Math 322 lecture: Determinants

November 15, 2019

Introducing determinants

Turn to page 210 in the notes.

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The goal of section 7.1 is to assign a single real number to each square matrix. This single number should serve as a kind of a "magic" number that tells you lots of useful information about the matrix.

Introducing determinants

To get started, let's start with a 2×2 matrix

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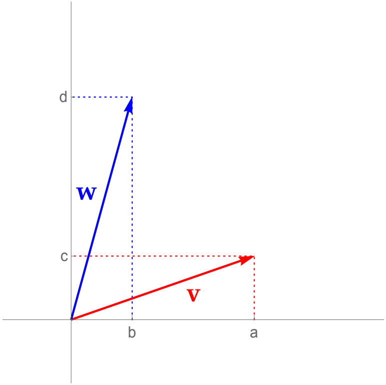
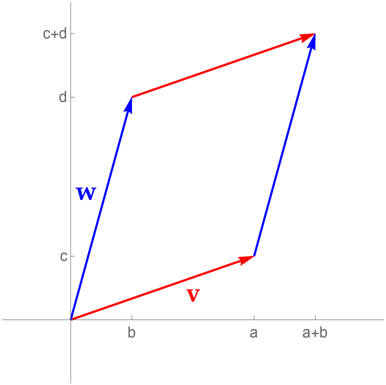
The magic number I'm thinking of will be the area of a parallelogram whose sides are the columns of A.

An area calculation

To compute this area, let's call the first column $\mathbf{v} = (a, c)$ and call the second column $\mathbf{w} = (b, d)$. Here are those vectors:

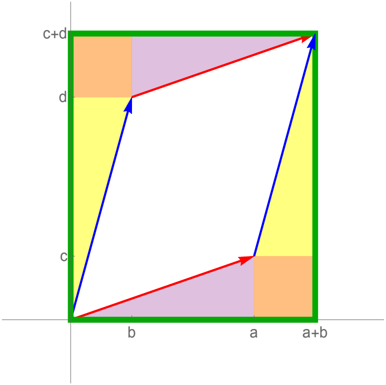


And here is the parallelogram they make:



Question: What is the area of this parallelogram?

To find the area, let's draw in a bunch of other stuff:



The area of the parallelogram is given by

$$\text{green area} - \text{orange area} - \text{yellow area} - \text{purple area}$$

An area calculation

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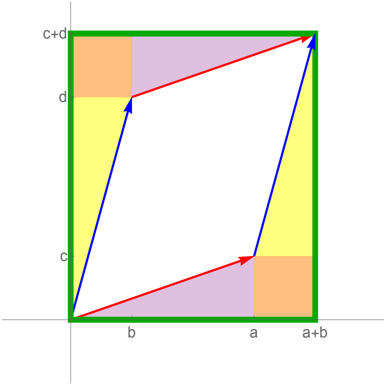
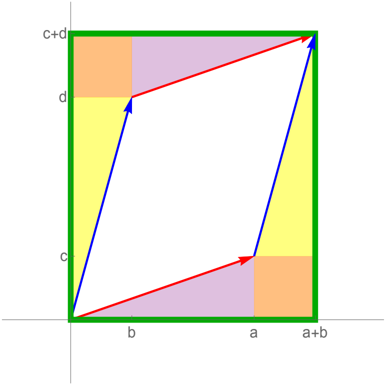


The area of the green rectangle is clearly $(a + b)(c + d)$.

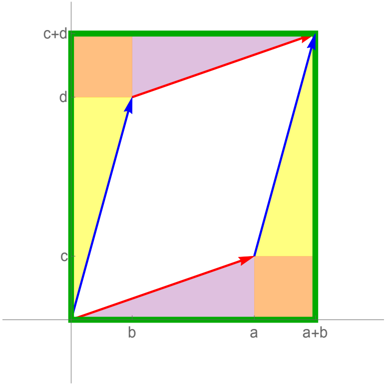
To find the area, let's draw in a bunch of other stuff:



The area of each orange rectangle is bc (the height is c and the width is b). There are two orange rectangles, so the total orange area is $2bc$.

An area calculation

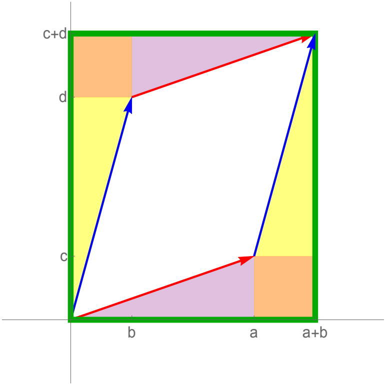
To find the area, let's draw in a bunch of other stuff:



NOTE: In my picture, the orange rectangles look like squares, but in general these are rectangles, not squares.

An area calculation

To find the area, let's draw in a bunch of other stuff:



Each yellow triangle has base b and height d, so they each have area $\frac{1}{2}bd$. There are two yellow triangles, so the total yellow area is $2\left(\frac{1}{2}bd\right) = bd$.

An area calculation

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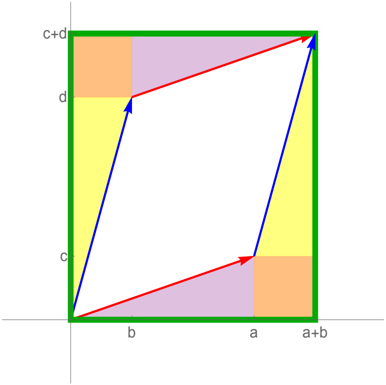

Each purple triangle has base a and height c, so they each have area $\frac{1}{2}ac$. There are two triangles, so the total purple area is $2\left(\frac{1}{2}ac\right) = ac$.

An area calculation

Putting this all together, the area of the parallelogram is

$$\begin{aligned}
&\text{green area} - \text{orange area} - \text{yellow area} - \text{purple area} \\
&= (a+b)(c+d) - 2bc - bd - ac \\
&= ac + ad + bc + bd - 2bc - bd - ac \\
&= ad - bc.
\end{aligned}$$

This formula motivates the formula in the gray box on the next page of the notes (turn to page 211).

2×2 determinants

Definition 7.1

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$. The **determinant** of A, denoted $\det A$, is the number $\det A = ad - bc$.

Notice that I use the notation $\det A$ or $\det(A)$ for the determinant of A. Some others use the notation $|A|$ for the determinant.

2×2 determinants

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DO NOT DO THIS. The reason is that often, you want to take the absolute value of the determinant. Using my notation, the absolute value of the determinant of A is $|\det A|$. Using the other notation, that would be $||A||$. But $||A||$ means something else! It denotes the norm of A, as defined back in Chapter 4.

2×2 determinants

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Make sure you understand that the determinant of a matrix is a number (i.e. a scalar), not some other type of object.

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One purpose of the determinant of a 2×2 matrix is to compute areas of parallelograms. If the sides of the parallelograms are the columns of 2×2 matrix A, then from the previous discussion we know

$$\text{area of the parallelogram} = |\det A|.$$

The sign of the determinant

The absolute value signs here are important. To see why, let's look at the next example on p. 211.

The sign of the determinant

Example

Find the determinant of $\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$.

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Solution

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The sign of the determinant

Example

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Solution

$$\det \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} = 2(1) - 3(3) = -7.$$

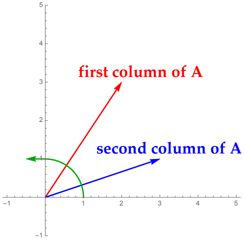
Remark: that means the area of the parallelogram formed by these vectors is $|-7| = 7$.

The sign of the determinant

Example

$$\det \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} = -7.$$

Notice that this solution is negative. The reason is this picture:



The sign of the determinant

As you go around counterclockwise from the positive x-axis (i.e. along the green arrow on the picture on the previous slide)... this is the normal way you would measure angles in standard position back in trig), *you hit the second column before the first one*.

This means the columns are sort of “backwards” from what would be “normal”. When the determinant of a 2×2 matrix is negative, that tells you that the second column comes before the first one as you go around counterclockwise like this. When it is positive, the first column comes before the second.

Determinants in general

Turn to page 212.

Determinants in general

Now let's move to square matrices of different sizes. In two dimensions, the natural notion of "size" is area, but in three dimensions, the natural notion of "size" is volume.

If you take a 3×3 matrix A and treat each column as a vector, you can make a box called a *parallelepiped* (spelling test upcoming) where all the sides and faces are parallel. A picture of a three-dimensional parallelepiped is at the top of page 212 (this is like a rectangular box, except that you don't have to have right angles).

In this picture, we are thinking of the three columns of the matrix as being the three solid black lines with arrows on them.

Determinants in general

The determinant of the 3×3 matrix A is $\pm$ the volume of this parallelepiped. If the vectors are written in the "right" order, the determinant is positive. If they are written in the "wrong" order, the determinant is negative. (You don't need to know what "right" or "wrong" order means in dimensions other than 2.)

Determinants in general

The same idea works for any size square $n \times n$ matrix (note: determinants of non-square matrices are not defined). Take the columns of the matrix; each column is a vector in $\mathbb{R}^n$. Make a n-dimensional parallelepiped out of those vectors, and find its "n-dimensional volume". The determinant of the matrix A is $\pm$ that volume.

(There is a more rigorous definition of determinant, but we won't talk about that.)

The main point is that the "number" in the determinant is the length/area/volume/size of some "box" whose sides come from n vectors in $\mathbb{R}^n$, and the sign of the determinant has to do with the order in which those vectors are put in a matrix as columns.

Properties of determinants

Now we're up to Theorem 7.2, which lists a bunch of important properties of determinants. Read this theorem carefully, and read the discussion of some of the proofs on p. 213. I may quiz you on these properties.

One property I want to say a bit more about is the last one:

Theorem 7.2 (part 9)

A is invertible if and only if $\det A \neq 0$.

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Remember earlier that we had a formula for 2×2 inverses that looked like this:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Theorem 7.2 (part 9)

A is invertible if and only if $\det A \neq 0$.

Note that the determinant appears in this formula:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Theorem 7.2 (part 9)

A is invertible if and only if $\det A \neq 0$.

In particular, a 2×2 matrix is invertible exactly when its determinant is nonzero, and we could write the previous formula as

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Theorem 7.2 (part 9)

A is invertible if and only if $\det A \neq 0$.

This should make some sense, given the way I defined determinants. To see why, recall that a square matrix is not invertible if it does not have full rank.

For a 2×2 matrix, that means its two columns are parallel.

Properties of determinants

Theorem 7.2 (part 9)

A is invertible if and only if $\det A \neq 0$.

If you take two parallel vectors in $\mathbb{R}^2$ and try to make a parallelogram out of them, you won't get much of a parallelogram (it will look like a line segment, since the vectors point in the same direction).

That means the area of this "parallelogram" is 0, i.e. $\det A = 0$.

Properties of determinants

The same idea holds for square matrices of any size. Whether or not $\det A = 0$ is a test for whether or not A has full rank.

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Properties of determinants

The same idea holds for square matrices of any size. Whether or not $\det A = 0$ is a test for whether or not A has full rank.

If $\det A = 0$, then A is not invertible and A does not have full rank (meaning that A has at least one linearly dependent row/column).

But if $\det A \neq 0$, you know immediately that A is invertible and A has full rank, so that all the rows/columns of A are linearly independent (and the 50 million other equivalent things we learned on page 176).

Properties of determinants

That having been said, if you want to determine whether or not an $n \times n$ matrix A is invertible, we now have two methods:

1. do row reductions on A and see if you get n pivots, or
2. find $\det A$ and see if the determinant is nonzero.

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It seems like the second method is "easier", but for large matrices, it actually takes <u>far</u> fewer computations to do the row reductions!

So determinants are useful, but should not be thought of as a magic bullet that solves all your problems (some of you may have seen determinants before... a reason I did not tell you about them until now is because I don't want you to overuse them).

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It seems like the second method is "easier", but for large matrices, it actually takes far fewer computations to do the row reductions!

So determinants are useful, but should not be thought of as a magic bullet that solves all your problems (some of you may have seen determinants before... a reason I did not tell you about them until now is because I don't want you to overuse them).

That finishes Section 7.1. Now let's turn to methods of computing determinants in Section 7.2.

Computing determinants

The determinant of a 1×1 matrix is essentially itself:

Determinant formula for 1×1 matrices

If $A = (a)_{1\times 1}$, then $\det A = a$.

We already discussed 2×2 matrices:

Determinant formula for 2×2 matrices

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det A = ad - bc$.

As practice, do the example in the middle of page 214. You should get $\det A = -1$.

Computing determinants

Now for 3×3 matrices. Without giving any motivation, the formula is in Theorem 7.3:

Determinant formula for 3×3 matrices

Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in M_3(\mathbb{R})$. Then

$$\det A = aei + cdh + bfg - bdi - ceg - afh.$$

Notice that there are six terms in this formula, three of which are added and three of which are subtracted. Each term is the product of three entries in the matrix.

Instead of memorizing this formula, we use a trick to find 3×3 determinants called the Rule of Sarrus (after the guy who found it).

The Rule of Sarrus for 3 × 3 determinants

Suppose you want to find the determinant of this matrix:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

The Rule of Sarrus for 3×3 determinants

To do this, first copy the first two columns of the matrix to the right like this:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

The Rule of Sarrus for 3×3 determinants

This produces three diagonals that go downward, like this:

$$A = \left(\begin{array}{ccccc} a & & b & & c \\ & \searrow & & \searrow & \\ d & & e & & f \\ & & & \searrow & \\ g & & h & & i \end{array}\right) \begin{array}{ccccc} & & a & & b \\ \searrow & & & & \\ & & d & & e \\ \searrow & & & \searrow & \\ & & g & & h \end{array}$$

The Rule of Sarrus for 3 × 3 determinants

Multiply along these diagonals like so:

$$A = \left(\begin{array}{ccccc} a & & b & & c \\ & \searrow & & \searrow & \\ d & & e & & f \\ & & & \searrow & \\ g & & h & & i \end{array}\right) \begin{array}{ccccc} \searrow & & a & & b \\ & & & & \\ & & d & & e \\ \searrow & & & \searrow & \\ & & g & & h \end{array}$$

$$aei \qquad bfg \qquad cdh$$

The Rule of Sarrus for 3×3 determinants

Add the three products you get (call this the "bottom sum"):

$$A = \begin{pmatrix} a & \searrow & b & \searrow & c \\ d & & e & & f \\ g & & h & \searrow & i \end{pmatrix} \begin{matrix} & & a & & b \\ \searrow & & d & & e \\ \searrow & & g & \searrow & h \end{matrix}$$

$$aei + bfg + cdh = \text{bottom sum}$$

The Rule of Sarrus for 3×3 determinants

There are also three diagonals that go upward, like this:

$$A = \left(\begin{array}{ccccc} a & & b & & c \\ & & & \nearrow & \\ d & & e & & f \\ & \nearrow & & \nearrow & \\ g & & h & & i \end{array}\right) \begin{array}{cccc} & a & & b \\ \nearrow & & \nearrow & \\ & d & & e \\ \nearrow & & & \\ & g & & h \end{array}$$

$$aei \quad + \quad bfg \quad + \quad cdh = \text{bottom sum}$$

The Rule of Sarrus for 3 × 3 determinants

Multiply along the diagonals and add to get the "top sum":

$$\begin{array}{ccccc} & & ceg & + \quad afh \quad + & bdi = \text{top sum} \end{array}$$

$$A = \left(\begin{array}{ccccc} a & & b & & c \\ & & & \nearrow & \\ d & & e & & f \\ & \nearrow & & \nearrow & \\ g & & h & & i \end{array} \right) \begin{array}{cccc} & a & & b \\ \nearrow & & \nearrow & \\ & d & & e \\ \nearrow & & & \\ & g & & h \end{array}$$

$$aei \quad + \quad bfg \quad + \quad cdh = \text{bottom sum}$$

The Rule of Sarrus for 3×3 determinants

The Rule of Sarrus is that the determinant of A is obtained by subtracting the top sum from the bottom sum:

$$\begin{array}{ccccc} & & ceg & + \; afh & + \; bdi = \text{top sum} \end{array}$$

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

$$aei \; + \; bfg \; + \; cdh = \text{bottom sum}$$

Rule of Sarrus

$$\begin{aligned} \det A &= \text{bottom sum} - \text{top sum} \\ &= (aei + bfg + cdh) - (ceg + afh + bdi) \end{aligned}$$

(Compare this with the formula in Theorem 7.3; it's the same.)

The Rule of Sarrus for 3×3 determinants

Try the Rule of Sarrus out on the matrix A at the top of page 215. You should get (don't go forward until you've worked this out)...

The Rule of Sarrus for 3×3 determinants

Try the Rule of Sarrus out on the matrix A at the top of page 215. You should get (don't go forward until you've worked this out)...

$$\det A = -29.$$

(If you don't get -29, find your error and fix it. As a hint, the bottom sum should be 4 and the top sum should be 33.)

A word of warning

WARNING: There is no Rule of Sarrus for 4×4 and larger matrices.

Reason: The determinant formula for a 4×4 matrix has 24 terms (each of which is the product of 4 entries from the matrix), but a "Rule of Sarrus" for 4×4 matrices would only give 8 of those terms (four downward diagonals and four upward diagonals).

A word of warning

I REPEAT: There is no Rule of Sarrus for 4×4 and larger matrices.

A word of warning

JUST IN CASE YOU ARE ASLEEP AT THE WHEEL: There is no Rule of Sarrus for 4×4 and larger matrices.

A word of warning

Question

How then, do you find the determinant of a 4×4 or larger matrix?

We will discuss this, starting halfway down the page on p. 215, when I return next week.