why or why not?

6. Consider the following three vectors in $\mathbb{R}^4$:
$$\mathbf{u} = (1, 1, 0, 0) \quad \mathbf{v} = (1, -1, 2, 0) \quad \mathbf{w} = (0, 1, 0, -2).$$
   (a) Prove that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set.
   (b) Is there a vector $\mathbf{a} \in \mathbb{R}^4$ so that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{a}\}$ is a basis of $\mathbb{R}^4$? If so, find such an $\mathbf{a}$. If not, explain why not.

7. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^n$ are linear transformations and that $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are three bases of $\mathbb{R}^n$. Let P be the change of basis matrix from $\mathcal{A}$ to $\mathcal{B}$, and let Q be the change of basis matrix from $\mathcal{A}$ to $\mathcal{C}$.
   (a) In terms of P and Q, what is the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$?
   (b) Let A be the matrix of T with respect to $\mathcal{A}$ and let B be the matrix of S with respect to $\mathcal{B}$. In terms of A, B, P and Q, what is the matrix of the linear transformation $TST : \mathbb{R}^n \to \mathbb{R}^n$ with respect to the basis $\mathcal{C}$?

8. Consider the subspace $W \subseteq \mathbb{R}^3$ defined by
$$W = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + 3z = 0\}.$$
   (a) What is a one-word geometric description of W?
   (b) Find the dimension of W; give a basis of W.
   (c) Find a matrix A such that $W = N(A)$.
   (d) Find a matrix B such that $W = C(B)$.
   (e) Find the dimension of $W^\perp$; find a basis for $W^\perp$.
   (f) Find the intersection of W with the line l in $\mathbb{R}^3$ whose parametric equations are
$$\begin{cases} x = 3 + t \\ y = 2 - t \\ z = 1 + 2t \end{cases}$$

1. (a) FALSE; every linear system has either zero, one or infinitely many solutions.
   (b) TRUE; by a basic result about linear independence.
   (c) FALSE; the standard matrix of T is $q \times p$, not $p \times q$.
   (d) FALSE; $f(0) = 2 \neq 0$ so f is not linear.
   (e) FALSE; $(1, 1)$ is in the set and $(1, -1)$ is in the set, but $(1, 1) + (1, -1) = (2, 0)$ is not in the set. So the set is not closed under addition and is therefore not a subspace.
   (f) FALSE; H must also be nonempty.
   (g) FALSE; $\mathbb{P}_2$ has dimension 3 but $\mathbb{R}^2$ has dimension 2 so these vector spaces cannot be isomorphic.
   (h) FALSE; the pivot columns of A (but not necessarily B) form a basis for $C(A)$.
   (i) TRUE; this set is the kernel of the linear transformation $T : \mathbb{P}_2 \to \mathbb{R}$ defined by $T(p) = p(1)$ (this map is linear because it is an evaluation); kernels of linear transformations are always subspaces.
   (j) FALSE; the system may have no solution (a simple example is the system of one equation $0x + 0y = 1$).

2. (a) Let $z_1 = a + ib$ and $z_2 = c + id$ so that $\overline{z_1} = a - ib$ and $\overline{z_2} = c - id$. Also, we see $z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i(ac + bd)$ so $\overline{z_1 z_2} = (ac - bd) - i(ac + bd)$. Finally,

$$\overline{z_1} \cdot \overline{z_2} = (a - ib)(c - id) = (ac - bd) - i(ac + bd) = \overline{z_1 z_2}$$

   as desired.

   (b) Proceed by induction on n; the base case $n = 2$ is part (a) of this question. Assume

$$\overline{z_1 ... z_k} = \overline{z_1} \cdot ... \cdot \overline{z_k};$$

   then by multiplying both sides by $\overline{z_{k+1}}$ we get

$$\overline{z_1 ... z_k} \cdot \overline{z_{k+1}} = \overline{z_1} \cdot ... \cdot \overline{z_k} \cdot \overline{z_{k+1}}.$$

   By the base case, the left-hand side is equal to $\overline{z_1 \cdot ... \cdot z_{k+1}}$, so by induction we are done.

3. (a) i. Using the elementary properties of the inner product, we see

$$\begin{aligned} \langle \mathbf{w}, 2\mathbf{v} + i\mathbf{x} \rangle &= \overline{2} \langle \mathbf{w}, \mathbf{v} \rangle + \overline{i} \langle \mathbf{w}, \mathbf{x} \rangle \\ &= 2\overline{\langle \mathbf{v}, \mathbf{w} \rangle} + (-i)(2i) \\ &= 2(2 + i) + 2 \\ &= 6 + 2i. \end{aligned}$$

ii. Using the definition of distance, we see

$$\begin{aligned}\text{dist}(\mathbf{v},\mathbf{w}) = ||\mathbf{v}-\mathbf{w}|| &= \sqrt{\langle \mathbf{v}-\mathbf{w}, \mathbf{v}-\mathbf{w}\rangle} \\ &= \sqrt{\langle \mathbf{v},\mathbf{v}\rangle - \langle \mathbf{v},\mathbf{w}\rangle - \langle \mathbf{w},\mathbf{v}\rangle + \langle \mathbf{w},\mathbf{w}\rangle} \\ &= \sqrt{||\mathbf{v}||^2 - (2-i) - \overline{(2-i)} + ||\mathbf{w}||^2} \\ &= \sqrt{4-2+i-2-i+4} \\ &= \sqrt{4} = 2.\end{aligned}$$

(b) Let $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d, v_1, v_2, w_1 and $w_2 \in \mathbb{C}$. Then the left-hand side of the equation we are to prove is:

$$\begin{aligned}\langle A\mathbf{v},\mathbf{w}\rangle &= \left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right\rangle \\ &= \left\langle \begin{pmatrix} av_1+bv_2 \\ cv_1+dv_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right\rangle \\ &= (av_1+bv_2)\overline{w_1} + (cv_1+dv_2)\overline{w_2} \\ &= av_1\overline{w_1} + bv_2\overline{w_1} + cv_1\overline{w_2} + dv_2\overline{w_2}.\end{aligned}$$

The right-hand side of the equation is

$$\begin{aligned}\langle \mathbf{v}, A^H\mathbf{w}\rangle &= \left\langle \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right\rangle \\ &= \left\langle \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} \overline{a}w_1+\overline{c}w_2 \\ \overline{b}w_1+\overline{d}w_2 \end{pmatrix}\right\rangle \\ &= v_1\overline{(\overline{a}w_1+\overline{c}w_2)} + v_2\overline{(\overline{b}w_1+\overline{d}w_2)} \\ &= v_1(a\overline{w_1}+c\overline{w_2}) + v_2(b\overline{w_1}+d\overline{w_2} \\ &= av_1\overline{w_1} + cv_1\overline{w_2} + bv_2\overline{w_1} + dv_2\overline{w_2}\end{aligned}$$

which is the same as the left-hand side.

4. (a) Observe that $TS = 0$ if and only if the image of S is a subset of the kernel of T. Translating this statement into one about matrices, we see that $B \in V$ if and only if $C(B) \subseteq N(A)$. So begin by finding $N(A)$; perform row reductions on A to transform it into its echelon form. The sequence of row operations is (1) add Row 1 to Row 2, (2) add 2 times Row 2 to Row 3. We obtain

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

so $A\mathbf{x} = \mathbf{0}$ if and only if $x_1 = 2x_3, x_2 = 2x_3$ which holds when $\mathbf{x} \in \{x_3(2,2,1) : x_3 \in \mathbb{R}\}$. So $N(A)$ is the span of the single vector $(2,2,1)$.

Returning to the problem, $B \in V$ if and only if $C(B) \subseteq N(A) = \text{Span}(2,2,1)$. This occurs if and only if the columns of B are all multiples of $(2,2,1)$.

(b) The zero matrix belongs to V, clearly, so V is nonempty.

Suppose B_1 and B_2 are the standard matrices of transformations T_1 and T_2, respectively. If B_1 and B_2 are in V, then $T_1 \circ S = 0$ and $T_2 \circ S = 0$. So $(T_1 + T_2) \circ S = T_1 \circ S + T_2 \circ S = 0 + 0 = 0$ and therefore $B_1 + B_2$, the standard matrix of $T_1 + T_2$, is in V. So V is closed under addition.

Last, suppose B is the standard matrix of transformation T. Let $c \in \mathbb{R}$; if $B \in V$ then $T \circ S = 0$ so $(cT) \circ S = c(T \circ S) = 0$ so cB, the standard matrix of cB, is in V. Therefore V is closed under scalar multiplication.

Since V is nonempty, closed under addition, and closed under scalar multiplication, V is a subspace.

(c) From part (a), a matrix B belongs to V if and only if its columns are all multiples of $(2, 2, 1)$, i.e. B is of the form

$$\begin{pmatrix} 2a & 2b & 2c \\ 2a & 2b & 2c \\ a & b & c \end{pmatrix}$$

where a, b and c are real numbers. Therefore, V, the set of such B, is

$$\begin{aligned} V &= \{\begin{pmatrix} 2a & 2b & 2c \\ 2a & 2b & 2c \\ a & b & c \end{pmatrix} : a, b, c \in \mathbb{R}\} \\ &= \{a \begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R}\}. \end{aligned}$$

The three matrices in the above expression are clearly linearly independent (since they share no common nonzero entries), therefore they form a basis of V. To state this explicitly, a basis is

$$\mathcal{B} = \left\{ \begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

5. First, carry out the following row operations to put the matrix in reduced row-echelon form: add Row 3 to Row 4, then multiply Row 4 by $1/2$, then multiply Row 3 by -1, then add -2 times Row 4 to Row 2, then add Row 3 to Row 2. The resulting reduced row-echelon form is

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right).$$

(a) From the reduced row-echelon form, we obtain the equations $x_1 + 2x_3 = 2$, $x_2 = 6$, $x_4 = 3$ and $x_5 = -2$. Solving for x_1 in terms of the free variable x_3, we see that the solution set is

$$\{(2 - 2x_3, 6, x_3, 3, -2) : x_3 \in \mathbb{R}\} = \{(2, 6, 0, 3, -2) + x_3(-2, 0, 1, 0, 0) : x_3 \in \mathbb{R}\}.$$

(b) No; here's why: since A has four pivot columns (i.e. as many columns as rows), $\dim C(A) = 4$. Since $C(A) \subseteq \mathbb{R}^4$, we conclude $C(A) = \mathbb{R}^4$. Therefore any $\mathbf{c} \in \mathbb{R}^4$ must belong to $C(A)$ and so $A\mathbf{x} = \mathbf{c}$ has at least one solution.

(c) No; if $A\mathbf{x} = \mathbf{c}$ has solution $\mathbf{x_p}$, then for any $\mathbf{n} \in N(A)$, $\mathbf{x_p} + \mathbf{n}$ is also a solution. Since $N(A) \neq \{\mathbf{0}\}$, there are infinitely many choices for $\mathbf{n}$ and consequently $A\mathbf{x} = \mathbf{c}$ must have infinitely many solutions.

6. (a) Let A be a matrix whose columns are $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ respectively, and perform row operations to put this matrix in echelon form:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(A sequence of row operations is (1) add -1 times Row 1 to Row 2, (2) swap Row 2 and Row 3, (3) multiply Row 3 by 1/2, (4) add 2 times Row 3 to Row 4.) It is clear from the echelon form that A has three pivots, hence three linearly independent columns; therefore $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent.

(b) Such an $\mathbf{a}$ is guaranteed to exist by the Basis Extension Theorem. Let $W =$ Span($\mathbf{u}$,$\mathbf{v}$,$\mathbf{w}$). We need to find any $\mathbf{a}$ which is not in W. Then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{a}\}$ will be a linearly independent set of four vectors in $\mathbb{R}^4$ and will therefore be a basis. There are a variety of ways to do this:

Solution # 1: Find a nonzero vector $\mathbf{a} = (a_1, a_2, a_3, a_4) \in W^\perp$. By homework problem, $\mathbf{a} \in W^\perp$ if and only if $\mathbf{a}$ is orthogonal to $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Translating this into equations, this means

$$\begin{cases} a_1 + a_2 = 0 \\ a_1 - a_2 + 2a_3 = 0 \\ a_2 - 2a_4 = 0 \end{cases} .$$

Solving this system via row reductions or old-fashioned algebra, we see that $\mathbf{a} = (-2a_4, 2a_4, 2a_4, a_4)$ where a_4 is free. Choose any nonzero value of a_4 to obtain a vector in $W^\perp$, for example when $a_4 = 1$ we get $\mathbf{a} = (-2, 2, 2, 1)$.

Solution # 2: Let A be as in part (a) and find an $\mathbf{a}$ such that the system $A\mathbf{x} = \mathbf{a}$ has no solution. To do this, write the augmented matrix $(A \,|\, \mathbf{a})$ and perform the same row operations on this augmented matrix as were performed in part (a):

$$(A \,|\, \mathbf{a}) = \left(\begin{array}{ccc|c} 1 & 1 & 0 & a_1 \\ 1 & -1 & 1 & a_2 \\ 0 & 2 & 0 & a_3 \\ 0 & 0 & -2 & a_4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & a_1 \\ 0 & 2 & 0 & a_3 \\ 0 & 0 & 1 & a_2 \\ 0 & 0 & 0 & 2a_2 + a_4 \end{array}\right)$$

So any choice of $\mathbf{a} = (a_1, a_2, a_3, a_4)$ where $2a_2 + a_4 \neq 0$ works; an easy choice is $(0, 0, 0, 1)$.

7. (a) To change from $\mathcal{B}-$coordinates to $\mathcal{C}-$ coordinates, we can first change from $\mathcal{B}-$coordinates to $\mathcal{A}-$coordinates by multiplying by P^{-1}, then change from $\mathcal{A}-$coordinates to $\mathcal{C}-$coordinates by multiplying by Q. So the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is QP^{-1}.

   (b) To find the matrix of TST relative to $\mathcal{C}$, we need to start with $\mathcal{C}-$coordinates of $\mathbf{x}$ in $\mathbb{R}^n$. To find the $\mathcal{C}-$coordinates of $TST(\mathbf{x})$ based on the given information, we need to:
   1. Multiply by Q^{-1} to find the $\mathcal{A}-$coordinates of $\mathbf{x}$.
   2. Multiply by A to obtain the $\mathcal{A}-$coordinates of $T(\mathbf{x})$.
   3. Multiply by P to change the $\mathcal{A}-$ coordinates of $T(\mathbf{x})$ into $\mathcal{B}-$ coordinates.
   4. Multiply by B to find the $\mathcal{B}-$coordinates of $ST(\mathbf{x})$.
   5. Multiply by P^{-1} to change the $\mathcal{B}-$coordinates of $ST(\mathbf{x})$ into $\mathcal{A}-$ coordinates.
   6. Multiply by A to obtain the $\mathcal{A}-$coordinates of $TST(\mathbf{x})$.
   7. Multiply by Q to change the $\mathcal{A}-$coordinates of $TST(\mathbf{x})$ into $\mathcal{C}-$ coordinates.

   The product of these matrices (from right to left) gives the matrix of TST relative to $\mathcal{C}$; the answer is $QAP^{-1}BPAQ^{-1}$.

8. (a) W is a plane.

   (b) Since W is a plane, it has dimension 2; a basis of W is therefore given by any two linearly independent (i.e. nonparallel) vectors in W. For example, one basis is
   $$\mathcal{B} = \{(3, 0, -1), (2, -1, 0)\}.$$

   (c) Let $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$; then $(x, y, z) \in N(A)$ if and only if $x + 2y + 3z = 0$, the condition that defines W.

   (d) Take the basis vectors from part (b) and place them in a matrix B as columns; W will be equal to $C(B)$ for such a matrix. Given the choice of basis I made in part (b), I would obtain
   $$B = \begin{pmatrix} 3 & 2 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

   (e) We know that $\dim W + \dim W^\perp = \dim \mathbb{R}^3$, i.e. that $2 + \dim W^\perp = 3$, so $\dim W^\perp = 1$. From part (c), $W = N(A)$ so $W^\perp = [N(A)]^\perp = R(A)$ which is spanned by $(1, 2, 3)$. So $W^\perp$ has a basis consisting of the one vector $(1, 2, 3)$.

   (f) Take the equations which define l and plug them in for x, y, z in the equations which define the plane W to obtain
   $$(3 + t) + 2(2 - t) + 3(1 + 2t) = 0.$$
   Solving for t, we obtain $t = -2$ and from the parametric equations of l, plugging in $t = -2$ gives the point $(x, y, z) = (1, 4, -3)$.