Old Math 324 Exams

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Last updated to include exams from Fall 2017

Contents

Chapter 1

General information about these exams

These are the exams I gave in Fall 2017 when teaching Math 324 at Ferris. Each exam is followed by what I believe are valid solutions (there may be some number of computational errors or typos in these answers).

Exam 1 covered Chapters 1 and 2 in my Math 324 lecture notes; Exam 2 covered Chapters 3 and 4, and Exam 3 covered Chapters 4 and 5. The final exam was cumulative.

Chapter 2

Exams from Fall 2017

2.1 Fall 2017 Exam 1

1. Construct a truth table for the proposition $(P \vee Q) \Rightarrow (\sim P \vee \sim Q)$.

2. Write a useful denial of each of the following statements:

   a) Tom Cruise is an actor and Bobby Flay is a chef.
   b) There exists $x \in \mathbb{R}$ such that x is not normal.
   c) If Mike is a rabbit, then Mike likes to eat carrots.

3. a) Write the converse of the statement "If Jenny eats pie, then Peter eats cake".
   b) Write the contrapositive of the statement "If Jenny eats pie, then Peter eats cake."

4. Given the three open sentences

   $D(x) =$ "x is dangerous"; $K(x) =$ "x is a koala"; $A(x) =$ "x is from Australia";

   write each statement below in symbolic form. (Assume that the universe of discourse for x is the set of all animals.)

   a) All animals are dangerous.
   b) There is exactly one dangerous koala.
   c) There is a non-dangerous Australian animal.
   d) Dangerous koalas cannot be from Australia.

5. Prove that the sum of an even number and an odd number is odd.

6. Prove that for any integer n, either $4 \mid n^2$ or $4 \mid (n^2 - 1)$.

7. Prove that there do not exist integers p and q such that $\left(\frac{p}{q}\right)^2 = 2$.

8. Let m be an integer. Prove $m^3 - 1$ is even if and only if m is odd.

9. Define a function $f : \mathbb{R} \to \mathbb{R}$ to be **even** if for all $x \in \mathbb{R}$, $f(-x) = f(x)$. For each function f below, determine (with proof) whether or not f is even:

   a) $f(x) = (x-2)^2$

   b) $f(x) = x^2 - 2$

10. **(Bonus)** Let $n \in \mathbb{Z}$. Prove that $3 \mid (2^n - 1)$ if and only if n is even.

    Hint: You may assume that given any three consecutive integers, exactly one of them is divisible by 3.

Solutions

1. Here is the truth table:

P	Q	$\sim P$	$\sim Q$	$P \vee Q$	$\sim P \vee \sim Q$	$(P \vee Q) \Rightarrow (\sim P \vee \sim Q)$
T	T	F	F	T	F	F
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	F	T	T

2. a) Tom Cruise is not an actor, or Bobby Flay is not a chef.
   b) For all $x \in \mathbb{R}$, x is normal.
   c) Mike is a rabbit who does not like to eat carrots.

3. a) If Peter eats cake, then Jenny eats pie.
   b) If Peter does not eat cake, then Jenny does not eat pie.

4. a) $\forall x, D(x)$.
   b) $\exists! x : (D(x) \wedge K(x))$.
   c) $\exists x : (\sim D(x) \wedge A(x))$.
   d) $(D(x) \wedge K(x)) \Rightarrow \sim A(x)$.

5. PROOF: Let $x \in \mathbb{Z}$ be even, so that $x = 2k$ for $k \in \mathbb{Z}$. Let $y \in \mathbb{Z}$ be odd, so that $y = 2l + 1$ for $l \in \mathbb{Z}$. The sum of x and y is therefore

$$x + y = 2k + 2l + 1 = 2(k + l) + 1$$

which is odd. □

6. PROOF: We proceed in two cases, depending on whether n is even or odd:

   Case 1: n is even, i.e. $n = 2k$ for $k \in \mathbb{Z}$. Then $n^2 = (2k)^2 = 4k^2$ so $4 \mid n^2$ as wanted.

   Case 2: n is odd, i.e. $n = 2k+1$ for $k \in \mathbb{Z}$. Then $n^2 = (2k+1)^2 = 4k^2+4k+1 = 4(k^2+k)+1$ and therefore $n^2 - 1 = 4(k^2+k)$ so $4 \mid (n^2 - 1)$ as desired. □

7. PROOF: Suppose not, i.e. that there exists $p, q \in \mathbb{Z}$ such that $\left(\frac{p}{q}\right)^2 = 2$. WLOG $\frac{p}{q}$ is in lowest terms. Now rewriting the equation $\left(\frac{p}{q}\right)^2 = 2$, we get $p^2 = 2q^2$ so p^2 is even. By a result from class (or the notes), that means p is even, so we can write $p = 2k$ where $k \in \mathbb{Z}$. Then

$$2q^2 = p^2 = (2k)^2 = 4k^2$$

so $q^2 = 2k^2$, so q^2 is even. By the same result from class, that means q is even. But since both p and q are even, $\frac{p}{q}$ isn't in lowest terms (2 divides both p and q), a contradiction. Thus no such p and q can exist. $\square$

8. PROOF: ($\Rightarrow$) Assume m is even, i.e. $m = 2k$ for $k \in \mathbb{Z}$. Then

$$m^3 - 1 = (2k)^3 - 1 = 8k^3 - 1 = 8k^3 - 2 + 1 = 2(4k^3 - 1) + 1$$

so $m^3 - 1$ is odd. By contraposition, we are done.

($\Leftarrow$) Assume m is odd, i.e. $m = 2k + 1$ for $k \in \mathbb{Z}$. Then, using the hint,

$$m^3 - 1 = (2k + 1)^3 - 1 = 8k^3 + 12k^2 + 6k + 1 - 1 = 2(4k^3 + 6k^2 + 3k)$$

so $m^3 - 1$ is even, as wanted. $\square$

9. a) **Claim:** $f(x) = (x - 2)^2$ is not even.
   PROOF: Notice that $f(-1) = (-1 - 2)^2 = 9$ but $f(1) = (1 - 2)^2 = 1$. Since $f(-1) \neq f(1)$, f is not even. $\square$

   b) **Claim:** $f(x) = x^2 - 2$ is even.
   PROOF: Observe that for any $x \in \mathbb{R}$, $f(-x) = (-x)^2 - 2 = x^2 - 2 = f(x)$. Thus f is even. $\square$

10. PROOF: ($\Leftarrow$) Suppose n is even. Then $n = 2k$ for $k \in \mathbb{Z}$. Thus

$$2^n - 1 = 2^{2k} - 1 = (2^k + 1)(2^k - 1).$$

Now, of the three consecutive integers $2^k - 1$, 2^k and $2^k + 1$, by the hint, one must be divisible by 3. Clearly 2^k isn't divisible by 3, so either $3 \mid (2^k - 1)$ or $3 \mid (2^k + 1)$. Either way, 3 divides the product $(2^k - 1)(2^k + 1) = 2^n - 1$ (by the result from class which says that $a|b$ and $b|c$ implies $a|c$).

($\Rightarrow$) Suppose n is odd; then $n = 2k + 1$ for $k \in \mathbb{Z}$. Thus

$$2^n - 1 = 2^{2k+1} - 1 = 2(2^{2k}) - 1 = 2(2^{2k} - 1) + 1.$$

The number $2k$ is even, so by the ($\Leftarrow$) part of this proof, $3|(2^{2k} - 1)$, so $3 \,|(2(2^{2k} - 1))$ as well. By the hint, 3 cannot divide the next consecutive integer, which is $2(2^{2k} - 1) + 1 = 2^n - 1$. $\square$.

2.2 Fall 2017 Exam 2

1. Classify the following statements as true or false.

   a) The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$ is surjective.

   b) The function $f : [0,1] \to [0,3]$ given by $f(x) = 3x$ is injective.

   c) If $f : A \to B$ is a bijection, then the inverse function $f^{-1} : B \to A$ must also be a bijection.

   d) For any sets P, Q and R, $(P \cap Q) \cap R = P \cap (Q \cap R)$.

   e) For any sets E and F, $(E \cup F)^C = E^C \cup F^C$.

   f) For any sets G and H, $G \cap H \subseteq G$.

   g) For any sets X, Y and Z, if $X \cup Z \subseteq Y \cup Z$, then $X \subseteq Y$.

   h) For any set E, the relation $\simeq$ is an equivalence relation on the power set 2^E, where $\simeq$ is defined by:

   $$A \simeq B \Leftrightarrow A \cap B \neq \emptyset.$$

   i) 19 and 7 belong to the same congruence class modulo 3.

   j) The relation $\{(3,2), (5,-8), (-4,7), (2,-7)\}$ is a function.

2. Consider the following sets:

   $$A = \{0,1,2\} \qquad B = \{2,4,6,8\} \qquad C = \{7,8,9,10\}$$

   and the following functions:

   $$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \text{ given by } f(m,n) = 2^m 3^n$$

   $$g : \mathbb{R} \to \mathbb{R} \text{ given by } g(x) = x + 2$$

   $$h : \mathbb{R} \to \mathbb{R} \text{ given by } h(x) = x^2$$

   Describe each of the following sets by giving a list of their elements.

   a) $B \cup C$

   b) $A \cap B^C$

   c) $A \times \emptyset$

   d) $g^{-1}(B)$

   e) $g^{-1}(B) \cap g(A)$

   f) $f(A \times \{1\})$

   g) $h([0,5])$

h) $h^{-1}(9)$

i) $(h \circ g)(A)$

j) $h^{-1}(h(A))$

3. Let A, B and C be sets. Prove that if $A \subseteq B$, then $A - C \subseteq B - C$.

4. Let $f : A \to B$. Prove or disprove: for any two sets $E \subseteq B$ and $F \subseteq B$,

$$f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F).$$

5. Let $g : \mathbb{R} - \{3\} \to \mathbb{R} - \{1\}$ be defined by

$$g(x) = \frac{x+1}{x-3}.$$

Prove that g is a bijection, by computing the inverse of g and proving that your formula for the inverse of g is correct.

6. Suppose $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow C$ are surjective functions. Prove that $g \circ f$ is also surjective.

Solutions

1. a) **FALSE:** $-1 \notin Range(f)$, so f is not surjective.
   b) **TRUE:** if $f(x) = f(y)$, then $3x = 3y$ so $x = y$.
   c) **TRUE:** f^{-1} is also invertible (its inverse is f), so it must be a bijection.
   d) **TRUE:** this is the associative law for intersections.
   e) **FALSE:** DeMorgan's Law says $(E \cup F)^C = E^C \cap F^C$.
   f) **TRUE:** taking intersection makes a set smaller.
   g) **FALSE:** for $X = \{1\}; Y = \emptyset; Z = \{1\}$, $X \cup Z \subseteq Y \cup Z$ but $X \not\subseteq Y$.
   h) **FALSE:** Let $A = \{1\}$, $B = \{1, 2\}$ and $C = \{2\}$. $A \simeq B$ and $B \simeq C$ but $A \not\simeq C$ so $\simeq$ is not transitive.
   i) **TRUE:** $3 \mid (19 - 7)$.
   j) **TRUE:** none of the inputs are related to more than one different output.

2. a) $B \cup C = \{2, 4, 6, 7, 8, 9, 10\}$.
   b) $A \cap B^C = A - B = \{0, 1\}$.
   c) $A \times \emptyset = \emptyset$.
   d) $g^{-1}(B) = \{0, 2, 4, 6\}$.
   e) $g^{-1}(B) \cap g(A) = \{0, 2, 4, 6\} \cap \{2, 3, 4\} = \{2, 4\}$.
   f) $f(A \times \{1\}) = f(\{(0, 1), (1, 1), (2, 1)\}) = \{3, 6, 12\}$.
   g) $h([0, 5]) = [0, 25]$.
   h) $h^{-1}(9) = \{-3, 3\}$.
   i) $(h \circ g)(A) = h(g(A)) = h(\{2, 3, 4\}) = \{4, 9, 16\}$.
   j) $h^{-1}(h(A)) = h^{-1}(\{0, 1, 4\}) = \{-2, -1, 0, 1, 2\}$.

3. PROOF Suppose $x \in A - C$. Then $x \in A \cap C^C$, so $x \in C^C$. Also, $x \in A$ and since $A \subseteq B$, this means $x \in B$. Thus $x \in B \cap C^C = B - C$. □

4. PROOF ($\subseteq$) Let $x \in f^{-1}(E \cup F)$. That means $f(x) \in E \cup F$.
   Case 1: $f(x) \in E$. Thus $x \in f^{-1}(E) \subseteq f^{-1}(E) \cup f^{-1}(F)$.
   Case 2: $f(x) \in F$. Thus $x \in f^{-1}(F) \subseteq f^{-1}(E) \cup f^{-1}(F)$.
   In either case, $x \in f^{-1}(E) \cup f^{-1}(F)$, as wanted.

   ($\supseteq$) Let $x \in f^{-1}(E) \cup f^{-1}(F)$.
   Case 1: $x \in f^{-1}(E)$. Therefore $f(x) \in E$ so $f(x) \in E \cup F$ so $x \in f^{-1}(E \cup F)$.
   Case 2: $x \in f^{-1}(F)$. Therefore $f(x) \in F$ so $f(x) \in E \cup F$ so $x \in f^{-1}(E \cup F)$.
   In either case, $x \in f^{-1}(E \cup F)$ as wanted. □

5. Set $y = \frac{x+1}{x-3}$ and solve for x using algebra to get $x = \frac{3y+1}{y-1}$. Thus $g^{-1}(x) = \frac{3x+1}{x-1}$.

   PROOF (THAT g AND g^{-1} ARE INVERSES) We compute $(g \circ g^{-1})(x)$ and $(g^{-1} \circ g)(x)$ and show that both are equal to x:

$$(g \circ g^{-1})(x) = g(g^{-1}(x)) = g\left(\frac{3x+1}{y-1}\right) = \frac{\frac{3x+1}{x-1}+1}{\frac{3x+1}{x-1}-3} = \frac{3x+1+x-1}{3x+1-3(x-1)} = \frac{4x}{4} = x;$$

$$(g^{-1} \circ g)(x) = g^{-1}(g(x)) = g^{-1}\left(\frac{x+1}{x-3}\right) = \frac{3\left(\frac{x+1}{x-3}\right)+1}{\frac{x+1}{x-3}-1} = \frac{3(x+1)+x-3}{x+1-(x-3)} = \frac{4x}{4} = x. \square$$

6. PROOF Let $z \in C$. Since g is surjective, there exists $y \in B$ such that $g(y) = z$. Since f is surjective, there exists $x \in A$ such that $f(x) = y$. For this x, $(g \circ f)(x) = g(f(x)) = g(y) = z$ so $z \in Im(g)$. Thus $g \circ f$ is surjective. $\square$

2.3 Fall 2017 Exam 3

1. In each item, you are given the description of a set. Characterize the set as finite, countably infinite, or uncountable:

   a) The integers $\mathbb{Z}$
   b) The rational numbers $\mathbb{Q}$
   c) The irrational numbers $\mathbb{R} - \mathbb{Q}$
   d) The power set of a finite set
   e) The union of a finite set and a countably infinite set
   f) The intersection of a finite set and a countably infinite set
   g) A union of countably many countably infinite sets
   h) The Cartesian product of a countable number of sets, each of which has cardinality 2

2. Prove that for all $n \in \mathbb{N}$,

$$0 \cdot 0! + 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1.$$

3. Prove that for all $n \in \mathbb{N}$,

$$3 \mid (5^n - 2^n).$$

4. Prove that for all $n \in \mathbb{N}$, when n^2 is divided by 8, the remainder is 0, 1 or 4.

5. Let x_n be the sequence defined by setting $x_0 = 3$ and $x_1 = 4$, and for all $n \geq 1$, setting

$$x_{n+1} = x_n + 6x_{n-1}.$$

   Prove that for all $n \in \mathbb{N}$,

$$x_n = 2 \cdot 3^n + (-2)^n.$$

6. Choose one of (a) or (b):

   a) Prove the **Fundamental Theorem of Arithmetic**, which says that every natural number greater than or equal to 2 can be written as a product of prime numbers.
   b) Prove the **Inclusion-Exclusion Law**, which says that for any two finite sets A and B,

$$\#(A \cup B) = \#(A) + \#(B) - \#(A \cap B).$$

Solutions

1. a) $\mathbb{Z}$ is **countably infinite**.

   b) $\mathbb{Q}$ is **countably infinite**.

   c) $\mathbb{R} - \mathbb{Q}$ is **uncountable** (if they were countable, $\mathbb{R}$ would be the union of two countable sets, hence countable).

   d) The power set of a finite set is **finite** (a set of size n has 2^n subsets).

   e) The union of a finite set and a countably infinite set is **countably infinite**.

   f) The intersection of a finite set and a countably infinite set is **finite**.

   g) A union of countably many countably infinite sets is **countably infinite**.

   h) The Cartesian product of a countable number of sets, each of which has cardinality 2 is **uncountable** (each element would be an infinite string of elements from the set of cardinality 2, which makes this set equinumerous with the set X we proved is uncountable in the notes).

2. We proceed by induction on n.

   Base case: when $n = 0$ we have

   $$0 \cdot 0! = 0 = 1 - 1 = (0+1)! - 1.$$

   Induction step: suppose the result is true when $n = k$. Then

   $$\begin{aligned} 0 \cdot 0! + \cdots + (k+1) \cdot (k+1)! &= [0 \cdot 0! + \cdots k \cdot k!] + (k+1) \cdot (k+1)! \\ &= [(k+1)! - 1] + (k+1) \cdot (k+1)! \quad \text{(by the IH)} \\ &= (k+1)!(1 + (k+1)) - 1 \\ &= (k+2)(k+1)! - 1 \\ &= (k+2)! - 1. \end{aligned}$$

   By induction, the result is true. □

3. We proceed by induction on n.

   Base case: when $n = 1$ we have $5^1 - 2^1 = 3$ so $3 \mid (5^1 - 2^1)$ as wanted.

   Induction step: suppose $3 \mid (5^k - 2^k)$. That means $5^k - 2^k = 3x$ where $x \in \mathbb{Z}$. Multiplying both sides by 5, we get $5^{k+1} - 5 \cdot 2^k = 3(5x)$. Adding $3 \cdot 2^k$ to both sides, we get $5^{k+1} - 2^{k+1} = 3(5x) + 3(2^k) = 3(5x + 2^k)$ so $3 \mid (5^{k+1} - 2^{k+1})$ as desired.

   By induction, we are done. □

4. By the Division Theorem, every $n \in \mathbb{N}$ can be written as $n = 4q + r$ where $r \in \{0, 1, 2, 3\}$. We argue in cases based on the value of r:

   Case 1: $r = 0$. Then $n = 4q$ so $n^2 = 16q^2 = 8(2q^2)$ so when n^2 is divided by 8, the remainder is 0.

   Case 2: $r = 1$. Then $n = 4q+1$ so $n^2 = (4q+1)^2 = 16q^2+8q+1 = 8(2q^2+q)+1$ so when n^2 is divided by 8, the remainder is 1.

   Case 3: $r = 2$. Then $n = 4q+2$ so $n^2 = (4q+2)^2 = 16q^2+16q+4 = 8(2q^2+2q)+4$ so when n^2 is divided by 8, the remainder is 4.

   Case 4: $r = 3$. Then $n = 4q + 3$ so $n^2 = (4q + 3)^2 = 16q^2 + 24q + 9 = 8(2q^2 + 3q + 1) + 1$ so when n^2 is divided by 8, the remainder is 1.

   In all cases, the remainder is 0, 1 or 4, so the result is true. □

5. We proceed by strong induction on n.

   Base cases: when $n = 0$, $x_0 = 3 = 2 \cdot 3^0 + (-2)^0$ as desired.

   When $n = 1$, $x_1 = 4 = 2 \cdot 3^1 + (-2)^1$ as desired.

   Induction step: Suppose the result is true for all $k \leq n$. Then

$$\begin{aligned} x_{n+1} &= x_n + 6x_{n-1} \\ &= 2 \cdot 3^n + (-2)^n + 6\left[2 \cdot 3^{n-1} + (-2)^{n-1}\right] \quad \text{(by the IH)} \\ &= 2 \cdot 3^n + (-2)^n + 4 \cdot 3^n - 3 \cdot (-2)^n \\ &= 6 \cdot 3^n - 2(-2)^n \\ &= 2 \cdot 3^{n+1} + (-2)^{n+1}. \end{aligned}$$

   By induction, we are done. □

6. a) Let E be the set of natural numbers greater than or equal to 2 that cannot be written as a product of primes. Suppose $E \neq \emptyset$; then by the WOP E has a least element x. If x is prime, then x can be written as a product of primes, so $x \notin E$, a contradiction. But if x is composite, then $x = ab$ where $1 < a < x$, $1 < b < x$. Since neither a nor b can lie in E, a and b can be written as product of primes:

$$a = p_1 \cdots p_m \qquad b = q_1 \cdots q_n$$

   But then $x = ab = p_1 \cdots p_m q_1 \cdots q_n$ is a product of primes, so $x \notin E$, a contradiction. Either way, E must be empty, proving the result. □

   b) Notice that $A \cup B$ is the disjoint union of the sets A and $B - A$. Therefore, by a result from class,

$$\#(A \cup B) = \#(A) + \#(B - A).$$

Therefore

$$\#(A \cup B) = \#(A) + \#(B - A) + \#(A \cap B) - \#(A \cap B).$$

Now, observe that B is the disjoint union of $A \cap B$ and $B - A$, so $\#(B) = \#(A \cap B) + \#(B - A)$. Substituting this into the previous equation gives

$$\#(A \cup B) = \#(A) + \#(B) - \#(A \cap B)$$

as wanted. □

2.4 Fall 2017 Final Exam

1. Write a truth table for the compound proposition $(P \vee Q) \Leftrightarrow (\sim P \wedge R)$.

2. a) Write the contrapositive of the statement "if a number is perfect, then it is even".

   b) Write a useful denial of the statement "if a number is perfect, then it is even".

   c) Write a useful denial of the statement "for every $x \in \mathbb{R}$, there exists $y \in \mathbb{N}$ such that $(x, y) \in E$".

3. Suppose you are trying to write a proof by induction of this statement:

   For all $n \in \mathbb{N}$, $[\![n]\!] \vdash E_n \oplus F$.

   a) Write the fact you would be trying to establish when you are doing the base case of your proof:

   b) Write the induction hypothesis you would start with, when writing the inductive step of your proof:

   c) Write the statement you would be trying to establish, when you get to the end of the inductive step of your proof:

4. In this problem, assume the universe of discourse is $\{a, b, c, d, e, f, g, h\}$ and that sets X, Y and Z are given by

   $$X = \{a, b, c, d, e\} \qquad Y = \{d, e\} \qquad Z = \{b, d, f, h\}$$

   List the elements in each of these sets:

   a) X^C

   b) $Y \cup \emptyset$

   c) $X \cap (Y \cup Z)$

   d) $Z - X$

   e) Y^2

   f) $X \triangle Z$

5. Classify the following statements as true or false:

   a) If $2 = 3$, then $5 = 8$.

   b) "If P, then Q" is logically equivalent to "P or not Q".

   c) "If P, then Q" is logically equivalent to "If not Q, then not P".

   d) For any two sets A and B, $A \times B = B \times A$

e) For any three sets E, F and G, $E \cap (F \cup G) = (E \cap F) \cup (E \cap G)$.
f) The relation $R = \{(x, y) : x + y \text{ is even}\}$ is an equivalence relation on $\mathbb{Z}$.
g) The relation $\{(3, 2), (7, 5), (8, 1), (4, 4), (2, 5)\}$ is a function.
h) The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - x$ is injective.
i) The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - x$ is surjective.
j) For any function f and any two subsets E and F of the codomain of f, $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$.
k) Every nonempty subset of $\mathbb{N}$ has a least element.
l) $\mathbb{R}$ is countable.
m) The Cartesian product of countably many finite sets, each of whom contain 2 elements, is countable.

6. Complete a total of eight of the sixteen proofs below, with the caveat that you must complete at least one and at most two from each category (A, B, C, D, E).

Category A:

A1. Prove that $\sqrt{2}$ is irrational.
A2. Let $x, y \in \mathbb{Z}$. Prove that if xy is even, then x or y is even.
A3. Let $n \in \mathbb{Z}$. Prove that if 7 does not divide n^2, then 7 does not divide n.

Category B:

B1. Let A, B and C be sets. Prove $(A \cup B) - (A \cup C) \subseteq A \cup (B - C)$.
B2. Let E, F and G be sets. Prove $(E \cap F) \times G = (E \times G) \cap (F \times G)$.
B3. Let R be an equivalence relation on a set E. Let $x, y \in E$ and suppose $x \, R \, y$. Prove $[x] = [y]$ (where $[z]$ denotes the R-equivalence class of z).

Category C:

C1. Prove the function $f(x) = (e^x + 1)^3$ is a bijection from $\mathbb{R}$ to $(1, \infty)$.
C2. Let $f : A \to B$ and $g : B \to C$. Prove that if $g \circ f$ is injective, then f is injective.
C3. Let $f : X \to Y$, and let A and B be subsets of X. Prove $f(A \cup B) = f(A) \cup f(B)$.

Category D:

D1. Let F be the set of all <u>finite</u> subsets of $\mathbb{N}$. Prove that F is countable.
D2. Prove that the union of two countably infinite sets is countable.

D3. Prove the Inclusion-Exclusion Law, which says that for two finite sets A and B, $\#(A \cup B) = \#(A) + \#(B) - \#(A \cap B)$.

Category E:

E1. Prove that for all natural numbers $n \geq 3$, $2n + 1 < 2^n$.

E2. Prove that for all $n \in \mathbb{N}$,

$$0 \cdot 2^0 + 1 \cdot 2^1 + 2 \cdot 2^2 + ... + n \cdot 2^n = 2 + (n-1)2^{n+1}.$$

E3. Prove that the union of finitely many finite sets is finite (you may assume that the union of two finite sets is finite).

E4. Let F_n denote the n^{th} Fibonacci number (i.e. $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$). Prove

$$\sum_{j=0}^{n} F_j = F_{j+2} - 1.$$

Solutions

1. Here is the completed truth table:

P	Q	R	$P \vee Q$	$\sim P$	$\sim P \wedge R$	$(P \vee Q) \Leftrightarrow (\sim P \wedge R)$
T	T	T	T	F	F	F
T	T	F	T	F	F	F
T	F	T	T	F	F	F
T	F	F	T	F	F	F
F	T	T	T	T	T	T
F	T	F	T	T	F	F
F	F	T	F	T	T	F
F	F	F	F	T	F	T

2. a) If a number is not even, then it is not perfect.
   b) There is a number which is perfect, but not even.
   c) There exists $x \in R$ such that for all $y \in \mathbb{N}$, $(x, y) \notin E$.

3. a) $[\![0]\!] \vdash E_0 \oplus F$.
   b) $[\![k]\!] \vdash E_k \oplus F$.
   c) $[\![k+1]\!] \vdash E_{k+1} \oplus F$.

4. a) $X^C = \{f, g, h\}$.
   b) $Y \cup \emptyset = Y = \{d, e\}$.
   c) $X \cap (Y \cup Z) = X \cap \{b, d, e, f, h\} = \{b, d\}$.
   d) $Z - X = \{f, h\}$.
   e) $Y^2 = \{(d, d), (d, e), (e, d), (e, e)\}$.
   f) $X \triangle Z = \{a, c, e, f, h\}$.

5. a) **TRUE** (vacuously).
   b) **FALSE**: "If P, then Q" is logically equivalent to "Q or not P".
   c) **TRUE**: a conditional is logically equivalent to its contrapositive.
   d) **FALSE**: the order of an ordered pair matters.
   e) **TRUE** (this is a distributive law for set operations).
   f) **TRUE** (this is the same relation as equivalence modulo 2)
   g) **TRUE** since each input has at most one output.
   h) **FALSE** since $f(0) = f(1) = 0$.
   i) **TRUE** this function is cubic, so its tails point toward $+\infty$ and $-\infty$, so its range includes every real number.
   j) **TRUE**: inverses preserve both union and intersection.
   k) **TRUE** (this is the Well-Ordering Property of $\mathbb{N}$).

l) **FALSE**: $\mathbb{R}$ is uncountable.

m) **FALSE**: such a product space would contain infinite sequences of elements of the sets; such a set is uncountable for the same reasons the set X discussed in class is uncountable.

A1. Suppose not, i.e. $\sqrt{2} = \frac{p}{q}$ in lowest terms where $p, q \in \mathbb{Z}$. Then $2q^2 = p^2$, so p^2 is even, so p is even. Thus $p = 2s$ for $s \in \mathbb{Z}$ and consequently $2q^2 = (2s)^2 = 4s^2$ so $q^2 = 2s^2$. Thus q^2 is even, so q is even. Therefore 2 divides both p and q, so $\frac{p}{q}$ wasn't in lowest terms after all, a contradiction. Therefore $\sqrt{2} \notin \mathbb{Q}$. □

A2. We prove the contrapositive: suppose x and y are odd. Then $x = 2m + 1$ and $y = 2n + 1$ for $m, n \in \mathbb{Z}$. Thus $xy = (2m + 1)(2n + 1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1$ is odd. By contraposition, we are done. □

A3. We prove the contrapositive: suppose $7 \mid n$. Then $n = 7k$ for $k \in \mathbb{Z}$. Thus $n^2 = (7k)^2 = 49k^2 = 7(7k^2)$ so $7 \mid n^2$. By contraposition, we are done. □

B1. Let $x \in (A - C) \cup (B - C)$.

Case 1: $x \in A - C = A \cap C^C$. Then, since $x \in A$, $x \in A \cup B$. Also, $x \in C^C$ so $x \in (A \cup B) - C$ as desired.

Case 2: $x \in B - C = B \cap C^C$. Then, since $x \in B$, $x \in A \cup B$. Also, $x \in C^C$ so $x \in (A \cup B) - C$ as desired.

In either case, $x \in (A \cup B) - C$. □

B2. ($\subseteq$) Let $x \in (E \cap F) \times G$. Then $x = (a, b)$ where $a \in E \cap F$ and $b \in G$. Then, since $a \in E$ and $b \in G$, $x = (a, b) \in E \times G$. Similarly, since $a \in F$ and $b \in G$, $x = (a, b) \in F \times G$ so $x \in (E \times G) \cap (F \times G)$ as desired.

($\supseteq$) Let $x = (a, b) \in (E \times G) \cap (F \times G)$. Since $x \in E \times G$, $a \in E$ and $b \in G$. Also, since $x \in F \times G$, $a \in F$ and $b \in G$. Thus $a \in E \cap F$ so $x = (a, b) \in (E \cap F) \times G$. □

B3. ($\subseteq$) Let $z \in [x]$. This means xRz. Since xRy, by transitivity zRy, i.e. $z \in [y]$. This proves $[x] \subseteq [y]$.

($\supseteq$) Let $z \in [y]$. This means yRz. Since xRy, by transitivity zRx, i.e. $z \in [x]$. This proves $[y] \subseteq [x]$. □

C1. Define $f^{-1} : (1, \infty) \to \mathbb{R}$ by $f^{-1}(y) = \ln(\sqrt[3]{y} - 1)$. We have

$$f^{-1}(f(x)) = \ln\left(\sqrt[3]{(e^x + 1)^3} - 1\right) = \ln(e^x + 1 - 1) = \ln e^x = x$$

and

$$f(f^{-1}(y)) = \left(e^{\ln\left(\sqrt[3]{y} - 1\right)} + 1\right)^3 = (\sqrt[3]{y} - 1 + 1)^3 = (\sqrt[3]{y})^3 = y$$

so f and f^{-1} are inverse functions. Since f is invertible, it is a bijection. □

C2. Suppose x_1 and x_2 in X are such that $f(x_1) = f(x_2)$. Applying g to both sides, we get $g(f(x_1)) = g(f(x_2))$, i.e. $(g \circ f)(x_1) = (g \circ f)(x_2)$. But since $g \circ f$ is injective, $x_1 = x_2$. Thus f is injective. $\square$

C3. ($\subseteq$) Let $x \in f(A \cup B)$. Then $x = f(a)$ where $a \in A \cup B$.

Case 1: $a \in A$. Then $x = f(a) \in f(A)$.

Case 2: $a \in B$. Then $x = f(a) \in f(B)$.

In either case, $x \in f(A) \cup f(B)$.

($\supseteq$) Let $x \in f(A) \cup f(B)$.

Case 1: $x \in f(A)$. Thus $x = f(a)$ for $a \in A$. Since $a \in A \cup B$, $x \in f(A \cup B)$.

Case 2: $x \in f(B)$. Thus $x = f(b)$ for $b \in B$. Since $b \in A \cup B$, $x \in f(A \cup B)$.

In either case, $x \in f(A \cup B)$. $\square$

D1. For each $n \in \mathbb{N}$, let F_n be the set of subsets of $\mathbb{N}$ of cardinality n. The function $f : \mathbb{N}^n \to F$ given by $f(x_1, x_2, ..., x_n) = \{x_1, ..., x_n\}$ maps onto all of F_n, so F_n is countable for each n. But $F = \bigcup_{n=0}^{\infty} F_n$ is therefore a countable union of countable sets, so F is countable. $\square$

D2. Let A and B be countably infinite; then there are bijections $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$. Define $h : \mathbb{N} \to A \cup B$ by "alternating" values of f and g. More precisely, set

$$h(n) = \begin{cases} f\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ g\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd} \end{cases}.$$

h is the union of two surjections, so is surjective. Since we have established a surjective function $h : \mathbb{N} \to A \cup B$, $A \cup B$ is countable. $\square$

D3. Notice that A is the disjoint union of $A \cap B$ and $A - B$; similarly B is the disjoint union of $A \cap B$ and $B - A$. Therefore

$$\#(A) = \#(A \cap B) + \#(A - B) \quad \text{and} \quad \#(B) = \#(A \cap B) + \#(B - A).$$

Adding the left- and right-hand sides of these equations gives

$$\#(A) + \#(B) = \#(A \cap B) + \#(A - B) + \#(A \cap B) + \#(B - A)$$

so by subtracting $\#(A \cap B)$ from both sides, we get

$$\#(A) + \#(B) - \#(A \cap B) = \#(A \cap B) + \#(A - B) + \#(B - A).$$

But the three sets on the right-hand side of this equation are disjoint and have union $A \cup B$, so the result follows:

$$\#(A) + \#(B) - \#(A \cap B) = \#(A \cup B). \ \square$$

E1. Induction on n:

Base case: when $n = 3$, $2(3) + 1 = 7 < 8 = 2^3$.

Induction step: Suppose $k \geq 3$ and $2k+1 < 2^k$. Since $k \geq 3$, $2 < 2^k$ so by adding the smaller and larger terms of both inequalities, we get $2k + 1 + 2 < 2^k + 2^k$, i.e. $2k + 3 < 2 \cdot 2^k$, i.e. $2(k + 1) + 1 < 2^{k+1}$. By induction, we are done. □

E2. *Base case:* when $n = 0$, $0 \cdot 2^0 = 0 = 2 - 2 = 2 + (-1)2^1$.

Induction step: Suppose $\sum_{j=0}^{k} j2^j = 2 + (k - 1)2^{k+1}$. Then

$$\begin{aligned}
\sum_{j=0}^{k+1} j\, 2^j &= \sum_{j=0}^{k} j2^j + (k + 1)2^{k+1} \\
&= \left[2 + (k - 1)2^{k+1}\right] + (k + 1)2^{k+1} \quad \text{(by the IH)} \\
&= 2 + (k - 1 + k + 1)2^{k+1} \\
&= 2 + 2k\, 2^{k+1} \\
&= 2 + k\, 2^{k+2} \\
&= 2 + [(k + 1) - 1]2^{(k+1)+1} \text{ as wanted.}
\end{aligned}$$

By induction on n, we are done. □

E3. Let n be the number of sets in the union. The proof is by induction on n (the base case $n = 2$ can be assumed):

Induction step: Suppose the union of any k finite sets is finite. Then, let $A_1, ..., A_{k+1}$ be finite sets. We have

$$\bigcup_{j=1}^{k+1} A_j = \left(\bigcup_{j=1}^{k} A_j\right) \cup A_{k+1}.$$

By the IH, the set $\bigcup_{j=1}^{k} A_j$ is finite, so $\bigcup_{j=1}^{k+1} A_j$ is the union of two finite sets, which is finite by the base case. By induction, we are done. □

E4. The proof is by strong induction on n:

Base cases: when $n = 0$, $\sum_{j=0}^{0} F_j = F_0 = 0 = 1 - 1 = F_2 - 1$.

When $n = 1$, $\sum_{j=0}^{1} F_j = F_0 + F_1 = 0 + 1 = 2 = 3 - 1 = F_3 - 1$.

Induction step: Suppose that for all $k \leq n$, $\sum_{j=0}^{k} F_j = F_{k+2} - 1$. Then

$$\begin{aligned}
\sum_{j=0}^{n+1} F_j &= \sum_{j=0}^{n} F_j + F_{n+1} \\
&= F_{n+2} - 1 + F_{n+1} \quad \text{(by the IH)} \\
&= [F_{n+2} + F_{n+1}] - 1 \\
&= F_{n+3} - 1 \quad \text{(by the definition of the Fibonacci numbers)} \\
&= F_{(n+1)+2} - 1 \text{ as wanted.}
\end{aligned}$$

By induction, we are done. □