Quiz 1: Mathematical induction

Let f_n be the sequence of Fibonacci numbers; that is, $f_0 = 1$, $f_1 = 1$, and for all $n \geq 1$, $f_{n+1} = f_n + f_{n-1}$. So

$$f_0 = 1, f_1 = 1, f_2 = 2, f_3 = 3, f_4 = 5, f_5 = 8, \text{ etc.}$$

Prove that for all $n \geq 0$,

$$\sum_{j=0}^{n} f_j^2 = f_n f_{n+1}.$$

Hint: You will need to find a relationship between f_n^2 and the three Fibonacci numbers f_{n-1}, f_n, and f_{n+1}.

Quiz 2: Fields, vector spaces and subspaces

1. True or False?

   TRUE FALSE The empty set $\emptyset$ is a subspace of every vector space.

   TRUE FALSE $\mathbb{C}$ is a vector space over $\mathbb{R}$.

   TRUE FALSE $\mathbb{C}$ is a vector space over $\mathbb{C}$.

2. Let $V = M_2(\mathbb{R})$ be the vector space of 2×2 matrices with real entries. Is the set

$$\left\{ W = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : ad = bc \right\}$$

   a subspace of V? Prove your answer.

3. Let $V = \mathbb{P}_n$ be the vector space of polynomials of degree $\leq n$ with real coefficients. Is the set W of polynomials of odd degree (and degree $\leq n$) a subspace of V? Prove your answer.

Quiz 3: Dimension, span, linear independence and basis

1. Classify the following statements as "always true" (AT), "sometimes true" (ST), or "never true" (NT).

   AT ST NT Five vectors span $\mathbb{R}^4$.

   AT ST NT Five vectors span $\mathbb{R}^5$.

   AT ST NT Five vectors span $\mathbb{R}^6$.

   AT ST NT Five vectors in $\mathbb{R}^4$ are linearly independent.

   AT ST NT Five vectors in $\mathbb{R}^5$ are linearly independent.

   AT ST NT Five vectors in $\mathbb{R}^6$ are linearly independent.

   AT ST NT Given a vector $\mathbf{v} \in \mathbb{R}^4$, that vector is an element of a basis.

AT ST NT Given four linearly independent vectors in $\mathbb{R}^5$, those vectors comprise a basis.

AT ST NT Given a basis of $\mathbb{R}^5$, a vector $\mathbf{v}$ can be written as a linear combination of the basis elements in two distinct ways.

AT ST NT Given V, a finite-dimensional vector space, and W, a subspace of V which is not V itself, $\dim(W) < \dim(V)$.

2. Find parametric equations of the line in $\mathbb{R}^4$ passing through the point $(1, -2, 4, 0)$ and parallel to the line passing through the points $(-3, 0, 1, 5)$ and $(2, -1, 1, 3)$.

3. Let W be the subspace of $M_2(\mathbb{R})$ consisting of 2×2 matrices where the sum of the entries in the top row is equal to the sum of the entries in the bottom row. Find a basis of W, and prove that your answer is a basis.

Quiz 4: Inner products and orthogonality

1. Let $V = \mathbb{R}^3$, endowed with the usual inner product. Let $\mathbf{v} = (0, -4, 3)$ and let $\mathbf{w} = (-1, 2, 2)$. Find the projection of $\mathbf{v}$ onto $\mathbf{w}$ and the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$.

2. Let V be a vector space with some inner product $<,>$. Suppose $\mathbf{v}_1, ..., \mathbf{v}_n$ are a *pairwise orthogonal set* of vectors in V (this means that $\mathbf{v}_j \perp \mathbf{v}_k$ for all $j \neq k$. Prove that the vectors $\mathbf{v}_1, ..., \mathbf{v}_n$ form a linearly independent set.

Quiz 5: Linear transformations

In each question, you are given a function T from one vector space to another. Prove whether or not T is a linear transformation. If T is linear, find a basis of the kernel of T and a basis of the image of T.

1. Let $\mathbf{v} = (a, b)$ be a fixed nonzero vector in $\mathbb{R}^2$; let $<,>$ be dot product and define $T : \mathbb{R}^2 \to \mathbb{R}$ by $T(\mathbf{x}) = <\mathbf{x}, \mathbf{v}>$.

2. Let $V_1 = V_2 = M_2(\mathbb{R})$; define $T : V_1 \to V_2$ by $T(A) = A - A^T$.

Quiz 6: Systems of linear equations

1. In this question, assume A is a 4×6 matrix of rank 4 and assume B is an 11×8 matrix of rank 8. The entries of both matrices are all real numbers. Answer the following questions; no justification is required.

   (a) What is the dimension of $N(A)$?
   (b) What is the dimension of $R(A)$?
   (c) What is the dimension of $N(A^T)$?
   (d) $C(A)$ is a subspace of $\mathbb{R}^d$ for what d?
   (e) How many solutions does the system $A\mathbf{x} = \mathbf{0}$ have?
   (f) How many solutions does the system $B\mathbf{x} = \mathbf{0}$ have?
   (g) How many free variables are present in the system $A\mathbf{x} = \mathbf{b}$?
   (h) If $T : \mathbb{R}^a \to \mathbb{R}^b$ is the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, what is b?
   (i) If T is as in part (h), the kernel of T is a subspace of $\mathbb{R}^d$ for what d?
   (j) Is there a vector $\mathbf{b}$ such that the system $B\mathbf{x} = \mathbf{b}$ is inconsistent?

2. Here is the augmented matrix $(A\,\mathbf{b})$ corresponding to a system of linear equations $A\mathbf{x} = \mathbf{b}$, and the reduced row-echelon form of this augmented matrix:

$$\begin{pmatrix} -1 & -2 & -3 & 1 & -5 & -7 \\ 2 & 4 & -3 & 1 & 4 & -4 \\ 1 & 2 & 1 & 0 & 3 & 3 \\ 0 & 0 & 4 & -3 & 6 & 8 \\ -2 & -4 & 2 & 0 & 6 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   (a) Find the solution set of the system $A\mathbf{x} = \mathbf{b}$.
   (b) Write a basis for the column space of A.
   (c) Write a basis for the null space of A.
   (d) Write a basis for the row space of A.

Quiz 7: Matrix inverses

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with standard matrix

$$A = \begin{pmatrix} 0 & 4 & 3 \\ 2 & -2 & -1 \\ 0 & 3 & 2 \end{pmatrix}.$$

   Show that T is invertible, and find the standard matrix of T^{-1}.

2. Does there exist a matrix $A \in M_2(\mathbb{R})$ other than the identity such that $A = A^{-1}$? If so, give an example of such an A. If not, prove that no such matrix can exist.

3. An invertible matrix $A \in M_n(\mathbb{R})$ is called *orthogonal* if $A^T = A^{-1}$. Show that the product of two orthogonal matrices (of the same size) is an orthogonal matrix.

Quiz 1

Start with the equation $f_{n+1} = f_n + f_{n-1}$ and multiply both sides by f_n to obtain $f_{n+1}f_n = f_n^2 + f_n f_{n-1}$. Then subtract $f_n f_{n-1}$ from both sides to obtain

$$f_n^2 = f_{n+1}f_n - f_n f_{n-1}, \tag{1}$$

which holds for all $n \geq 1$. Now we prove the statement by induction on n:

Base Case ($n = 0$):

$$\sum_{j=0}^{0} f_j^2 = f_0^2 = 1^2 = 1 = 1 \cdot 1 = f_0 f_1.$$

Inductive Step: Let $k \geq 0$ and assume the inductive hypothesis, that is

$$\sum_{j=0}^{k} f_j^2 = f_k f_{k+1}.$$

Then

$$\begin{aligned}\sum_{j=0}^{k+1} f_j^2 &= \sum_{j=0}^{k} f_j^2 + f_{k+1}^2 \\ &= f_k f_{k+1} + f_{k+1}^2 \quad \text{(by IH)} \\ &= f_k f_{k+1} + (f_{k+2}f_{k+1} - f_k f_{k+1}) \quad \text{(by equation (1) with } n = k+1) \\ &= f_{k+1}f_{k+2}.\end{aligned}$$

By induction we are done.

Quiz 2

1. The first statement is FALSE; by definition subspaces are nonempty so $\emptyset$ is never a subspace of any vector space. The second statement is TRUE (proven in class); the third statement is TRUE (every field is a vector space over itself).

2. W is not a subspace of V because it is not closed under addition; for example, let $w_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and let $w_2 = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$; both w_1 and w_2 belong to W since $ad = bc = 0$ for both matrices but $w_1 + w_2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ which does not satisfy $ad = bc$ since $2 \cdot 1 \neq 1 \cdot 1$.

3. W is not a subspace of V because it is not closed under addition: $f(x) = x^2 - x^3 \in W$, $g(x) = x^3 \in W$ but $(f+g)(x) = x^2 \notin W$. Just as well, W does not contain the zero function (0 has degree zero or degree $-\infty$ depending on your perspective) so it cannot be a subspace.

Quiz 3

1. (a) SOMETIMES TRUE: It depends on what the five vectors are; for example, the same vector repeated five times doesn't span $\mathbb{R}^4$ but $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, $(0,0,0,1)$ do span $\mathbb{R}^4$.
   (b) SOMETIMES TRUE: Same reasoning as previous problem.
   (c) NEVER TRUE: Since $\dim(\mathbb{R}^6) = 6$, there must be at least 6 vectors in any spanning set.
   (d) NEVER TRUE: Since $\dim(\mathbb{R}^4) = 4$, any set of more than 4 vectors is linearly dependent.

(e) SOMETIMES TRUE: If the five vectors form a basis, this is true, but the same vector repeated five times does not form a linearly independent set.

(f) SOMETIMES TRUE: The vectors $(1, 0, 0, 0, 0, 0)$, $(0, 1, 0, 0, 0, 0)$, $(0, 0, 1, 0, 0, 0)$, $(0, 0, 0, 1, 0, 0)$, $(0, 0, 0, 0, 1, 0)$ are linearly independent, but the same vector repeated five times forms a linearly dependent set.

(g) SOMETIMES TRUE: If $\mathbf{v} = \mathbf{0}$, the vector is not part of a basis (the zero vector is never part of a linearly independent set), but otherwise the statement is true by the Basis Extension Theorem.

(h) NEVER TRUE: Since $\dim(\mathbb{R}^5) = 5$, every basis of $\mathbb{R}^5$ consists of five vectors.

(i) NEVER TRUE: This was proven in class.

(j) ALWAYS TRUE: We know from a result in class that $\dim(W) \leq \dim(V)$. If $\dim(W) = \dim(V)$, then any basis for W is also a basis of V, hence $W = V$. So if $W \neq V$, then $\dim(W) < \dim(V)$ *Note:* This is no longer true if V is infinite-dimensional.

2. Parametric equations for a line (or plane) are never unique, so there are multiple answers to this question. We know the line passes through $\mathbf{p} = (1, -2, 4, 0)$; we need to find a direction vector for the line; since it is parallel to the line through $(-3, 0, 1, 5)$ and $(2, -1, 1, 3)$ it has direction vector $\mathbf{v} = (2, -1, 1, 3) - (-3, 0, 1, 5) = (5, -1, 0, -2)$. So parametric equation for the line come from the components of the vector equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$; this gives

$$\begin{cases} x_1 = 1 + 5t \\ x_2 = -2 - t \\ x_3 = 4 \\ x_4 = -2t \end{cases}$$

3. Bases are never unique, so there are multiple correct answers to this question. That said, the *number* of elements in a basis is always the same, no matter the answer. To solve this, first observe that W does not contain all 2×2 real matrices since $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \notin W$. So $W \neq M_2(\mathbb{R})$ and therefore $\dim(W) < \dim(M_2(\mathbb{R})) = 4$ so $\dim(W) \leq 3$. Consider the three matrices in W:

$$M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}; \quad M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}; \quad M_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

I claim these three matrices are linearly independent. To show this, suppose we have constants c_1, c_2, c_3 such that $c_1 M_1 + c_2 M_2 + c_3 M_3 = 0$. This means

$$c_1 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Looking at this matrix equation entry-by-entry, this implies that c_1, c_2, c_3 must satisfy the system of equations

$$\begin{cases} c_1 + c_3 = 0 \\ c_2 = 0 \\ c_1 = 0 \\ c_2 + c_3 = 0 \end{cases};$$

the only solution to this system is $c_1 = c_2 = c_3 = 0$ so M_1, M_2 and M_3 are linearly independent. This implies that the dimension of W is at least 3, since we already knew $\dim(W) \leq 3$ we now know $\dim(W) = 3$ and since any linearly independent set of three vectors form a basis in a three-dimensional space, $\mathcal{B} = \{M_1, M_2, M_3\}$ is a basis of W.

Quiz 4

1. First, $< \mathbf{v}, \mathbf{w} >= 0 \cdot (-1) + (-4) \cdot 2 + 3 \cdot 2 = -2$ and $< \mathbf{w}, \mathbf{w} >= -1(-1) + 2 \cdot 2 + 2 \cdot 2 = 9$; therefore
$$\text{proj}_{\mathbf{w}}\mathbf{v} = \frac{< \mathbf{v}, \mathbf{w} >}{< \mathbf{w}, \mathbf{w} >}\mathbf{w} = \frac{-2}{9}(-1, 2, 2) = \left(\frac{2}{9}, \frac{-4}{9}, \frac{-4}{9}\right).$$
Next, observe $||\mathbf{v}|| = \sqrt{< \mathbf{v}, \mathbf{v} >} = \sqrt{0^2 + (-4)^2 + 3^2} = 5$ and $||\mathbf{w}|| = \sqrt{< \mathbf{w}, \mathbf{w} >} = \sqrt{9} = 3$. Now if θ is the angle between the vectors,
$$\cos\theta = \frac{< \mathbf{v}, \mathbf{w} >}{||\mathbf{v}|| \cdot ||\mathbf{w}||} = \frac{-2}{5 \cdot 3} = \frac{-2}{15}.$$

2. Suppose there are scalars $c_1, ..., c_n$ such that
$$c_1\mathbf{v}_1 + ... + c_n\mathbf{v}_n = \mathbf{0}.$$
Take the inner product of both sides of this equation with $\mathbf{v}_k$ to get
$$\left\langle \sum_{j=1}^{n} c_j\mathbf{v}_j, \mathbf{v}_k \right\rangle =< \mathbf{0}, \mathbf{v}_k > .$$
The right-hand side of this equation is clearly zero, so the left-hand side must also be zero, hence by linearity of the inner product,
$$\left\langle \sum_{j=1}^{n} c_j\mathbf{v}_j, \mathbf{v}_k \right\rangle = \sum_{j=1}^{n} c_j < \mathbf{v}_j, \mathbf{v}_k >= 0.$$
By assumption, all the $< \mathbf{v}_j, \mathbf{v}_k >$ are zero unless $j = k$ so the above equation reduces to
$$c_k < \mathbf{v}_k, \mathbf{v}_k >= 0.$$
Since $\mathbf{v}_k$ is part of a basis, it cannot be the zero vector so by positive defniteness $< \mathbf{v}_k, \mathbf{v}_k > \neq 0$ and therefore $c_k = 0$. Since the choice of k is arbitrary, we have $c_k = 0$ for all k, $1 \leq k \leq n$. Hence the vectors are linearly independent.

Quiz 5

1. Write $\mathbf{x} = (x_1, x_2)$; to check whether or not T is linear we need to check whether or not
$$T(r\mathbf{x} + s\mathbf{y}) = rT(\mathbf{x}) + sT(\mathbf{y}). \quad (2)$$
The left-hand side of (2) is
$$< r\mathbf{x} + s\mathbf{y}, \mathbf{v} >=< (rx_1 + sy_1, rx_2 + sy_2), (a, b) >= a(rx_1 + sy_1) + b(rx_2 + sy_2).$$
The right-hand side of (2) is
$$r < \mathbf{x}, \mathbf{v} > +s < \mathbf{y}, \mathbf{v} >= r(ax_1 + bx_2) + s(ay_1 + by_2) = a(rx_1 + sy_1) + b(rx_2 + sy_2)$$
which is the same as the left-hand side, so T is linear.

To find the kernel of T, observe
$$\begin{aligned} (x, y) \in \ker(T) &\Leftrightarrow < (x, y), (a, b) >= 0 \\ &\Leftrightarrow ax + by = 0 \\ &\Leftrightarrow (x, y) = (-bc, ca) \text{ for some } c \in \mathbb{R} \\ &\Leftrightarrow (x, y) \in \text{Span}(-b, a) \end{aligned}$$

so a basis for $\ker(T)$ is $(-b, a)$; i.e. $\dim(\ker(T)) = 1$.

To find the image of T, observe that for any $r \in \mathbb{R}$, the equation $ax + by = r$ always has a solution if a and b are not both zero; if $a \neq 0$, a solution is $(x, y) = (r/a, 0)$; if $b \neq 0$ a solution is $(x, y) = (0, r/b)$. So every r is equal to $T(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{R}^2$ so $\operatorname{im}(T) = \mathbb{R}$ and has basis $\{1\}$ (i.e. T is onto).

2. To check whether or not T is linear we need to show whether or not

$$T(rA + sB) = rT(A) + sT(B). \tag{3}$$

The left-hand side of (3) is

$$(rA + sB) - (rA + sB)^T = rA + sB - (rA)^T - (sB)^T = rA - rA^T + sB - sB^T;$$

the right-hand side of (3) is

$$r(A - A^T) + s(B - B^T) = rA - rA^T + sB - sB^T$$

which is the same as the left-hand side; therefore T is linear.

To find the kernel of T, observe

$$\begin{aligned} T(A) = 0 &\Leftrightarrow A - A^T = 0 \\ &\Leftrightarrow A = A^T \\ &\Leftrightarrow A = \begin{pmatrix} r & s \\ s & t \end{pmatrix} \text{ for some } r, s, t \in \mathbb{R} \\ &\Leftrightarrow A = r\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + s\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + t\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore

$$\ker(T) = \operatorname{Span}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right);$$

i.e. $\ker(T)$ has dimension 3 (the three matrices in the spanning set are clearly linearly independent, hence form a basis of $\ker(T)$).

To find the image of T, write $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and observe that

$$\begin{aligned} T(A) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} &= \begin{pmatrix} 0 & a_{12} - a_{21} \\ a_{21} - a_{12} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & a_{12} - a_{21} \\ -(a_{12} - a_{21}) & 0 \end{pmatrix} \\ &= (a_{12} - a_{21})\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

Therefore

$$\operatorname{im}(T) = \operatorname{Span}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

i.e. $\operatorname{im}(T)$ has dimension 1.

Quiz 6

1. (a) $\dim N(A) = n - r = 6 - 4 = 2$.

(b) $\dim R(A) = r = 4$.

(c) $\dim N(A^T) = m - r = 4 - 4 = 0$.

(d) $d = 4$; the number of rows of A.

(e) Infinitely many, since $\dim N(A) > 0$.

(f) One (only $\mathbf{x} = \mathbf{0}$), since $N(B) = \{\mathbf{0}\}$ (we know this because $\dim N(B) = 8 - 8 = 0$).

(g) Two (the number of free variables is the dimension of $N(A)$).

(h) $b = 4$, the number of rows of A.

(i) $d = 6$, the number of columns of A.

(j) Yes; since B has only 8 columns, the column space of B has dimension at most 8, so $C(B) \neq \mathbb{R}^{11}$. For any $\mathbf{b} \notin C(B)$, $B\mathbf{x} = \mathbf{b}$ has no solution.

2. (a) From looking at the rref form of the augmented matrix, the pivot variables are x_1, x_3 and x_4; the free variables are x_2 and x_5. Solving for the pivot variables in terms of the free variables, we get the solution

$$\{(1,0,2,0,0) + x_2(-2,1,0,0,0) + x_5(-3,0,0,2,1) : x_2, x_5 \in \mathbb{R}\}.$$

(b) A basis of the column space consists of the pivot columns of A:

$$\{(-1,2,1,0,-2), (-3,-3,1,4,2), (1,1,0,-3,0)\}.$$

(c) By looking at the solution set to $A\mathbf{x} = \mathbf{b}$ found in part (a), it is clear that a basis for $N(A)$ is

$$\{(-2,1,0,0,0), (-3,0,0,2,1)\}.$$

(d) A basis for the row space consists of the pivot rows of the rref form of A, i.e.

$$\{(1,2,0,0,3), (0,0,1,0,0), (0,0,0,1,-2)\}.$$

Quiz 7

1. If A is an invertible matrix, then T is invertible with standard matrix A^{-1}. Use the Gauss-Jordan method to find A^{-1}:

$$(A\,|\,I) \;=\!=\; \left(\begin{array}{ccc|ccc} 0 & 4 & 3 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 2 & -2 & -1 & 0 & 1 & 0 \\ 0 & 4 & 3 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}\cdot R_1,\, -1\cdot R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & -1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 3 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-3\cdot R_2 + R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & -3 & 0 & 4 \end{array}\right)$$

$$\xrightarrow{-1\cdot R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 & 0 & -4 \end{array}\right)$$

$$\xrightarrow{-1\cdot R_3+R_2,\frac{1}{2}\cdot R_3+R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & \frac{3}{2} & \frac{1}{2} & -2 \\ 0 & 1 & 0 & -2 & 0 & 3 \\ 0 & 0 & 1 & 3 & 0 & -4 \end{array}\right)$$

$$\xrightarrow{R_2+R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 0 & -2 & 0 & 3 \\ 0 & 0 & 1 & 3 & 0 & -4 \end{array}\right)$$

Since A row reduces to the identity matrix, it is invertible and $A^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 1 \\ -2 & 0 & 3 \\ 3 & 0 & -4 \end{pmatrix}$; this is the standard matrix of T^{-1}.

2. *Easy solution:* Let $A = -I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $A^{-1} = (-I)^{-1} = -(I^{-1}) = -I = A$. There are other matrices you could come up by trial and error as well.

   More involved solution: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $A = A^{-1}$, then $A^2 = AA^{-1} = I$ so

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

   This produces the system of equations

$$\begin{cases} a^2+bc = 1 \\ ab+bd = b(a+d) = 0 \\ ac+dc = c(a+d) = 0 \\ bc+d^2 = 1 \end{cases};$$

   any solution (a, b, c, d) to this system produces a matrix A which satisfies $A = A^{-1}$. From the second equation, we see either $b = 0$ or $a = -d$; in the first case we have $a^2 = d^2 = 1$ so $a = \pm 1, d = \pm 1$ and $c = 0$. This leads to the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

   In the second case ($a = -d$), from the first and fourth equations we see $c = \frac{1-a^2}{b} = \frac{1-d^2}{b}$. So another family of matrices with $A = A^{-1}$ are those of the form

$$\{\begin{pmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{pmatrix} : a, b \in \mathbb{R}, b \neq 0\}.$$

3. Let A and B be orthogonal. Then

$$\begin{aligned} (AB)^T &= B^T A^T \\ &= B^{-1}A^{-1} \quad \text{(since } A \text{ and } B \text{ are both orthogonal)} \\ &= (AB)^{-1}. \end{aligned}$$

   Therefore AB is orthogonal.