# Exercises from Chapter 1

## Exercises from Section 1.1

1. (a) Construct a truth table for the compound proposition $(\sim P \vee Q) \wedge (P \vee \sim Q)$.
   (b) Based on the truth table above, is the proposition in part (a) a tautology, a contradiction, or neither?

2. For each proposition, write a useful denial:
   (a) Carson Daly is the host of The Voice, or Wolf Blitzer is a CNN anchor.
   (b) Jon Stark's mother is Lyanna Stark, and his father is Rhaegar Targaryen.

3. Let $P =$ "19 is an even number"; let $Q =$ "19 is a perfect square"; and let $R =$ "19 is a multiple of 9". Write each of the following statements in symbolic form (i.e. "$P \wedge Q$", etc.)
   (a) $19$ is an even number and a multiple of $9$.
   (b) $19$ is a perfect square, or $19$ is both even and a multiple of $9$.
   (c) $19$ is a perfect square if $19$ is even.
   (d) $19$ is an even number if and only if $19$ is a perfect square.
   (e) If $19$ is a perfect square which is a multiple of $9$, then $19$ is even.
   (f) If $19$ is not a perfect square, then $19$ is either even or not a multiple of $9$.

4. Prove the first **Distributive Law**, which says that for propositions $P$, $Q$ and $R$,
$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R).$$

5. Use the common logical equivalences given in Theorem 1.10 to find a simpler formula which is logically equivalent to $(P \wedge Q) \vee (P \wedge \sim Q)$.

6. (★) Use the common logical equivalences given in Theorem 1.10 to find a simpler formula which is logically equivalent to $(P \wedge R) \vee (\sim R \wedge (P \vee Q))$.

## Exercises from Section 1.2

1. Construct a truth table for the compound proposition $(P \wedge \sim Q) \Rightarrow (P \vee R)$.

2. Prove that the statements $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ and $(P \vee Q) \Rightarrow R$ are logically equivalent.

3. Let $R =$"it rains"; let $W =$"it is windy"; and let $S =$"the sun is shining". Write each of the following statements in symbolic form:
   (a) It it is sunny, then it doesn't rain.
   (b) Rain is sufficient to guarantee the sun is not shining.
   (c) It is sunny only if it is not raining.
   (d) If the sun is shining and it is windy, then it is not raining.

(e) Wind is a necessary condition for rain and no sunshine.
(f) Rain and wind occur at the same times.
(g) Either rain is implied by wind, or both rain and wind are implied by the absence of sun.

4. Find a formula involving the connectives $\sim$ and $\Rightarrow$ which is logically equivalent to $P \wedge Q$. Prove that your formula is logically equivalent to $P \wedge Q$.

5. (a) Write the contrapositive of the statement "If the UNC basketball team loses, Dr. McClendon is very upset."
   (b) Write the conditional "All insects are terrifying" as an "if, then" statement.
   (c) Write the converse of the statement you wrote in part (b).
   (d) Write a useful denial of the statement you wrote in part (b).

6. Consider the statement "If 3 is an eigenvalue of $A$, then $\det(A - 3I) = 0$."
   (a) Write this statement in disjunctive normal form.
   (b) Write the inverse of this statement.
   (c) Write the inverse of the converse of this statement.

7. Prove Lemma 1.16 from the lecture notes, which says that $P \Leftrightarrow Q$ is logically equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

8. Determine whether the "if" in each of the following English sentences actually means "if" or "if and only if":
   (a) If you behave in church, Mom will buy you an ice cream afterwards.
   (b) If a light bulb turns on, then the light switch works.
   (c) If at least twelve people show up, then we will play soccer.

## Exercises from Section 1.3

1. Give a roster for the each of these sets (some research using Google may be required):
   (a) $W$ is the set of wives of Henry VIII, King of England.
   (b) $S$ is the set of states which border Kentucky.
   (c) $T = \{x \in \mathbb{R} : x^2 + 8x = 33\}$.
   (d) $N$ is the set of NBA players who have scored at least $40,000$ points in their career.
   (e) $E = \{3x + 1 : x \in \mathbb{Z}\}$.

2. Write a definition, using either set-builder notation or English, which describes each of the following sets (again, some research using Google may be required):
   (a) {Argentina, Bolivia, Brazil, Chile, Colombia, Ecuador, Guyana, Paraguay, Peru, Suriname, Uruguay, Venezuela}

(b) {*The King's Speech*, *The Artist*, *Argo*, *12 Years a Slave*, *Birdman*, *Spotlight*, *Moonlight*, *The Shape of Water*, *Green Book*}
(c) $\{1, 9, 25, 49, 81, 121, 169, ...\}$
(d) $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, ...\}$

In Questions 3-6, consider the following sets:

$$A = \{0, 1, 2, 3\} \qquad B = \{\{0, 1\}, \{2, 3\}\} \qquad C = \{0, 1\}$$

$$D = \{x \in \mathbb{R} : x^2 = x\} \qquad E = \{-1, 0, 1\}$$

In each part of Questions 3-6, classify each the following statement as "TRUE" if it uses valid notation and is true; classify the statement as "FALSE" if it uses valid notation but is false; classify it as "NONSENSE" if the notation of the statement doesn't make sense.

3. (a) $2 \in \emptyset$ (b) $\{2\} \in \emptyset$ (c) $\emptyset \in \{2\}$ (d) $2 \in A$ (e) $\{2\} \in A$ (f) $\{0, 2\} \subseteq A$

4. (a) $E \subseteq A$ (b) $E \supseteq A$ (c) $A \not\subseteq C$ (d) $\{0, 1\} \subseteq C$ (e) $C = D$ (f) $0, 1 \in E$

5. (a) $C = E$ (b) $B \in A$ (c) $B \subseteq A$ (d) $B \in 2^A$ (e) $B \subseteq 2^A$ (f) $0, 1 \subseteq E$

6. (a) $E \in 2^E$ (b) $\{\emptyset\} \in 2^D$ (c) $\{\emptyset\} \subseteq A$ (d) $\{\emptyset\} \subseteq 2^A$ (e) $0 \in B$ (f) $\{0, 1\} \subseteq B$

7. Let $U = \{a, b, c, d\}$. List all the elements of the power set $2^U$.

8. Let $V = \{x, \{x\}\}$. (Carefully) list all the elements of $2^V$.

## Exercises from Section 1.4

1. Classify each of the following statements as true or false:

   (a) $\exists\, x \in \mathbb{R} : 2^x = x$.
   (b) $\exists!\, x \in \mathbb{R} : x^2 = x$.
   (c) $\forall\, x \geq 1, x^2 \geq x$.
   (d) $\forall\, x \in \mathbb{R}\, \exists\, y \in \mathbb{R} : y > x$.
   (e) $\forall\, x \in \mathbb{N}, x^2 - 3x + 37$ is a prime number.
   (f) $\exists\, x \in \mathbb{R} : \forall\, y \in \mathbb{R}, y > x$.
   (g) $\exists!\, x \in \mathbb{R} : \forall\, y \in \mathbb{R}, xy = x$.

2. Consider these three statements:

   (i). $\forall\, x\, \exists\, y : 4x - y = 0$

(ii). $\forall\, x\, \exists\, y : x - 2y = 0$

(iii). $\exists\, x\; :\; \forall\, y\, \exists z : y + z = x$

(a) Classify each of the statements (i), (ii) and (iii) as true or false, if the universe of discourse for all variables is $\mathbb{N}$.

(b) Classify each of the statements (i), (ii) and (iii) as true or false, if the universe of discourse for all variables is $\mathbb{Z}$.

(c) Classify each of the statements (i), (ii) and (iii) as true or false, if the universe of discourse for all variables is $\mathbb{Q}$.

(d) Classify each of the statements (i) and (ii) as true or false, if the universe of discourse for $x$ is $\mathbb{Z}$ but the universe of discourse for $y$ is $\mathbb{Q}$.

(e) Classify each of the statements (i) and (ii) as true or false, if the universe of discourse for $x$ is $\mathbb{Q}$ but the universe of discourse for $y$ is $\mathbb{Z}$.

3. In each part of this problem, you are given a statement. Determine if that statement is a proposition or an open sentence. If it is a proposition, classify it as true or false. If it is an open sentence, describe its truth set (assume any universe of discourse is $\mathbb{R}$ unless otherwise specified).

   (a) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{Z}$ such that $y \leq x < y + 1$.

   (b) $\forall x \in \mathbb{R}$, $\sin x < y$.

   (c) There is $y \in \mathbb{R}$ such that $x^2 + y^2 \leq 36$.

   (d) $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x$, $x < \delta$ implies $20x < \varepsilon$.

4. Write a useful denial of each statement:

   (a) There exists a largest element of $E$.

   (b) The sum of the measures of the angles in any triangle is $180°$.

   (c) $\forall\, g, h$, $\phi(gh) = \phi(g)\phi(h)$.

   (d) $\forall\, a, b\, \exists\, c$ s.t. $\forall\, d\, \exists\, e, f$ s.t. $P(a, b, c, d, e, f)$.

## Exercises from Section 1.5

In Questions 1 and 2, assume the universe is the set $\{1, 2, 3, 4, 5\}$, and let $A = \{1, 3, 5\}$, $B = \{2\}$ and $C = \{1, 2, 3\}$. In each part of Questions 1 and 2, give a roster for each set:

1. (a) $B^C$ (b) $A \cup B$ (c) $B \cup C^C$ (d) $(A \cap B)^C \cup C$ (e) $A \cap B \cap C$ (f) $(A \cup C)^C \cup (B \cap C)^C$

2. (a) $A - C$ (b) $A \triangle C$ (c) $A \times B$ (d) $A \times 2^B$ (e) $B^2$ (f) $A^2 - C^2$

3. In this question, assume the universe is $\mathbb{R}$ and let $E = [-4, \infty)$; $F = (-\infty, -4]$; $G = (0, 3)$; and $H = (2, \infty)$. Describe each set:

(a) $E \cap F$
(b) $E \cup F$
(c) $E \cap F^C$
(d) $(F \cup H)^C$
(e) $G \cap H^C$
(f) $F \cap G$
(g) $E \cup G \cup H$
(h) $[-8, \infty) - E$

In Problems 4-7, suppose:
- $P(x)$ is an open sentence whose truth set is $E$;
- $Q(x)$ is an open sentence whose truth set is $F$; and
- $R(x)$ is an open sentence whose truth set is $G$.

4. In each part of this problem, you are given a compound open sentence. Describe the truth set of that open sentence in terms of $E$, $F$ and/or $G$ using set operations.

   (a) $P(x) \vee \sim Q(x)$
   (b) $(P(x) \wedge \sim R(x)) \vee (Q(x) \wedge R(x))$
   (c) $P(x) \wedge (Q(y) \vee \sim R(y))$

5. In each part of this problem, you are given a set. Use logical connectives to construct an open sentence (built from $P(x)$, $Q(x)$ and/or $R(x)$) whose truth set is the one given in the problem.

   (a) $E^C \cap F$
   (b) $(E \cup F) \cap G^C$
   (c) $E \triangle (F \cap G)$

6. In each part of this problem, you are given a proposition involving the open sentences $P(x)$, $Q(x)$ and/or $R(x)$. Write a proposition involving sets $E$, $F$ and/or $G$ (with appropriate set operations) that is logically equivalent to the given proposition:

   (a) $R(x) \Rightarrow P(x)$
   (b) $P(x) \Rightarrow (Q(x) \wedge \sim R(x))$
   (c) $(P(x) \wedge Q(y)) \Leftrightarrow (Q(x) \wedge R(y))$

7. In each part of this problem, you are given a proposition involving sets $E$, $F$ and/or $G$. Write a proposition involving the open sentences $P(x)$, $Q(x)$ and/or $R(x)$ (with appropriate logical connectives) which is logically equivalent to the given proposition:

   (a) $F = G^C$
   (b) $(E \cup F) \subseteq G$
   (c) $E = F^C \cap G$
   (d) $(E \times F) \supseteq G$

8. ($\bigstar$) Prove the **Commutative Law (for unions)**, which says that for any two sets $E$ and $F$,
$$E \cup F = F \cup E.$$

9. Prove that **unions distribute across intersections**, i.e. that for any three sets $E$, $F$ and $G$,
$$E \cup (F \cap G) = (E \cup F) \cap (E \cup G).$$

10. (★) Prove the second **DeMorgan Law (for sets)**, which says that for any two sets $E$ and $F$,
$$(E \cap F)^C = E^C \cup F^C.$$

## Exercises from Section 1.6

1. Prove **Reductio ad absurdum**, which says $[\sim P \Rightarrow (Q \wedge \sim Q)] \Rightarrow P$.
2. (★) Prove **modus tollens**, which says $[(P \Rightarrow Q) \wedge \sim Q] \Rightarrow \sim P$.
3. Prove **hypothetical syllogism**, which says $[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$.

   *Remark:* The statement "$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$" is often shorthanded as $P \Rightarrow Q \Rightarrow R$, which by hypothetical syllogism reduces to $P \Rightarrow R$.
4. In each part of this problem, you are given a list of statements which you are to assume. Determine if a conclusion can be drawn from the given statements; if so, write the conclusion. If not, write "no conclusion possible".
   (a) If a function is differentiable, then it is continuous. $f$ is a differentiable function.
   (b) We will eat venison tonight if I shoot a deer. I do not shoot a deer.
   (c) If the moon is made of Jello, we can eat it. We cannot eat the moon.
   (d) Tigger bounces on his tail if and only if Eeyore is sad. Eeyore is sad.
   (e) If a group is abelian, then all of its subgroups are normal. $G$ is a group that has a non-normal subgroup.
   (f) If a group is abelian, then all of its subgroups are normal. $G$ is a group that has a normal subgroup.
   (g) If the sun rises in the west, then $q^2 > 8$ or chocolate sauce will rain from the sky. The sun rises in the west and $q^2 = 3$.
5. Below, you are given a list of statements that you are to assume are true.
   (a) For each given statement, translate the statement symbolically as a conditional (your answer should be something like "$B \Rightarrow C$" or "$\sim A \Rightarrow \sim K$", etc.) Be sure to define any letters you use here.
   (b) Symbolically write the contrapositive of each symbolic expression you wrote down in part (a).
   (c) Write down a chain of implications connecting all the letters you defined in part (a), that uses each letter exactly once. (Your answer should be something like "$A \Rightarrow \sim B \Rightarrow K \Rightarrow L \Rightarrow \sim Q \Rightarrow X$".)
   (d) Use hypothetical syllogism (described in Problem 3 above) repeatedly to deduce the simplest conclusion that can be drawn from the statements; write this conclusion symbolically and write it in English, using the simplest possible verbiage.

   - No birds, other than ostriches, are 9 feet high.
   - No birds in this aviary belong to anyone but me.
   - No ostrich eats frogs.

- I own no birds less than 9 feet high.

6. Determine the simplest conclusion that can be deduced from the following statements, writing your conclusion using plain English.
   - Promise breakers are untrustworthy.
   - Wine drinkers are very communicative.
   - A man who keeps his promises is honest.
   - No teetotalers are pawnbrokers.
   - One can always trust a very communicative person.

7. Determine the simplest conclusion that can be deduced from the following statements, writing your conclusion using plain English.
   - Animals that do not kick are always unexcitable.
   - Donkeys have no horns.
   - A buffalo can always toss one over a gate.
   - No animals that kick are easy to swallow.
   - No hornless animal can toss one over a gate.
   - All animals are excitable, except buffaloes.

## Exercises from Chapter 2

From this point forward, proofs must be typeset using Overleaf (or LaTeX).

### Exercises from Section 2.1

1. Prove that every integer is even or odd, and that no integer is both even and odd.

   *Remark:* your proof is going to be wrong unless you are a prodigy or unless you cheat (although you may well think you have proven these statements). We don't have the machinery developed yet to prove this–the point of this exercise is for you to try, and then for us to learn by examining what's wrong with your attempts.

2. Prove that if $x \in \mathbb{Z}$ is odd and $y \in \mathbb{Z}$ is even, then $x + y - xy$ is odd.

3. Prove that for $a, b \in \mathbb{Z}$, if $6 \mid a$ and $9 \mid b$, then $18 \mid (21a + 10b)$.

4. Here's a "proof" that is wrong. Explain the error in the "proof":

   **Claim:** For $a, b, c \in \mathbb{Z}$, if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.

   PROOF: Since $a \mid b$, there is an integer $q$ such that $b = aq$. Similarly, since $a \mid c$, there is an integer $q$ such that $c = aq$. Therefore $b + c = aq + aq = 2aq = (2q)a$. Since $2q \in \mathbb{Z}$, $a \mid (b + c)$ as wanted. □

### Exercises from Section 2.2

1. Prove that if $x$ is a positive real number, then $x + \frac{9}{x} \geq 6$.

2. Prove that the sum of any three consecutive numbers is divisible by 3.

   *Hint:* If you let $n$ be the smallest of the three consecutive integers, what are the other two integers (in terms of $n$)?

3. Let $x \in \mathbb{R}$. Prove that if $x^3 + 5x^2 < 0$, then $4 - 3x > 10$.

4. Prove that for $a, b \in \mathbb{R}$, if $a < b$ then $a < \frac{a+b}{2} < b$.

5. Prove $\{4n + 1 : n \in \mathbb{Z}\} \subseteq \{2n + 3 : n \in \mathbb{Z}\}$.

6. (★) Let $A, B$ be sets. Prove $2^A \cup 2^B \subseteq 2^{A \cup B}$.

### Exercises from Section 2.3

1. (a) Let $x \in \mathbb{R}$. Prove that $|x| > 7$ implies $(x - 4)^2 > 9$.

   (b) Rewrite what you proved in part (a) as a statement about the relationship between two sets.

2. Prove that for any integer $n$, $\frac{n^2+n}{2}$ is an integer.

3. Prove that for any two real numbers $a$ and $b$ with $b \neq 0$, $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.

   *Hint:* if you don't remember, $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.

4. Prove that for any real number $x$, $|x + 3| + |x - 1| \geq 4$.

5. Suppose six people are in a room. Prove that either (at least) three of the six all know each other, or there are at least three of the six that are mutual strangers.

6. (★) Prove that the product of any five consecutive integers is divisible by 120.

### Exercises from Section 2.4

1. Let $m, n \in \mathbb{Z}$. Prove that if $mn$ is odd, then both $m$ and $n$ are odd.

2. Let $a, b, c \in \mathbb{Z}$. Prove that $a \nmid bc$ implies $a \nmid b$.

### Exercises from Section 2.5

1. Prove that if $p, q \in \mathbb{Z}$, then $\left(\frac{p}{q}\right)^2 \neq 2$.

   *Hint:* You may assume WLOG that $\frac{p}{q}$ is in lowest terms. You may **not** assume $\sqrt{2}$ is irrational, because that is essentially what you are being asked to prove.

2. (★) Prove that if $p, q \in \mathbb{Z}$, then $\left(\frac{p}{q}\right)^2 \neq 3$.

3. Prove that $\log_2 3$ cannot be written as $\frac{p}{q}$ where $p, q \in \mathbb{Z}$.

4. Let $A, B$ and $C$ be sets. Suppose $A \cap C \subseteq B$ and $x \in C$. Prove $x \notin A - B$.

## Exercises from Section 2.6

1. Let $x, y \in \mathbb{Z}$. Prove $x$ and $y$ have the same parity if and only if $x + y$ is even.
2. **Prove or disprove:** for every integer $n$, $15 \mid n$ if and only if $3 \mid n$ and $5 \mid n$.
3. **Prove or disprove:** for every integer $n$, $60 \mid n$ if and only if $6 \mid n$ and $10 \mid n$.

## Exercises from Section 2.7

In each of Problems 1-5, you are given two sets $A$ and $B$. For each problem, you are to:

- determine the most specific relationship between $A$ and $B$ that holds (the four choices are $A \subseteq B$; $B \subseteq A$; $A = B$; or "no relationship");
- formulate that relationship as a proposition (a.k.a. claim);
- prove that proposition.

1. $A = 12\mathbb{Z}$; $B = 36\mathbb{Z}$.
2. In this question, assume $E$ and $F$ are arbitrary sets.
   $A = (E - F)^C$; $B = E^C \cup F$.
3. $A = \{n^2 : n \in \mathbb{Z}\}$; $B = 4\mathbb{Z} \cup (4\mathbb{Z} + 1)$.
4. In this question, assume $P \subseteq R$ and $Q \subseteq S$.
   $A = P - Q$; $B = R - S$.
5. In this question, assume $C$ is an arbitrary set.
   $A = (B \cap C) \cup (B - C)$; $B = B$.
6. Call a function $f : \mathbb{R} \to \mathbb{R}$ **nice** if for any $x, y \in \mathbb{R}$, $f(x + y) = f(x)f(y)$.
   (a) Prove that the constant function $f(x) = 1$ is nice.
   (b) **Prove or disprove:** there exists a non-constant nice function.

## Exercises from Section 2.8

1. Prove that **intersecting a set makes it smaller**, i.e. for any sets $E$ and $F$, $E \cap F \subseteq E$.
2. Let $E, F$ and $G$ be sets. Prove that **intersection with a third set preserves subset relationship**, i.e.
   $$E \subseteq F \;\Rightarrow\; E \cap G \subseteq F \cap G.$$
3. Choose (a) or (b):
   (a) Prove that for any set $E$, $E \cap \emptyset = \emptyset$.
   (b) Prove that for any set $E$, $E \cap E = E$.
4. Choose (a) or (b):

(a) Prove that for any set $E$, $E \cup E = E$.
(b) Prove that for any set $E$, $E \cup E^C = U$, where $U$ is the universal set.

5. (★) Let $E$ be any set. Prove that $E \triangle E = \emptyset$.

6. Let $E, F$ and $G$ be sets. Prove that **products distribute over intersection**, which says
$$E \times (F \cap G) = (E \times F) \cap (E \times G).$$

7. (★) Let $E$ be any set. Prove $E \times \emptyset = \emptyset$.

8. Let $E, F, G$ and $H$ be sets. Prove the **interchange of product with intersection** law, which says
$$(E \times F) \cap (G \times H) = (E \cap G) \times (F \cap H).$$

## Exercises from Sections 2.9 and 2.10

A function $f : \mathbb{R} \to \mathbb{R}$ is called **multiplicative** if for any $x, y \in \mathbb{R}$, $f(xy) = f(x)f(y)$.

1. **Prove or disprove:** The function $f(x) = x^4$ is multiplicative.

2. **Prove or disprove:** The function $f(x) = 2x^4$ is multiplicative.

3. (★) Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is multiplicative, but which is neither constant, nor of the form $f(x) = x^n$ for any constant $n$. (You need to prove your example is multiplicative.)

4. (a) Write a definition of what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be **decreasing** (this should be similar to the definition of increasing given in the notes).
   (b) Without using calculus, prove that the function $f(x) = 2 - \sqrt[3]{x}$ is decreasing.

5. A subset $E$ of $\mathbb{R}^2$ is called **symmetric** if, for all $x, y \in \mathbb{R}$, $(x, y) \in E$ if and only if $(y, x) \in E$. Determine, with proof, whether or not the following subsets of $\mathbb{R}^2$ are symmetric:
   (a) $\{(x, y) \in \mathbb{R}^2 : x + y = 5\}$
   (b) $\{(x, y) \in \mathbb{R}^2 : x^2 + xy + y^4 = 1\}$
   (c) $\{(x, y) \in \mathbb{R}^2 : x^3 + x^2 y + y^2 x = 10\}$

## Exercises from Sections 2.11 and 2.12

1. **Prove or disprove:** There exists a prime number between 80 and 90.

2. **Prove or disprove:** There is a smallest positive real number.

3. **Prove or disprove:** There exists a unique differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfying these equations:
$$\begin{cases} f'(x) = 3x^2 f(x) \\ f(0) = 1 \end{cases}$$
*Hints:* For the existence part, try the function $f(x) = e^{x^3}$. For the uniqueness part, suppose $f$ and $g$ satisfy both equations, and consider $h = \frac{f}{g}$. Prove something about $h$, which will allow you to conclude that $f = g$.

4. **Prove or disprove:** The equation $x = \cos x$ has a solution in the interval $(0, \frac{\pi}{2})$.

5. Let $a < b$ be real numbers. Prove that for any continuous function $f : [a, b] \to [a, b]$, there is a number $x \in [a, b]$ such that $f(x) = x$.

   *Hint:* Apply the Intermediate Value Theorem to a function which has a formula built from $f$.

# Exercises from Chapter 3

## Exercises from Section 3.1

1. Let $S$ be the relation from $\mathcal{A}$, the set of English letters, to $\mathbb{N}$ defined by letting $(x, n) \in S$ if the letter $x$ appears at least $n$ times in the sentence

   `PACK MY BOXES WITH FIVE DOZEN LIQUOR JUGS.`

   (a) True or false: $(\mathtt{I}, 1) \in S$.
   (b) True or false: $(\mathtt{T}, 2) \in S$.
   (c) True or false: $\forall x \in \mathcal{A}, \forall n \in \{1, 2, 3, ...\}$, if $(x, n) \in S$, then $(x, n-1) \in S$.
   (d) List the elements in the range of $S$.
   (e) List all $n \in \mathbb{N}$ such that $(\mathtt{E}, n) \in S$.
   (f) List all $x \in \mathcal{A}$ such that $(x, 3) \in S$.

## Exercises from Section 3.2

1. Determine, with proof, whether or not the relation $R$ is an equivalence relation on the set $E = \{1, 2, 3, 4, 5\}$, where

$$R = \{(1, 1), (1, 3), (3, 1), (3, 3)\}.$$

   (Be sure to correctly formulate your assertion as a claim, before proving it.)

2. Determine, with proof, whether or not the relation $\sim$ is an equivalence relation on the set $\mathbb{R}^2$, where

$$(a, b) \sim (c, d) \Leftrightarrow a + 2b = c + 2d.$$

## Exercises from Section 3.3

1. Prove Lemma 3.7 from the lecture notes, which says that if $R$ is an equivalence relation on a set $E$, then:

   1. If $x\,R\,y$, then $[x]_R = [y]_R$.
   2. If $x \not R\, y$, then $[x]_R \cap [y]_R = \emptyset$.

2. Let $E = \{w, x, y, z\}$ and consider the equivalence relation $*$ on $2^E$ defined by $A * B$ if and only if $A$ and $B$ have the same number of elements. List the elements in $[\{w, y, z\}]_*$.

3. Let $R$ be an equivalence relation on set $E$ and suppose $F \subseteq E$.
   (a) Prove $S = R \cap F^2$ is an equivalence relation on $F$.
   (b) Prove that for every $x \in F$, $[x]_S = [x]_R \cap F$.

### Exercises from Section 3.5

1. List five integers $x$, at least one of which is negative, such that $x \equiv 4 \mod 11$.
2. Let $m \in \{2, 3, 4, ...\}$. Prove that for integers $a_1, b_1, a_2$ and $b_2$, if $a_1 \equiv_m b_1$ and $a_2 \equiv_m b_2$, then $(a_1 + a_2) \equiv_m (b_1 + b_2)$.
3. Let $m \in \{2, 3, 4, ...\}$. Prove that for integers $a_1, b_1, a_2$ and $b_2$, if $a_1 \equiv_m b_1$ and $a_2 \equiv_m b_2$, then $a_1 a_2 \equiv_m b_1 b_2$.

### Exercises from Section 3.6

1. Prove the relation $R_{\mathbb{Z}}$ introduced in Definition 3.13 is an equivalence relation on $\mathbb{N}^2$.
2. (★) Prove that if $m$ and $n$ are different natural numbers, then (1) $(0, m)$ and $(0, n)$ are not in the same $R_{\mathbb{Z}}$-equivalence class and (2) except for when $m = n = 0$, $(0, m)$ and $(n, 0)$ are not in the same $R_{\mathbb{Z}}$-equivalence class.
3. (★) Prove that there are no other $R_{\mathbb{Z}}$-equivalence classes other than equivalence classes of the form $[(n, 0)]$ and $[(0, n)]$ where $n \in \mathbb{N}$.
4. Prove the relation $R_{\mathbb{Q}}$ introduced in Definition 3.14 is an equivalence relation on $\mathbb{Z}^2$.

## Exercises from Chapter 4

### Exercises from Section 4.1

1. Prove that if $f : A \to B$ is any function with $Dom(f) = A$, then the relation $R_f$ on $A$ defined by
$$(x, y) \in R_f \Leftrightarrow f(x) = f(y)$$
is an equivalence relation on $A$.
2. Prove that if $E$ is any set and $R$ is an equivalence relation on $E$, then there is a function $f$, whose domain is $E$, such that $x \, R \, y \Leftrightarrow f(x) = f(y)$.
3. In this problem, we prove that addition is well-defined on $\mathbb{Z}$. Recall that $\mathbb{Z}$ was defined to be a quotient space on $\mathbb{N}^2$. We define **addition** on $\mathbb{Z}$ by defining the sum $S$ of integers $(a, b)$ and $(x, y)$ to be
$$S\left([(a, b)]_{R_{\mathbb{Z}}}, [(x, y)]_{R_{\mathbb{Z}}}\right) = [(a + x, b + y)]_{R_{\mathbb{Z}}}.$$
What you have to prove that this function $S$ is well-defined.
4. (★) Prove that multiplication is well-defined on $\mathbb{Q}$.
   *Hint:* You have to start by rigorously defining the product of two rational numbers by a function, similar to what was done in the previous problem.

## Exercises from Section 4.3

1. Throughout this problem, assume $f : \mathbb{R} \to \mathbb{R}^2$ is the function $f(x) = (x+1, x-1)$. Compute each indicated quantity (if the quantity does not exist, say so):

   (a) $f(3)$
   (b) $f(2) + f(4)$
   (c) $f(2+4)$
   (d) $f^{-1}(3)$
   (e) $f^{-1}(0, -2)$
   (f) $f^{-1}(1, 1)$
   (g) $f^{-1}(3, 5) + 4$
   (h) $f(3) + 1$

Throughout Problems 2-4, assume $f : \mathbb{R} \to \mathbb{R}$ is the function $f(x) = |x|$, assume $A = [-3, 5]$ and $B = [-1, 9]$. Compute each indicated quantity (if the quantity does not exist, say so):

2. (a) $f^{-1}(5)$
   (b) $f^{-1}(-3)$
   (c) $f^{-1}(A)$
   (d) $f(A)$
   (e) $f^{-1}(A)$
   (f) $f(f^{-1}(2))$

3. (a) $f(f^{-1}(-1))$
   (b) $f^{-1}(f(-4))$
   (c) $f(f^{-1}(B))$
   (d) $f(A \cup B)$
   (e) $f(A \cap B)$
   (f) $f(A) \cap B$

4. (a) $f(B) \cap f^{-1}(A)$
   (b) $f(A) \cup B$
   (c) $f^{-1}(A - f(B))$
   (d) $f^{-1}\left[B \cap \left(f(f^{-1}(B) \cap f(A))\right)\right]$

5. Let $f : A \to B$ and suppose $E, F \subseteq A$. Prove that the **image of intersection is contained in intersection of image**, i.e.
$$f(E \cap F) \subseteq f(E) \cap f(F).$$

6. Let $f : A \to B$ and suppose $E, F \subseteq B$. Prove that the **preimages preserve unions**, i.e.
$$f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F).$$

7. Let $f : A \to B$ and suppose $E, F \subseteq B$. Prove that the **preimages preserve intersections**, i.e.
$$f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F).$$

8. Let $f : A \to B$ and suppose $E \subseteq A$. Prove that the $f$**, then** $f^{-1}$ **makes a set bigger**, i.e.
$$E \subseteq f^{-1}\left(f(E)\right).$$

9. Let $f : A \to B$ and suppose $E \subseteq B$. Prove that the $f^{-1}$**, then** $f$ **makes a set smaller**, i.e.
$$f\left(f^{-1}(E)\right) = E \cap Im(f).$$

## Exercises from Section 4.4

1. Let $E$ and $F$ be sets. Prove $\mathbb{1}_{E \cap F} = \mathbb{1}_E \mathbb{1}_F$.

2. Let $E$ and $F$ be sets. Prove $E$ is a subset of $F$ if and only if $\mathbb{1}_E(x) \leq \mathbb{1}_F(x)$ for all $x$.

3. Evaluate each of the following expressions:

(a) $\delta_{4,4}$
(b) $\delta_{(5,3),(3,5)}$
(c) $\mathbb{1}_{\mathbb{Q}}(\pi)$
(d) $I_{\mathbb{R}}(-4)$
(e) $I(3x)$
(f) $i(-2)$, where $i : \mathbb{Z} \to \mathbb{R}$ is the inclusion map
(g) $\pi_3(1, 5, -2, 1)$
(h) $\pi_{R_2}(13)$

**Exercises from Section 4.6**

1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = 7 + 2x^3$.
   (a) Determine (with proof) whether or not $f$ is surjective.
   (b) Determine (with proof) whether or not $f$ is injective.

2. Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x) = \frac{x^2}{x^2+1}$.
   (a) Determine (with proof) whether or not $g$ is surjective.
   (b) Determine (with proof) whether or not $g$ is injective.

3. Let $E$ be any nonempty set. Prove that there exists an injection from $E$ to its power set $2^E$.

4. Let $E$ be any nonempty set. Prove that there is no surjective map from $E$ to $2^E$.
   *Hint:* To prove this, suppose not, i.e. that $f : E \twoheadrightarrow 2^E$ is surjective. Now define the subset $B$ of $E$ by
   $$B = \{x \in E : x \notin f(x)\}.$$
   Keep going from here (use the fact $f$ is surjective); you will achieve a contradiction.

5. Prove Theorem 4.24 from the lecture notes, which says that if $f : A \twoheadrightarrow B$ is onto, then for any $E \subseteq B$, $f(f^{-1}(E)) = E$.

6. Prove Theorem 4.25 from the lecture notes, which says that if $g : A \twoheadrightarrow B$ is surjective and $f : B \twoheadrightarrow C$ is surjective, then $f \circ g$ is surjective.

7. Prove Theorem 4.28 from the lecture notes, which says that if $f : A \hookrightarrow B$ is injective, then for any $E \subseteq Dom(f)$, $f^{-1}(f(E)) = E$.

8. Prove Theorem 4.29 from the lecture notes, which says that if $g : A \hookrightarrow B$ is injective and $f : B \hookrightarrow C$ is injective, then $f \circ g$ is injective.

9. (★) Prove that the direct product of two injections is injective.

10. (★) Prove that any projection map (defined on the product of two non-empty sets) is surjective.

### Exercises from Section 4.7

1. Consider the function $f : \mathbb{R} \to (0,1)$ defined by $f(x) = \frac{e^x}{e^x+1}$. Prove that $f$ is a bijection, find a formula for the inverse of $f$, and prove that your formula for the inverse is correct (by showing that $f \circ f^{-1}(x) = x$ and $f^{-1} \circ f(x) = x$).

2. Let $E$ be any set. Consider the function $g : 2^E \to 2^E$ defined by $g(E) = E^C$. Prove that $g$ is a bijection, find a formula for the inverse of $g$, and prove that your formula for the inverse is correct.

3. (a) Write down a bijection between the intervals $[3, 6]$ and $[18, 25]$. Prove that your function is a bijection by writing down a formula for its inverse, and proving that your formula for the inverse is correct.
   (b) Using your answer to part (a) as a template, write down a bijection between the intervals $[a, b]$ and $[c, d]$. (You do not need to prove that your function is a bijection.)

4. Describe a bijection between $\mathbb{R}$ and the interval $(0, \infty)$.

5. List all the permutations of the set $\{0, 1, 2\}$.

6. (★) Construct a bijection between the two sets $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$. (You need to prove that the function you define is a bijection.)

7. (★) Construct a bijection (proof needed) between the open interval $(0, 1)$ and the closed interval $[0, 1]$.

## Exercises from Chapter 5

### Exercises from Section 5.1

1. Explain (i.e. prove) why $<$ ("less than") is not an order relation on $\mathbb{R}$.

2. Let $E = \{1, 2, 3, 4\}$ and define the relation $\prec$ on $2^E$ by setting $A \prec B$ if and only if $B$ has at least as many elements as $A$.
   (a) List all the sets $A$ such that $A \prec \{1, 3\}$.
   (b) Find a set $A$ such that $A \prec B$ for all $B \in 2^E$. (This $A$ is the **smallest** element in $2^E$ relative to $\prec$.)
   (c) Find a set $A$ such that $B \prec A$ for all $B \in 2^E$. (This $A$ is the **largest** element in $2^E$ relative to $\prec$.)
   (d) Is $\prec$ a partial ordering on $2^E$? Explain (i.e. prove your answer).
   (e) Is $\prec$ a well ordering of $2^E$? Explain.

### Exercises from Section 5.2

1. (a) Prove that for any two positive natural numbers $p$ and $q$, the fraction $\frac{p}{q}$ can be written in lowest terms.

   *Hint:* Prove this by contradiction. Suppose not; then let $A$ be the set of natural numbers which are numerators of fractions which cannot be written in lowest terms. By assumption, $A \neq \emptyset$. Apply the WOP to $A$ to find the least element of $A$; call this element $a$. By definition of $A$, there is some fraction $\frac{a}{b}$ which cannot be written in lowest terms. That means $a$ and $b$ have a common factor. Keep going from here and derive a contradiction.

   (b) Prove that for any two integers $p$ and $q$ (with $q \neq 0$), the fraction $\frac{p}{q}$ can be written in lowest terms.

   *Hint:* Use cases depending on whether or not $p$ and/or $q$ is positive, negative or zero (of course, $q$ cannot be zero). Each case should follow from what you proved in Problem 1.

Earlier this semester I told you that we could assume that every integer is either even or odd, and not both. This fact isn't obvious, but you can prove it with the WOP (and we'll do exactly that in the next few problems):

2. Prove $0$ is not odd.

3. Prove that for any $n \in \mathbb{N}$, if $n+1$ is even then $n$ is odd.

4. Prove that for any $n \in \mathbb{N}$, if $n+1$ is odd then $n$ is even.

5. Prove that there are no natural numbers which are both even and odd.

   *Hint:* Prove this by contradiction; at some point, you will need the WOP. The results of the previous three problems may also be useful.

6. Prove that there are no integers which are both even and odd.

   *Hint:* Suppose $n$ is an integer which is even and odd. Consider two cases, where either $n \geq 0$ or $n < 0$. In both cases, derive a contradiction from what you proved in Problem 5.

7. Prove that there are no natural numbers which are neither even nor odd.

8. Prove that there are no integers which are neither even nor odd.

9. (★) Prove that there are no natural numbers between $0$ and $1$.

### Exercises from Section 5.3

1. Prove that for all $n \in \mathbb{N}$, $\sum\limits_{j=0}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.

2. (★) Find and prove a formula for $\sum\limits_{j=0}^{n} \frac{1}{j(j+1)}$.

3. Prove that for all $n \in \mathbb{N}$, $5 \,|\, (7^n - 2^n)$.

   *Note:* In this problem, a proof by induction is required (no modular arithmetic tricks are allowed).

4. Consider the sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers defined recursively by setting $a_0 = 1$ and setting, for all $n \geq 0$, $a_{n+1} = \sqrt{1+a_n}$.

   (a) Use a calculator to find decimal approximations of $a_0, a_1, a_2, a_3$ and $a_4$.

   (b) Prove that the sequence $\{a_n\}$ is increasing, i.e. for all $n \in \mathbb{N}$, $a_n \leq a_{n+1}$.

   (c) Prove that for all $n \in \mathbb{N}$, $a_n \leq 2$.

5. Recall that for natural numbers $n$ and $k$, the **binomial coefficient** $\binom{n}{k}$ (pronounced "$n$ choose $k$") is defined to be the number

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

   (As a reminder, $0! = 1! = 1$ and for all $n \geq 2$, $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$.) Prove that for all $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

   *Hint:* A result we proved in Chapter 2 of the lecture notes may be useful.

6. The following "proof" is wrong. Explain why it is wrong, and then disprove the claim.

   **Claim:** For any $n \in \mathbb{N}$, $1+2+3+...+n = \frac{n^2+n+1}{2}$.

   PROOF We proceed by induction. Assume $1+2+...+k = \frac{k^2+k+1}{2}$. Then

$$\begin{aligned} 1+2+...+(k+1) &= (1+2+...+k)+(k+1) \\ &= \frac{k^2+k+1}{2} + k + 1 \quad \text{(by the IH)} \\ &= \frac{k^2+k+1+2k+2}{2} \\ &= \frac{k^2+3k+3}{2} \\ &= \frac{(k+1)^2+(k+1)+1}{2}. \end{aligned}$$

   By induction, we are done. □

7. In this problem, we will prove that $e$ is not a rational number. (When submitting your proof, don't divide it into parts (a)-(f); submit it as a single proof that $e$ is irrational.)

   *Hint:* I suggest the following steps:

   (a) First, from Calculus 2, we know that $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$, so by plugging in $x = 1$ to this series, we know $e = \sum\limits_{n=0}^{\infty} \frac{1}{n!}$.

(b) Next, suppose $e$ is rational, i.e. $e = \frac{p}{q}$ where $p$ and $q$ are positive integers. Define

$$z = q!\left(e - \sum_{n=0}^{q} \frac{1}{n!}\right)$$

and prove that $z$ is an integer.

*Hint:* Write the stuff inside the parentheses with a common denominator of $q!$.

(c) Explain why $z > 0$.

(d) Show that whenever $n > q+1$, $\frac{q!}{n!} < (q+1)^{q-n}$. Induction may be useful here.

(e) Use part (d) to explain why $z < 1$.

(f) Explain the contradiction you have obtained, and finish up the proof.

8. Prove the **Binomial Theorem**, which says that for all $x, y \in \mathbb{R}$ and all $n \in \mathbb{N}$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

9. Prove that for all $n \in \mathbb{N}$,

$$\int_0^\infty x^n e^{-x}\,dx = n!.$$

10. (★) Consider a tournament where every team plays the other exactly once (in each game, one team wins and the other loses–no ties). Define a team $x$ to be **awesome** if for every other team $z \neq x$, either $x$ defeated $z$ or there is a third team $y$ such that $x$ defeated $y$ and $y$ defeated $z$. Prove that in any such tournament, there is always at least one awesome team.

## Exercises from Section 5.4

1. Prove that there are no perfect squares whose ones digit is 7.

   *Hint:* By the Division Theorem, every integer $n$ can be written as $n = 5b + q$ where $q \in \{0, 1, 2, 3, 4\}$. This leads to five cases (depending on the value of $q$); show that in none of these cases can $n^2$ end with the digit 7. (If $n^2$ ends with a 7, what is the remainder when $n^2$ is divided by 5?)

2. Prove that the sum of the cubes of any three consecutive integers is a multiple of 9.

## Exercises from Section 5.5

1. Prove that for each natural number $n \geq 2$, $3^n > 1 + 2^n$.

2. Let $F_n$ be the $n^{th}$ Fibonacci number. Prove that for all $n \geq 1$,

$$F_{n-1}F_{n+1} - (F_n)^2 = (-1)^n.$$

3. Let $F_n$ be the $n^{th}$ Fibonacci number. Prove that for all $n \in \mathbb{N}$,

$$\sum_{j=0}^{n} (F_j)^2 = F_n F_{n+1}.$$

4. (★) Prove Theorem 5.34 from the lecture notes (the generalized strong form of PMI).

5. Let $F_n$ be the $n^{th}$ Fibonacci number. Prove that for all $n \in \mathbb{N}$,

$$F_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n.$$

6. (★) Let $F_n$ be the $n^{th}$ Fibonacci number. Prove that for all $m, n \in \mathbb{N}$, where $n \geq 1$,

$$F_{m+n} = F_{n-1} F_m + F_n F_{m+1}.$$

7. (★) **Prove or disprove:** If $F_n$ is the $n^{th}$ Fibonacci numbers, then for every $n \in \mathbb{N}$, the three numbers $F_n F_{n+3}$, $2F_{n+1} F_{n+2}$ and $\left(F_{n+1}^2 + F_{n+2}^2\right)$ are lengths of three sides of a right triangle (because the three numbers satisfy the Pythagorean theorem).

8. (★) Construct the sequence $d_1, d_2, ...$ as follows: let $d_1 = 1$, and for each $n \geq 1$, set $d_{n+1} = d_n + n \cdot n!$. Find and prove a formula for $d_n$.