Note: Problems 1-12 are unrelated to material we cover during lecture; these questions are based on the handout "All about sets" distributed with the syllabus on the first day of class (this handout is available at my website).

In Problems 1-3, consider the following sets:

$$E = \{1, 3, 5, 7\}$$

$$F = \{0, 2, 4, 6\}$$

$$G = \{1, 2, 4, 5, 7\}$$

$$H = \{x : 0 \leq x \leq 7 \text{ and } x \text{ is an integer}\}$$

1. Classify the following statements as true or false:

   (a) $2 \in G$
   (b) $2 \notin E$
   (c) $2 \in F \cap H$
   (d) $E \in H$
   (e) $E \subseteq H$
   (f) $H \subseteq G$
   (g) $E \cap F = \emptyset$
   (h) $E \cup F = H$

2. Describe the set by giving a list of its elements:

   (a) $E \cup F$
   (b) $E \cap F$
   (c) $(E \cap F) \cap G$
   (d) $(E \cup F) \cap G$

3. Which two of the sets E, F, G and H are disjoint?

4. (a) Suppose A and B are two arbitrary sets with $A \subseteq B$.
      i. In this situation, what is $A \cap B$?
      ii. What is $A \cup B$?

   (b) Let S be any set.

      i. What is $S \cup S$?
      ii. What is $S \cap S$?
      iii. What is $S \cap \emptyset$?
      iv. What is $S \cup \emptyset$?

5. Let E, F and G be any three sets. Classify the following statements as true or false:

   (a) $E \cap F = F \cap E$
   (b) $E \cup F = F \cup E$
   (c) $E \cap (F \cap G) = (E \cap F) \cap G$
   (d) $E \cup (F \cup G) = (E \cup F) \cup G$
   (e) $E \cap (F \cup G) = (E \cap F) \cup G$
   (f) $E \cap (F \cup G) = (E \cap F) \cup (E \cap G)$

   Hint: For each question, draw a Venn diagram and figure out which region(s) of the Venn diagram go with the left- and right-hand sides of the equation. If the same regions go with both sides, the statement is true; otherwise, the statement is false.

6. In this problem, you are given various subsets of $\mathbb{R}^2$. For each subset, sketch a picture of $\mathbb{R}^2$ and indicate (by shading, or color-coding, etc.) which points belong to the set. Please draw a different picture for each problem:
   (a) $A = \{(x, y) : y = 2x\}$;
   (b) $B = \{(x, y) : x \geq 1\}$;
   (c) $A \cap B$ (where A and B are as in the previous two parts);
   (d) $C = \{(x, y) : x + y = 3\}$.

7. Same directions as the previous problem:
   (a) $A \cup C$ (where A and C are as in the previous problem);
   (b) $A \cap C$ (where A and C are as in the previous problem);
   (c) $D = \{(x, y) : x^2 = y^2\}$;
   (d) $E = \{(x, y) : y < x^2\}$.

8. In this problem, let E be the set of all even integers, let D be the set of all odd integers, let P be the set of positive integers, and let N be the set of all negative integers.
   (a) Is the statement $E + E = 2E$ true or false? Explain.
   (b) Describe each of the following sets (i.e. they are one of E, D, P, N or $\mathbb{Z}$):

$$E + D \qquad D + D \qquad D + P \qquad P + P \qquad -P \qquad E + 3$$

9. In this problem, consider the following intervals in $\mathbb{R}$:

$$A = [0, 3) \qquad \text{(i.e. } A = \{x : 0 \leq x < 3\}\text{)}$$
$$B = [-5, -1]$$
$$C = [2, \infty)$$

   Write each of the sets below in interval notation:

   (a) $4A$
   (b) $-3C$
   (c) $A + B$
   (d) $A + C$
   (e) $C + C + C$
   (f) $A - A$
   (g) $C - 3$
   (h) $B + 4$

10. In this problem, consider the following subsets of $\mathbb{R}^2$:

$$A = \{(x, y) : y = x\}$$
$$B = \{(x, y) : y = 3x + 6\}$$
$$C = \{(x, y) : |x| \leq 1\}$$
$$D = \{(x, y) : y \geq 3\}$$
$$E = \{(x, y) : x \geq 2\}$$

   For each given set below, sketch a picture of $\mathbb{R}^2$ and indicate (by shading, etc.) which points belong to the set. Please draw a different picture for each problem:

(a) $3A$
(b) $2B$
(c) $-C$
(d) $D + E$

11. Same directions and sets as the previous problem:

(a) $A + E$
(b) $A + B$
(c) $A + (0, 1)$
(d) $A + (-3, 4)$

12. Consider the following subsets of $\mathbb{R}^2$:

$$A = \{(x, y) \in \mathbb{R}^2 : x \geq 2\}$$

$$B = \{(x, y) \in \mathbb{R}^2 : y = \frac{1}{2}x\}$$

$$C = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$$

$$D = \{(x, y) \in \mathbb{R}^2 : x = 0\}$$

$$E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

For each subset of $\mathbb{R}^2$ given below, sketch a picture of $\mathbb{R}^2$ and indicate (by shading, etc.) which points belong to the set. Please draw a different picture for each problem.

(a) $B + (2, 3)$
(b) $C - (0, 1)$
(c) $C + (0, 5)$
(d) $D + (3, 2)$
(e) $D + (4, 0)$
(f) $A + B$
(g) $B + D$
(h) $-E$
(i) $4E$
(j) $D + E$

13. (a) Let W be a subset of a vector space V and let $\mathbf{v} \in V$. Based on your answers to parts (a)-(e) of the preceding question, describe in your own words what you think the difference is between the sets W and $W + \mathbf{v}$.

Hint: One or more of the words "rotate", "reflect", "shift", "translate", "stretch", "shrink" may be useful in describing what is going on.

(b) Let E be the pentagon in $\mathbb{R}^2$ whose vertices are $(-1, 0)$, $(0, 1)$, $(2, 1)$, $(2, -1)$ and $(0, -1)$. Sketch a picture of $E + (-5, -2)$, indicating the coordinates of the vertices of this object.

14. Classify the following statements as true or false (no explanation is required). The sets $A, ..., E$ in this problem are the same ones as in Problem 12.

(a) $A = A + A$
(b) $B = 2B$
(c) $C = 2C$
(d) $B = B + B$

(e) $D = D - D$
(f) $E = 2E$
(g) $cD = D$ for any constant c
(h) $0A = D$

15. Which one or ones of the sets A, B, C, D, E from Problem 12 contain the zero vector?

16. Which one or ones of the sets A, B, C, D, E from Problem 12 are "closed under addition"? (A set is called **closed under addition** if the sum of any two elements from that set is also in that set.)

17. Which one or ones of the sets A, B, C, D, E from Problem 12 are "closed under scalar multiplication"? (A set is called **closed under scalar multiplication** if the product of any scalar times any element of the set must also be in that set.)

18. Compute the following quantities, if they make sense. If they don't make any sense, just write "nonsense".

(a) $(2, -3, 1) + (-7, 5, 2)$
(b) $3(-1, 0, 4) - 2(1, 3, 0)$
(c) $2(1, 4, -2, 5) + (2, 3, 0, 1, 0)$
(d) $4(1, 2) + 3(0, -1) + (5, 6)$

In problems 19 through 24, let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}; \quad C = \begin{pmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{pmatrix};$$

$$D = \begin{pmatrix} 3 & -2 \\ 2 & 4 \end{pmatrix}; \quad E = \begin{pmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{pmatrix}; \quad F = \begin{pmatrix} -4 & 5 \\ 2 & 3 \end{pmatrix};$$

$$\mathbf{x} = (2, 1, -1); \quad \mathbf{y} = (1, -3).$$

19. Compute each of the following quantities, if they make sense. If they don't make any sense, just write "nonsense".

(a) the size of A
(b) the size of C
(c) the size of B^T
(d) the diagonal entries of E
(e) the trace of C
(f) $tr(F)$

20. Compute each of the following quantities, if they make sense. If they don't make any sense, just write "nonsense".

(a) $b_{1,2}$
(b) $b_{2,1}$
(c) $2c_{1,3} - c_{2,3}$
(d) the second row of A
(e) the third column of C
(f) the third row of D

21. Compute each of the following quantities, if they make sense. If they don't make any sense, just write "nonsense".

(a) $A + B$
(b) $C + E$
(c) $2F - 3D$
(d) A^T
(e) $(A^T)^T$
(f) $(2 + 3)B$

22. Compute each of the following quantities, if they make sense. If they don't make any sense, just write "nonsense".

(a) $2(3B)$
(b) $D - D^T$
(c) $(2A)^T$
(d) $(B + A^T)^T$
(e) $(B + F)^T$
(f) $C + I$ (you should know what I is)

23. Compute each of the following quantities, if they make sense. If they don't make any sense, just write "nonsense".

(a) AB
(b) $(3A)B$
(c) BF
(d) FB
(e) $A(C + E)$
(f) F^3

24. Compute each of the following quantities, if they make sense. If they don't make any sense, just write "nonsense".

(a) $ABDF$
(b) DB^T
(c) DA^T
(d) $A\mathbf{x}$
(e) $\mathbf{x}A$
(f) $\mathbf{y}^T F\mathbf{y}$

25. (a) Give an example of two 2×2 matrices A and B such that $AB \neq BA$ (verify that these products are unequal).
(b) Give an example of two 2×2 matrices A and B such that:
- neither A nor B are diagonal matrices;
- $B \neq cA$ for any scalar c; and
- $AB = BA$ (verify that the products are equal).

26. Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ be vectors in some vector space V. Define the **span** of this collection of vectors to be the set of linear combinations of those vectors, i.e.

$$Span(\mathbf{v}_1, ..., \mathbf{v}_n) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_n : c_1, c_2, ..., c_n \in \mathbb{R}\}.$$

Prove that the span of any collection of vectors is a subspace.

Hint: a generic element of the span is a vector $\mathbf{w}$ where $\mathbf{w} = c_1\mathbf{v}_1 + ... + c_n\mathbf{v}_n$.

27. Let V be a vector space and let W_1 and W_2 be two subspaces of V. Prove that W_1+W_2 is also a subspace of V.

    Hint: Let $W = W_1+W_2$. A generic element of W is a vector $\mathbf{w}$ such that $\mathbf{w} = \mathbf{w}_1+\mathbf{w}_2$ where $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$.

28. Sketch a picture of each of the following subsets of $\mathbb{R}^2$ (please draw a different picture for each set):

    (a) $Span((2,3))$
    (b) $Span((-4,-1))$
    (c) $Span((1,0),(1,1))$
    (d) $Span((4,2),(2,1))$
    (e) $Span((0,0))$

29. Sketch a picture of each of the following subsets of $\mathbb{R}^2$ (please draw a different picture for each set):

    (a) $(2,-1) + Span((1,1))$
    (b) $(5,3) + Span((-2,0))$
    (c) $Span((2,-2)) + Span((-3,3))$
    (d) $Span((3,0)) + Span((1,-1))$

30. Give an example of a subset of $\mathbb{R}^2$ which contains the zero vector and is closed under scalar multiplication, but is **not** a subspace of $\mathbb{R}^2$.

In each part of Problems 31-34, you are given a vector space V and a subset W of V. Determine whether or not W is a subspace of V, and write a rigorous proof to support your assertion.

31. (a) $V = \mathbb{R}^3$; $W = \{(x,y,z) : x+y+z = 0\}$
    (b) $V = \mathbb{R}^3$; $W = \{(x,y,z) : x+y+z = 1\}$
    (c) $V = \mathbb{R}^2$; $W = \{(x,y) : y = \frac{1}{2}x\}$

32. (a) $V = \mathbb{R}^2$; $W =$ the $x-$axis
    (b) $V = \mathbb{R}^2$; $W = \{(x,y) : x+y \geq 1\}$
    (c) $V = \mathbb{R}^4$; $W = \{(w,x,y,z) : x = 2w \text{ and } y = -3w\}$

33. (a) $V = C(\mathbb{R},\mathbb{R})$; $W =$ the set of functions which are equal to their derivative
    (b) $V = C(\mathbb{R},\mathbb{R})$; $W =$ the set of functions f which are of the form $a\sin x + b\cos x$ for constants $a, b \in \mathbb{R}$
    (c) $V = M_{23}(\mathbb{R})$; $W =$ the set of 2×3 matrices such that the entries in the first row sum to zero

34. (a) $V = M_2(\mathbb{R})$; $W =$ the set of 2×2 matrices which equal to (-1) times their transpose
    (b) $V = M_2(\mathbb{R})$; $W =$ the set of 2×2 matrices which have at least one entry which is zero

(c) $V = M_2(\mathbb{R})$; $W = \{A \in M_{22}(\mathbb{R}) : tr(A) = 0\}$

35. Write parametric equations for each of the given lines:

    (a) The line in $\mathbb{R}^2$ that passes through the points $(-3, 5)$ and $(2, -6)$.
    (b) The $y-$axis in $\mathbb{R}^2$.
    (c) The line in $\mathbb{R}^2$ whose Cartesian equation is $y = 5x + 4$.
    Hint: find any two points on this line and then proceed as in part (a).

36. Write parametric equations for each of the given lines:

    (a) The line in $\mathbb{R}^3$ which has direction vector $(2, -3, 0)$ and passes through the point $(1, -3, 4)$.
    (b) The line in $\mathbb{R}^3$ passing through the points $(0, 1, -5)$ and $(4, -2, -1)$.
    (c) The line in $\mathbb{R}^6$ passing through the points $(-1, 1, 2, 0, 1, 4)$ and $(2, -1, 0, 3, 1, -2)$.

37. Write parametric equations for each of the given planes:

    (a) The plane in $\mathbb{R}^3$ that passes through the points $(1, 2, 3)$, $(-1, 4, 2)$ and $(3, 2, 2)$.
    (b) The $xy-$plane in $\mathbb{R}^3$ (this is the set of points whose $z-$coordinate is zero).
    Hint: first find any three noncollinear points in this plane and then proceed as in part (a).
    (c) The plane in $\mathbb{R}^3$ that passes through the points $(0, 5, -2)$, $(-5, 3, 2)$ and $(4, 1, -1)$.

38. Here are the parametric equations for two lines in $\mathbb{R}^3$:

$$\begin{cases} x = 2 + 3t \\ y = 1 - t \\ z = 4 + 7t \end{cases} ; \quad \begin{cases} x = 3 - 7t \\ y = -2 + 5t \\ z = -6 - 4t \end{cases}$$

    Show that these two lines intersect in a point. Find the coordinates of this point.

    Hint: If two lines intersect, they must meet at the same (x, y, z), but the t doesn't have to be the same for both lines (think about why this is).

39. Here are the parametric equations for two lines in $\mathbb{R}^3$:

$$\begin{cases} x = 3t \\ y = 2 - t \\ z = -1 + t \end{cases} ; \quad \begin{cases} x = 1 + 4t \\ y = -2 + t \\ z = -3 - 3t \end{cases}$$

    Show that these two lines do not intersect.

40. Two airplanes fly along straight lines. At time t, plane 1 is at $(75, 50, 25) + t(5, 10, 1)$ and plane 2 is at $(60, 80, 34) + t(10, 5 - 1)$.

    (a) Do the flight paths of these planes intersect? Explain.
    (b) Do the planes crash into one another? Explain.

41. Given each of the following vector spaces V and lists $\mathcal{S}$ of vectors, determine whether or not $\mathcal{S}$ is a linearly independent set. Give a quick reason for your answer.

   (a) $V = \mathbb{R}^2$; $\mathcal{S} = \{(1,3)\}$.
   (b) $V = \mathbb{R}^2$; $\mathcal{S} = \{(2,5),(-4,-10)\}$.
   (c) $V = \mathbb{R}^2$; $\mathcal{S} = \{(-3,7),(4,-10)\}$.
   (d) $V = \mathbb{R}^4$; $\mathcal{S} = \{(1,1,1,1),(2,2,2,2),(1,3,5,8),(0,-2,5,7)\}$.
   (e) $V = \mathbb{R}^3$; $\mathcal{S} = \{(1,2,7),(-1,7,5),(10,-3,6),(8,4,-1)\}$.

42. Same directions as the previous problem:

   (a) $V = \mathbb{R}^7$; $\mathcal{S} = \{(2,3,8,-5,0,0,0)\}$.
   (b) $V = \mathbb{R}^4$; $\mathcal{S} = \{(1,2,3,4),(5,6,7,8),(0,0,0,0)\}$
   (c) $V = \mathbb{R}^3$; $\mathcal{S} = \{(0,1,4),(0,0,1),(0,7,-3)\}$.
   (d) $V = \mathbb{R}^3$; $\mathcal{S} = \{(0,1,2),(0,-3,-6),(2,5,-8)\}$.

43. Give an example of three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, all belonging to the same vector space, such that $\{\mathbf{u},\mathbf{v}\}$, $\{\mathbf{u},\mathbf{w}\}$ and $\{\mathbf{v},\mathbf{w}\}$ are all linearly independent sets, but the set $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is linearly independent.

44. Let $V = C^\infty(\mathbb{R},\mathbb{R})$ be the vector space whose elements are functions from $\mathbb{R}$ to $\mathbb{R}$ which are differentiable infinitely many times (this is a real vector space since it is closed under addition and scalar multiplication and contains zero).

   (a) In this vector space, is the set of functions $\{\cos^2 x, \sin^2 x, 1\}$ linearly independent? Why or why not?
   Hint: think of an identity relating $\cos^2 x$ and $\sin^2 x$.
   (b) In this vector space, is the set of functions $\{\sin 2x, \sin x \cos x\}$ linearly independent? Why or why not?
   Hint: do a Google search to find an identity relating these functions.
   (c) In this vector space, is the set of functions $\{\cos^2 x, \sin 2x, 1\}$ linearly independent? Why or why not?
   Hint: do a Google search to find an identity relating some of these functions.

45. In this problem we will determine whether or not the set of functions

$$\{e^x, e^{-x}, 1\}$$

linearly independent.

Note: These functions are linearly independent if and only if no identity of the form $c_1 e^x + c_2 e^{-x} = 1$ exists. Now, you're probably not aware of such an identity, but just because you aren't aware of such an identity doesn't mean that one doesn't exist. To prove that no such identity exists (i.e. to prove that the functions are linearly independent), we will carry out the following steps:

(a) Suppose we write 0 (the constant function 0) as a linear combination of $\{e^x, e^{-x}, 1\}$:

$$c_1 e^x + c_2 e^{-x} + c_3 1 = 0 \tag{1}$$

Show that it must be the case that $c_1 + c_2 + c_3 = 0$.

Hint: if the above equation holds as an equality between functions, then it is supposed to hold for all particular values of x. Plug in $x = 0$ and see what happens.

(b) Differentiate both sides of equation (1) and subsequently (by plugging in $x = 0$) show that $c_1 - c_2 = 0$.

(c) Differentiate both sides of the equation again, and show that $c_1 + c_2 = 0$.

(d) Solve the equations obtained in parts (a), (b) and (c) to explain why $c_1 = c_2 = c_3 = 0$ (and thus why the three functions $\{e^x, e^{-x}, 1\}$ are linearly independent).

Note: In Math 330, you learn a more general technique to prove that a collection of functions is linearly independent: you compute the Wronskian of those functions, and if there is any x where the Wronskian is nonzero, then the functions are linearly independent. Essentially, what this problem is doing is verifying that the Wronskian of $\{e^x, e^{-x}, 1\}$ is nonzero when $x = 0$.

46. Suppose f and g are functions such that $f(17) = 2, f'(17) = 1, g(17) = 3$ and $g'(17) = 2$. Are f and g linearly independent? Why or why not?

47. Given each of the vector spaces (or subspaces) W, find a basis of W, and the dimension of W. (You may assume without proof that these are all subspaces.)

    (a) $W = \mathbb{R}^4$

    (b) $W = M_{42}(\mathbb{R})$ (4×2 matrices with real entries)

    (c) $W = \mathbb{P}_3$ (this is notation for the set of polynomials whose degree is ≤ 3).

48. Given each of the vector spaces (or subspaces) W, find a basis of W, and the dimension of W. (You may assume without proof that these are all subspaces.)

    (a) $W = Span((2,3), (12,18), (-2,-3))$

    (b) $W = \{(x,y,z) \in \mathbb{R}^3 : 2x + y - 3z = 0\}$.

    (c) $W =$ the line in $\mathbb{R}^4$ with parametric equations $x_1 = 4t, x_2 = -3t, x_3 = 2t, x_4 = 0$.

49. Given each of the vector spaces (or subspaces) W, find a basis of W, and the dimension of W. (You may assume without proof that these are all subspaces.)

    (a) $V = C(\mathbb{R}, \mathbb{R})$; W is the subspace of V consisting of functions f satisfying the differential equation $f'(x) = f(x)$.

    (b) $W = \left\{ \begin{pmatrix} 2s - 5t \\ s \\ 4t \end{pmatrix} : s, t \in \mathbb{R} \right\}$

(c) $W =$ the set of 2×2 matrices which are equal to their transpose

50. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^4$; let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^3$, and let A be a 4×3 matrix. Determine whether each of the following quantities are a **scalar**, a **vector** (in which case you should say if it belongs to $\mathbb{R}^3$ or $\mathbb{R}^4$), a **matrix** (in which case you should give its size), or **nonsense**.

    Note: For the purposes of this problem, matrices with a single column should be called vectors.

    (a) $\mathbf{x} \cdot \mathbf{x}$
    (b) $\mathbf{x} \cdot \mathbf{x} \cdot \mathbf{x}$
    (c) $(\mathbf{x} \cdot \mathbf{x}) \cdot \mathbf{x}$
    (d) $(\mathbf{x} \cdot \mathbf{x})\mathbf{x}$
    (e) $(\mathbf{x} + 2\mathbf{y}) \cdot \mathbf{y}$
    (f) $\mathbf{v} \cdot A\mathbf{x}$
    (g) $A\mathbf{v} \cdot A\mathbf{w}$
    (h) $\mathbf{x} \cdot A^T\mathbf{y}$

51. Same directions as the previous problem (where $\mathbf{v}$, $\mathbf{w}$, A, $\mathbf{x}$ and $\mathbf{y}$ are the same types of objects as in the previous problem):

    (a) $(\mathbf{v} \cdot \mathbf{w})A\mathbf{x}$
    (b) $A((\mathbf{x} \cdot \mathbf{y})A\mathbf{w} \cdot \mathbf{y})$
    (c) $dist(\mathbf{x}, \mathbf{x})$
    (d) $||2\mathbf{y} + \mathbf{x}||$
    (e) $||\mathbf{y}||\mathbf{y}$
    (f) $||\mathbf{w}||\mathbf{x}$
    (g) $dist(\mathbf{w}, 2\mathbf{v})A$
    (h) $dist(\mathbf{x}, \mathbf{w})\mathbf{y} \cdot \mathbf{x}$

52. Let $\mathbf{v} = (-2, 2, -1)$, let $\mathbf{w} = (0, 1, 2)$ and let $\mathbf{x} = (3, -1, -1)$. Compute the following:

    (a) $\mathbf{v} \cdot \mathbf{w}$
    (b) $\mathbf{x} \cdot 2\mathbf{w}$
    (c) $\mathbf{x} \cdot (\mathbf{v} - \mathbf{x})$
    (d) $dist(\mathbf{v}, \mathbf{w})$
    (e) $||\mathbf{v}||$
    (f) $||\mathbf{v} + 2\mathbf{w} - \mathbf{x}||$
    (g) $\frac{\mathbf{v}}{||\mathbf{v}||}$
    (h) $dist(3\mathbf{v}, 2\mathbf{x} - \mathbf{w})$

53. Suppose $\mathbf{v}, \mathbf{w}, \mathbf{x}$ are vectors in some dot product space V such that $\mathbf{v} \cdot \mathbf{w} = 4$, $||\mathbf{v}|| = 5$, $\mathbf{v} \cdot \mathbf{x} = -3$ and $\mathbf{x} \cdot \mathbf{x} = 20$. Compute the following:

    (a) $-\mathbf{x} \cdot 2\mathbf{v}$
    (b) $\mathbf{v} \cdot (4\mathbf{x} - 3\mathbf{w})$
    (c) $\mathbf{v} \cdot \mathbf{v}$
    (d) $\mathbf{w} \cdot \frac{1}{||\mathbf{w}||^2}\mathbf{w}$
    (e) $||\mathbf{x}||$
    (f) $(\mathbf{v} + \mathbf{x}) \cdot (\mathbf{v} + \mathbf{x})$
    (g) $2\mathbf{x} \cdot \mathbf{w} - x \cdot 2\mathbf{w}$
    (h) $\mathbf{v} \cdot ||\mathbf{v}||\mathbf{v}$

54. Consider the dot product space $C([-1, 1], \mathbb{R})$, and let $f(x) = x$ and $g(x) = x^2$. Compute the following:

(a) $f \cdot g$
(b) $f \cdot f$
(c) $||f||$
(d) $||f + g||$
(e) $dist(f, g)$
(f) $dist(f, 2)$

55. Find all real numbers k such that the vectors $(1, 2, k)$ and $(2, 4, 3)$ are distance 3 apart.

56. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. Suppose that for *some* nonzero $\mathbf{v} \in \mathbb{R}^3$, $\mathbf{v} \cdot \mathbf{a} = \mathbf{v} \cdot \mathbf{b}$. Is it necessarily the case that $\mathbf{a} = \mathbf{b}$? Justify your answer (this means prove the result if it is true, and provide a counterexample if it is false).

57. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. Suppose that for *every* $\mathbf{v} \in \mathbb{R}^3$, $\mathbf{v} \cdot \mathbf{a} = \mathbf{v} \cdot \mathbf{b}$. Prove that $\mathbf{a} = \mathbf{b}$.

    Hint: Write $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$. Make some clever choices of $\mathbf{v}$ and use the hypothesis of this problem to show that $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$ (which means $\mathbf{a} = \mathbf{b}$).

58. Let $\mathbf{v} = (-2, 0, 1, 3)$ and $\mathbf{w} = (1, 5, -2, 3)$.

    (a) Find $\pi_{\mathbf{w}}(\mathbf{v})$, the projection of $\mathbf{v}$ onto $\mathbf{w}$.
    (b) Find the projection of $\mathbf{w}$ onto $\mathbf{v}$.
    (c) Find $\pi_{\mathbf{w}^\perp}(\mathbf{v})$, the component of $\mathbf{v}$ orthogonal to $\mathbf{w}$.
    (d) Find the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$.
    (e) Find a unit vector in the same direction as $\mathbf{w}$.
    (f) Find a vector of length 8 in the same direction as $\mathbf{w}$.
    Hint: Start with the answer to part (e).
    (g) Determine whether the vectors $\mathbf{v}$ and $(1, 1, 5, -1)$ are orthogonal.
    (h) If $(3, 1, 4x, 2x) \perp \mathbf{v}$, find the value of x.

59. Without using any methods from calculus, find the point on the line $y = 3x$ which is closest to the point $(8, 3)$.

    Hint: This has something to do with projections. Draw a picture and identify some vectors to help you figure out what to compute.

60. Use the Gram-Schmidt procedure to convert each of the following sets of vectors into an orthonormal basis of the space they span:

    (a) $\{(3, -4, 5), (-3, 14, -7)\}$
    (b) $\{(1, -4, 0, 1), (7, -7, -4, 1), (6, 3, 6, -3)\}$
    (c) $\{(1, -1, 0, 1, 1), (3, -3, 2, 5, 5), (5, 1, 3, 2, 8)\}$

61. Let $\mathbf{v} = (1, 3, 5)$ and let $W = Span((1, 3, -2), (5, 1, 4))$.

    (a) Compute the projection of $\mathbf{v}$ onto W.
    (b) Compute the component of $\mathbf{v}$ orthogonal to W.

(c) Find the distance between $\mathbf{v}$ and W.

(d) Find the element of W which is closest to $\mathbf{v}$.

62. Let $V = C([0,1],\mathbb{R})$. Consider the subspace W of V consisting of functions of the form $f(x) = mx + b$, where m and b are constants.

    (a) Find a basis of W.

    (b) Use the Gram-Schmidt procedure to convert your answer to part (a) into an orthonormal basis of W.

    (c) Let $g(x) = x^2$. Find the projection of g onto W.
    Note: The answer to part (c) should be thought of as the linear function which best approximates the function $g(x) = x^2$ on the interval $[0,1]$.

63. Find the linear function which best approximates the function $f(x) = x^3 - x^2$ on the interval $[0,1]$.

64. Find the linear function which best approximates the function $f(x) = x^3 - 4x^2 + 4x$ on the interval $[0,3]$.

65. Suppose an object sits on the plane in $\mathbb{R}^3$ that is described as the set of points satisfying the equation $8x - 5y + z = 0$. If a force is applied to the object in the direction $\mathbf{f} = (2,-1,-3)$, in what direction will the object move? (Ignore the effect of friction, and assume that the force is sufficiently large to move the object.)

    Hint: This has something to do with projections.

66. In each part of this problem, you are given a vector space V and a subspace W of V. Describe the orthogonal complement $W^\perp$ (by "describe", I mean give the dimension of $W^\perp$ and write down a basis of $W^\perp$).

    (a) $V = \mathbb{R}^2$; $W = Span((3,5))$

    (b) $V = \mathbb{R}^2$; $W =$ the $y-$axis

    (c) $V = \mathbb{R}^2$; $W = \{(x,y) : 3x - 7y = 0\}$

    (d) $V = \mathbb{R}^3$; $W = Span((2,1,-3),(0,2,-5))$

    (e) $V = \mathbb{R}^3$; $W = Span((2,1,-3))$

    (f) $V = \mathbb{R}^3$; $W = \{(x,y,z) : 2x + 3y + z = 0\}$
    Hint: First, find a basis of W.

67. In each part of this problem, you are given a vector space V and a subspace W of V. Describe the orthogonal complement $W^\perp$.

    (a) $V = \mathbb{R}^4$; $W = Span((1,0,0,0),(0,1,0,0),(0,0,0,1))$

    (b) $V = \mathbb{R}^4$; $W = Span((0,3,0,0))$

    (c) $V = \mathbb{R}^4$; $W = Span((1,-2,5,2))$

(d) $V = \mathbb{R}^4$; $W = Span((1,0,-1,0),(-4,-1,0,1))$

68. Prove (using vectors and dot products) that the diagonals of a rhombus are perpendicular.

    Hint: Draw a rhombus and think of the sides as vectors; give them names like **v** and **w**. (You should only need use two letters for the sides, even though there are four sides.) Since the shape you drew is a rhombus, what is assumed to be true about **v** and **w**? Draw the diagonals of the rhombus, figure out what they are in terms of **v** and **w**, and then show they are orthogonal.

69. Prove (using vectors and dot products) that if the diagonals of a parallelogram have the same length, then the parallelogram is a rectangle.

70. Let $S = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ be a set of nonzero vectors in an inner product space such that for all $i \neq j$, $\mathbf{v}_i \perp \mathbf{v}_j$. Prove that S is a linearly independent set of vectors.

    Hint: Suppose $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_n = \mathbf{0}$. Now take the dot product of both sides of this equation with $\mathbf{v}_1$, and use the hypothesis of orthogonality (and some other stuff) to explain why $c_1 = 0$. Proceed from there.

71. (a) Write the normal equation of the plane in $\mathbb{R}^3$ passing through the points $(2,1,-3)$, $(4,-3,2)$ and $(1,-5,1)$.

    (b) The line in $\mathbb{R}^3$ passing through $(1,0,2)$ and $(3,-1,5)$ intersects the plane described in part (a) in one point. Find the coordinates of this point.

72. For each of the following functions T, determine (with proof) whether or not T is a linear transformation from V_1 to V_2.

    (a) $V_1 = V_2 = \mathbb{R}^3$; $T(x,y,z) = (x-y,0,0)$.

    (b) $V_1 = V_2 = \mathbb{R}^3$; $T(x,y,z) = (x^2,y,z)$.

    (c) $V_1 = \mathbb{R}^3$; $V_2 = \mathbb{R}^2$; $T(x,y,z) = (x+y+2z, y-2z)$.

73. For each of the following functions T, determine (with proof) whether or not T is a linear transformation from V_1 to V_2.

    (a) $V_1 = V_2 = \mathbb{R}^2$; T reflects points through the line $y = x$.

    (b) $V_1 = C^\infty(\mathbb{R},\mathbb{R})$; $V_2 = \mathbb{R}$; $T(f) = f'(0)$.

    (c) $V_1 = C([0,1],\mathbb{R})$; $V_2 = \mathbb{R}$; $T(f) = \int_0^1 (f(x))^2\,dx$.

74. For each of the following functions T, determine (with proof) whether or not T is a linear transformation from V_1 to V_2.

    (a) $V_1 = M_n(\mathbb{R})$; $V_2 = \mathbb{R}$; $T(A) = tr(A)$.

    (b) $V_1 = V_2 = M_2(\mathbb{R})$; T is defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & 0 \\ 0 & c+d \end{pmatrix}.$$

(c) $V_1 = M_{32}(\mathbb{R})$; $V_2 = \mathbb{R}^2$; $T(A)$ is the first row of A.

75. (a) Suppose A is a 7×3 matrix whose entries are real numbers. Let T be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. What vector space is the domain of T? To what vector space do the outputs of T belong?

    (b) How many rows and columns must a matrix A have in order to define a linear transformation from $\mathbb{R}^4$ into $\mathbb{R}^5$ by the formula $T(\mathbf{x}) = A\mathbf{x}$?

76. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that $T(1,0) = (-1,3,7)$ and $T(0,1) = (0,-2,-2)$.

    (a) What is $T(x,y)$ for an arbitrary vector $(x,y) \in \mathbb{R}^2$?

    (b) Find $T(2,3)$ and $T(-4,1)$.

77. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T(-1,2) = (2,1)$ and $T(1,3) = (-2,4)$.

    (a) Find $T(x,y)$ for an arbitrary $(x,y) \in \mathbb{R}^2$.

    (b) Find $T(2,5)$.

78. Find the standard matrix of each of the following linear transformations.

    (a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ where T is projection onto the vector $(2,1,-2)$;

    (b) $T : \mathbb{R}^4 \to \mathbb{R}^2$ where $T(x_1,x_2,x_3,x_4) = (x_1 - 5x_3 + x_4, 0)$;

    (c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where T stretches vectors by a factor of 2, then rotates the plane $\pi/3$ radians counterclockwise;

79. Find the standard matrix of each of the following linear transformations.

    (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where T rotates vectors by an angle of $2\pi/3$ radians clockwise;

    (b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ described by $T(1,0,0) = (1,2,3)$; $T(0,2,0) = (4,4,8)$; $T(0,0,1) = (0,0,1)$;

    (c) $T : \mathbb{R}^3 \to \mathbb{R}^3$ where T reflects points in $\mathbb{R}^3$ through the $xy-$plane.

80. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation that first reflects points through the $x-$axis, then reflects points through the $y-$axis.

    (a) Find the standard matrix of T.

    (b) Show that T is a rotation, and find the angle T rotates points by.

81. A transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called an **orthogonal transformation** if T "preserves dot product", i.e. $T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

    (a) Prove that if A is the standard matrix of T, then $A^T A = I$.

(b) Prove that if $A \in M_n(\mathbb{R})$ is such that $A^T A = I$, then the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ is orthogonal.

Note: This means that one can define a square matrix A to be **orthogonal** if $A^T A = I$.

(c) Prove that a rotation in $\mathbb{R}^2$ (by any angle θ) is an orthogonal transformation.

82. (a) Prove that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal, and if $\mathbf{x} \perp \mathbf{y}$, then $T(\mathbf{x}) \perp T(\mathbf{y})$.

(b) Prove that "orthogonal transformation preserve norms", i.e. if $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal, then $||T(\mathbf{x})|| = ||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{R}^n$.

(c) Prove that "orthogonal transformation preserve angles", i.e. if $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal, then the angle between $\mathbf{x}$ and $\mathbf{y}$ is the same as the angle between $T(\mathbf{x})$ and $T(\mathbf{y})$.

83. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that $T(1,0,0) = (1,-1)$, $T(0,1,0) = (2,0)$ and $T(0,0,1) = (0,1)$. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto the vector $(3,4)$, and let $R : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation $R(x,y) = (x+2y, y, -x-y)$. Find the standard matrix of the linear transformation RST.

84. For each of the following linear transformations $T : V_1 \to V_2$:

- Find the dimension of the kernel of T, and give a basis of $ker(T)$;
- Find the dimension of the image of T, and give a basis of $im(T)$.

You do not need to prove that the transformations T are linear.

(a) $V_1 = V_2 = \mathbb{R}^3$; $T(x,y,z) = (x-y,0,0)$.

(b) $V_1 = \mathbb{R}^3$; $V_2 = \mathbb{R}^2$; $T(x,y,z) = (x+y+2z, y-2z)$.

(c) $V_1 = V_2 = \mathbb{R}^2$; T reflects points through the line $y = x$.

(d) $V_1 = \mathbb{R}^2$; $V_2 = \mathbb{R}^4$; $T(x,y) = (x+3y, -2x-6y, 6x+18y, 0)$.

85. Same directions as the previous question:

(a) $V_1 = V_2 = \mathbb{P}^3$ (the space of polynomials of degree ≤ 3); $T(f) = f''$.

(b) $V_1 = \mathbb{P}^3$; $V_2 = \mathbb{R}$; $T(f) = f(0) + f(1)$.

(c) $V_1 = V_2 = M_2(\mathbb{R})$; T is defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & 0 \\ 0 & c+d \end{pmatrix}.$$

(d) $V_1 = M_{32}(\mathbb{R})$; $V_2 = \mathbb{R}^2$; $T(A)$ is the first row of A.

86. Let $T : V_1 \to V_2$ be a linear transformation.

Prove that if T is injective, then for any linearly independent set $\{\mathbf{v}_1, .., \mathbf{v}_n\}$ of vectors in V_1, the set $\{T(\mathbf{v}_1), ..., T(\mathbf{v}_n))\}$ is a linearly independent set of vectors in V_2.

Hint: Start by assuming that there are scalars $c_1, ..., c_n$ such that

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + ... + c_nT(\mathbf{v}_n) = \mathbf{0}.$$

Show that the scalars must all be equal to zero. Use the fact that T is linear, and use the fact that T is injective (make it clear in your proof where you use these facts).

87. Let $T : V_1 \to V_2$ be a linear transformation.

    Prove that if T is surjective, then for any set $\{\mathbf{v}_1, .., \mathbf{v}_n\}$ of vectors which span V_1, the set $\{T(\mathbf{v}_1), ..., T(\mathbf{v}_n))\}$ spans V_2.

    Hint: Let $\mathbf{w} \in V_2$ (the goal is to write $\mathbf{w}$ as a linear combination of the vectors $T(\mathbf{v}_j)$). First, use the fact that T is surjective to find a $\mathbf{v} \in V_1$ such that $T(\mathbf{v}) = \mathbf{w}$. Then, use the fact that the vectors $\mathbf{v}_j$ span V_1 to do something with $\mathbf{v}$. Proceed from there.

88. Suppose A is a 5×8 matrix.
    (a) Which vector space is $C(A)$ a subspace of? (I'm thinking of the answer as being $\mathbb{R}^n$ for some n.)
    (b) Which vector space is $R(A)$ a subspace of?
    (c) Which vector space is $N(A)$ a subspace of?
    (d) Which vector space is $N(A^T)$ a subspace of?

89. (a) Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Is it possible that for every $\mathbf{b}$, the equation $T(\mathbf{x}) = \mathbf{b}$ has no solution? Give a specific example if this is possible; if this is impossible, explain why not.
    (b) Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Is it possible that for every $\mathbf{b}$, the equation $T(\mathbf{x}) = \mathbf{b}$ has exactly one solution $\mathbf{x}$? Give a specific example if this is possible; if this is impossible, explain why not.
    (c) Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Is it possible that for every $\mathbf{b}$, the equation $T(\mathbf{x}) = \mathbf{b}$ has exactly two solutions? Give a specific example if this is possible; if this is impossible, explain why not.
    (d) Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Is it possible that for every $\mathbf{b}$, the equation $T(\mathbf{x}) = \mathbf{b}$ has infinitely many solutions? Give a specific example if this is possible; if this is impossible, explain why not.

90. For each given matrix, find the dimension of, and bases for, each of the four fundamental subspaces of the matrix.

    (a) $\begin{pmatrix} 2 & -3 & 6 \\ 3 & 0 & 9 \end{pmatrix}$

    (b) $\begin{pmatrix} 1 & -2 & 5 \\ -1 & 2 & -5 \\ 2 & -4 & 10 \\ 1 & -2 & 5 \end{pmatrix}$

    (c) $\begin{pmatrix} 2 & 0 & 5 & 3 & -1 \end{pmatrix}$

    (d) $\begin{pmatrix} 2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & -2 & 6 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

91. Let $T(\mathbf{x}) = A\mathbf{x}$, where A is a 9×9 matrix with 6 linearly independent columns.

(a) What vector space is the domain of T?
(b) To what vector space do the outputs of T belong?
(c) $ker(T)$ is a subspace of what vector space?
(d) $im(T)$ is a subspace of what vector space?
(e) Find the dimension of $ker(T)$.
(f) Find the dimension of $im(T)$.
(g) Is there a vector $\mathbf{b}$ such that the equation $T(\mathbf{x}) = \mathbf{b}$ has no solution? Explain.
(h) Is there a vector $\mathbf{b}$ such that the equation $T(\mathbf{x}) = \mathbf{b}$ has exactly one solution? Explain.
(i) Is there a vector $\mathbf{b}$ such that the equation $T(\mathbf{x}) = \mathbf{b}$ has infinitely many solutions? Explain.
(j) Is T injective? Explain.
(k) Is T surjective? Explain.
(l) Is T bijective? Explain.
(m) Is T invertible? Explain.
(n) Does A^{-1} exist? Explain.

92. Let $T(\mathbf{x}) = A\mathbf{x}$, where A is a 9×9 matrix with 9 linearly independent columns. Answer the same questions (a)-(n) that were asked in Problem 91.

93. Let $T(\mathbf{x}) = A\mathbf{x}$, where A is a 4×8 matrix with 4 linearly independent columns. Answer the same questions (a)-(n) that were asked in Problem 91.

94. Let $T(\mathbf{x}) = A\mathbf{x}$, where A is a 4×8 matrix with 2 linearly independent columns. Answer the same questions (a)-(n) that were asked in Problem 91.

95. Let $T(\mathbf{x}) = A\mathbf{x}$, where A is a 10×7 matrix with 5 linearly independent columns. Answer the same questions (a)-(n) that were asked in Problem 91.

96. Let $T(\mathbf{x}) = A\mathbf{x}$, where A is a 10×7 matrix with 7 linearly independent columns. Answer the same questions (a)-(n) that were asked in Problem 91.

97. Find the inverse of each of the following matrices (if the matrix is not invertible, say so and explain why):

(a) (4) (b) (0) (c) $\begin{pmatrix} 2 & 3 \\ -5 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 5 \\ 2 & 0 \end{pmatrix}$

98. Find the inverse of each of the following matrices (if the matrix is not invertible, say so and explain why):

(a) $\begin{pmatrix} 10 & 5 \\ -6 & -3 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

99. Assume that the following chart shows the number of grams of nutrients per ounce of food indicated:

	BEEF	POTATO	CABBAGE
PROTEIN	20	5	1
FAT	4	7	12
CARBOHYDRATES	15	20	5

Suppose the army desires to use these three delectable foods to feed new recruits a dinner providing 305 grams of protein, 365 grams of fat, and 575 grams of carbohydrates. Write a system of equations which, when solved, will figure out how much of each food should be prepared for each recruit. (Be sure to define any variables which are used in your problem.)

100. Consider the following system of linear equations:

$$\begin{cases} x - 2y + 3z - \frac{w}{2} - 2 = 5 - 3x + z \\ -\sqrt{3}x + y + 5z + \pi^2 w = -4 \\ 2(x - 3z) = 6 \end{cases}$$

(a) Write the system as a matrix equation $A\mathbf{x} = \mathbf{b}$ (i.e. what is A? What is $\mathbf{x}$? What is $\mathbf{b}$?)

(b) Write the system as a vector equation.

(c) Essentially, this system is asking whether some vector (say $\mathbf{v}$) is in the span of some other vectors (say $\mathbf{w}_1$, $\mathbf{w}_2$, ...). In this context, what is the $\mathbf{v}$? What are the $\mathbf{w}_1$, $\mathbf{w}_2$, ...?

101. Suppose A is some matrix with $N(A) = Span((1, 3, 0, -2), (1, 0, 0, -2))$. Suppose also that $A(1 - 3, 0, 2) = (2, 0, 3, 1, 4)$.

(a) How many equations, and how many variables are in the system $A\mathbf{x} = (2, 0, 3, 1, 4)$?

(b) What are the dimensions of the four fundamental subspaces of A?

(c) Describe all solutions to the equation $A\mathbf{x} = (2, 0, 3, 1, 4)$.

102. Suppose A is some matrix with $N(A) = \{0\}$. Suppose also that $A(1 - 3, 0, 2) = (8, 1, -2, 5)$.

(a) How many equations, and how many variables are in the system $A\mathbf{x} = (8, 1, -2, 5)$?

(b) What are the dimensions of the four fundamental subspaces of A?

(c) Describe all solutions to the equation $A\mathbf{x} = (8, 1, -2, 5)$.

103. Suppose A is some matrix with $N(A) = \{0\}$. Suppose also that $A(2, 5) = (1, 1, 1, 1, 1)$.

(a) How many equations, and how many variables are in the system $A\mathbf{x} = (1, 1, 1, 1, 1)$?

(b) What are the dimensions of the four fundamental subspaces of A?

(c) Describe all solutions to the equation $A\mathbf{x} = (1, 1, 1, 1, 1)$.

104. Use Gaussian elimination to row reduce the following matrix to a row-echelon form, then to a reduced row-echelon form. In this problem, and only in this problem, you are required to perform Gaussian elimination by hand and show all your steps.

$$A = \begin{pmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 3 \end{pmatrix}$$

105. Find the reduced row-echelon form of the matrix below:

Note: Henceforth, to perform Gaussian elimination I recommend that you use a calculator, or *Mathematica*, or an online row reducer like the ones at

`http://www.math.purdue.edu/~dvb/matrix.html` or
`http://www.math.odu.edu/~bogacki/cgi-bin/lat.cgi?c=roc`

$$\begin{pmatrix} 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 6 & 6 & 0 & 20 & 19 \end{pmatrix}$$

106. Solve this system:

$$\left\{ \begin{array}{rcrcrcr} x & - & 2y & + & z & = & 7 \\ 2x & - & y & + & 4z & = & 17 \\ 3x & - & 2y & + & 2z & = & 14 \end{array} \right.$$

107. Solve this system:

$$\left\{ \begin{array}{rcrcrcr} x & + & 2y & - & z & = & 3 \\ x & + & 3y & + & z & = & 5 \\ 3x & + & 8y & + & 4z & = & 17 \end{array} \right.$$

108. Solve this system:

$$\left\{ \begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & 5x_3 & & & = & 1 \\ x_1 & - & 3x_2 & + & 3x_3 & - & x_4 & = & -4 \\ & & x_2 & & & & 2x_4 & = & -5 \\ -2x_1 & + & 2x_2 & + & 2x_3 & + & 3x_4 & = & -2 \end{array} \right.$$

109. Solve this system:

$$\left\{ \begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & + & x_4 & = & 7 \\ & & 2x_2 & - & x_3 & - & 7x_4 & = & 6 \\ & & & - & 3x_3 & + & 2x_4 & = & -6 \end{array} \right.$$

110. Solve this system:

$$\left\{ \begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 1 \\ 2x & + & 3y & + & 4z & = & 0 \\ 3x & + & 4y & + & 5z & = & 1 \end{array} \right.$$

111. Solve this system:

$$\begin{cases} x_1 - 2x_2 - 3x_3 + 5x_4 - 2x_5 = 4 \\ \qquad\qquad 2x_3 - 6x_4 + 3x_5 = 2 \\ \qquad\qquad\qquad\qquad 5x_5 = 10 \end{cases}$$

112. Solve three of these four problems:

(a) Determine if the following three planes in $\mathbb{R}^3$ have at least one common point of intersection: $x + 2y + z = 4$, $y - z = 1$ and $x + 3y = 0$.

(b) Find the unique polynomial of degree at most 3 which goes through the points $(1, 1)$,$(2, 3)$,$(3, 6)$ and $(4, 10)$.

Hint: write the polynomial as $y = ax^3 + bx^2 + cx + d$; substitute in the given points to get a system of four equations in the four variables (a, b, c, d).

(c) Express $(1, -2, 5)$ as a linear combination of $(1, -3, 2)$, $(2, -4, -1)$ and $(1, -5, 7)$, if it can be done; if not, explain why this problem is impossible.

(d) Let W be the subspace of $\mathbb{R}^4$ spanned by $(1, 2, -5, 2)$ and $(0, 1, 3, -1)$. Find a basis of $W^\perp$.

113. Assume the matrices A and B given below are row equivalent. Find a basis for $C(A)$, a basis for $R(A)$, and a basis for $N(A)$.

$$A = \begin{pmatrix} -1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{pmatrix}; B = \begin{pmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

114. Assume the matrices A and B given below are row equivalent. Find a basis for $C(A)$, a basis for $R(A)$, and a basis for $N(A)$.

$$A = \begin{pmatrix} -1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{pmatrix}; B = \begin{pmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

115. In each of the following, determine (a) if the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution ("nontrivial" means a solution other than $\mathbf{x} = \mathbf{0}$) and (b) if the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for every choice of $\mathbf{b}$.

(a) A is a 3×3 matrix with 3 pivot columns.
(b) A is a 2×4 matrix with rank 2.
(c) A is an 8×6 matrix with $1-$dimensional null space.
(d) A is a 5×3 matrix with 3 pivot columns.
(e) A is a 4×4 matrix with rank 3.

116. Consider the system of equations $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 1 & -2 & 4 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & 4 & -4 & 3 \\ 3 & 2 & 4 & -2 \\ 1 & 0 & 2 & 2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 4 \\ 2 \\ -11 \\ -2 \\ -2 \end{pmatrix}.$$

For each j, let $\mathbf{a}_j$ denote the j^{th} column of A.

(a) Find the solution set of this system.
(b) Find bases for the column space of A, the row space of A, and the null space of A.
(c) Find the dimensions of $R(A)$, $C(A)$, $N(A)$ and $N(A^T)$.
(d) Find the rank of A.
(e) Let $T : \mathbb{R}^a \to \mathbb{R}^b$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. What are a and b? Find bases for the kernel and image of T, and the dimensions of the kernel and image of T.
(f) Is $\mathbf{b}$ in the span of the columns of A? If so, write $\mathbf{b}$ as a linear combination of the columns of A. If not, explain why not.
(g) Find a vector which is not in the column space of A.
(h) Is there any vector $\mathbf{v} \in \mathbb{R}^5$ for which $A\mathbf{x} = \mathbf{v}$ has exactly one solution? If so, find such a vector $\mathbf{v}$. If not, explain why not.
(i) Is the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ linearly independent? Why or why not?
(j) Is the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ linearly independent? Why or why not?
(k) Is the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{b}\}$ linearly independent? Why or why not?

117. Use the Gauss-Jordan method to find the inverse of this matrix (if the inverse exists). In this problem, and only in this problem, you are required to perform this method by hand and show all your steps.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

118. Use appropriate technology to find the inverse of this matrix (if the inverse exists).

$$\begin{pmatrix} 3 & 10 & 3 & 8 \\ 3 & -2 & 8 & 7 \\ 2 & 1 & 4 & -5 \\ 5 & 11 & 7 & 3 \end{pmatrix}$$

119. Use appropriate technology to find the inverse of this matrix (if the inverse exists).

$$\begin{pmatrix} 3 & 2 & -1 & 4 \\ 3 & 0 & -2 & 1 \\ 1 & 1 & 4 & 0 \\ 0 & -3 & 1 & -2 \end{pmatrix}$$

120. Find the least-squares solution (a decimal approximation is okay; keep at least four decimal places) of the following system of equations:

$$\begin{cases} w - 2x + y - 2z = 4 \\ 2w + x + 4z = 5 \\ -3x - 2y + 5z = -2 \\ -w - x - y + 3z = 0 \\ w + 4y + z = 1 \\ 2x - y + 6z = -3 \end{cases}$$

121. Suppose you obtain a bunch of data points (x, y, z) which are supposed to fit a model of the form

$$z = \alpha + \beta x^2 + \gamma y^3$$

where α, β, γ are constants. Suppose that the eight data points you obtain are:

$$(1, 2, -1) \quad (0, 2, 0) \quad (-2, 3, -1) \quad (1, 5, 2)$$

$$(-1, -1, -3) \quad (-4, 0, -6) \quad (0, -2, 7) \quad (2, 2, 7)$$

(a) Write down the matrix equation $A\mathbf{x} = \mathbf{b}$ you would solve to find α, β, γ (i.e. what is A? What is $\mathbf{x}$? What is $\mathbf{b}$?)
(b) Write down a formula in terms of A and/or $\mathbf{b}$ which gives the least-squares solution $\widehat{\mathbf{x}}$.
(c) Write down the formula in terms of A, $\mathbf{b}$ and/or $\widehat{\mathbf{x}}$ which gives the corresponding $\widehat{\mathbf{b}}$.
(d) Compute the least-squares solution $\widehat{\mathbf{x}}$ to the matrix equation of part (a) (a decimal approximation is okay; keep at least four decimal places).
(e) Compute the corresponding $\widehat{\mathbf{b}}$.
(f) Compute $||\mathbf{b} - \widehat{\mathbf{b}}||$.
(g) What is the significance of the quantity you computed in part (f)? In other words, if that quantity is small, what does that say about the least-squares solution? If it is big, what does that say about the least-squares solution?
(h) What is the model that best fits the data?
(i) Use the model of part (h) to predict the value of z when $x = 4$ and $y = 5$.

122. Suppose you are looking at a curve on an oscilloscope which for theoretical reasons should be the graph of some function of the form

$$y = a\cos x + b\sin x + c\cos 2x + d\sin 2x$$

where a, b, c and d are constants. If you see the following (x, y) points on your oscilloscope:

$$(0, 0) \quad (.2, -1.6) \quad (.7, 5.1) \quad (1, 8) \quad (1.3, 8.7) \quad (1.8, 4.4) \quad (2.2, -1.5)$$

use least-squares to compute (decimal approximations to) the values of a, b, c and d which best fit this data.

123. Suppose you are looking at data which is supposed to fit an exponential equation, i.e. a model of the form

$$y = Ce^{kx}$$

where C and k are constants. Suppose your data points are $(2,5)$, $(3,8)$ and $(4,17)$. Use least-squares to find (decimal approximations to) the values of C and k which best fit this model.

How to proceed: First, take the natural log of both sides of the model to obtain what is called a "log-log" equation. Then, use log rules to rewrite the log-log equation as a linear equation in terms of the variables $\ln C$ and k. Then use least-squares to compute $\ln C$ and k; last, find C.

124. Find the projection of $(2,1,-2,4,-1)$ onto the subspace spanned by the three vectors $(1,2,-1,0,4)$, $(1,-5,-1,8,2)$ and $(0,0,2,3,-2)$.

125. Find the determinant of each given matrix, if it exists (you can use a computer or calculator, but you should be able to compute this by hand):

(a)
$$\begin{pmatrix} -3 \end{pmatrix}$$

(b)
$$\begin{pmatrix} 1 & 2 \end{pmatrix}$$

(c)
$$\begin{pmatrix} -5 & 2 \\ 8 & -7 \end{pmatrix}$$

(d)
$$\begin{pmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{pmatrix}$$

(e)
$$\begin{pmatrix} -1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 5 & -4 \end{pmatrix}$$

126. Find the determinant of each given matrix, if it exists (you can use a computer or calculator, but you should be able to compute this by hand):

(a)
$$\begin{pmatrix} -3 & 2 & 4 \\ -1 & 5 & 2 \\ 1 & -3 & -2 \end{pmatrix}$$

(b)
$$\begin{pmatrix} -3 & 1 & 1 \\ -1 & 2 & -1 \\ 4 & 0 & -2 \\ 1 & 7 & -3 \end{pmatrix}$$

(c)
$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ -2 & 3 & 3 & -1 \\ 1 & 0 & 5 & 2 \end{pmatrix}$$

(d)
$$\begin{pmatrix} 0 & 0 & -2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

127. Suppose that A, B and C are 4×4 matrices such that $\det A = 2$, $\det B = \frac{-2}{3}$ and $\det C = 0$.

   (a) Find $\det(4A)$.
   (b) Find $\det(B^{-1})$.
   (c) Find $2\det(3A)$.
   (d) Find $\det(BAB)$.
   (e) Which one or ones of the matrices A, B and C are invertible?
   (f) Is the matrix AC invertible? Is the matrix CA invertible?

128. Let $A \in M_n(\mathbb{R})$ be an orthogonal matrix (see Problem 81). What are the possible values of $\det A$? Explain.

129. Suppose that A and B are $n \times n$ square matrices.

   (a) Prove that if A and B have rank n, then AB also has rank n.
   Hint: use determinants.
   (b) Prove that if AB has rank n, then both A and B must both have rank n.
   Hint: use determinants.

130. (This problem must be done by hand.) Let $A = \begin{pmatrix} 2 & 6 \\ 7 & 1 \end{pmatrix}$.

   (a) Find the characteristic polynomial of this matrix.
   (b) Find the eigenvalues and eigenvectors of this matrix.

131. Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 2 & -1 & 1 \\ 2 & -1 & -8 \\ 4 & -2 & -7 \end{pmatrix}$.

132. (a) Suppose a 2×2 matrix has trace -2 and determinant -15.
      i. Find the eigenvalues of the matrix.
      ii. Is the matrix diagonalizable? Why or why not?
   (b) Suppose a 3×3 matrix has determinant -32 and trace 6. If the matrix has two eigenvalues, one of multiplicity one and one of multiplicity two, find the eigenvalues of the matrix (stating which one is of which multiplicity).

133. Suppose a 3×3 matrix A has the following eigenvalues and eigenvectors:

$$\lambda = 2 \leftrightarrow \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \lambda = -1 \leftrightarrow \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \qquad \lambda = 4 \leftrightarrow \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$

   Find A.

134. Compute the exact value of A^{25} if

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}.$$

135. Compute the exact value of e^A if

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}.$$

136. Let $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

   (a) Show that 0 is the only eigenvalue of B.
   (b) Prove that B is not diagonalizable by showing that it does not have two linearly independent eigenvectors.
   (c) Compute B^2.
   (d) Compute B^n for all $n \geq 2$.
   (e) Compute $\exp(B)$.

137. Suppose $\{x_n\}$ is a sequence defined by setting $x_0 = 2$; $x_1 = -3$ and for $n \geq 2$, setting $x_n = -10x_{n-2} + 7x_{n-1}$.

   (a) Find x_2, x_3 and x_4.
   (b) Find the exact value of x_{2000}.

138. Denote the owl population and wood rat population in a certain ecosystem at time k by $\mathbf{x}_k = (O_k, R_k)$. Suppose biologists determine that these populations evolve by the equations

$$\begin{cases} O_{k+1} = .5\,O_k + .4\,R_k \\ R_{k+1} = -p\,O_k + 1.1\,R_k \end{cases}$$

where $p > 0$ is some unknown parameter representing the rate of deaths of rats due to predation by owls. Suppose initially that there are 250 owls and 2000 rats in the ecosystem.

   (a) Show that this set of recursive relations is equivalent to the matrix equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for some matrix A. Find A.
   (b) Explain how you would compute the number of rats in the ecosystem when $k = 100$. (You don't actually have to do the problem, but your explanation should be thorough.)

139. Suppose that the number of foxes $f(t)$ and number of rabbits $r(t)$ in a forest at time t is modeled by the following system of differential equations:

$$\begin{cases} f'(t) = & 4f(t) + r(t) \\ r'(t) = & -2f(t) + r(t) \end{cases}$$

If there are initially 13 foxes and 7 rabbits in the forest, find formulas for $f(t)$ and $r(t)$.

140. In a certain type of electrical circuit (containing a battery, resistor, inductor and capicator in series), Kirchoff's Law says that the charge on the capacitor at time t is a function $Q(t)$ which satisfies

$$L\,Q''(t) + R\,Q'(t) + \frac{1}{C}Q(t) = E(t)$$

where L, R and C are constants and $E(t)$ is a function. Suppose that $L = 2$, $R = 10$, $C = \frac{1}{12}$ and $E(t) = 0$. Find the general solution $Q(t)$ of this equation.

Remarks: the values of L, R and C we are using are not realistic; the problem is that for more realistic L, R and C the eigenvalues are imaginary numbers (to learn a bit about how this works, take Math 330). Moreover, the situation $E(t) = 0$ is also unrealistic, but we don't know enough to solve this for general $E(t)$... if you are interested in how to solve this in more realistic situations, let me know.

141. Find a function $y = y(x)$ which satisfies $y(0) = 2$, $y'(0) = -3$, $y''(0) = 1$ and

$$y''' - y'' - 4y' + 4y = 0.$$