1. (a) Use the Gauss-Jordan method to find the inverse of this matrix. Show all the steps in your row reductions.

$$A = \begin{pmatrix} 3 & 4 & -2 \\ 1 & 0 & 1 \\ -2 & -3 & 2 \end{pmatrix}$$

(b) Let A be as in part (a). Use your answer to part (a) to find the solution set of the system $A\mathbf{x} = (3, -2, -5)$. (To receive credit, it must be clear how you are using your answer to part (a).)

2. Below, you are given the augmented matrix corresponding to a linear system of equations, together with the reduced row-echelon form of that augmented matrix:

$$\left(\begin{array}{ccccc|c} 3 & -1 & 2 & 4 & -3 & 2 \\ 2 & 1 & 7 & -4 & -1 & -3 \\ -1 & -8 & -29 & 32 & -4 & 21 \\ 4 & -3 & -3 & 12 & -5 & 7 \end{array}\right) \xrightarrow{\text{row ops}} \left(\begin{array}{ccccc|c} 1 & 0 & \frac{9}{5} & 0 & -\frac{4}{5} & -\frac{1}{5} \\ 0 & 1 & \frac{17}{5} & -4 & \frac{3}{5} & -\frac{13}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(a) If you think of this system of equations as a matrix equation $A\mathbf{x} = \mathbf{b}$, what is $\mathbf{b}$?

(b) If you think of this system of equations as a functional equation $T(\mathbf{x}) = \mathbf{b}$, what is the domain of T?

(c) How many linearly independent columns does A have?

(d) Find the solution set of $A\mathbf{x} = \mathbf{b}$.

(e) Find a basis for the row space of A.

(f) Find a basis for the null space of A.

3. Let x_n and y_n denote the number of female geese and male geese living in a pond at time n. Suppose that for every n,

$$\begin{cases} x_{n+1} &= \frac{8}{5}x_n + \frac{1}{5}y_n \\ y_{n+1} &= \frac{6}{5}x_n + \frac{7}{5}y_n \end{cases}.$$

If at time 0, there are 2 female geese and 5 male geese in the pond, find the number of male geese living in the pond at time 100.

4. Throughout this problem, let $\mathbf{v} = (4, 1, -3)$ and let $\mathbf{w} = (2, 0, 1)$.

(a) Compute $4\mathbf{v} + 5\mathbf{w}$.

(b) Compute $\mathbf{v} \cdot (\mathbf{v} + \mathbf{w})$.

(c) Find a unit vector in the same direction as $\mathbf{v}$.

(d) Compute $\mathbf{w} \times \mathbf{v}$.

(e) Compute the distance between $\mathbf{v}$ and $\mathbf{w}$.

(f) Find parametric equations of the line containing $\mathbf{v}$ and $\mathbf{w}$.

(g) Find the normal equation of the plane containing $\mathbf{v}$, $\mathbf{w}$ and $(1, 6, -2)$.

5. Throughout this problem, let W be the subspace of $\mathbb{R}^6$ which has orthonormal basis
$$\left\{\left(\frac{1}{3}, 0, \frac{-2}{3}, \frac{2}{3}, 0, 0\right), \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, 0, 0, 0\right), (0,0,0,0,0,1)\right\}$$
and let $\mathbf{v} = (9, 12, -6, 3, -7, 11)$.

   (a) Compute the projection of $\mathbf{v}$ onto W.

   (b) Compute the projection of $\mathbf{v}$ onto $W^\perp$.

6. Throughout this problem, let S and T be the following linear transformations:

   - $S : \mathbb{R}^2 \to \mathbb{R}^3$ satisfies $S(1, 0) = (1, 2, 1)$ and $S(0, 1) = (-3, 1, 0)$;
   - $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $T(1, 0) = (3, -4)$ and $T(0, 1) = (2, -5)$.

   (a) Compute $T(3, -2)$.

   (b) Is S surjective? Explain.

   (c) Is S injective? Explain.

   (d) Is T invertible? If so, find the standard matrix of T^{-1}. If not, explain why not.

   (e) Which of the two transformations $T \circ S$ or $S \circ T$ is defined? For the transformation that is defined, find its standard matrix.

7. In this problem, let A, B and M be the following matrices:
$$A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \; B = \begin{pmatrix} 5 & -2 \\ -6 & 1 \end{pmatrix} \; M = \begin{pmatrix} 1 & 4 & -2 \\ -3 & 1 & -5 \\ 2 & 0 & -3 \end{pmatrix}$$

   (a) Compute A^2B.

   (b) Compute $\det M$.

   (c) Compute $\det 10M$.

   (d) Compute the eigenvalues and eigenvectors of B.

8. Classify the following statements as true or false:

   (a) If a 3×3 matrix A has eigenvalues 3, 4 and -2, then A is diagonalizable.

   (b) If a 3×3 matrix A has eigenvalues 3, 4 and -2, then the equation $A\mathbf{x} = (-5, 7, 11)$ has exactly one solution.

(c) If $A \in M_{mn}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^m$, then the least-squares solution of $A\mathbf{x} = \mathbf{b}$ is given by $\widehat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$

(d) If A is an $m \times n$ matrix, then the row space of A and the null space of A are orthogonal complements.

(e) If W is a subspace of V, then $\dim W \leq \dim V$.

(f) If A and B are square matrices of the same size, then $tr(AB) = tr(A)\, tr(B)$.

(g) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, then for any matrix $A \in M_2(\mathbb{R})$, $T(A\mathbf{x}) = AT(\mathbf{x})$.

(h) If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, then $(3\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (3\mathbf{w})$.

(i) If A and B are invertible matrices of the same size, then $(AB)^{-1} = A^{-1}B^{-1}$.

(j) If $\mathbf{v}$, $\mathbf{w}_1$ and $\mathbf{w}_2$ are vectors in $\mathbb{R}^n$, then $\pi_{\mathbf{w}_1+\mathbf{w}_2}(\mathbf{v}) = \pi_{\mathbf{w}_1}(\mathbf{v}) + \pi_{\mathbf{w}_2}(\mathbf{v})$.

9. In each part of this problem, a set W is described.

- If W is a subspace of $\mathbb{R}^n$ for some n, say so, identify the vector space W is a subspace of, and find $\dim W$.
- If W is not a subspace, but is an affine subspace of $\mathbb{R}^n$ for some n, say so, identify the vector space W is an affine subspace of, and find $\dim W$.
- If W is not an affine subspace of $\mathbb{R}^n$ for any n, say so.

(a) $W = Span(1, 2, 3, 4)$.

(b) $W = Span((1, 2, 3, 4))$.

(c) W is the set of vectors orthogonal to both $(6, 7, 3)$ and $(-2, 4, -5)$.

(d) W is a hyperplane in $\mathbb{R}^6$ which does not contain $\mathbf{0}$.

(e) W is the null space of A, where A is a 7×9 matrix with rank 4.

(f) $W = \{(x, y) : 3x + 4y = 7\}$.

(g) $W = \{(x, y, z) : x = y = z\}$.

(h) W is the solution set of $A\mathbf{x} = (3, 5, 8)$, where $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

(i) $W = Span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, where $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ is a basis of $\mathbb{R}^5$.

(j) $W = ker(T)$, where $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(\mathbf{v}) =$ the projection of $\mathbf{v}$ onto $(1, -6, 4)$.

1. (a) Perform row reductions on the augmented matrix $(A\,|\,I)$:

$$(A\,|\,I) = \left(\begin{array}{ccc|ccc} 3 & 4 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & -3 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 4 & -2 & 1 & 0 & 0 \\ -2 & -3 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[2R_1+R_3]{-3R_1+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 4 & -5 & 1 & -3 & 0 \\ 0 & -3 & 4 & 0 & 2 & 1 \end{array}\right) \xrightarrow{R_3+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & -3 & 4 & 0 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{3R_2+R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 & -1 & 4 \end{array}\right) \xrightarrow[-R_3+R_1]{R_3+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & -4 \\ 0 & 1 & 0 & 4 & -2 & 5 \\ 0 & 0 & 1 & 3 & -1 & 4 \end{array}\right)$$

This last matrix is $(I\,|A^{-1})$, so $A^{-1} = \begin{pmatrix} -3 & 2 & -4 \\ 4 & -2 & 5 \\ 3 & -1 & 4 \end{pmatrix}$.

(b) Since A is invertible, the one and only solution to $A\mathbf{x} = (3,-2,-5)$ is

$$\mathbf{x} = A^{-1}(3,-2,-5) = \begin{pmatrix} -3 & 2 & -4 \\ 4 & -2 & 5 \\ 3 & -1 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix} = \begin{pmatrix} 7 \\ -9 \\ -9 \end{pmatrix}.$$

2. (a) $\mathbf{b} = (2,-3,21,7)$.

(b) $T : \mathbb{R}^5 \to \mathbb{R}^4$, so the domain is $\mathbb{R}^5$.

(c) This is the number of pivots, which is 2.

(d) Solve $A\mathbf{x} = \mathbf{b}$ using the rref form. Writing $\mathbf{x} = (v,w,x,y,z)$, we have the system

$$\begin{cases} v + \frac{9}{5}x - \frac{4}{5}z = -\frac{1}{5} \\ w + \frac{17}{5}x - 4y + \frac{3}{5}z = -\frac{13}{5} \end{cases} \Rightarrow \begin{cases} v = -\frac{1}{5} - \frac{9}{5}x + \frac{4}{5}z \\ w = -\frac{13}{5} - \frac{17}{5}x + 4y - \frac{3}{5}z \end{cases}$$

Substituting, we obtain the solution set

$$\begin{aligned} \mathbf{x} &= \{\left(-\frac{1}{5} - \frac{9}{5}x + \frac{4}{5}z, -\frac{13}{5} - \frac{17}{5}x + 4y - \frac{3}{5}z, x, y, z\right) : x,y,z \in \mathbb{R}\} \\ &= \{\left(-\frac{1}{5}, -\frac{13}{5}, 0,0,0\right) + x\left(\frac{-9}{5}, \frac{-17}{5}, 1,0,0\right) + y(0,4,0,1,0) + z\left(\frac{4}{5}, \frac{-3}{5}, 0,0,1\right) : x,y,z \in \mathbb{R}\} \\ &= \left(-\frac{1}{5}, -\frac{13}{5}, 0,0,0\right) + Span(\left(\frac{-9}{5}, \frac{-17}{5}, 1,0,0\right), (0,4,0,1,0), \left(\frac{4}{5}, \frac{-3}{5}, 0,0,1\right)). \end{aligned}$$

(e) A basis for $R(A)$ consists of the pivot rows of $rref(A)$:

$$\{\left(1, 0, \frac{9}{5}, 0, -\frac{4}{5}\right), \left(0, 1, \frac{17}{5}, -4, \frac{3}{5}\right)\}$$

(f) From the work in part (d), we can conclude

$$N(A) = Span(\left(\frac{-9}{5}, \frac{-17}{5}, 1, 0, 0\right), (0, 4, 0, 1, 0), \left(\frac{4}{5}, \frac{-3}{5}, 0, 0, 1\right)).$$

The three vectors in the spanning set form a basis, since we know $\dim N(A) = n - r = 5 - 2 = 3$.

3. Writing $A = \begin{pmatrix} \frac{8}{5} & \frac{1}{5} \\ \frac{6}{5} & \frac{7}{5} \end{pmatrix}$ and $\mathbf{x}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$, we have $\mathbf{x}_{100} = A^{100}\mathbf{x}_0 = A^{100}\begin{pmatrix} 2 \\ 5 \end{pmatrix}$. To compute this, diagonalize A by finding eigenvalues and eigenvectors. The characteristic polynomial of A is

$$\det(A - \lambda I) = (\frac{8}{5} - \lambda)(\frac{7}{5} - \lambda) - \frac{6}{25} = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$$

so the eigenvalues are $\lambda = 2$ and $\lambda = 1$. To find the corresponding eigenvectors, set $\mathbf{x} = (x, y)$ and solve $A\mathbf{x} = \lambda\mathbf{x}$ to get

$$\lambda = 2: \begin{cases} \frac{8}{5}x + \frac{1}{5}y &= 2x \\ \frac{6}{5}x + \frac{7}{5}y &= 2y \end{cases} \Rightarrow \tfrac{1}{5}y = \tfrac{2}{5}x \Rightarrow y = 2x \Rightarrow (1, 2)$$
$$\lambda = 1: \begin{cases} \frac{8}{5}x + \frac{1}{5}y &= x \\ \frac{6}{5}x + \frac{7}{5}y &= y \end{cases} \Rightarrow y = -3x \Rightarrow y = -3x \Rightarrow (1, -3)$$

Thus $A = S\Lambda S^{-1}$ where $S = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. That means

$$\begin{aligned} \mathbf{x}_{100} &= A^{100}\mathbf{x}_0 \\ &= S\Lambda^{100}S^{-1}\begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 2^{100} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 2^{100} & 1 \\ 2 \cdot 2^{100} & -3 \end{pmatrix}\frac{1}{-5}\begin{pmatrix} -3 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ &= -\frac{1}{5}\begin{pmatrix} 2^{100} & 1 \\ 2 \cdot 2^{100} & -3 \end{pmatrix}\begin{pmatrix} -11 \\ 1 \end{pmatrix} \\ &= -\frac{1}{5}\begin{pmatrix} -11 \cdot 2^{100} + 1 \\ -22 \cdot 2^{100} - 3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{5}(11 \cdot 2^{100} - 1) \\ \frac{1}{5}(4 \cdot 2^{100} - 9) \end{pmatrix}. \end{aligned}$$

The number of female geese at time 100 is therefore $\frac{1}{5}(11 \cdot 2^{100} - 1)$.

4. (a) $4\mathbf{v} + 5\mathbf{w} = (16, 4, -12) + (10, 0, 5) = (26, 4, -7)$.

   (b) $\mathbf{v} \cdot (\mathbf{v} + \mathbf{w}) = (4, 1, -3) \cdot (6, 1, -2) = 24 + 1 + 6 = 31$.

   (c) The unit vector is $\frac{1}{||\mathbf{v}||}\mathbf{v} = \frac{1}{\sqrt{4^2+1^2+(-3)^2}}\mathbf{v} = \frac{1}{\sqrt{26}}(4, 1, -3) = \left(\frac{4}{\sqrt{26}}, \frac{1}{\sqrt{26}}, \frac{-3}{\sqrt{26}}\right)$.

   (d) $\mathbf{w} \times \mathbf{v} = ((-3)0 - 1(1), 1(4) - 2(-3), 2(1) - 0(4)) = (-1, 10, 2)$.

   (e) $dist(\mathbf{v}, \mathbf{w}) = ||\mathbf{v} - \mathbf{w}|| = ||(2, 1, -4)|| = \sqrt{2^2 + 1^2 + (-4)^2} = \sqrt{21}$.

   (f) The line has direction vector $\mathbf{v} - \mathbf{w} = (2, 1, -4)$ and passes through the point $\mathbf{v} = (4, 1, -3)$. Thus one set of parametric equations for the line is

$$\mathbf{x} = \mathbf{v} + t(\mathbf{v} - \mathbf{w}) \Leftrightarrow \begin{cases} x = 4 + 2t \\ y = 1 + t \\ z = -3 - 4t \end{cases}$$

   (g) Two vectors in the plane are $\mathbf{v} - \mathbf{w} = (2, 1, -4)$ and $\mathbf{v} - (1, 6, -2) = (3, -5, -1)$. So a normal vector to the plane is $\mathbf{n} = (2, 1, -4) \times (3, -5, -1) = (-21, -10, -13)$. Set $d = \mathbf{n} \cdot \mathbf{w} = (-21, -10, 13) = -55$; then the plane has normal equation $\mathbf{n} \cdot \mathbf{x} = d$, i.e. $(-21, -10, -13) \cdot (x, y, z) = -55$. Writing this out, the plane has equation $-21x - 10y - 13z = -55$.

5. (a) Denote the given orthonormal basis of W by $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$. Using the projection formula, we get

$$\begin{aligned} \pi_W(\mathbf{v}) &= (\mathbf{v} \cdot \mathbf{x}_1)\mathbf{x}_1 + (\mathbf{v} \cdot \mathbf{x}_2)\mathbf{x}_2 + (\mathbf{v} \cdot \mathbf{x}_3)\mathbf{x}_3 \\ &= 9\left(\frac{1}{3}, 0, \frac{-2}{3}, \frac{2}{3}, 0, 0\right) + 12\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, 0, 0, 0\right) + 11(0, 0, 0, 0, 0, 1) \\ &= (3, 0, -6, 6, 0, 0) + (8, 8, 4, 0, 0, 0) + (0, 0, 0, 0, 0, 11) \\ &= (11, 8, -2, 6, 0, 11). \end{aligned}$$

   (b) Subtract the answer from part (a) from $\mathbf{v}$:

$$\begin{aligned} \pi_{W^\perp}(\mathbf{v}) &= \mathbf{v} - \pi_W(\mathbf{v}) \\ &= (9, 12, -6, 3, -7, 11) - (11, 8, -2, 6, 0, 11) \\ &= (-2, 4, -4, -3, -7, 0). \end{aligned}$$

6. (a) $T(3, -2) = 3T(1, 0) - 2T(0, 1) = 3(3, -4) - 2(2, -5) = (9, -12) - (4, -10) = (5, -2)$.

   (b) Since S maps a 2-dimensional space into a space of dimension greater than 2, S cannot be surjective.

   (c) Note that $im(S) = Span((1, 2, 1), (-3, 1, 0))$, so $im(S)$ contains two linearly independent vectors. Thus $rank(S) = \dim im(S) \geq 2$. That means $\dim ker(S) \leq 2 - 2 = 0$, meaning $\dim ker(S) = 0$, meaning S is injective.

(d) The standard matrix of T is $\begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -4 & -5 \end{pmatrix}$. Since the determinant of this matrix is $3(-5)-2(-4) = -7 \neq 0$, this matrix is invertible, meaning T is invertible. The standard matrix of T^{-1} is

$$\begin{pmatrix} 3 & 2 \\ -4 & -5 \end{pmatrix}^{-1} = \frac{1}{-7}\begin{pmatrix} -5 & -2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{7} & \frac{2}{7} \\ -\frac{4}{7} & -\frac{3}{7} \end{pmatrix}.$$

(e) Since T is given by a 2×2 matrix and S is given by a 3×2 matrix, $S \circ T$ is defined. Its standard matrix is

$$\begin{pmatrix} 1 & -3 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -4 & -5 \end{pmatrix} = \begin{pmatrix} 15 & 17 \\ 2 & -1 \\ 3 & 2 \end{pmatrix}.$$

7. (a) By usual matrix multiplication,

$$A^2B = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -6 & 1 \end{pmatrix} = \begin{pmatrix} -49 & 7 \\ -3 & 4 \end{pmatrix}.$$

(b) Using the Rule of Sarrus,

$$\begin{aligned} \det M &= [1(1)(-3) + (4)(-5)2 + (-2)(-3)0] - [2(1)(-2) + 0(-5)1 + (-3)(-3)4] \\ &= [-3-40] - [-4+36] \\ &= -43 - 32 \\ &= -75. \end{aligned}$$

(c) Since M is 3×3, $\det 10M = 10^3 \det M = 1000(-75) = -75000$.

(d) Start with the eigenvectors. The characteristic polynomial is $p_B(\lambda) = \det(B - \lambda I) = (5-\lambda)(1-\lambda) - 12 = \lambda^2 - 6\lambda - 7 = (\lambda - 7)(\lambda+1)$ so the eigenvalues are $\lambda = 7$ and $\lambda = -1$. Now for the eigenvectors. Set $\mathbf{x} = (x, y)$ and solve $A\mathbf{x} = \lambda\mathbf{x}$ to get

$$\begin{aligned} \lambda = 7: &\quad \begin{cases} 5x - 2y &= 7x \\ -6x + y &= 7y \end{cases} \Rightarrow -y = x \Rightarrow (1,-1) \\ \lambda = -1: &\quad \begin{cases} 5x - 2y &= -x \\ -6x + y &= -y \end{cases} \Rightarrow -2y = -6x \Rightarrow y = 3x \Rightarrow (1,3). \end{aligned}$$

8. (a) **TRUE**. Any $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

(b) **TRUE**. $\det A = 3(4)(-2) = -24 \neq 0$, so A is invertible, meaning $A\mathbf{x} = \mathbf{b}$ has exactly one solution for any $\mathbf{b}$.

(c) **TRUE**. The formula is indeed $\widehat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$.

(d) **TRUE**. This is one part of the Fundamental Theorem of Linear Algebra.

(e) **TRUE**. This follows from the Exchange Lemma.

(f) **FALSE**. For a counterexample, set $A = B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $tr(AB) = 2$ but $tr(A)\, tr(B) = 0 \cdot 0 = 0$.

(g) **FALSE**. For a counterexample, set $T(x, y) = (x, 2y)$ and let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $T(A(1, 0)) = T(1, 1) = (1, 2)$ but $AT(1, 1) = A(1, 2) = (3, 3)$.

(h) **TRUE**. This is what is called "bilinearity of dot product".

(i) **FALSE**. The order reverses: $(AB)^{-1} = B^{-1}A^{-1}$, not $A^{-1}B^{-1}$.

(j) **FALSE**. For a counterexample, set $\mathbf{v} = \mathbf{w}_1 = \mathbf{w}_2 = (1, 0)$. Then if $\pi_{\mathbf{w}_1+\mathbf{w}_2}(\mathbf{v}) = \pi_{(2,0)}(1, 0) = (1, 0)$ but $\pi_{\mathbf{w}_1}(\mathbf{v}) + \pi_{\mathbf{w}_2}(\mathbf{v}) = \pi_{(1,0)}(1, 0) + \pi_{(1,0)}(1, 0) = (1, 0) + (1, 0) = (2, 0)$.

9. (a) W is a subspace of $\mathbb{R}$ with $\dim W = 1$. (This W is the span of four elements of $\mathbb{R}$, all of which are parallel to one another.)

(b) W is a subspace of $\mathbb{R}^4$ with $\dim W = 1$. (This W is the span of one nonzero element of $\mathbb{R}^4$.)

(c) W is a subspace of $\mathbb{R}^3$ with $\dim W = 1$, since $W = Span((6, 7, 3), (-2, 4, -5))^{\perp}$.

(d) W is an affine subspace of $\mathbb{R}^6$ with $\dim W = 6 - 1 = 5$.

(e) W is a subspace of $\mathbb{R}^9$, with $\dim W = n - r = 9 - 4 = 5$.

(f) W is an affine subspace of $\mathbb{R}^2$ with $\dim W = 1$. (W is a line in $\mathbb{R}^2$ not passing through $\mathbf{0}$.)

(g) W is a subspace of $\mathbb{R}^3$ with $\dim W = 1$. ($W = \{(x, x, x)\} = Span((1, 1, 1))$.)

(h) W is an affine subspace of R^4 with $\dim W = 4 - 3 = 1$. (In general, $\dim W = \dim N(A) = n - r$ where A is $m \times n$ and has rank r.)

(i) W is a subspace of $\mathbb{R}^5$ with $\dim W = 3$ (since the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent).

(j) W is a subspace of $\mathbb{R}^3$ of dimension 2. (The rank of this T is 1, since $im(T) = Span((1, -6, 4))$, so $\dim ker(T) = n - r = 3 - 2 = 1$.)