1. Below, you are given the augmented matrix corresponding to a linear system of equations, together with the reduced row-echelon form of that augmented matrix:

$$(A\,|\,\mathbf{b}) = \left(\begin{array}{ccc|c} 0 & 1 & -2 & 3 \\ 2 & -3 & 1 & 0 \\ 4 & -7 & 4 & -3 \\ -6 & 13 & -11 & 12 \end{array}\right) \xrightarrow{\text{row ops}} \left(\begin{array}{ccc|c} 1 & 0 & \frac{-5}{2} & \frac{9}{2} \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = \text{rref}(A\,|\,\mathbf{b})$$

   (a) Write down the system of linear equations which correspond to the original matrix.
   (b) Solve the system $A\mathbf{x} = \mathbf{b}$.
   (c) Find a basis for the column space of A.
   (d) Is $\mathbf{b}$ in the span of the columns of A? Why or why not?
   (e) Find a basis for the null space of A.

2. Suppose that data obtained in an experiment is supposed to fit a model of the form

$$z = a + bx + cx^2 + dy$$

   where a, b, c and d are constants.

   (a) Set up a linear system which can be used to solve for a, b, c and d if the data points (of the form (x, y, z) obtained are $(2, 1, 5)$, $(-3, 1, 2)$, $(-2, 0, 2)$, $(1, 4, 3)$ and $(3, 5, 10)$. In particular, what are A, $\mathbf{x}$ and $\mathbf{b}$?
   (b) Write down the formula (in terms of A, $\mathbf{x}$ and/or $\mathbf{b}$) which computes the least-squares solution $\widehat{\mathbf{x}}$. (You do not actually have to compute $\widehat{\mathbf{x}}$.)

3. (a) If $\mathbf{v} = (2, -1, 5)$ and $\mathbf{w} = (0, 1, -2)$, compute $2\mathbf{v} - \mathbf{w}$.
   (b) Compute $(3, -1, 4) \cdot (2, 0, -5)$.
   (c) Compute the projection of $(-11, 3)$ onto $(2, 7)$.
   (d) Compute the distance between the vectors $(2, -1, 4, 3)$ and $(-4, 0, 7, -1)$.

4. (a) Find the transpose of the matrix $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -2 & 4 \end{pmatrix}$.
   (b) Find AB if $A = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & 1 \\ -2 & 5 & 3 \end{pmatrix}$.
   (c) Find the inverse of the matrix $\begin{pmatrix} 8 & -5 \\ -4 & 3 \end{pmatrix}$.
   (d) Find the determinant of the matrix $\begin{pmatrix} 2 & -5 & 1 \\ 0 & 4 & -1 \\ -3 & 3 & 2 \end{pmatrix}$.

5. Let $A = \begin{pmatrix} 3 & 4 \\ 1 & 6 \end{pmatrix}$.

   (a) Compute the exact value of A^{3000} (show all your work).

   (b) Compute e^A.

6. (a) Find the point of intersection (if there is one) of the two lines in $\mathbb{R}^3$ whose parametric equations are

$$\begin{cases} x = 3t \\ y = 1 + 2t \\ z = -1 - t \end{cases} \qquad \begin{cases} x = -2 + 4t \\ y = 7 - t \\ z = -3 + t \end{cases}$$

   (b) Write the normal equation of the plane in $\mathbb{R}^3$ whose parametric equations are

$$\begin{cases} x = -1 + 2s - 3t \\ y = 1 - s + t \\ z = 3 - 3s + 5t \end{cases} .$$

7. Suppose:

   - A is an invertible 3×3 matrix;
   - B is a 3×4 matrix;
   - C is a 4×3 matrix;
   - D is a 1×3 matrix;
   - $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are vectors in $\mathbb{R}^3$;
   - $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^4$; and
   - k, l and m are scalars.

   Determine whether the following expressions are a **matrix** (in which you should give its size), a **vector** (in which case you should give the vector space to which the vector belongs), a **scalar**, or **nonsense**:

   (a) $tr(A)$

   (b) $\mathbf{v} \cdot (\mathbf{vw})$

   (c) $\mathbf{x} \times \mathbf{y}$

   (d) $\mathbf{v} \times \mathbf{w}$

   (e) $||\mathbf{v} - \mathbf{w}|| \, ||\mathbf{x} + \mathbf{y}||$

   (f) $||\mathbf{x}||\mathbf{x}$

   (g) $kB^TA^{-1}m$

   (h) $(D\mathbf{x})A^2(\mathbf{z} + 3k\mathbf{y})$

   (i) $\det(A)B$

   (j) $\det(AB)$

   (k) $D\mathbf{z}m$

   (l) $m\mathbf{z}D$

8. Classify the following statements as true or false:
   (a) The vectors $(2, 1, -5)$ and $(3, 3, 2)$ are orthogonal.
   (b) $\mathbb{R}^4$ is a four-dimensional subspace of $\mathbb{R}^5$.
   (c) The function $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (2x, x - y, x + y)$ is a linear transformation.
   (d) The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which projects vectors onto the $y-$axis is injective.
   (e) The set of vectors lying on the plane $3x - 2y + 4z = 2$ is a subspace of $\mathbb{R}^3$.
   (f) The following set is linearly independent: $\{(1, 2, 1), (2, -5, 4), (3, -1, 2), (4, 4, -7)\}$
   (g) For square matrices A and B of the same size, $\det(A+B) = \det(A) + \det(B)$.
   (h) If $\mathbf{v}, \mathbf{w}$ and $\mathbf{x}$ are in $\mathbb{R}^n$, then $\mathbf{v} \cdot (\mathbf{w} + \mathbf{x}) = \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{x}$.
   (i) The vectors $(0, 0, 0, 0)$ and $(2, -3, 4, -1)$ are parallel.
   (j) The transformation $T : C(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ defined by $T(f) = f(0)$ is a linear transformation.

9. Fill in the blank with the word "always", "sometimes" or "never" to make the statement correct:
   (a) A 2×3 matrix is _____ diagonalizable.
   (b) If $T : V_1 \to V_2$ and $S : V_2 \to V_3$ are linear transformations, then the composition $S \circ T$ is _____ linear.
   (c) A system of 5 linear equations in 3 variables _____ has exactly one solution.
   (d) $|\mathbf{v} \cdot \mathbf{w}|$ is _____ less than or equal to $||\mathbf{v}|| \, ||\mathbf{w}||$.
   (e) The zero vector is _____ part of a basis.
   (f) Given a set of three vectors in $\mathbb{R}^4$, that set _____ spans $\mathbb{R}^4$.
   (g) Given a set of four vectors in $\mathbb{R}^4$, that set _____ spans $\mathbb{R}^4$.
   (h) Given a set of five vectors in $\mathbb{R}^4$, that set _____ spans $\mathbb{R}^4$.
   (i) A matrix with eigenvalue 0 is _____ invertible.
   (j) A set of one nonzero vector is _____ linearly independent.

10. Answer the following questions:
    (a) If W is an eight-dimensional subspace of $\mathbb{R}^{13}$, what is the dimension of $W^\perp$?
    (b) If a 10×7 matrix has 5 linearly independent columns, what is the dimension of the null space of this matrix?
    (c) If a 5×8 matrix is the matrix of a surjective linear transformation, what is the rank of the matrix?
    (d) If the eigenvalues of a matrix are $2, 2, -3$ and 1, what is the trace of the matrix?
    (e) How many vectors are there in a basis of a three-dimensional subspace of $\mathbb{R}^7$?

1. (a) If you call the variables x, y and z the system is

$$\begin{cases} y - 2z = 3 \\ 2x - 3y + z = 0 \\ 4x - 7y + 4z = -3 \\ -6x + 13y - 11z = 12 \end{cases}$$

(b) From the rref form, we see

$$\begin{cases} x - \frac{5}{2}z = \frac{9}{2} \\ y - 2z = 3 \end{cases} \Rightarrow \begin{cases} x = \frac{9}{2} + \frac{5}{2}z \\ y = 3 + 2z \end{cases}$$

Thus the solution set is

$$(x, y, z) = (\frac{9}{2} + \frac{5}{2}z, 3+2z, z) = (\frac{9}{2}, 3, 1) + z(\frac{5}{2}, 2, 1) = \left(\frac{9}{2}, 3, 1\right) + Span\left(\frac{5}{2}, 2, 1\right).$$

(c) Such a basis consists of the pivot columns of A; it is $\{(0, 2, 4, -6), (1, -3, -7, 13)\}$.

(d) Yes, because $A\mathbf{x} = \mathbf{b}$ has at least one solution.

(e) From the answer to (b), we see that since the solution is always of the form $\mathbf{x}_p + N(A)$, $N(A) = Span(\frac{5}{2}, 2, 1)$ so a basis for $N(A)$ is the single vector $(\frac{5}{2}, 2, 1)$.

2. (a) Plugging each point in for (x, y, z), we obtain the system

$$\begin{cases} a + 2b + 4c + 1 = 5 \\ a - 3b + 9c + d = 2 \\ a - 2b + 4c = 2 \\ a + b + c + 4d = 3 \\ a + 3b + 9c + 5d = 10 \end{cases}$$

Therefore we are trying to solve $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 1 & 2 & 4 & 1 \\ 1 & -3 & 9 & 1 \\ 1 & -2 & 4 & 0 \\ 1 & 1 & 1 & 4 \\ 1 & 3 & 9 & 5 \end{pmatrix}; \quad \mathbf{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}; \quad \mathbf{b} = \begin{pmatrix} 5 \\ 2 \\ 2 \\ 3 \\ 10 \end{pmatrix}.$$

Set up a linear system which can be used to solve for a, b, c and d if the data points (of the form (x, y, z) obtained are $(2, 1, 5)$, $(-3, 1, 2)$, $(-2, 0, 2)$, $(1, 4, 3)$ and $(3, 5, 10)$. In particular, what are A, $\mathbf{x}$ and $\mathbf{b}$?

(b) As always, $\widehat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$.

3. (a) $2\mathbf{v} - \mathbf{w} = 2(2,-1,5) - (0,1,-2) = (4,-3,12)$.
   (b) $(3,-1,4)\cdot(2,0,-5) = 3(2) - 1(0) + 4(-5) = 6 - 20 = -14$.
   (c) $proj_{(2,7)}(-11,3) = \frac{(2,7)\cdot(-11,3)}{(2,7)\cdot(2,7)}(2,7) = \frac{-1}{53}(2,7) = \left(\frac{-2}{53}, \frac{-7}{53}\right)$.
   (d) $||(2,-1,4,3)-(-4,0,7,-1)|| = ||(6,-1,-3,4)|| = \sqrt{(6,-1,-3,4)\cdot(6,-1,-3,4)} = \sqrt{36+1+9+16} = \sqrt{62}$.

4. (a) $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -2 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & -2 \\ 1 & 4 \end{pmatrix}$.
   (b) $AB = \begin{pmatrix} 1(3)+(-3)(-2) & 1(0)-3(5) & 1(1)-3(3) \\ 2(3)-2 & 2(0)+5 & 2+3 \end{pmatrix} = \begin{pmatrix} 9 & -15 & -8 \\ 4 & 5 & 5 \end{pmatrix}$.
   (c) $\begin{pmatrix} 8 & -5 \\ -4 & 3 \end{pmatrix}^{-1} = \frac{1}{8(3)-(-5)(-4)}\begin{pmatrix} 3 & 5 \\ 4 & 8 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & 5 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{5}{4} \\ 1 & 2 \end{pmatrix}$.
   (d) Repeat the first two columns to the right of the matrix; multiply along the diagonals and then add/subtract to get

$$\det\begin{pmatrix} 2 & -5 & 1 \\ 0 & 4 & -1 \\ -3 & 3 & 2 \end{pmatrix} = 16 - 15 + 0 - (-12) - (-6) - 0 = 19.$$

5. (a) First, find the eigenvalues of A. $\det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 4 \\ 1 & 6-\lambda \end{pmatrix} = (3-\lambda)(6-\lambda) - 4 = \lambda^2 - 9\lambda + 14 = (\lambda-7)(\lambda-2)$ so the eigenvalues are $\lambda = 2$ and $\lambda = 7$.
   Next, eigenvectors: when $\lambda = 2$, $A\mathbf{x} = \lambda\mathbf{x}$ gives $3x+4y = 2x$ and $x+6y = 2y$, i.e. $x = -4y$, so an eigenvector is $(-4,1)$. When $\lambda = 7$, $A\mathbf{x} = \lambda\mathbf{x}$ gives $3x+4y = 7x$ and $x+6y = 7y$, i.e. $x = y$, so an eigenvector is $(1,1)$. Therefore

$$A = S\Lambda S^{-1} = \begin{pmatrix} -4 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}\begin{pmatrix} -4 & 1 \\ 1 & 1 \end{pmatrix}^{-1}.$$

   Now

$$A^{3000} = S\Lambda^{3000}S^{-1} = \begin{pmatrix} -4 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2^{3000} & 0 \\ 0 & 7^{3000} \end{pmatrix}\frac{1}{-5}\begin{pmatrix} 1 & -1 \\ -1 & -4 \end{pmatrix}$$
$$= \frac{-1}{5}\begin{pmatrix} -4\cdot 2^{3000} - 7^{3000} & 4\cdot 2^{3000} - 4\cdot 7^{3000} \\ 2^{3000} - 7^{3000} & -2^{3000} - 4\cdot 7^{3000} \end{pmatrix}.$$

   (b) Using much of the work from part (a),

$$e^A = Se^{\Lambda}S^{-1} = \begin{pmatrix} -4 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^2 & 0 \\ 0 & e^7 \end{pmatrix}\frac{1}{-5}\begin{pmatrix} 1 & -1 \\ -1 & -4 \end{pmatrix}$$
$$= \frac{-1}{5}\begin{pmatrix} -4\cdot e^2 - e^7 & 4e^2 - 4e^7 \\ e^2 - e^7 & -e^2 - 4e^7 \end{pmatrix}.$$

6. (a) We need to change the t in one equation to s, then solve for the intersection point. We get

$$\begin{cases} 3t = -2 + 4s \\ 1 + 2t = 7 - s \\ -1 - t = -3 + s \end{cases}$$

Solving the second equation for s, we get $s = 6 - 2t$. Plugging this into the third equation, we get $-1-t = -3+6-2t$, i.e. $t = 4$ (so by back-substitution in the equation $x = 6 - 2t$, $s = -2$). These values of s and t work in the last two equations, but not in the first one, so these lines do not intersect.

(b) Two direction vectors for the plane are $(2, -1, 3)$ and $(-3, 1, 5)$. To obtain a normal vector, take the cross product: $\mathbf{n} = (2, -1, 3) \times (-3, 1, 5) = (-8, -19, -1)$. One point on the plane is $(-1, 1, 3)$; set $d = \mathbf{n} \cdot (-1, 1, 3) = 8 - 19 - 3 = -14$. So the equation of the plane is $\mathbf{n} \cdot \mathbf{x} = d$, i.e. $-8x - 19y - z = -14$. (Any multiple of this equation is also a valid solution.)

7. (a) $tr(A)$ is the sum of the diagonal entries, which is a **scalar**.

(b) $\mathbf{vw}$ is nonsense, so the whole thing is **nonsense**.

(c) $\mathbf{x} \times \mathbf{y}$ is a **vector** in $\mathbb{R}^3$.

(d) $\mathbf{v} \times \mathbf{w}$ is **nonsense** (there is no cross product of vectors in $\mathbb{R}^4$).

(e) $||\mathbf{v} - \mathbf{w}||\,||\mathbf{x} + \mathbf{y}||$ is the product of two scalars, hence a **scalar**.

(f) $||\mathbf{x}||\mathbf{x}$ is a scalar times a vector which is a **vector** in $\mathbb{R}^3$.

(g) $k(B^T)_{4\times3}(A^{-1})_{3\times3}m$ is a 4×3 **matrix**.

(h) First, $\mathbf{z} + 3k\mathbf{y}$ is a vector in $\mathbb{R}^3$. Next, $D\mathbf{x} = D_{1\times3}\mathbf{x}_{3\times1}$ is a 1×1 matrix, hence a scalar. Then $(D\mathbf{x})_{scalar}(A^2)_{3\times3}(\mathbf{z} + 3k\mathbf{y})_{3\times1}$ is a 3×1 matrix, i.e. a **vector** in $\mathbb{R}^3$.

(i) $\det(A)B$ is a scalar times a matrix which is a 3×4 **matrix**.

(j) $\det(AB)$ is **nonsense** since nonsquare matrices do not have determinants.

(k) $D_{1\times3}\mathbf{z}_{3\times1}m$ is a 1×1 matrix, i.e. a **scalar**.

(l) $m\mathbf{z}_{3\times1}D_{1\times3}$ is a 3×3 **matrix**.

8. (a) $(2,1,-5)\cdot(3,3,2) = 6+3-10 = -1 \neq 0$ so this is FALSE.
   (b) $\mathbb{R}^4$ consist of vectors with four components, but subspaces of $\mathbb{R}^5$ are sets of vectors with five components. Therefore this is FALSE.
   (c) You can check $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x})+T(\mathbf{y})$ and $T(r\mathbf{x}) = rT(\mathbf{x})$ so this T is linear; the answer is TRUE.
   (d) $T(1,0) = 0$ so $\ker(T) \neq \{\mathbf{0}\}$ so this transformation is not injective, so this statement is FALSE.
   (e) $\mathbf{0}$ is not in this set, so this statement is FALSE.
   (f) These are four vectors in the three-dimensional space $\mathbb{R}^3$ (i.e. too many to be lin. indep.), so this statement is FALSE.
   (g) If you try virtually any examples of matrices, you will see that this is FALSE.
   (h) This is a property of dot products; it is TRUE.
   (i) $\mathbf{0}$ is parallel to every vector, so this is TRUE.
   (j) Evaluation of a function is linear, so this is TRUE.

9. (a) A 2×3 matrix is NEVER diagonalizable (it isn't square).
   (b) If $T : V_1 \to V_2$ and $S : V_2 \to V_3$ are linear transformations, then the composition $S\circ T$ is ALWAYS linear (theorem from class).
   (c) A system of 5 linear equations in 3 variables SOMETIMES has exactly one solution (because the null space could have dimension 0 or dimension greater than 0).
   (d) $|\mathbf{v}\cdot\mathbf{w}|$ is ALWAYS less than or equal to $||\mathbf{v}||\,||\mathbf{w}||$ (this is the Cauchy-Schwarz Inequality).
   (e) The zero vector is NEVER part of a basis (it is never part of a lin. indep. set).
   (f) Given a set of three vectors in $\mathbb{R}^4$, that set NEVER spans $\mathbb{R}^4$ (there aren't enough vectors to span).
   (g) Given a set of four vectors in $\mathbb{R}^4$, that set SOMETIMES spans $\mathbb{R}^4$ (depending on what those vectors are).
   (h) Given a set of five vectors in $\mathbb{R}^4$, that set SOMETIMES spans $\mathbb{R}^4$ (it depends on what those vectors are).
   (i) A matrix with eigenvalue 0 is NEVER invertible (because its determinant is the product of the eigenvalues which must be 0).
   (j) A set of one nonzero vector is ALWAYS linearly independent (fact from class).

10. (a) $\dim W^\perp = \dim \mathbb{R}^{13} - \dim W = 13 - 8 = 5$.
   (b) We have $m = 10$, $n = 7$ and $r = 5$. So $\dim N(A) = n - r = 2$.
   (c) If the transformation is surjective, we have $r = m = 5$.
   (d) The trace is the sum of the eigenvalues: $2 + 2 - 3 + 1 = 2$.
   (e) Since the dimension is 3, there are 3 vectors in any basis of that subspace.