Ordinary Differential Equations

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Contents

Contents **2**

1 First-order equations: theory **4**
1.1 What is a differential equation? 4
1.2 Linear versus nonlinear 10
1.3 A first example: exponential growth and decay 20
1.4 Qualitative analysis of first-order ODEs (slope fields) 23
1.5 Numerical techniques: Euler's method 29
1.6 Existence and uniqueness of solutions 38
1.7 Autonomous equations; equilibria and phase lines 41
1.8 Bifurcations 51
1.9 Summary of Chapter 1 58

2 First-order equations: solution techniques **60**
2.1 Solution of first-order homogeneous linear equations 60
2.2 Solution of first-order linear equations by integrating factors 65
2.3 Solution of first-order linear equations by undetermined coefficients 70
2.4 Separation of variables 76
2.5 Exact equations 82
2.6 Applications of first-order equations 87
2.7 Summary of Chapter 2 98
2.8 Exam 1 Review 100

3 First-order linear systems **108**
3.1 Language for systems of ODEs 108
3.2 Euler's method for systems 112
3.3 Existence and uniqueness of solutions 116
3.4 Matrices and matrix operations 117
3.5 Matrix operations on *Mathematica* 129

3.6 First-order linear systems of ODEs . . . 131
3.7 The structure of the solution set of a linear system . . . 133
3.8 Autonomous systems; slope fields and phase planes . . . 144
3.9 First-order, constant-coefficient homogeneous systems . . . 151
3.10 A crash course in complex numbers . . . 172
3.11 Complex eigenvalues . . . 180
3.12 Repeated eigenvalues . . . 187
3.13 Summary of linear, const.-coeff., homogeneous linear systems . . . 191
3.14 Non-homogeneous systems . . . 196
3.15 Classification of equilibria . . . 198
3.16 The trace-determinant plane . . . 204
3.17 Applications of first-order systems . . . 207
3.18 Exam 2 Review . . . 213

4 Higher-order linear equations **221**
4.1 Reduction of order . . . 221
4.2 n^{th}-order, linear, constant-coefficient equations . . . 225
4.3 Variation of parameters . . . 232
4.4 Applications of higher-order equations and systems . . . 236
4.5 Final Exam Review . . . 250

A Homework **258**
A.1 Problems from Chapter 1 . . . 261
A.2 Problems from Chapter 2 . . . 271
A.3 Problems from Chapter 3 . . . 279
A.4 Problems from Chapter 4 . . . 297
A.5 Selected answers to the homework problems . . . 301
A.6 Extra credit problems . . . 308

B *Mathematica* information **310**
B.1 General *Mathematica* principles . . . 310
B.2 *Mathematica* quick reference guides . . . 313
B.3 Graphing functions with *Mathematica* . . . 317
B.4 Solving equations with *Mathematica* . . . 323
B.5 Matrix operations on *Mathematica* . . . 324
B.6 Complex numbers in *Mathematica* . . . 326
B.7 Slope fields for first-order equations . . . 327
B.8 Euler's method for first-order equations . . . 328
B.9 Picard's method for first-order equations . . . 330
B.10 Euler's method for first-order 2×2 and 3×3 systems . . . 330
B.11 Slope fields for first-order 2×2 systems . . . 333

Index **335**

Chapter 1

First-order equations: theory

1.1 What is a differential equation?

Consider an object of mass 20 kg, that is falling through the Earth's atmosphere (gravitational constant is 9.8 m/sec^2, drag coefficient 3 N sec/m), near sea level. Let's try to formulate an equation which describes the velocity of the object.

The setup on the previous page is an example of an *ordinary differential equation*. More precisely:

Definition 1.1 *An* **ordinary differential equation (ODE)** *is an equation involving an independent variable t and a function $y = y(t)$, together with the derivatives of y with respect to t.*

Every such equation can be written as

$$\Phi(t, y, y', y'', y'''...) = 0$$

where Φ is some function of t, y and the derivatives of y with respect to t.

The point of this course is to learn how to solve as many ODEs (and systems of ODEs) as possible, and learn how to analyze ODEs that are "too complicated" to solve.

Notation: Throughout this course, the independent variable will be called t. This is because in real-world applications of differential equations, the independent variable usually represents time. Letters such as x, y or z are almost always taken to be functions of t, i.e. most of the time x means $x(t)$; y means $y(t)$; etc.

Examples of ODEs

Example 1: $y' + ty = 3t$ (i.e. $y' + ty - 3t = 0$)

Unwritten assumption: y is a function of t, i.e. $y = y(t)$.

Goal: find explicit equation relating y and t (containing no derivatives), i.e. find $y = y(t)$.

This equation can also be written as

$$y'(t) + ty(t) = 3t$$
or
$$\frac{dy}{dt} + ty = 3t$$
or
$$\dot{y} + ty = 3t$$
etc.

Question: Is $y(t) = e^{-t}$ a solution of the ODE in Example 1?

Question: Is $y(t) = 3 + e^{-t^2/2}$ a solution of the ODE in Example 1 (the ODE was $y' + ty = 3t$)?

Example 2: $\cos y^{(7)} - y^{(4)} e^{ty''-3t^2} + 2y''(y')^3 - 4t^2 y''' + (t^2 - 3ty)y'' + 2y'y = 3ty' - 4$

Example 3: $x'' + c_1 x' + c_0 x = 0$ (describes motion of damped oscillator)

Assumption: $x = x(t)$ is a function of t

Ambiguity: c_1 and c_0: are they constants, or functions of t? In general, letters near the beginning of the alphabet connote constants (and in this example, c_1 and c_0 are constants). However, in the grand scheme of things, letters like this could be constants or could be functions of t ($c_1(t)$ and $c_0(t)$).

This equation can also be written as $\ddot{x} + c_1 \dot{x} + c_0 x = 0$

Example 4: $\frac{dy}{dt} = 3y$

Assumption: $y = y(t)$

In Example 4,

- $y = e^{3t}$ is a solution because $\frac{dy}{dt} = \frac{d}{dt}(e^{3t}) = 3e^{3t} = 3y$
- $y = 2e^{3t}$ is a solution because $\frac{dy}{dt} = \frac{d}{dt}(2e^{3t}) = 6e^{3t} = 3 \cdot 2e^{3t} = 3y$
- $y = Ce^{3t}$ is a solution for any constant C because $\frac{dy}{dt} = \frac{d}{dt}(Ce^{3t}) = 3Ce^{3t} = 3y$
- $y = t^2$ is not a solution because $\frac{dy}{dt} = 2t$ but $3y = 3t^2$.

Question: Are there any solutions to Example 4 other than $y = Ce^{3t}$?

Some ODEs are particularly easy to solve. Suppose $\frac{dy}{dt} = g(t)$ (i.e. there's only a t on the right-hand side). Then

$$y = y(t) = \int g(t)\,dt$$

by the Fundamental Theorem of Calculus (in particular, notice that there will be infinitely many solutions, parameterized by a single constant C).

Example 5:

$$\frac{dy}{dt} = 6t^2$$

Example 6:

$$y'' = \cos 2t$$

These constants (the C in Example 4, the C in Example 5, and the C and D in Example 6) are typical of solutions to ODEs, because solving an ODE is akin to performing indefinite integration. For a general ODE (not just one of the form $\frac{dy}{dt} = g(t)$), we expect arbitrary constants in the description of the solution. We will prove in this course that for the most common class of ODEs, **the number of constants in the solution is equal to the highest order of derivative occurring in the equation**, i.e.

$$y''' - 3y = t^2 + 3 \Rightarrow \text{there will be } 3 \text{ arbitrary constants in the answer}$$
$$y + t\frac{d^6y}{dt^6} = t^3 - \frac{d^2y}{dt^2} \Rightarrow \text{there will be } 6 \text{ arbitrary constants in the answer}$$
$$y^{(5)} + 3 = t \Rightarrow \text{there will be } 5 \text{ arbitrary constants in the answer}$$

With that in mind, we make the following definition:

Definition 1.2 *The* **order** *of a differential equation is the highest order of derivative that appears in the equation. An ODE whose order is n is called n^{th}* **order**.

So in an ODE of order n, we expect n arbitrary constants in the answer.

Sometimes you know additional information which allows you to solve for the constant(s):

Example 7: Suppose a bug travels along a line with velocity at time t given by $v(t) = 2t - 4$. If at time 0, the bug is at position 7, what is the bug's position at time t?

In many real-world applications of ODEs, you are given a point $(t_0, y(t_0))$ that the solution of the ODE is to pass through (for example, in Example 7, you are given $t_0 = 0, y_0 = y(t_0) = 7$).

Definition 1.3 *A point (t_0, y_0) through which a solution to an ODE must pass is called an* **initial value**. *(In this context $y_0 = y(t_0)$.) An ODE, together with an initial value, is called an* **initial value problem (IVP)**. *A solution of an initial value problem is called a* **particular solution** *of the ODE; the set of all particular solutions of the ODE is called the* **general solution** *of the ODE.*

Example: $y' = 2t$ is a first-order ODE, whose general solution is $y = t^2 + C$.

Example:

$$\begin{cases} y' = 2t \\ y(0) = 1 \end{cases}$$

is an initial value problem, whose only particular solution is $y = t^2 + 1$.

General rule of thumb: If you have an initial value problem, start by solving the ODE (i.e. find the general solution). Then plug in the initial value(s) to solve for the constant(s), and write the particular solution.

Why do we call these equations "ordinary"?

Notice that in Definition 1.1 above, there is one independent variable t. In multi-variable calculus (Math 320), you learn about functions of more than one variable, like

$$f(s,t) = s^2 - 2s^2t + 5t^3.$$

The natural kind of differentiation you do with these functions is to compute *partial derivatives* $f_s = \frac{\partial f}{\partial s}$ and $f_t = \frac{\partial f}{\partial t}$ (see Section 2.5 for more on these). Differential equations involving partial derivatives are called **partial differential equations (PDEs)**. An example of a PDE would be something like

$$\frac{\partial f}{\partial s} - \frac{\partial f}{\partial t} = s^2 - st \qquad \text{or} \qquad \frac{1}{2}\frac{\partial^2 f}{\partial s^2} + \frac{1}{2}\frac{\partial^2 f}{\partial t^2} = 1;$$

in this case the assumption is that f is some function of two independent variables s and t (i.e. $f = f(s,t)$.

PDEs are **much harder** to study than ODEs, in general. Your professor, for example, knows next to nothing about PDEs.

We call differential equations "ordinary" to make clear that they are **not** PDEs, i.e. that there is one independent variable and that there are therefore no partial derivatives being taken.

1.2 Linear versus nonlinear

Numerical equations can be classified into different types:

EQUATION	CLASS
$5x + 2 = 7(x - 1)$	linear
$2x^2 + 4x - 3 = x(x - 1)$	quadratic
$x^4 - 3x^2 + x - 4 = 0$	polynomial
$\frac{3}{x+1} + \frac{x}{x-2} = \frac{2}{x}$	rational
$2 \sin x + 1 = 0$	trigonometric
$5e^{x-2} = 3$	exponential
$\ln x + \ln(x + 2) = 1$	logarithmic
$\vdots$	$\vdots$

Each class of equation is studied with its own techniques. (For example, to solve a quadratic you set one side equal to zero and factor or use the quadratic formula; to solve a rational function, you find a common denominator; etc.)

Question: What is the most important class of numerical equations to learn how to solve?

Question: Why is this class particularly important to study? There are three reasons:

1. Ubiquity:

2. Simplicity:

3. "Approximability":

Similarly, ODEs can be classified into various types; *the most important distinction to make when classifying an ODE is to determine whether or not that ODE is "linear"*.

Whether or not a differential equation is "linear" has a lot to do with the subject of linear algebra. In fact, linear algebra is essentially concerned with studying what is meant by the word "linear" in a very general sense.

First, remember what we know about lines from high school: they all (other than vertical lines) have an equation of the form

$$y = mx + b$$

What operations are required to describe (the right-hand side of) this equation?

1.

2.

Based on this observation, if we are going to define "linear" in a general sense, we probably need to assume that there is some notion of each of the two operations above.

Vector spaces

A general setting in which we can add objects and multiply objects by real numbers is called a *vector space*; elements of a vector space are called *vectors*. More precisely:

Definition 1.4 *Informally, a* **(real) vector space** V *is a set of objects called* **vectors** *such that:*

1. *you can add two vectors in* V*, and the sum is always a vector in* V*;*
2. *you can multiply a vector in* V *by a real number (in this context the real number is called a* **scalar***), and the result is always a vector in* V*;*
3. *these methods of addition and scalar multiplication obey a bunch of reasonable "laws" (like the commutative property, the associative property, the distributive property, the fact that multiplying a vector by* 1 *gives the same vector, etc.)*

Exactly what laws need to be obeyed in condition (3) of this definition are spelled out in Chapter 1 of my Linear Algebra (Math 322) Lecture Notes.

Examples of vector spaces

Pretty much any set of objects where you can (a) add the objects to one another, and (b) multiply the objects by real numbers, is a vector space. Here are some prototypical examples:

1. *Real numbers:* $\mathbb{R}$ is a real vector space (where the addition and scalar multiplication are the usual numerical operations).

2. *Ordered pairs:* $\mathbb{R}^2$ is the set of ordered pairs (x, y) of real numbers. To add two ordered pairs, add them coordinate-wise. For example,

$$(2,5) + (-3,2) = (2 + (-3), 5 + 2) = (-1,7).$$

   Similarly, to multiply an ordered pair by a scalar, multiply coordinate-wise (i.e. multiply each coordinate by the scalar). For example,

$$3(-1,7) = (-3,21).$$

3. *Ordered n-tuples ("traditional" vectors):* For any $n \in \mathbb{N}$, $\mathbb{R}^n = \{(x_1, ..., x_n) : x_j \in \mathbb{R} \text{ for all } j\}$ is a real vector space, where the addition and scalar multiplication are defined coordinate-wise.

4. *Matrices:* The set of $m \times n$ matrices (this means m rows and n columns) with elements in $\mathbb{R}$, denoted $M_{mn}(\mathbb{R})$, is a real vector space where the addition and scalar multiplication are defined entry-wise. (Notation: the set of square $n \times n$ matrices with entries in $\mathbb{R}$ is denoted $M_n(\mathbb{R})$ rather than $M_{nn}(\mathbb{R})$.) Matrix spaces are discussed in more detail later in this course and in Chapter 2 of my Math 322 lecture notes.

So among the things that can be thought of as vectors are numbers, ordered pairs, ordered n-tuples (these are "traditional" vectors), and matrices. In this course, we are most concerned with a different vector space, where the vectors are *functions*:

Definition 1.5 *Let $C^\infty(\mathbb{R}, \mathbb{R})$ denote the set of functions from $\mathbb{R}$ to $\mathbb{R}$ which are* **infinitely differentiable***, i.e. functions f for which the n^{th} derivative of f exists for all n. Define the following operations on $C^\infty(\mathbb{R}, \mathbb{R})$:*

Addition: *Given $f, g \in C^\infty(\mathbb{R}, \mathbb{R})$, define $f + g \in C^\infty(\mathbb{R}, \mathbb{R})$ by setting $(f + g)(t) = f(t) + g(t)$;*

Scalar multiplication: *Given $f \in C^\infty(\mathbb{R}, \mathbb{R})$ and $c \in \mathbb{R}$, define $cf \in C^\infty(\mathbb{R}, \mathbb{R})$ by setting $(cf)(t) = c\, f(t)$.*

These are the usual definitions of addition and scalar multiplication of functions that you already know. For example, if $f(t) = \sin t$ and $g(t) = 4 \sin 2t$, then

$$(f + g)(t) = \sin t + 4 \sin 2t \qquad (3f)(t) = 3 \sin t \qquad (2f - 5g)(t) = 2 \sin t - 20 \sin 2t$$

Theorem 1.6 *$C^\infty(\mathbb{R}, \mathbb{R})$, with the operations defined above, is a vector space.*

Linear operators

Definition 1.7 *Let V be a vector space. A function $T : V \to V$ is called an* **operator (on V)**. *A function $T : V \to V$ is called a* **linear operator (on V)** *(or just* **linear***) if it preserves the vector space operations on V, i.e.*

T **preserves addition:** $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ *for any* $\mathbf{v}, \mathbf{w} \in V$*; and*

T **preserves scalar multiplication:** $T(c\mathbf{v}) = c\,T(\mathbf{v})$ *for any* $c \in \mathbb{R}$ *and any* $\mathbf{v} \in V$.

Notation: "$f : A \to B$" means f is a function whose inputs are in set A and whose outputs are in set B.

Note: In linear algebra, you learn about functions called *linear transformations*. A linear operator is the same thing as a linear transformation, except that for an operator, the domain and range are the same vector space.

Motivating Example: What are the linear operators on the vector space $\mathbb{R}$?

- T, defined by $T(t) = 5t$, is a linear operator:

- More generally, multiplication by any fixed constant a is linear. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $T(t) = at$:

  $$T \text{ preserves addition: } T(s+t) = a(s+t) = as + at = T(s) + T(t);$$

  $$T \text{ preserves scalar multiplication: } T(ct) = a(ct) = c(at) = cT(t).$$

  Since T preserves addition and scalar multiplication, it is a linear operator.

- T, defined by $T(t) = \sqrt{t}$, is not linear:

- T, defined by $T(t) = \frac{3}{2}t + \frac{5}{2}$, is not linear.

Theorem 1.8 *The **only** linear operators on $\mathbb{R}$ are multiplication by constants. In other words, if $T : \mathbb{R} \to \mathbb{R}$ is linear, then $T(t) = at$ for some constant $a \in \mathbb{R}$.*

PROOF Suppose $T : \mathbb{R} \to \mathbb{R}$ is linear. Let $a = T(1)$. Then for any $t \in \mathbb{R}$,

$$T(t) = T(t1) = tT(1) = ta = at. \square$$

Question: How do you decide whether a numerical equation is linear?

Definition 1.9 *A numerical equation is called* **linear** *if there is a linear operator T on $\mathbb{R}$ and a real number $b \in \mathbb{R}$ such that the equation can be written in the form*

$$T(t) = b.$$

(The goal here is to solve for t.) Equivalently, this means that there are real numbers a and b so that the equation can be rewritten as

$$at = b.$$

The first line of this definition will generalize to ODEs, but we need to start with a different vector space (other than $\mathbb{R}$).

Question: Let $V = C^\infty(\mathbb{R}, \mathbb{R})$. What operators on this vector space are linear?

Example 1: operator T on $C^\infty(\mathbb{R}, \mathbb{R})$ defined by $T(y) = t^2 y$.

Example 1, generalized: Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Then the operator T on $C^\infty(\mathbb{R}, \mathbb{R})$ defined by $T(y) = f(t)y$ is linear:

$$\begin{aligned} T(x+y) = T(x(t)+y(t)) = f(t)(x(t)+y(t)) &= f(t)x(t) + f(t)y(t) \\ &= T(x(t)) + T(y(t)) \\ &= T(x) + T(y). \end{aligned}$$

$$T(cy) = T(cy(t)) = f(t)cy(t) = c[f(t)y(t)] = cT(y(t)) = cT(y).$$

Since T preserves addition and scalar multiplication, it is a linear operator.

So multiplication by any function is a linear operator on $C^\infty(\mathbb{R}, \mathbb{R})$. This is analogous to multiplication by any fixed constant being a linear operator on $\mathbb{R}$ (see Theorem 1.8). But there are other linear operators on $C^\infty(\mathbb{R}, \mathbb{R})$ (see Example 3 below).

Example 2: operator T defined by $T(y) = y^2$ is not linear:

$$T(x+y) = (x+y)^2 \neq x^2 + y^2 = T(x) + T(y).$$

Example 3: operator D (D is for differentiation) defined by $D(y) = y'$ is linear:

$$D(x+y) = (x+y)' = x' + y' = D(x) + D(y);$$

$$D(cy) = (cy)' = c\,y' = cD(y).$$

You can make more complicated linear operators out of simpler ones by adding them, multiplying by scalars, and composing them:

Lemma 1.10 *Suppose T_1 and T_2 are linear operators on vector space V. Then:*

Sums of linear operators are linear: *T_1+T_2 is a linear operator on V (where $(T_1+T_2)(\mathbf{v}) = T_1(\mathbf{v}) + T_2(\mathbf{v})$);*

Scalar multiples of linear operators are linear: *for any constant $c \in \mathbb{R}$, cT_1 is a linear operator on V (where $(cT_1)(\mathbf{v}) = c\,T_1(\mathbf{v})$);*

Compositions of linear operators are linear: *$T_2 \circ T_1$ is a linear operator on V (where $T_2 \circ T_1(\mathbf{v}) = T_2(T_1(\mathbf{v}))$).*

PROOF See Section 5.4 of my Math 322 lecture notes. □

Example: Suppose $D : C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R})$ is differentiation ($D(y) = y'$), and $T : C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R})$ is multiplication by t^2 ($T(y) = t^2 y$).

Lemma 1.10 has the following important consequence:

Theorem 1.11 *For any collection of* $n+1$ *functions* $p_0, p_1, p_2, ..., p_n : \mathbb{R} \to \mathbb{R}$*, the function* $T : C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R})$ *defined by*

$$T(y) = \sum_{j=0}^{n} p_j(t) y^{(j)}(t) = p_0 y + p_1 y' + p_2 y'' + p_3 y''' + ... + p_n y^{(n)}$$

is a linear operator.

Recall: for a function $y : \mathbb{R} \to \mathbb{R}$, $y^{(j)}$ denotes the j^{th} derivative of y; in particular $y^{(0)} = y$ and $y^{(1)} = y'$.

PROOF From Example 3 earlier in this section, the differentiation operator D defined by $D(y) = y'$ is linear, and from Example 1 (generalized) of this section, multiplication by a fixed function is linear. T is made up of a sum of compositions of these types of operators, so T is linear by Lemma 1.10. □

Definition 1.12 *An operator* T *on* $C^\infty(\mathbb{R}, \mathbb{R})$ *is called a* **linear differential operator** *(or just a* **differential operator***) if it can be written as*

$$T(y) = \sum_{j=0}^{n} p_j y^{(j)} = p_0 y + p_1 y' + p_2 y'' + p_3 y''' + ... + p_n y^{(n)}$$

for functions $p_0, p_1, p_2, ..., p_n : \mathbb{R} \to \mathbb{R}$*. We say the linear differential operator is* n^{th} **order** *if* $p_n \neq 0$ *but* $p_j = 0$ *for all* $j > n$*.*

Example: Consider the third-order linear differential operator T defined by setting $p_0(t) = \cos t$, $p_1(t) = t^3$, $p_2(t) = 0$ and $p_3(t) = t$. Write the formula for $T(y)$ and compute $T(y)$ where $y = t^4$.

We are now in position to define what it means for a differential equation to be linear:

Definition 1.13 *An ODE is called* **linear** *if there is a linear differential operator* T *and a function* $q \in C^\infty(\mathbb{R}, \mathbb{R})$ *such that the equation can be written as*

$$T(y) = q.$$

In other words, an ODE is linear if there are functions $p_0, p_1, p_2, ..., p_n : \mathbb{R} \to \mathbb{R}$ *and an infinitely differentiable function* $q : \mathbb{R} \to \mathbb{R}$ *such that the equation can be written as*

$$p_0 y + p_1 y' + p_2 y'' + p_3 y''' + ... + p_n y^{(n)} = q$$

i.e. the equation is of the form

$$p_0(t)y(t) + p_1(t)y'(t) + p_2(t)y''(t) + p_3(t)y'''(t) + ... + p_n(t)y^{(n)}(t) = q(t).$$

An ODE is called **nonlinear** *if it is not linear.*

(Compare the first line of this definition with the first line of Definition 1.9. In general, a linear equation is any equation of the form $T(\mathbf{x}) = \mathbf{b}$ where T is a linear transformation, and $\mathbf{b}$ is given.)

Note: A linear ODE is n^{th}-order if and only if the corresponding differential operator has order n.

More vocabulary

Definition 1.14 *A linear ODE is called* **homogeneous** *if the equation can be written as* $T(y) = 0$ *for some linear differential operator* T*. Equivalently, this means the function* q *as in Definition 1.13 is zero and the equation looks like*

$$p_0 y + p_1 y' + p_2 y'' + p_3 y''' + ... + p_n y^{(n)} = 0$$

for functions $p_0, p_1, p_2, ..., p_n : \mathbb{R} \to \mathbb{R}$.

A linear ODE is called **constant-coefficient** *if the functions* p_j *as in Definition 1.13 are all constants (the function* q *need not be constant).*

Example: Classify each of the following equations as linear or nonlinear; it it is linear, give its order, determine whether or not it is homogeneous, and determine whether or not it is constant-coefficient:

(a) $y' + t^2 y = t^3$

(b) $y''' + t^2 y''' = y''$

(c) $y'' - 3y' + 5y = 6t$

(d) $y' = y^2 + t$

Generally speaking:

- linear equations are easier to work with/solve than nonlinear equations;
- the smaller the order, the easier the equation is to solve;
- homogeneous equations are easier to solve than non-homogeneous equations;
- constant-coefficient equations are easier than those which do not have constant coefficients.

1.3 A first example: exponential growth and decay

Suppose that you have an ODE which is "as easy as possible". It should therefore be linear, first-order, homogeneous, and constant-coefficient.

Since it is linear and first-order, it must look like

Since it is homogeneous,

Since it is constant-coefficient,

This is the world's most common class of ODEs, because this class represents situations where the rate of change of quantity y is proportional to y itself. Examples from the real-world include:

- the amount of money in an account which earns interest compounded continually;
- the population of bacteria in a petri dish;
- the amount of radioactive isotopes of a substance like U-238;
- the processing power of the world's most powerful computer, expressed as a function of time.

The ODE $y' = ry$ is easy to solve:

Because this particular ODE occurs so often, we don't repeat this method of solution over and over. We just memorize the answer:

Theorem 1.15 (Exponential growth/decay) *Every linear, first-order, homogeneous, constant-coefficient ODE can be written in the form*

$$y' = ry \qquad \textit{a.k.a.} \qquad \frac{dy}{dt} = ry$$

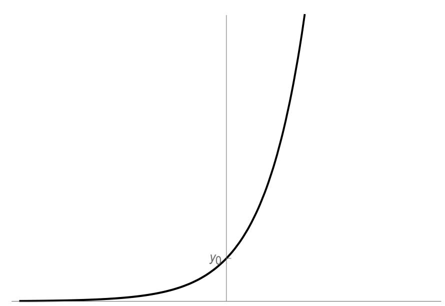
where $r \in \mathbb{R}$ is a nonzero constant. The general solution of this ODE is

$$y = y_0 e^{rt}$$

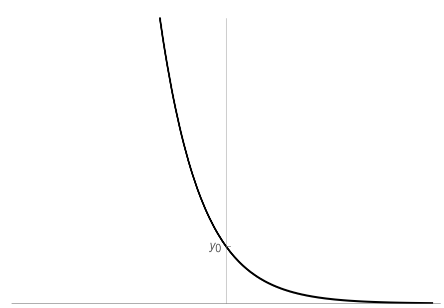
where $y_0 = y(0)$, the value of y when $t = 0$. In this setting, r is called the **rate** *or* **proportionality constant***.*

If you graph the solution curve $y = y(t)$, there are two possibilities (really three, but we'll ignore the $r = 0$ situation: in that case, since $y' = 0$, y is constant):

1. When $r > 0$, this model is called **exponential growth**. The graph of the solution $y(t) = y_0 e^{rt}$ looks like this (assuming $y_0 > 0$):



2. When $r < 0$, this model is called **exponential decay**. The graph of the solution $y(t) = y_0 e^{rt}$ looks like this (assuming $y_0 > 0$):



Example: Experiments have shown that the rate at which a radioactive element decays is directly proportional to the amount present. (Radioactive elements are chemically unstable elements that decay, or decompose, into stable elements as time passes.) Suppose that if you start with 40 grams of a radioactive substance, in 12 years you will have 20 grams of radioactive substance left.

1. Write down the initial value problem which models the situation.
2. What is the rate of the model?
3. How much radioactive substance will you have after 7 years?
4. How long will it take for you to only have 5 grams of radioactive substance left?

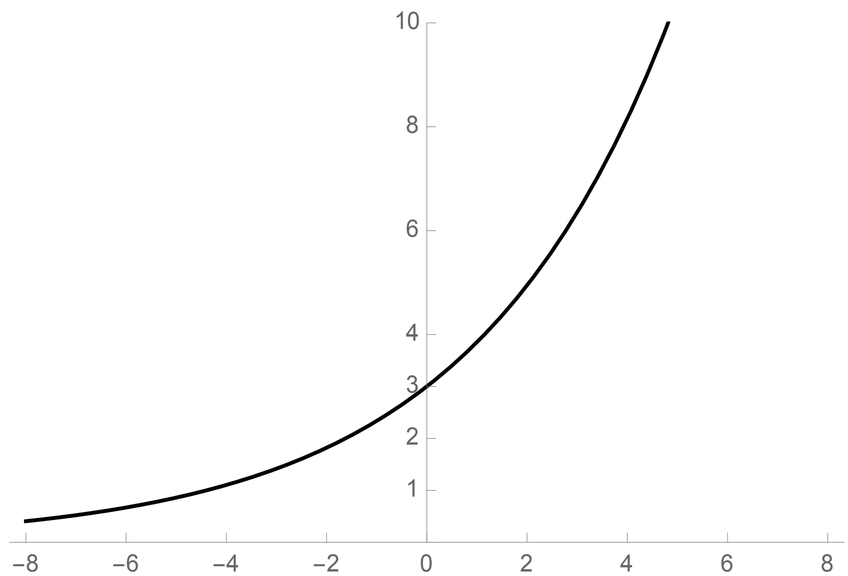
1.4 Qualitative analysis of first-order ODEs (slope fields)

Let's consider the initial value problem

$$\begin{cases} y' = \frac{1}{4}y \\ y(0) = 3 \end{cases}$$

From the previous section on exponential growth, we know the particular solution of this IVP is

and the graph of this solution is given below:

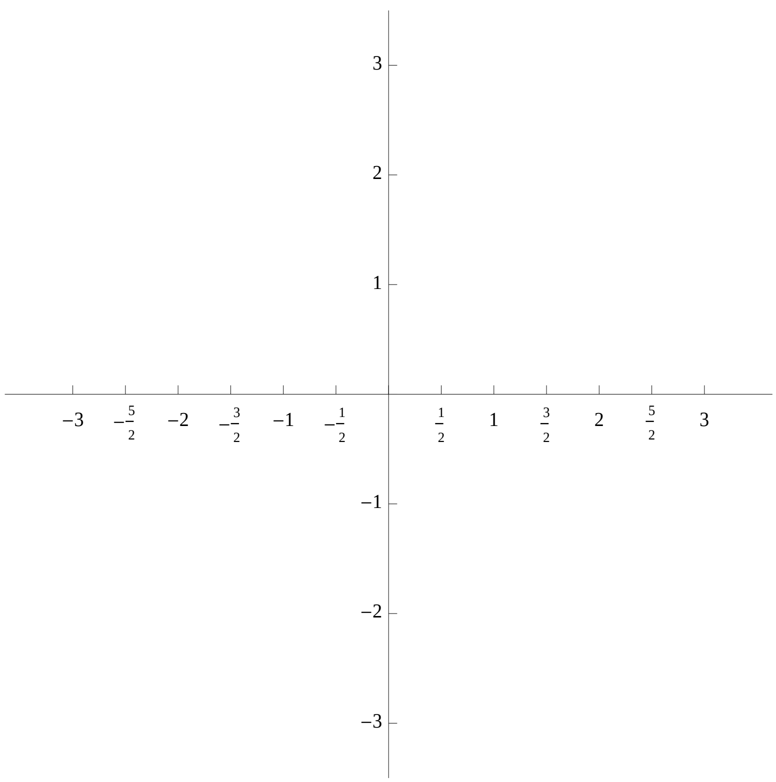


Suppose we draw "mini-tangent lines" to this curve at various points on the curve, and measure the slopes of these mini-tangent lines.

The observation that that the slope of any mini-tangent line to a solution of an ODE must be equal to y' suggests a graphical method of looking at any first-order ODE which can be rewritten as $y' = \phi(t, y)$.

Definition 1.16 *Consider a first-order ODE which is written in the form $y' = \phi(t, y)$ for some function ϕ. At each point (t, y) in the plane, imagine a "mini-tangent line" passing through the point (t, y) with slope $\phi(t, y)$. The collection of all these "mini-tangent lines" is called the* **slope field** *or* **vector field** *associated to the ODE.*

Example: $y' = t$

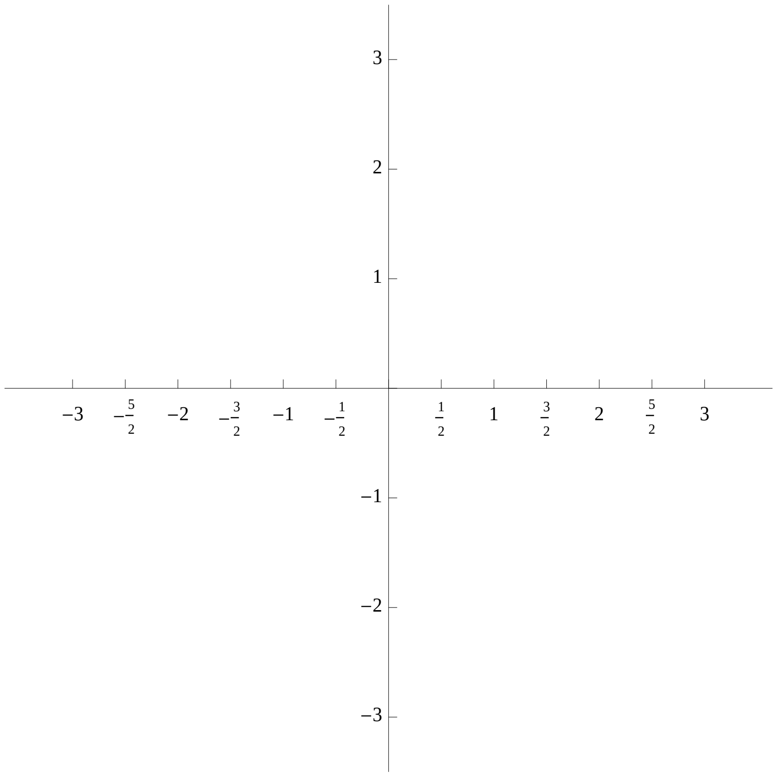


The slope field suggests what the graphs of solutions of the first-order ODE $y' = \phi(t, y)$ look like. If $y = y(t)$ is a solution, then the graph of $y(t)$ must be tangent to all of the mini-tangents at all the points (t, y) on the graph. Put another way,

*"the solutions have to have graphs which **flow with** the mini-tangents of the vector field"*

In the example $y' = t$, it appears that the solutions are parabolas. That is the case, because

Example: $y' = y - t$



Observations:

- there appear to be infinitely many different solutions to this ODE,
- but given any one initial value (t_0, y_0), there is one and only one particular solution which passes through that point.

Using *Mathematica* to draw slope fields

The computer algebra system *Mathematica* is an extremely useful tool for drawing slope fields. To sketch the slope field associated to ODE $y' = \phi(t, y)$, use the following code (which is explained below). The code is to be typed in one *Mathematica* cell and executed all at once.

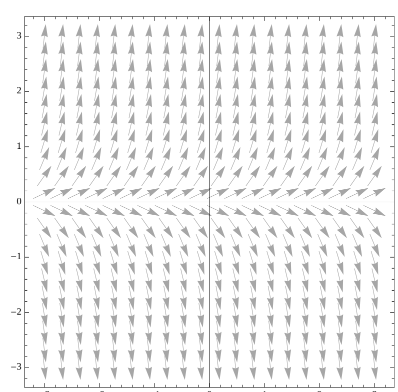
```
phi[t_,y_] := 3y;
VectorPlot[{1,phi[t,y]}, {t, -3, 3}, {y, -3, 3},
 VectorPoints -> 20, Axes -> True,
 VectorScale -> {Automatic, Automatic, None},
 VectorStyle -> Orange]
```

An explanation of the code (and things you can change):

1. The first line defines $\phi(t, y)$. For example, this code will produce the vector field for $y' = 3y$.
2. The second line controls the range of the picture; for example, this will sketch the vector field for t ranging from -3 to 3 and y ranging from -3 to 3.
3. The third line tells *Mathematica* how many arrows to draw in each direction, and to include the $t-$ and $y-$axes in the picture.
4. The fourth line controls the size of the arrows and is optional, but I think these choices make for a nice picture.
5. The last line tells *Mathematica* what color to draw the arrows.

Note: In principle, you don't type this code over and over. You can get this code from the file `slopefields.nb` (available on my webpage) and you simply copy and paste the cells into your *Mathematica* notebooks, editing the formula for $\phi(t, y)$, the number of vectors you want to draw, and the plot range as necessary.

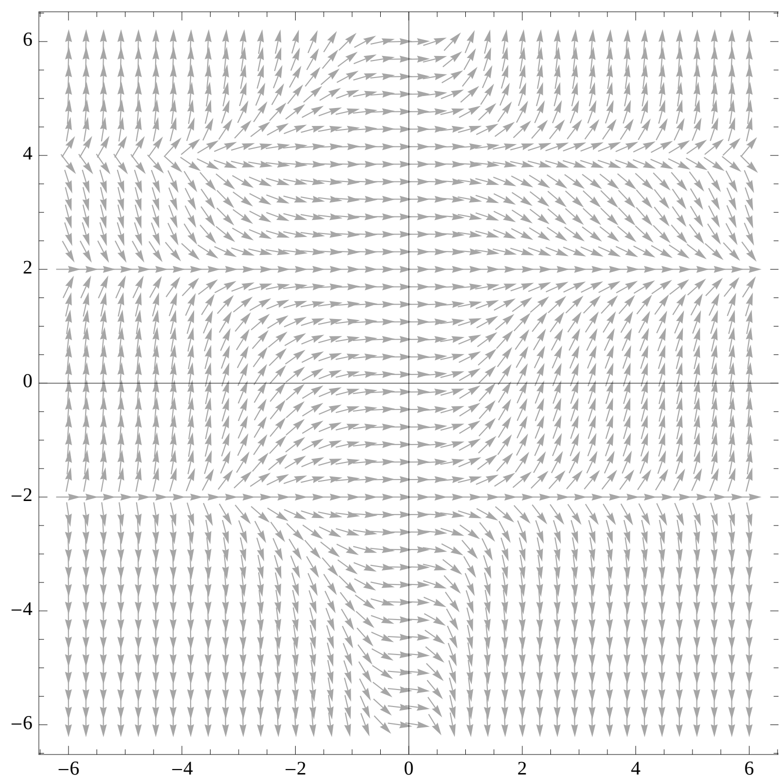
The code above produces this picture:



Reading pictures of slope fields

As mentioned earlier, solutions to an ODE must "flow with" the slope field of the ODE. This allows you to qualitatively study solutions to ODEs by examining a picture of the slope field associated to the ODE.

Example: Below is a picture of the vector field associated to some first-order ODE $y' = \phi(t, y)$:



1. Write the equations of two explicit solutions to this ODE:

2. On the above picture, sketch the graph of the solution satisfying $y(-2) = 1$.

3. Suppose $y(-2) = 0$. Estimate $y(2)$.

4. Suppose $y(-2) = 3$. Find $\lim\limits_{t \to \infty} y(t)$.

5. Suppose $y(2) = 0$. Find $\lim\limits_{t \to -\infty} y(t)$.

More *Mathematica* code

All this code can be found in the file `slopefields.nb`, downloadable from my website:

`mcclendonmath.com/330.html`

Code to sketch the slope field and several solution curves

The following code will sketch a slope field and also sketch several solution curves (passing through randomly chosen points). Execute all this in a single *Mathematica* cell:

```
phi[t_,y_] := 3y;
VectorPlot[{1,phi[t,y]}, {t, -3, 3}, {y, -3, 3},
 VectorPoints -> 20, Axes -> True,
 VectorScale -> {Automatic, Automatic, None},
 VectorStyle -> Orange,
 StreamPoints -> 35,
 StreamScale -> Full,
 StreamStyle -> {Blue, Thick}]
```

The first five lines are the same as the command described earlier; the sixth line directs *Mathematica* to sketch 35 solution curves at random locations on the picture. The last line tells *Mathematica* what color to draw the solution curves.

Code to sketch the slope field and a solution curve passing through a specific point

The following code (executed in a single cell) will sketch a slope field and sketch a single solution curve passing through a given point (t_0, y_0). In this case the initial value is $(-1, 2)$:

```
phi[t_,y_] := 3y;
VectorPlot[{1,phi[t,y]}, {t, -3, 3}, {y, -3, 3},
 VectorPoints -> 20, Axes -> True,
 VectorScale -> {Automatic, Automatic, None},
 VectorStyle -> Orange,
 StreamPoints -> {{-1,2}},
 StreamScale -> Full,
 StreamStyle -> {Blue, Thick}]
```

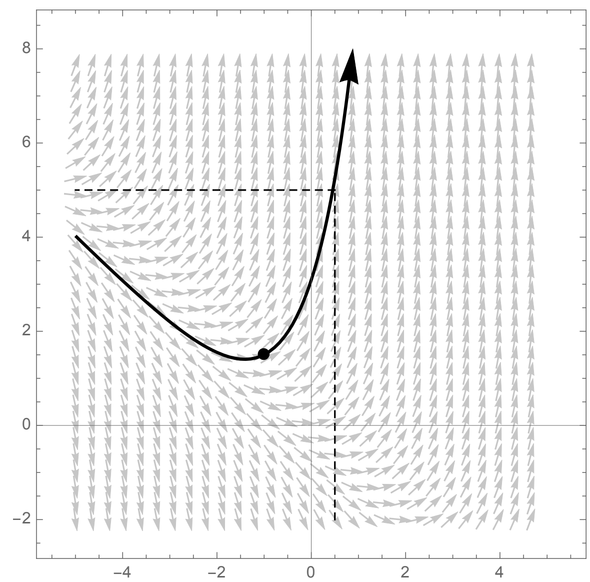
1.5 Numerical techniques: Euler's method

In the previous section we saw that the solution to an initial value problem of the form

$$\begin{cases} y' = \phi(t, y) \\ y(t_0) = y_0 \end{cases}$$

can be estimated by sketching a slope field for the ODE, and drawing a curve through the point (t_0, y_0) so that the "flows with" the slope field.

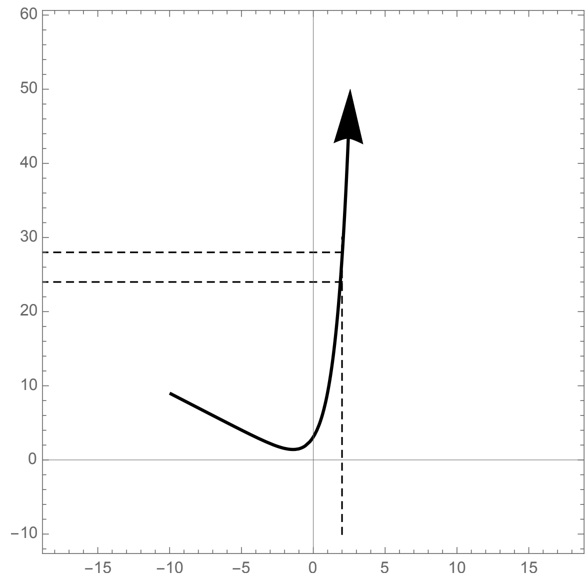
Example: $y' = t + y$; $(t_0, y_0) = (-1, 1.5)$.



You can see from the picture that for the solution $y = y(t)$ to this IVP, $y(.5) \approx 5$.

But, what is $y(2)$?

Suppose you wanted to know what $y(2)$ was, and tried to used *Mathematica* to draw a picture with an appropriate scale on the y-axis. You'd get something like this:



Based on this picture, a reasonable person might estimate that $y(2)$ is anything from 20 to 30.

Question: How could you estimate what $y(2)$ was without solving the ODE or relying on reading a picture of the slope field?

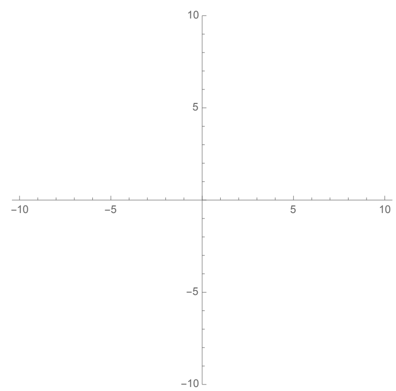
What we want is what is called a "numerical method" to approximate information about the ODE. In general, a **numerical method** is any computational method (by "computational", I mean a method that usually can be implemented on a computer by writing some appropriate code) which will approximate the solution to some problem. The study of numerical methods is called **numerical analysis**.

Euler's method

Goal: Given initial value problem of form

$$\begin{cases} y' = \phi(t, y) \\ y(t_0) = y_0 \end{cases}$$

estimate $y(t_n)$ for some value t_n.



How the method works: first, you pick a positive integer n and divide the interval $[t_0, t_n]$ into n equal subintervals. Then let Δt be the length of each subinterval.

Next, start at (t_0, y_0). Use tangent line approximation to estimate the change in y as t changes from t_0 to $t_1 = t_0 + \Delta t$:

This gives you a point (t_1, y_1) which probably isn't on the solution curve $y(t)$, but is pretty close to being on that curve.

Next, repeat the process as if you started at (t_1, y_1): use tangent line approximation to estimate the change in y as t changes from t_1 to $t_2 = t_1 + \Delta t$:

Repeat this over and over. In general, you obtain the point (t_{j+1}, y_{j+1}) from the point (t_j, y_j) by the equations

$$\begin{cases} t_{j+1} = t_j + \Delta t \\ y_{j+1} = y_j + \phi(t_j, y_j)\Delta t \end{cases}$$

Keep going until you get to t_n, the value at which you wanted to approximate the $y-$coordinate of the solution curve. To summarize:

Definition 1.17 *Given a first-order initial value problem of the form*

$$\begin{cases} y' = \phi(t, y) \\ y(t_0) = y_0 \end{cases},$$

given a number $t_n \neq t_0$, and given a natural number n, set $\Delta t = \frac{t_n - t_0}{n}$. Define a sequence of points (t_j, y_j) recursively by setting

$$\begin{cases} t_{j+1} = t_j + \Delta t \\ y_{j+1} = y_j + \phi(t_j, y_j)\Delta t \end{cases}$$

The y_n obtained by this method is called **the approximation to $y(t_n)$ obtained by Euler's method with n steps**. *n is called the* **number of steps** *and Δt is called the* **step size**.

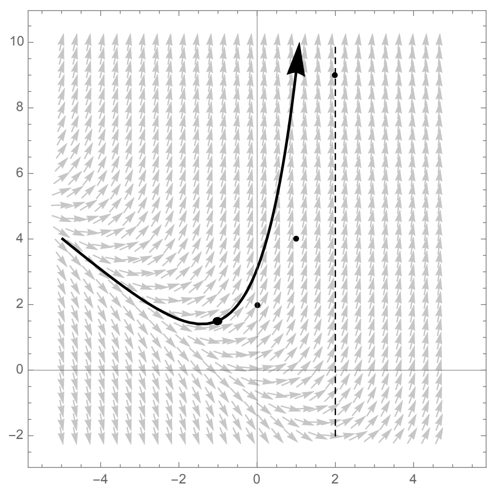
Example: Let $y(t)$ be the solution to the IVP $y' = t + y$; $(t_0, y_0) = (-1, 1.5)$. Use Euler's method with three steps to approximate $y(2)$.

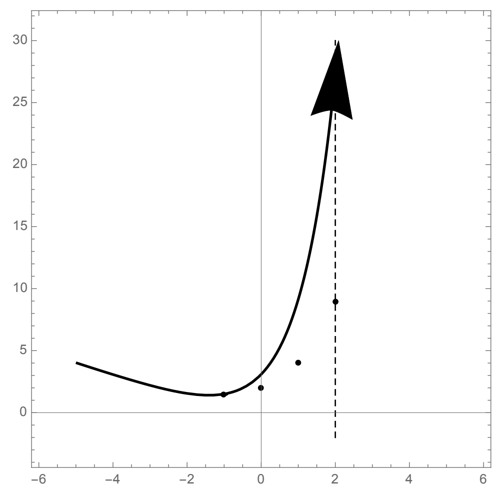
Having computed $(t_2, y_2) = (1, 4)$, repeat the procedure again to find (t_3, y_3):

$$\phi(t_2, y_2) = \phi(1, 4) = 1 + 4 = 5;$$

$$\begin{cases} t_3 & = t_2 + \Delta t = 1 + 1 = 2 \\ y_3 & = y_2 + \phi(t_2, y_2)\Delta t = 4 + 5(1) = 9 \end{cases}$$

Therefore $(t_3, y_3) = (2, 9)$ so from execution of Euler's method with 3 steps, $y(2) \approx 9$.





Why was our approximation so far off?

How might this approximation improve?

Suppose we repeated the same problem ($y' = t + y$; $(t_0, y_0) = (-1, 1.5)$; $y(2) = ?$) with 100 steps. That means

$$\Delta t = \frac{2 - (-1)}{100} = \frac{3}{100} = .03,$$

so we obtain

$$\begin{cases} t_1 &= t_0 + \Delta t = -1 + .03 = -.97 \\ y_1 &= y_0 + \phi(t_0, y_0)\Delta t = 1.5 + (.5)\,(.03) = 1.515 \end{cases}$$

$$\begin{cases} t_2 &= t_1 + \Delta t = -.97 + .03 = -.94 \\ y_2 &= y_1 + \phi(t_1, y_1)\Delta t = 1.515 + (-.97 + 1.515)(.03) = 1.53135 \end{cases}$$

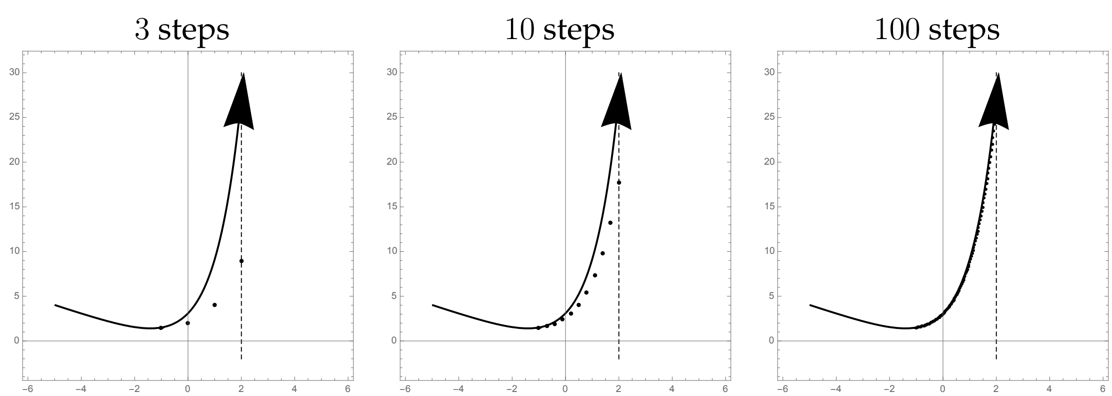
$$\vdots$$

$$\vdots$$

$$\begin{cases} t_{100} &= t_{99} + \Delta t = 1.97 + .03 = 2 \\ y_{100} &= y_{99} + \phi(t_{99}, y_{99})\Delta t = 25.8279 \end{cases}$$

Therefore $y(2) \approx 25.8279$.

Plots of points obtained by Euler's method for this example



So if your step size Δt is small enough (i.e. if you use enough steps), Euler's method does a really good job of producing a sequence of points which will approximate the solution curve $y(t)$.

Bad news: Doing this by hand would take too long.

Good news: We have computers that will perform Euler's method quickly and easily.

Mathematica code to implement Euler's method

I have written some *Mathematica* code which will implement Euler's method quickly. Using this code requires two steps. First, you need to type exactly this code (or copy and paste it from the `eulermethod.nb` file on my web page); this code defines a program called "euler":

```
euler[f_, {t_, t0_, tn_}, {y_, y0_}, steps_] :=
 Block[{told = t0, yold = y0, thelist = {{t0, y0}}, t, y, h},
  h = N[(tn - t0)/steps];
  Do[tnew = told + h;
   ynew = yold + h *(f /.{t -> told, y -> yold});
   thelist = Append[thelist, {tnew, ynew}];
   told = tnew;
   yold = ynew, {steps}];
  Return[thelist];]
```

The above code has to be run **once** each time you restart *Mathematica* (to define "euler"). Once it is executed, you can then implement Euler's method with the following type of command:

```
euler[3y, {t,1,3}, {y,-1}, 2]
```

This command performs Euler's method for the differential equation $y' = 3y$ where the initial point is $(t_0, y_0) = (1, -1)$. The procedure stops when $t = 3$ and uses 2 steps. If you are given a different ODE of the form $y' = \phi(t, y)$, you change the "$3y$" to whatever $\phi(t, y)$ is. If you are given a different initial value, you change the 1 and -1 to the coordinates of your initial value. You change the 3 to wherever you want Euler's method to stop, and you change the 2 to the number of steps you want.

The output you get in this situation is

```
{{1,-1},{2,-4},{3,-16}}
```

which means that the points coming from Euler's method are

$$(t_0, y_0) = (1, -1) \qquad (t_1, y_1) = (2, -4) \qquad (t_2, y_2) = (3, -16).$$

The great thing about *Mathematica* is that you can quickly perform Euler's method with a very large number of steps. If you run the command

```
euler[3y, {t,1,3}, {y,-1}, 400]
```

this will perform Euler's method with 400 steps. The bad news is that your output will be an enormous list (of 401 points). To get only the last point in the list (which is usually what you are most interested in), tweak this command as follows:

```
euler[3y, {t,1,3}, {y,-1}, 400][[401]]
```

The number in the double brackets should always be one more than the number of steps.

If you only care about a picture of the points you would get by implementing Euler's method, you can use a command like the one below. It plots all the points (t_n, y_n) obtained by running Euler's method using the same parameters as in the previous commands. If you use a large number of points, it will often appear as though they make a curve (since the points are really close together):

```
ListPlot[euler[3y, {t, 1,3}, {y, -1}, 400]]
```

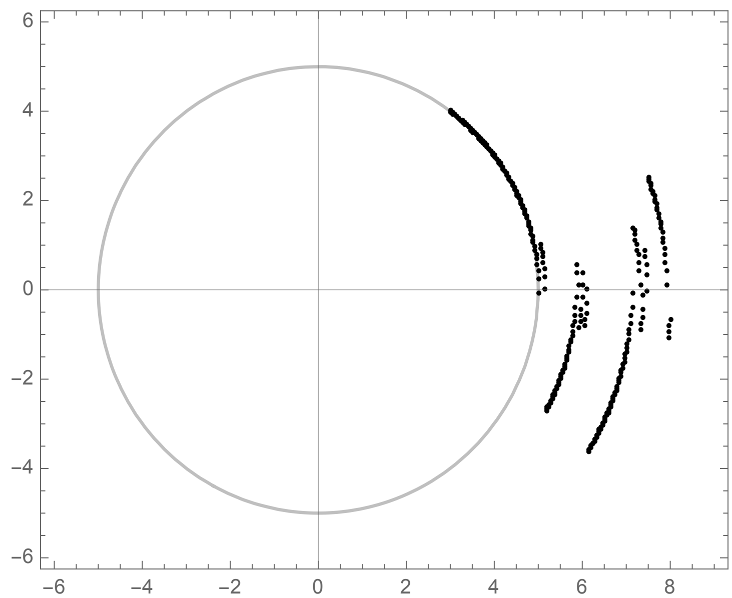
A potential pitfall with Euler's method

Example: $y' = \frac{-t}{y}$.

The solution to this ODE passing through $(3, 4)$ is $t^2 + y^2 = 25$. The graph of this solution is a...

Note that for this solution, y is not a function of t, because its graph does not pass the Vertical Line Test.

Suppose you tried Euler's method for this solution (starting at $(3, 4)$ with $\Delta t = .01$). You would get a sequence of points graphed here:



Suppose you wanted to estimate $y(t_n)$ where y is a solution of the IVP

$$\begin{cases} y' = \phi(t, y) \\ y(t_0) = y_0 \end{cases}.$$

You can use Euler's method to get an answer, but to know if the Euler's method computation is valid, you would have to know whether or not the solution even exists at $t = t_n$ (more generally, whether it exists for all $t \in [t_0, t_n]$), and you would have to know that the solution can be written where y is a function of t. We address this problem in the next section.

1.6 Existence and uniqueness of solutions

Recall that in the last section we were concerned about whether an IVP of the form $y' = \phi(t, y)$, $y(t_0) = y_0$ had a solution where y could be written as a function of t. More generally, we might be interested in *how many* solutions of this form might exist. Questions of this type in mathematics are called *existence/uniqueness questions*:

Existence:

Uniqueness:

Here is a very important theoretical result (used in many settings outside of ODEs) which settles these questions in the context we have been discussing. Recall from Math 320 (or look ahead to Section 2.5) that $\frac{\partial \phi}{\partial y}$ is the partial derivative of ϕ with respect to y (this is obtained from function $\phi(t, y)$ by treating t as a constant and differentating ϕ with respect to y, i.e.

$$\phi(t, y) = t^2 + 3ty^3 + ty^5 \qquad \Rightarrow \qquad \frac{\partial \phi}{\partial y} =$$

Theorem 1.18 (Existence/Uniqueness Theorem for first-order ODEs) *Suppose $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a function such that both ϕ and $\frac{\partial \phi}{\partial y}$ are continuous for all (t, y) is some rectangle in $\mathbb{R}^2$ containing (t_0, y_0). Then for some interval I of values of t containing t_0, the initial value problem*

$$\begin{cases} y' = \phi(t, y) \\ y(t_0) = y_0 \end{cases}$$

has one and only one solution which is of the form $y = f(t)$ for some function $f : \mathbb{R} \to \mathbb{R}$.

PROOF First, without loss of generality, we can assume that the initial value is $y(0) = 0$ (otherwise translate the axes so that the initial condition is at the origin).

Next, we rephrase the IVP of this theorem in a different way: suppose (for now) that there is some function $y = f(t)$ which solves this IVP. Then the function $\phi(t, y)$ can be rewritten as $\phi(t, f(t))$. This gives the equation

$$f'(t) = \phi(t, f(t)) \quad \text{a.k.a.} \quad f'(s) = \phi(s, f(s))$$

Integrate both sides of this equation from $s = 0$ to $s = t$ to get

$$\begin{aligned} \int_0^t f'(s)\, ds &= \int_0^t \phi(s, f(s))\, ds \\ f(t) - f(0) &= \int_0^t \phi(s, f(s))\, ds \\ f(t) &= \int_0^t \phi(s, f(s))\, ds \end{aligned}$$

The last line above is called the **integral equation** associated to the IVP; it is equivalent to the original IVP (in that it has exactly the same set of solutions as the original IVP). We solve this integral equation by a scheme somewhat like Euler's method; this scheme is called **Picard's method (of successive approximations)**:

Picard's method

Recall that our goal is to solve the integral equation

$$f(t) = \int_0^t \phi(s, f(s))\, ds.$$

Start by guessing what the solution is. Call this guess f_0. (If you aren't a good guesser, choose any function f, like the constant function $f_0(t) = 0$; the method will still work, but might take longer to get an accurate approximation.)

Now substitute this f_0 into the right-hand side of the integral equation and evaluate the right-hand side to obtain a function. Call this function f_1:

$$f_1(t) = \int_0^t \phi(s, f_0(s))\, ds$$

Repeat this procedure to obtain a sequence of functions $\{f_0, f_1, f_2, ...\}$

$$\begin{aligned} f_1(t) &= \int_0^t \phi(s, f_0(s))\, ds \\ f_2(t) &= \int_0^t \phi(s, f_1(s))\, ds \\ f_3(t) &= \int_0^t \phi(s, f_2(s))\, ds \\ \vdots\quad & \qquad\quad \vdots \\ f_{j+1}(t) &= \int_0^t \phi(s, f_j(s))\, ds \\ \vdots\quad & \qquad\quad \vdots \end{aligned}$$

Definition 1.19 *The sequence f_0, f_1, f_2, ... of functions so obtained are called the* **successive approximations** *or* **Picard approximations** *to the solution $y = f(t)$ of the original IVP.*

You can show that under the assumptions of this theorem,

$$f(t) = \lim_{j \to \infty} f_j(t)$$

exists for all t in some interval containing the initial value, and that the function f so obtained is the only solution of the IVP on that interval. (This is a technical argument using some difficult concepts from higher mathematics, and is omitted.) □

An example

Consider the initial value problem $y' = 2(1 + y)$, $y(0) = 0$. Find the solution of this IVP using Picard's method of successive approximations.

Picard approximations can also be computed using *Mathematica*; the relevant code is in Appendix A.9.

1.7 Autonomous equations; equilibria and phase lines

Another view of exponential growth/decay

Recall that the ODE $y' = ry$ has as its solutions $y = y_0 e^{rt}$.

Suppose we were only interested in determining (or estimating) the value of $y(t)$ for large t. In other words, we want to know

This is pretty easy to figure out in the context of exponential models:

Observe: There is a "special" solution to $y' = ry$ which is constant (i.e. its graph is a horizontal line).

This behavior generalizes to a large class of ODEs called *autonomous* equations: the "special" solutions of an autonomous ODE are called *equilibria* of that ODE; in general (but not always) each equilibrium solution of an ODE either "attracts" other solutions as $t \to \infty$, or "repels" them as $t \to \infty$.

Analysis of autonomous equations

Definition 1.20 *A first-order ODE is called* **autonomous** *if it is of the form*

$$y' = \phi(y)$$

for some function $\phi : \mathbb{R} \to \mathbb{R}$.

An IVP is called **autonomous** *if its differential equation is autonomous.*

Example: The exponential growth/decay equation $y' = ry$ is autonomous with $\phi(y) = ry$.

Example: $y' = 2y \cos y$

Non-example: $y' = y^2 t$

Consider an autonomous IVP:

$$\begin{cases} y' = \phi(y) \\ y(t_0) = y_0 \end{cases} . \tag{1.1}$$

Let's suppose that what we are really interested in is the long-term behavior of the solutions, i.e. we want to know

$$\lim_{t \to \infty} y(t).$$

Since there is no t in the formula for ϕ, and based on our discussion of exponential growth and decay, we expect that the answer to this question depends on what y_0 is.

First observation: Suppose y_0 is some number such that $\phi(y_0) = 0$. Then the constant function $y(t) = y_0$ is a solution of the autonomous IVP (1.1).

Definition 1.21 *Let $y' = \phi(y)$ be an autonomous ODE. A constant function which is a solution of this ODE is called an* **equilibrium (solution)** *of the ODE. (The plural of equilibrium is* **equilibria***.)*

Based on the discussion above, we can find equilibria by solving a numerical equation:

Theorem 1.22 *The constant function $y = y_0$ is an equilibrium solution of the autonomous ODE $y' = \phi(y)$ if and only if $\phi(y_0) = 0$.*

Once you find the equilibria of an autonomous ODE, you can conclude that the slope field associated to the autonomous ODE must look like this:

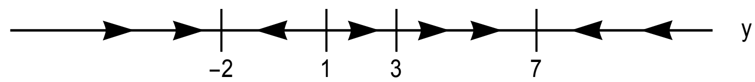
In fact, you know even more about the slope field associated to an autonomous ODE $y' = \phi(y)$:

Key observation: Since the slope of the mini-tangent line depends only on y (and not on t), that means

In other words, for an ODE of the form $y' = \phi(y)$, the slope field can be described by another picture which "ignores" the t:

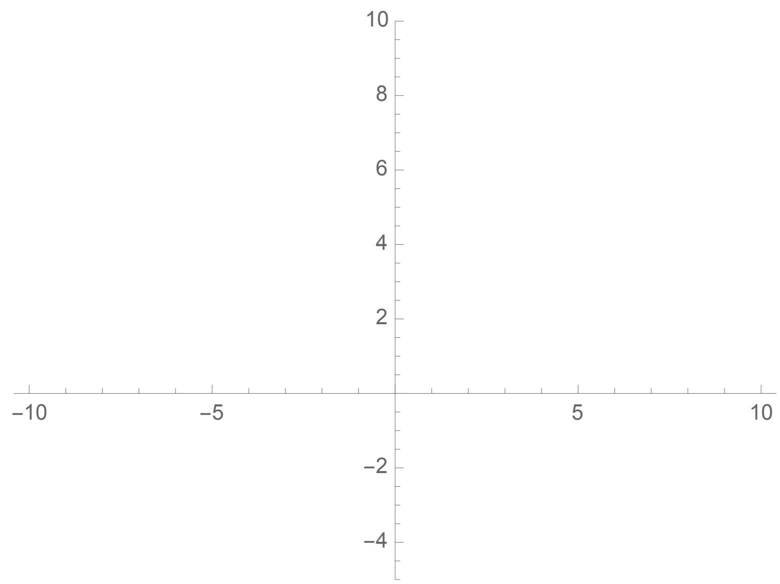
The picture above is called a **phase line** for the autonomous ODE $y' = \phi(y)$. It is a picture of a y-axis, with the equilibria indicated by dots or dashes, and arrows drawn between the equilibria which indicate the behavior of the solutions as $t \to \infty$.

Example: Suppose this is a phase line for an autonomous ODE:



(a) Write the equation of four explicit equations of this ODE.

(b) Sketch the slope field associated to this ODE.



(c) Suppose $y(0) = 4$. Find $\lim_{t \to \infty} y(t)$.

(d) Suppose $y(5) = 0$. Find $\lim_{t \to -\infty} y(t)$.

(e) Suppose $y(1) = 8$. Is $y(t)$ an increasing function, a decreasing function, or a constant function?

(f) Suppose $y(0) = y_0$. For what values of y_0 is $\lim_{t \to \infty} y(t) = 3$?

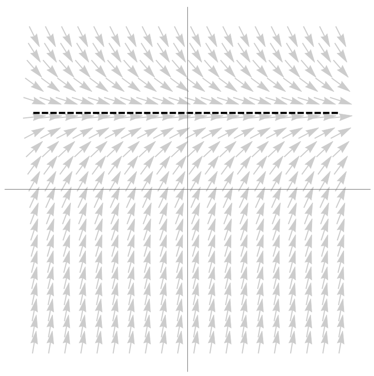
Classification of equilibria

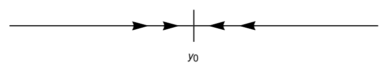
Definition 1.23 *An equilibrium solution $y = y_0$ of an autonomous ODE is called* **stable** *(or* **asymptotically stable** *or* **attracting** *or a* **sink***) if there is an open interval I of initial values containing y_0 such that if $y(t_0) \in I$,*

$$\lim_{t\to\infty} y(t) = y_0.$$

Example: In the preceding example, $y = -2$ and $y = 7$ are stable equilibria.

General pictures of stable equilibria (sinks):



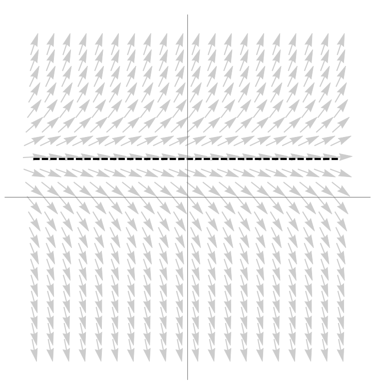


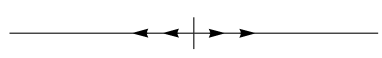
Definition 1.24 *An equilibrium solution $y = y_0$ of an autonomous ODE is called* **unstable** *(or* **asymptotically unstable** *or* **repelling** *or a* **source***) if there is an open interval I of initial values containing y_0 such that if $y(t_0) \in I$ but $y(t_0) \neq y_0$,*

$$\lim_{t\to\infty} y(t) \neq y_0.$$

Example: In the preceding example, $y = 1$ is an unstable equilibrium.

General pictures of unstable equilibria (sources):

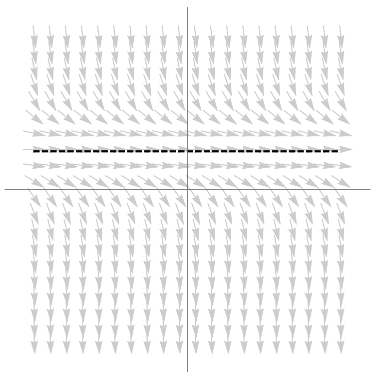


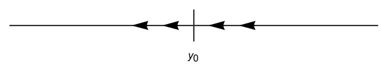


Definition 1.25 *An equilibrium solution $y = y_0$ of an autonomous ODE is called* **semistable** *(or* **neutral***) if it is neither stable nor unstable.*

Example: In the preceding example, $y = 3$ is a semistable equilibrium.

Example picture of semistable equilibria: In general, any equilibrium with pictures unlike either of the previous two cases is semistable. Here is an example of the kind of pictures you might see:





To find the equilibria of an autonomous ODE $y' = \phi(y)$, you set $\phi(y) = 0$ and solve for y (using Theorem 3.38). To classify the equilibria as stable, unstable or semistable, we use calculus:

Theorem 1.26 (Classification of equilibria) *Suppose $y = y_0$ is an equilibrium solution of autonomous ODE $y' = \phi(y)$, where ϕ is differentiable at y_0. Then:*

1. $\phi(y_0) = 0$;
2. *If $\phi'(y_0) < 0$, then y_0 is stable.*
3. *If $\phi'(y_0) > 0$, then y_0 is unstable.*
4. *If $\phi'(y_0) = 0$ and $\phi''(y_0) \neq 0$, then y_0 is semistable.*
5. *If $\phi'(y_0) = \phi''(y_0) = 0$, then you need to analyze the graph of ϕ near y_0 to classify y_0.*

PROOF Suppose y_0 is an equilibrium. Then $\phi(y_0) = 0$ by Theorem 1.22.

Case 1: Suppose the function ϕ is decreasing at y_0 (this occurs if $\phi'(y_0) < 0$). Then the graph of ϕ looks like

so the phase line looks like

and therefore y_0 is stable. This proves statement (2) of the theorem.

Case 2: Suppose the function ϕ is increasing at y_0 (this occurs if $\phi'(y_0) > 0$). Then the graph of ϕ looks like

so the phase line looks like

and therefore y_0 is unstable. This proves statement (3) of the theorem.

Case 3: Suppose the function ϕ has a local maximum or a local minimum at y_0 (this occurs if $\phi'(y_0) = 0$ but $\phi''(y_0) \neq 0$). Then the graph of ϕ looks like

so the phase line looks like

and therefore y_0 is semistable. This proves statement (4) and finishes the proof of the theorem. $\square$

Example: For each autonomous ODE, find the equilibrium solutions and classify them as stable, unstable or semistable. Then sketch the phase line associated to the ODE.

(a) $y' = y - 3$

(b) $y' = y^2 - 4y + 3$

(c) $y' = \sin y$

(d) $y' = y(y-2)^2$

Logistic models

Population biology models seek to determine or estimate the population y of a species in terms of the time t. Suppose the species reproduces at rate $r > 0$. This means that the rate of change in y should be something like r times y. This makes sense if the population is small. But if the population gets too big (say greater than some constant $L > 0$), there is not enough food in the ecosystem to support all of the organisms, so the population won't grow despite reproduction, because the organisms starve. A differential equation representing this type of situation is called a **logistic equation** and has the following form:

$$y' = r\,y\,(L - y)$$

In a logistic equation, r is a constant called the **rate of reproduction** and L is a constant called the **carrying capacity** or **limiting capacity** of the system.

Often L is set equal to 1, in which case y measures not the raw population of the species, but the fraction of the largest possible population of the species.

Note that r and L are always positive.

Let's analyze the logistic equation by finding and classifying its equilibria:

1.8 Bifurcations

Idea: In many applications of ODEs, you obtain a model which has some constants in it:

Ex. 1: Exponential models: $y' = ry$

Ex. 2: Logistic models: $y' = ry(L - y)$

These constants are often determined experimentally. For example, if a biologist studying the population of bacteria in a petri dish observes that at $t = 0$ there are 200 bacteria and that at time $t = 3$ there are 420 bacteria, the biologist would estimate the r as follows (see Section 1.3):

$$\begin{aligned} y(t) &= y_0 e^{rt} \\ y(t) &= 200e^{rt} \\ \Rightarrow\ 420 &= 200e^{r(3)} \\ \Rightarrow\ 2.1 &= e^{3r} \\ \Rightarrow\ r &= \frac{1}{3}\ln 2.1 \approx .2473. \end{aligned}$$

Potential problem(s):

For instance, if there were actually 445 bacteria at time $t = 2.95$, then the value of r (by the same calculation as above) would be $r \approx .2711$.

To get around this, scientists take lots of readings and compile the data to estimate r using what is called "least-squares" approximation. But this is still error-prone. Because of this error, it is good to sometimes think of the r (or the r and L in a logistic equation) as only being determined up to a small error.

Suppose you are interested in the long-term behavior of your model, i.e. what happens as $t \to \infty$. For an exponential model, suppose you estimate $r \approx 3$. Then

and even if your r is a little off, the same limit statement holds, so you can say with certainty that $y(t)$ will increase without bound as $t \to \infty$. So qualitatively, your estimates can be assured of accuracy.

BUT... if you estimate $r \approx .001$ and you figure that your r value might be off by as much as $.004$, then r might be positive or negative! Then (assuming $y_0 > 0$)

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y_0 e^{rt} = \begin{cases} & \\ & \\ & \end{cases}$$

so you cannot predict even the qualitative behavior of $y(t)$ for large t with any certainty.

In this section we are interested in studying families of autonomous ODEs that have an experimental constant (like r) in them (the constant is called a *parameter*). We want to determine whether or not small changes in the parameter wildly change the behavior of $y(t)$ for t large; equivalently, we want to find the values of the parameter for which the behavior of the family "qualitatively changes". These qualitative changes in the system are called **bifurcations**.

Remark on notation: In general, we describe an ODE with a parameter by writing

$$y' = \phi(y; r) \quad \text{or} \quad y' = \phi_r(y)$$

The semicolon or subscript tells you that the letter that follows is a parameter, and **not the independent variable** (i.e. $y = y(t)$, not $y = y(r)$).

Bifurcations of ODEs of the form $y' = \phi(y; r)$ can be classified into four types:

Saddle-node bifurcations (as r changes, a pair of equilibria appear/disappear - one equilibrium attracts and the other repels)

Pitchfork bifurcations (as r changes, one equilibrium changes behavior and turns into three equilibria - one on either side)

Transcritical bifurcations (as r changes, two equilibria "cross" and change behavior

Degenerate bifurcations anything that isn't in the other three classes (these are rare)

Saddle-node bifurcations

Example: $y' = y^2 - r$

Pitchfork bifurcations

Example: $y' = y^3 - ry$

Equilibria:

Stability/instability of equilibria: first, $\phi'(y) = \frac{d}{dy}\phi(y) = 3y^2 - r$.

- $y = 0$ (all r):

$$\phi'(0) = -r \quad \Rightarrow \quad \begin{cases} 0 \text{ is stable if } r > 0 \\ 0 \text{ is neutral if } r = 0 \\ 0 \text{ is unstable if } r < 0 \end{cases}$$

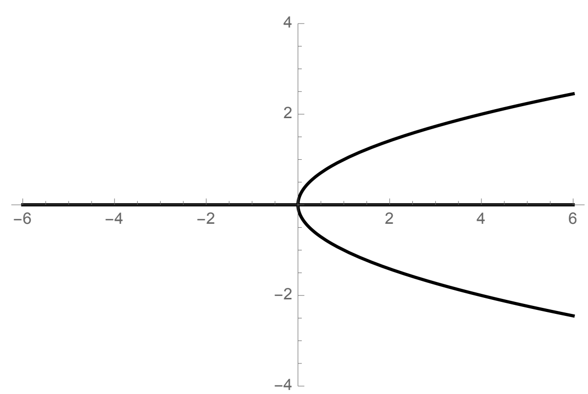
- $y = \sqrt{r}$ ($r > 0$):

$$\phi'(\sqrt{r}) = 3(\sqrt{r})^2 - r = 3r - r = 2r > 0 \quad \Rightarrow \quad \sqrt{r} \text{ is unstable if } r > 0$$

- $y = -\sqrt{r}$ ($r > 0$):

$$\phi'(-\sqrt{r}) = 3(-\sqrt{r})^2 - r = 3r - r = 2r > 0 \quad \Rightarrow \quad -\sqrt{r} \text{ is unstable if } r > 0$$

Bifurcation diagram:



Transcritical bifurcations

Example: $y' = y^2 - ry$

Equilibria: $y^2 - ry = 0 \;\Rightarrow\; y(y-r) = 0 \;\Rightarrow\; y = 0, y = r$

Stability/instability of equilibria: first, $\phi'(y) = \frac{d}{dy}\phi(y) = 2y - r$.

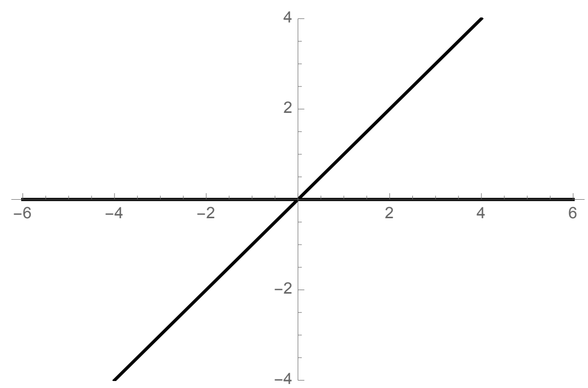
- $y = 0$ (all r):

$$\phi'(0) = -r \quad \Rightarrow \quad \begin{cases} 0 \text{ is stable if } r > 0 \\ 0 \text{ is neutral if } r = 0 \\ 0 \text{ is unstable if } r < 0 \end{cases}$$

- $y = r$ (all r):

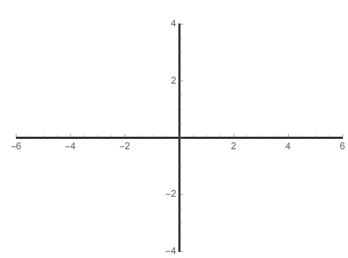
$$\phi'(r) = r \quad \Rightarrow \quad \begin{cases} r \text{ is unstable if } r > 0 \\ r \text{ is neutral if } r = 0 \\ r \text{ is stable if } r < 0 \end{cases}$$

Bifurcation diagram:

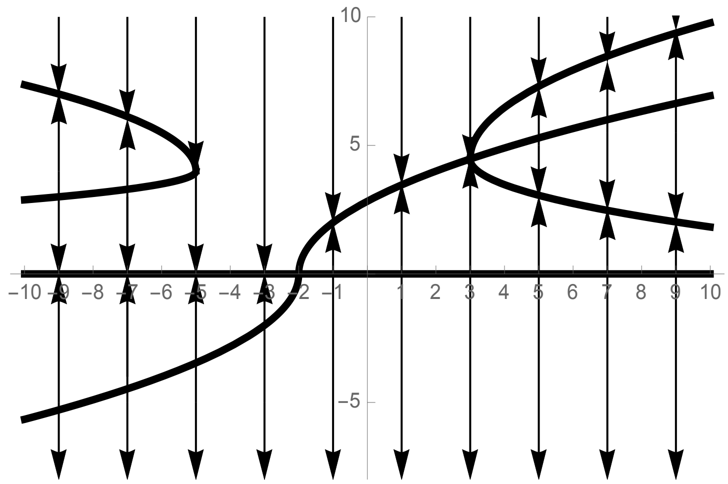


A second example: $y' = ry$

Equilibria: $y = 0$ (for all r); all y (if $r = 0$)
Stability/instability: $y = 0$ is stable if $r < 0$ and unstable if $r > 0$
Bifurcation diagram:



Example: The bifurcation diagram for a family of ODEs of the form $y' = \phi(y; r)$ is given below. For this system, find the locations of all bifurcations, and classify each bifurcation as a saddle-node, pitchfork or transcritical bifurcation:



Example: Consider the family of ODEs $y' = r \arctan y$. Find the locations of all bifurcations, and classify each bifurcation as a saddle-node, pitchfork, transcritical or degenerate bifurcation. Sketch the bifurcation diagram for this family.

Procedure to find bifurcations in family $y' = \phi(y; r)$:

1. Find the equilibria in terms of r (by solving $\phi(y; r) = 0$), keeping track of the values of r for which equilibrium exists.

2. For each equilibrium y_0 found in Step 1, classify the equilibrium by computing $\frac{d}{dy}\phi(y_0; r)$. Keep in mind that in general, this classification will depend on the value of r.

3. Sketch the bifurcation diagram (start by graphing the formulas for y in terms of r obtained in Step 1; then fill in with vertical arrows based on the classification in Step 2).

Example: Consider the family of ODEs $y' = e^y - r$. Find the locations of all bifurcations, and classify each bifurcation as a saddle-node, pitchfork, transcritical or degenerate bifurcation. Sketch the bifurcation diagram for this family.

1.9 Summary of Chapter 1

Background and vocabulary

An **ordinary differential equation (ODE)** is an equation involving an independent variable t and a function $y = y(t)$, together with derivatives of y with respect to t. The **order** of an ODE is the highest order of derivative that occurs in the equation. The goal is to solve for an equation relating y and t that has no derivatives in it, i.e. to find $y = y(t)$.

A point (t_0, y_0) through which an ODE must pass is called an **initial value**. An ODE together with an initial value is called an **initial value problem (IVP)**. A solution of an IVP is called a **particular solution**; the set of all particular solutions of an ODE is called the **general solution** of the ODE.

Generally, to solve an IVP you start by solving the ODE to obtain the general solution (which should have one or more arbitrary constants in it... you expect that the number of constants equals the order of the equation). Then you plug in the initial value(s) to solve for the constant(s).

An ODE is called **linear** if there are functions $p_0, p_1, ..., p_n : \mathbb{R} \to \mathbb{R}$ and a function $q : \mathbb{R} \to \mathbb{R}$ such that the equation can be written as

$$p_0 y + p_1 y' + p_2 y'' + ... + p_n y^{(n)} = q.$$

A linear equation is called **homogeneous** if $q = 0$.

Qualitative and numerical approaches to first-order ODEs

To study a first-order ODE qualitatively, sketch a picture of its **slope field** by drawing "mini-tangent lines" at every point in the plane (in practice, one does this with a computer). Solutions to the ODE must "flow with" the slope field.

To numerically approximate a solution to an ODE, use **Euler's method**: given IVP $y' = \phi(t, y)$; $y(t_0) = y_0$, to approximate $y(t_n)$, set $\Delta t = \frac{t_n - t_0}{n}$ and define points recursively by

$$\begin{cases} t_{j+1} = t_j + \Delta t \\ y_{j+1} = y_j + \phi(t_j, y_j)\Delta t \end{cases};$$

the y_n obtained by this method (usually implemented on a computer) is an approximation to $y(t_n)$. The larger n is, the better the approximation.

The **existence/uniqueness theorem** for first-order ODEs gives conditions under which an IVP $y' = \phi(t, y)$; $y(t_0) = y_0$ has one and only one solution. This theorem is proved using Picard's method of successive approximations, which come from the formula

$$f_{j+1}(t) = \int_0^t \phi(s, f_j(s))\, ds.$$

The f_j converge to the solution $y = f(t)$ as $j \to \infty$.

Autonomous ODEs and bifurcations

A first-order ODE is called **autonomous** if it is of the form $y' = \phi(y)$. A constant function which is a solution of this ODE is called an **equilibrium** of the ODE; to find the equilibia, set $\phi(y) = 0$ and solve for y. We classify equilibria y_0 of an autonomous system as follows:

- If $\phi'(y_0) < 0$, then y_0 is stable (it attracts on both sides as $t \to \infty$);
- If $\phi'(y_0) > 0$, y_0 is unstable (it repels on both sides as $t \to \infty$);
- If $\phi'(y_0) = 0$ but $\phi''(y_0) \neq 0$, y_0 is semistable (or neutral).

Having classified the equilibria of an autonomous ODE, we draw a **phase line** which describes the behavior of the solutions as $t \to \infty$.

Given a parameterized family of autonomous ODEs $y' = \phi(y; r)$, values of r at which qualitative changes of the solutions to the ODE occur are called **bifurcations**. There are three main types of bifurcations:

Saddle-node bifurcations: two equilibria appear or disappear
Pitchfork bifurcations: one equilibria turns into three (and reverses behavior)
Transcritical bifurcations: two equilibria cross each other and reverse behavior

Chapter 2

First-order equations: solution techniques

2.1 Solution of first-order homogeneous linear equations

Recall: A first-order ODE is linear if it of the form

$$p_1(t)y' + p_0(t)y = q(t)$$

for functions p_1, p_0 and q, where $p_1 \neq 0$. The equation is called **homogeneous** if $q = 0$ and is called **constant-coefficient** if p_1 and p_0 are constants.

Recall also: In Section 1.3, we studied the simplest ODE (first-order linear, constant-coefficient and homogeneous): the exponential growth/decay model:

$$y' = ry \text{ a.k.a. } y' - ry = 0 \qquad \Rightarrow \qquad y = y_0 e^{rt}$$

Example: Find the particular solution of the IVP $\begin{cases} 4y' + y = 0 \\ y(0) = 6 \end{cases}$.

Solution of first-order homogeneous linear equations

Any homogeneous first-order linear equation can be written in the form

$$p_1(t)y' + p_0(t)y = 0.$$

To solve an equation of this type, first divide through by $p_1(t)$ to make the equation have the form

$$y' + p(t)y = 0.$$

Then, write the y' in Leibniz notation, "separate the variables" and integrate both sides (more on this technique later in Chapter 2):

STEPS	GENERAL SITUATION	EXAMPLE
	$p_1(t)y'(t) + p_0(t)y(t) = 0$	$t^3y' + t^5y = 0$
Divide through by p_1		
	$y' + p(t)y = 0$	
Move py term to other side		
	$y' = -p(t)y$	
Write y' in Leibniz notation		
	$\frac{dy}{dt} = -p(t)y$	
Move y to left, dt to right (i.e. "separate the variables")		
	$\frac{1}{y}\,dy = -p(t)\,dt$	
Integrate both sides		
	$\int \frac{1}{y}\,dy = \int -p(t)\,dt$	
	$\ln y = \int -p(t)\,dt$	
Solve for y		
	$y = e^{-\int p(t)\,dt}$	

Theorem 2.1 (Solution of a first-order homogeneous linear equation) *The general solution of the first-order, homogeneous linear equation*

$$y' + p(t)y = 0$$

is

$$y = \exp\left(-\int p(t)\,dt\right) = e^{-\int p(t)\,dt}.$$

Note: general solutions of first-order equations should (and do) have arbitrary constants in them. The arbitrary constant in this general solution is the "$+C$" that appears when you calculate $\int p(t)\,dt$. This constant manifests itself as follows:

Therefore the general solution of a homogeneous, first-order linear equation is always a constant times one particular solution of that equation. More generally:

Theorem 2.2 (Solution of a homogeneous, first-order linear equation) *Let $y_h(t)$ be any nonzero solution of a homogeneous, first-order linear ODE*

$$p_1(t)y' + p_0(t)y = 0.$$

Then:

1. *the only solutions of that equation are of the form $Cy_h(t)$ for some constant C.*
2. *for any constant C, the function $Cy_h(t)$ is a solution of the same equation; and*

PROOF By the preceding discussion, the set of solutions is the set $Ke^{-P(t)}$ where K is an arbitrary constant and P is an antiderivative of $p(t) = \frac{p_0(t)}{p_1(t)}$. The result follows. □

How this theorem is applied: Suppose you have some homogeneous, linear first-order ODE and somehow, someway, you know that

$$y_h(t) = \sin t + e^{-t}$$

is a solution. Then you know

Subspaces

The theorem of the previous section can be rephrased in the language of linear algebra. Recall that a vector space is a set of objects which can be added to one another and multiplied by constants. If one vector space is a subset of another, we say that the first space is a *subspace* of the second. More precisely:

Definition 2.3 *Let V be a vector space and let $W \subseteq V$. We say W is a* **subspace (of V)** *if*

1. *W is closed under addition, i.e. for any two vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ in W, $\mathbf{w}_1 + \mathbf{w}_2 \in W$; and*
2. *W is closed under scalar multiplication, i.e. for any vector $\mathbf{w} \in W$ and any scalar r, $r\mathbf{w} \in W$.*

The most important example of a subspace is the set of multiples of a single vector.

Definition 2.4 *Let V be a vector space and let $\mathbf{v} \in V$ be a vector. The* **span** *of $\mathbf{v}$, denoted $Span(\mathbf{v})$, is the set of linear multiples of $\mathbf{v}$:*

$$Span(\mathbf{v}) = \{c\mathbf{v} : c \in \mathbb{R}\}$$

If $W \subseteq V$ is such that $W = Span(\mathbf{v})$, we say W is **spanned** *by $\mathbf{v}$.*

Examples of spans of a single vector:

- $V = \mathbb{R}^2$; $\mathbf{v} = (3, 2)$

- $V = \mathbb{R}^4$; $\mathbf{v} = (1, 1, 1, 1)$

- $V = C^\infty(\mathbb{R}, \mathbb{R})$; $y = \sin t$

Lemma 2.5 (Spans are subspaces) *The span of any vector is a subspace.*

PROOF Essentially, this is because the sum of two multiples of a vector is also a multiple of that vector, and because any constant times a multiple of a vector is also a multiple of that vector. For a precise proof, see Theorem 3.5 of my Math 322 lecture notes.

Lemma 2.6 *If W is spanned by $\mathbf{v}$, then W is also spanned by $c\mathbf{v}$ for any nonzero constant c.*

PROOF The set of multiples of $\mathbf{v}$ is the same as the set of multiples of $c\mathbf{v}$, so long as $c \neq 0$.

We can restate Theorem 2.2 as follows:

Theorem 2.7 *The set of solutions of a homogeneous, first-order linear ODE form a subspace of $C^\infty(\mathbb{R}, \mathbb{R})$ which is spanned by any one nonzero solution of the equation.*

Consequence: If you know one nonzero solution $y_h = y_h(t)$ of a homogeneous, first-order linear ODE, then you know them all. They are all of the form

A preview: The set of solutions to a homogeneous, n^{th}-order linear ODE will also be a subspace of $C^\infty(\mathbb{R}, \mathbb{R})$; this subspace will be spanned by n linearly independent functions rather than just one.

A last remark: Suppose $p(t)$ is constant. Then the first-order homogeneous linear equation is

2.2 Solution of first-order linear equations by integrating factors

To solve a general first-order linear equation, first write it in the form

$$p_1(t)y' + p_0(t)y = q(t)$$

and divide through by $p_1(t)$ to get

$$y' + \frac{p_0(t)}{p_1(t)}y = \frac{q(t)}{p_1(t)}.$$

After doing this, any first-order linear equation has the following standard form:

$$y' + p(t)y = q(t).$$

From this point, there are two methods of solution:

1. integrating factors (§2.2)
2. undetermined coefficients (§2.3)

Much of the time, either method works, but you should know both.

Integrating factors

Definition 2.8 *Given an ODE, an* **integrating factor** *for that equation is some expression μ (which may have a t and/or y in it) such that if you multiply through the equation by μ, the equation becomes "easier".*

Consider the first-order linear ODE

$$y'(t) + p(t)y(t) = q(t).$$

Now we look for an integrating factor of the form $\mu(t)$. After multiplying through by this function, the equation would be

$$\mu(t)y'(t) + p(t)\mu(t)y(t) = \mu(t)q(t). \tag{2.1}$$

Note that if $\mu(t)$ were some function such that

$$\mu'(t) = p(t)\mu(t),$$

then equation (2.1) would look like

$$\mu(t)y'(t) + \mu'(t)y(t) = \mu(t)q(t). \tag{2.2}$$

This works if $\mu'(t) = p(t)\mu(t)$, i.e. if μ satisfies the differential equation

$$\mu' - p(t)\mu = 0.$$

But this is a first-order, homogeneous linear equation, so we know from the previous section that the general solution is given by

$$\mu = \exp\left(-\int -p(t)\,dt\right) = e^{\int p(t)\,dt}$$

Since in this context, we don't need all the μs that work (we only need one μ), we can ignore the $+C$ in this integral.

This works! Here is the procedure we just went through:

Procedure to solve first-order linear ODEs via integrating factors

1. Rewrite the equation in the form $y'(t) + p(t)y(t) = q(t)$.

2. Multiply through both sides by the integrating factor $\mu(t) = e^{\left[\int p(t)\,dt\right]}$ (ignore the $+C$ in the integral). This yields the equation

$$\mu(t)y'(t) + \mu'(t)y(t) = \mu(t)q(t).$$

3. The left-hand side of the equation you get in Step 2 is always a derivative coming from the Product Rule. This makes the equation

$$\frac{d}{dt}\left[\mu(t)y(t)\right] = \mu(t)q(t).$$

4. Integrate both sides with respect to t to get $\mu(t)y(t) = \int \mu(t)q(t)\,dt$. (You need the $+C$ when doing this integral to get the general solution.)

5. Divide through by $\mu(t)$ to solve for y: $y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)\,dt\right]$.

You can solve any first-order linear ODE by this procedure (theoretically). The only drawback is that the integrals

$$\int p(t)\,dt \qquad \text{and} \int \mu(t)q(t)\,dt$$

have to be doable (and you don't always know if they are doable when you start).

Example 1: Find the general solution of $ty' + 2y = 4t^2$.

Example 2: Find the particular solution of the IVP

$$\begin{cases} y' + (\cos t)y = \cos t \\ y(\pi) = 0 \end{cases} .$$

Example 3: Find the general solution of $y' + y = 6t$.

2.3 Solution of first-order linear equations by undetermined coefficients

It turns out that the structure of the solutions of a non-homogeneous linear equation has a lot to do with an associated homogeneous equation:

Definition 2.9 *Given a first-order linear ODE $p_1(t)y' + p_0(t)y = q(t)$, the ODE*

$$p_1(t)y' + p_0(t)y = 0$$

is called the **corresponding homogeneous equation**.

Theorem 2.10 *Suppose y and $\widehat{y}$ are two solutions of the first-order linear ODE $p_1(t)y' + p_0(t)y = q(t)$. Then the function $y - \widehat{y}$ is a solution of the corresponding homogeneous equation $p_1(t)y' + p_0(t)y = 0$.*

PROOF The corresponding homogeneous equation is

$$p_1(t)y' + p_0(t)y = 0.$$

Plug the function $y - \widehat{y}$ into the left-hand side:

As a consequence, suppose y_p is any one particular solution of the ODE

$$p_1(t)y' + p_0(t)y = q(t).$$

If $y(t)$ is **any** solution of that ODE, then $y - y_p$ is a solution of the corresponding homogeneous equation, i.e.

$$y - y_p = Cy_h$$

where y_h is any nonzero solution of the corresponding homogeneous equation. Therefore

$$y(t) = y_p(t) + Cy_h(t)$$

where y_p is any solution of the original ODE, and y_h is any nonzero solution of the corresponding homogeneous equation. We have proven:

Theorem 2.11 (Solution of the non-homogeneous first-order linear equation) *Let $y_p(t)$ be any particular solution of the first-order, linear ODE*

$$p_1(t)y' + p_0(t)y = q(t).$$

Let $y_h(t)$ be any nonzero solution of the corresponding homogeneous equation

$$p_1(t)y' + p_0(t)y = 0$$

Then $y(t)$ is a solution of the original ODE if and only if

$$y(t) = y_p(t) + Cy_h(t),$$

where C is an arbitrary constant.

Restated in linear algebra language: This theorem says that the solution set of a first-order, linear ODE is an *affine subspace* of $C^\infty(\mathbb{R}, \mathbb{R})$ whose dimension is 1.

How this is applied: Suppose you have some first-order linear ODE and you know that $y_p(t) = t^2 - t$ is a solution of this ODE. Furthermore, suppose you know that $y_h(t) = e^{3t}$ is a solution of the corresponding homogeneous equation. Then the general solution of the ODE is

How does this theorem fit with the method of solution obtained via integrating factors? Recall from the previous section that the solution to first-order linear ODE

$$y'(t) + p_0(t)y(t) = q(t)$$

is given by

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)\,dt\right]$$

where $\mu(t) = \exp\left(\int_0^t p_0(s)\,ds\right)$ is the integrating factor.

Method of undetermined coefficients

Theorem 2.11 motivates an alternate way to solve first-order, linear ODEs. After writing the equation in the form

$$y' + p(t)y = q(t) \tag{2.3}$$

if you can find a solution y_h of the corresponding homogeneous equation $y' + p(t)y = 0$ (which we can do by the methods described in Section 2.1), and if you can find any one solution y_p of (2.3) (which we can sometimes do by "guessing"), then the general solution of (2.3) is

$$y = y_p + Cy_h.$$

This method of solving an ODE is called the **method of undetermined coefficients**.

Example: Find the general solution of the ODE $y' - 3y = e^t$ (without using integrating factors).

Example: Find the general solution of the ODE $y' + 2y = 8\cos t$ (without using integrating factors).

Method of undetermined coefficients

1. Start with an equation of the form $y'(t) + py(t) = q(t)$, where p is usually constant.

2. **If p is constant:** a solution of the corresponding homogeneous equation is $y_h(t) = e^{-pt}$ (from exponential growth/decay).
   If p is non-constant: solve the homogeneous equation $y'(t)+py(t) = 0$ for y_h by the method of §2.1.

3. Try to "guess" a particular solution y_p of the ODE based on what $q(t)$ is:

What you see in $q(t)$	What you should guess for y_p
linear	$At + B$
quadratic	$At^2 + Bt + C$
polynomial	polynomial (of same degree as q)
$\sin t$ and/or $\cos t$	$A\sin t + B\cos t$
$\sin 2t$ and/or $\cos 2t$	$A\sin 2t + B\cos 2t$
$\sin bt$ and/or $\cos bt$	$A\sin bt + B\cos bt$
$t\sin bt$ and/or $t\cos bt$	$At\sin bt + Bt\cos bt + C\sin bt + D\cos bt$
e^t	Ae^t
e^{bt}	Ae^{bt}
te^{bt}	$Ate^{bt} + Be^{bt}$

   WARNING: If your guess for y_p is part of y_h, you need to do something extra (which you learn in the HW).

4. Plug your guess into the ODE and try to find values of $A, B, C, ...$ which work. This tells you $y_p(t)$.

5. The general solution of the ODE is $y(t) = Cy_h(t) + y_p(t)$.

6. If given an initial value, plug it in and solve for C. Then write the particular solution.

2.4 Separation of variables

Question: Can you explicitly solve a first-order ODE $y' = \phi(t, y)$ for the solutions $y(t)$?

Answer:

> **Definition 2.12** *A first-order ODE is called* **separable** *if it can be rewritten in the form $f(y)y' = h(t)$ for functions f of y and h of t.*

In other words, an ODE is separable if one can **separate the variables**, i.e. put all the y on one side and all the t on the other side.

Theoretical solution of separable, first-order ODEs

Suppose you have a separable, first-order ODE. Then, by replacing the y' with $\frac{dy}{dt}$, it can be rewritten as

$$f(y)\,\frac{dy}{dt} = h(t).$$

Integrate both sides with respect to t to get

$$\int f(y)\,\frac{dy}{dt}\,dt = \int h(t)\,dt$$

On the left-hand side, perform the $u-$substitution $u = y(t)$, $du = \frac{dy}{dt}\,dt$ to get

$$\int f(u)\,du = \int h(t)\,dt$$

Assuming F and H are antiderivatives of f and h, respectively, we get

$$F(u) = H(t) + C$$

which, since $u = y = y(t)$, is equivalent to the solution

$$F(y) = H(t) + C.$$

Note: We only need a constant on one side of the equation, because the constants on the two sides can be combined into one.

A shortcut:

The following method involves the writing of things that are technically incorrect, but always works. Again start with a separable, first-order ODE. Again, replace the y' with $\frac{dy}{dt}$ to get

$$f(y)\,\frac{dy}{dt} = h(t).$$

Now, pretend that the $\frac{dy}{dt}$ is a fraction (**it isn't**) and "multiply" through by dt to get

$$f(y)\,dy = h(t)\,dt.$$

Now, integrate both sides:

$$\int f(y)\,dy = \int h(t)\,dt$$

This gives the same solution as before:

$$F(y) = H(t) + C.$$

The shortcut above suggests the following method to solve separable ODEs:

Procedure to solve separable ODEs

1. Write the derivative y' in Leibniz notation as $\frac{dy}{dt}$.
2. Separate the variables, i.e. put all the y (with the dy) on one side of the equation, and all the t (with the dt) on the other side of the equation.

   WARNING: The equation has to end up in the form $f(y)\,dy = h(t)\,dt$. (For example, you don't want something like $f(y)\,dy + y = h(t)\,dt$.)
3. Integrate both sides, putting the arbitrary constant on one side.
4. If given an initial value, plug it in and solve for C. Then write the particular solution.
5. If the problem asks for a solution of the form $y = y(t)$ or $y = f(t)$, solve the solution for y.

Example 1: Find the general solution of the ODE $y' = ty$.

Example 2: Find the particular solution of this initial value problem:

$$\begin{cases} y' = 2t - 2ty \\ y(1) = -2 \end{cases}$$

Example 3: Find the particular solution of this initial value problem:

$$\begin{cases} y' = \frac{y \cos t}{1+2y^2} \\ y(0) = 1 \end{cases}$$

Example 4: Find the particular solution of this initial value problem:

$$\begin{cases} y' = \frac{2-e^t}{3+2y} \\ y(0) = 0 \end{cases}$$

Find the value of t for which the solution of this IVP is maximized.

Example 5: Find the general solution of the logistic equation:

$$y' = ry(L - y)$$

We have proven:

Theorem 2.13 (Solution of the logistic equation) *The general solution of the logistic equation $y' = ry(L - y)$ is*

$$y = \frac{L}{1 + Ce^{-rLt}}.$$

Example 6: Find the general solution of this ODE.

$$y' = ty^3(1+t^2)^{-1/2}$$

Write your answer in the form $y = f(t)$.

2.5 Exact equations

Question: Suppose you are given a first-order ODE which is neither linear nor separable. Is there any hope of solving it?

Answer:

A review of some multivariable calculus

In multivariable calculus, you study (among other things) functions of two variables $z = \psi(s,t)$. For such a function, the natural method of differentiation is *partial differentiation*. Without going into much detail, here are some informal definitions:

Definition 2.14 *Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables, i.e. $z = \psi(s,t)$.*

1. *The* **partial derivative of ψ with respect to** s*, denoted ψ_s or $\frac{\partial \psi}{\partial s}$, is the expression obtained by treating t as a constant and differentiating the formula defining $\psi(s,t)$ with respect to s.*
2. *The* **partial derivative of ψ with respect to** t*, denoted ψ_t or $\frac{\partial \psi}{\partial t}$, is the expression obtained by treating s as a constant and differentiating the formula defining $\psi(s,t)$ with respect to t.*

Example: Let $\psi(s,t) = 2t^3 \cos s + e^t \sin s - 4t$.

$$\psi_s = \frac{\partial \psi}{\partial s} =$$
$$\psi_t = \frac{\partial \psi}{\partial t} =$$

Example: Suppose $\psi(s,t)$ is some function such that $\frac{\partial \psi}{\partial t} = 12s^2t^3 + 6t$. What can you say about $\psi(s,t)$?

The following theorem (or a more general version of it) is proven in multivariable calculus (Math 320):

Theorem 2.15 (Chain Rule) *Suppose $\psi : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function of variables t and y. Suppose further that both t and y are functions of a third variable s. Then*

$$\frac{d\psi}{ds} = \frac{\partial\psi}{\partial t}\frac{dt}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds}.$$

Suppose we let $s = t$ in this theorem. Then $\frac{dt}{ds} = \frac{dt}{dt} = 1$ so the formula in the Chain Rule becomes

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial y}\frac{dy}{dt} \qquad \text{a.k.a.} \qquad \frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial y}y' = \frac{d\psi}{dt}.$$

What does this have to do with ODEs?

Example: Find the general solution of $2t + y^2 + 2tyy' = 0$.

Bad news: This is not a separable equation.

Good news: This looks a bit like the equation at the above right.

Definition 2.16 *A first-order ODE is called* **exact** *if there is a function $\psi : \mathbb{R}^2 \to \mathbb{R}$ such that the equation has the form*

$$\frac{\partial\psi}{\partial t}(t,y) + \frac{\partial\psi}{\partial y}(t,y)y' = 0.$$

Theorem 2.17 (Characterization of exact equations) *A first order ODE of the form*

$$M(t,y) + N(t,y)y' = 0$$

is exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$.

PROOF ($\Rightarrow$) Suppose the equation is exact. Then there is a function $\psi = \psi(t,y) : \mathbb{R}^2 \to \mathbb{R}$ so that $M = \frac{\partial\psi}{\partial t}$ and $N = \frac{\partial\psi}{\partial y}$. Then

$$\frac{\partial M}{\partial y} = \frac{\partial^2\psi}{\partial y\partial t} \qquad \text{and} \qquad \frac{\partial N}{\partial t} = \frac{\partial^2\psi}{\partial t\partial y};$$

by a theorem from multivariable calculus (called the "Equality of Mixed Partials"), these two quantities are equal (because it doesn't matter the order in which you compute a mixed second-order partial derivative).

($\Leftarrow$) Now we assume $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$. Define

$$\psi(t,y) = \int M(t,y)\,dt + \int \left[N(t,y) - \int \frac{\partial}{\partial y} M(t,y)\,dt\right] dy.$$

Observe that for this ψ,

$$\begin{aligned}
\frac{\partial \psi}{\partial t} &= \frac{\partial}{\partial t}\left(\int M(t,y)\,dt + \int \left[N(t,y) - \int \frac{\partial}{\partial y} M(t,y)\,dt\right] dy\right) \\
&= \frac{\partial}{\partial t}\int M(t,y)\,dt \;+\; \int \left[\frac{\partial}{\partial t} N(t,y) - \frac{\partial}{\partial t}\int \frac{\partial}{\partial y} M(t,y)\,dt\right] dy \\
&= M(t,y) + \int \left[\frac{\partial}{\partial t} N(t,y) - \frac{\partial}{\partial y} M(t,y)\right] dy \\
&= M(t,y) + \int 0\,dy \qquad \text{(by the hypothesis } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}\text{)} \\
&= M(t,y)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \psi}{\partial y} &= \frac{\partial}{\partial y}\left(\int M(t,y)\,dt + \int \left[N(t,y) - \int \frac{\partial}{\partial y} M(t,y)\,dt\right] dy\right) \\
&= \int \frac{\partial}{\partial y} M(t,y)\,dt + N(t,y) - \int \frac{\partial}{\partial y} M(t,y)\,dt \\
&= N(t,y)
\end{aligned}$$

as desired. □

Remark: To find the ψ for a given exact equation, don't use the formula in the preceding theorem. Use the method outlined in forthcoming examples.

Theorem 2.18 (Solution of an exact equation) *The exact ODE*

$$\frac{\partial \psi}{\partial t}(t,y) + \frac{\partial \psi}{\partial y}(t,y)y' = 0$$

has general solution

$$\psi(t,y) = C$$

where C is an arbitrary constant.

PROOF By the preceding discussion, we know that the left-hand side of an exact equation can be rewritten as $\frac{d\psi}{dt}$, and since this is equal to zero, ψ must be a constant. $\square$

Back to the example: $2t + y^2 + 2tyy' = 0$.

New example: $2ty^3 - 2y^2 + 4t^2 + (3t^2y^2 - 4ty + 8y)y' = 0$

How to solve an exact equation

1. Write the equation in the form $M(t,y) + N(t,y)y' = 0$.
2. Check that the equation is exact by verifying that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$.
3. If the equation is exact, the solution is $\psi(t,y) = C$ where

$$\psi(t,y) = \int M(t,y)\,dt = \int N(t,y)\,dy.$$

   Keep in mind that when integrating M with respect to t, your answer includes a "$+A(y)$", and when you integrate N with respect to y, your answer includes a "$+B(t)$".
4. Choose $A(y)$ and $B(t)$ to reconcile two versions of ψ obtained by integration.
5. If given an initial value, plug it in and solve for C. Then write the particular solution.
6. If the problem asks for a solution of the form $y = y(t)$ or $y = f(t)$, solve the solution for y.

Example: $y' = \frac{1-2te^y}{t^2}$

2.6 Applications of first-order equations

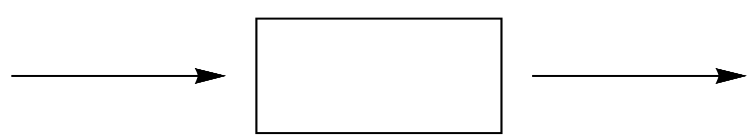
Compartmental models

Many mathematical models involve processes that can be divided into "compartments" or "stages". You build the model by listing the compartments, and then describing the interactions between the compartments. These are called **compartmental models** and generate ODEs (or systems of ODEs) which can be analyzed with the methods of this course.

General setup:

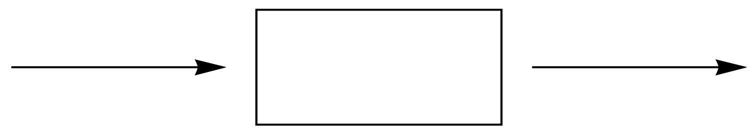
Example 1: Malthusian population model

Let $y(t)$ = population at time t; assume the birth and death rates are proportional to the population.

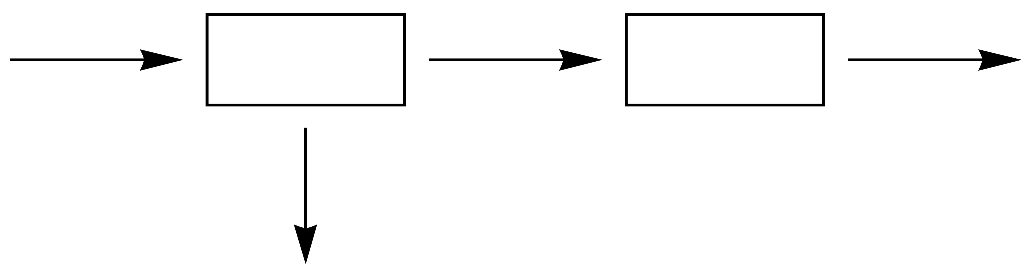


Example 2: Logistic population model

Let $y(t)$ = population at time t; assume the birth rate is proportional to the population; assume the death rate is proportional to the number of interactions between two living beings.



Examples 1 and 2 are models with one compartment. A population model with more than one compartment might look like this:



In this context, you would get a system of two ODEs

$$\begin{cases} x'(t) = & \text{birth rate - aging rate - death rate of young} \\ y'(t) = & \text{aging rate - death rate of old} \end{cases}$$

Under reasonable assumptions, you might get something like

$$\begin{cases} x'(t) = \lambda y - ax - \mu x \\ y'(t) = ax - \mu y \end{cases}$$

We don't know how to solve systems yet, but we'll return to these in Chapter 3.

Example 3: Mixing problems

Consider a large tank containing 1000 L of pure water, into which a brine solution of salt water (whose concentration is .1 kg / L) is poured at a constant rate of 6 L / min. The solution inside the tank is kept well mixed, and is flowing out of the tank at a rate of 6 L / min. Find the concentration of salt in the tank at time t.

Newtonian mechanics

Recall from physics the three Newtonian laws of motion:

1. When an object is subject to no external force, it continues with constant velocity.
2. When a body is subject to external forces, the time rate of change of the object's momentum (i.e. the mass times acceleration) is equal to the sum of the forces acting on it.
3. When one body interacts with a second, the force of the first body on the second is equal in magnitude, but opposite in direction to the force of the second body on the first.

Newton's Second Law is of particular use in modeling the motion of objects using ODEs: let $v = v(t)$ be the velocity of an object at time t. Then

Example: (from page 4 of these lecture notes) Consider an object of mass 20 kg, that is falling through the Earth's atmosphere (gravitational constant is 9.8 m/sec^2; drag coefficient 3 N sec/m), near sea level. Formulate an ODE which describes the velocity of the object, and solve it.

Heating and cooling

Goal: formulate (and solve) an ODE which models the temperature inside a building as a function of time.

Assumptions:

- The temperature $T(t)$ inside the building at time t will depend on the following things (and nothing else):
  -
  -
  -
- **Newton's Law of Cooling (or Heating):** the rate of change in temperature inside the building, absent other forces, is proportional to the temperature difference between the outside and the inside

Let $T(t)$ be the temperature inside the building at time t. We obtain the following ODE:

This equation is first-order, and linear. Let's solve it:

Theorem 2.19 (Heating and cooling model) *Suppose $T(t)$ is the temperature of a compartment at time t. If the outside temperature is given by $M(t)$, the compartment itself generates heat $H(t)$, and a furnace/air conditioner generates heat $U(t)$ (U is positive if you're heating and negative if you're cooling), then the temperature satisfies the ODE*

$$\frac{dT}{dt} = K\left[M(t) - T(t)\right] + H(t) + U(t)$$

where K is a constant. The solution of this ODE (obtained via integrating factors) is

$$T(t) = e^{-Kt}\left(\int e^{Kt}\left[KM(t) + H(t) + U(t)\right]\, dt\right).$$

Example 1: Suppose we have the simplest possible situation: no one in the building (so $H(t) = 0$), no A/C or furnace (so $U(t) = 0$), and a constant outside temperature $M(t) = M_0$. Find $T(t)$.

Example 2: Suppose the outside temperature t hours after midnight is

$$M(t) = 70 - 10\cos\left(\frac{\pi}{12}t\right).$$

Assuming there is no air conditioning or heating and that the constant from Newton's Law is $K = \frac{1}{4}$, find the temperature of the house at time t, if the temperature inside the house at midnight is 65.

In:
```
M[t_] = 70 - 10 Cos[Pi t / 12];
K = 1/4;
f[t_] = Integrate[E^(K t) K M[t] , t]
```

Out: $$-\frac{10e^{t/4}\left(-7(9+\pi^2)+9\cos\left(\frac{\pi t}{12}\right)+3\pi\sin\left(\frac{\pi t}{12}\right)\right)}{9+\pi^2}$$

Therefore the solution is $\frac{1}{\mu(t)} = e^{-Kt}$ times the previous output which is

$$T(t) = e^{-t/4}\left(f(t)+C\right).$$

To find C, plug in the initial value $t_0 = 0$, $T_0 = 65$ and solve for C. We do this with *Mathematica*:

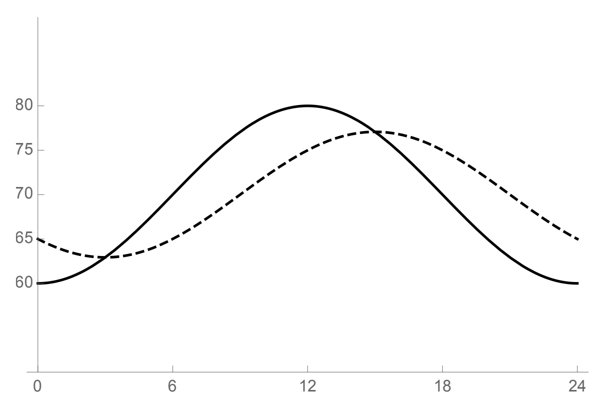
In:
```
Solve[65 == E^(-0/4) (f[0] + C), C]
```

Out: $$C = -\frac{5(-9+\pi^2)}{9+\pi^2}$$

Now we know what the temperature $T(t)$ is:

$$\begin{aligned} T(t) &= e^{-t/4}(f(t)+C) \\ &= e^{-t/4}\left(-\frac{10e^{t/4}\left[-7(9+\pi^2)+9\cos\left(\frac{\pi t}{12}\right)+3\pi\sin\left(\frac{\pi t}{12}\right)\right]}{9+\pi^2} - \frac{5(-9+\pi^2)}{9+\pi^2}\right). \end{aligned}$$

Below, the interior temperature is graphed with the dashed curve (the exterior temperature is the solid curve):



Electrical circuits

Kirchoff's laws govern the physics of electrical circuits. They are:

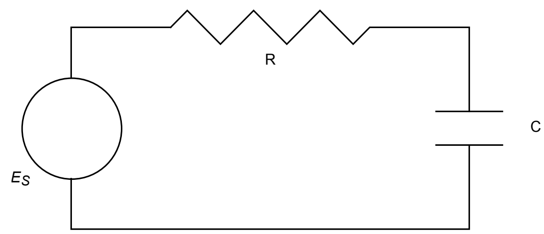
Current law: The sum of the current flowing into any junction point is zero.

Voltage law: The sum of the instantaneous changes in potential (i.e. the sum of the voltage drops) around any closed loop is zero.

Things that can be hooked up to form electrical circuits:

OBJECT	LETTER	UNIT OF MEASUREMENT	HOW IT IS DRAWN IN A DIAGRAM	HOW TO FIND THE VOLTAGE DROP
voltage source (battery or generator)	E	volt V		E_S = usually given/ controlled
resistor (reduces current flow)	R	ohm Ω		Ohm's Law: $E_R = RI_R$
inductor (coil)	L	henry H		Faraday's Law: $E_L = L\frac{dI}{dt}$
capacitor (stores energy)	C	farad F		$E_C = \frac{1}{C}q$ (q = charge) Also: voltage/ current relationship $I_C(t) = C\frac{dE_C}{dt}$

Example 1: RC circuit



Given: $E_S(t)$ = voltage from source as function of time; R = resistance, C = capacitance

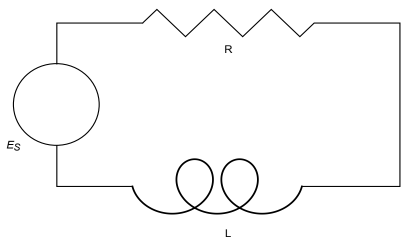
Goal: Describe $E_C(t)$ = voltage across the capacitor at time t

Kirchoff's voltage law:	$E_R(t) + E_C(t) = E_S(t)$	(1)
Ohm's Law: (I = current)	$E_R(t) = RI_R(t)$	(2)
Only one current in circuit:	$I_R(t) = I_C(t) = I(t)$	(3)
Plug equation (3) and voltage/current relationship into equation (2):	$E_R(t) = RI_C(t) = RC\frac{dE_C}{dt}$	(4)
Plug into equation (1):	$RC\frac{dE_C}{dt} + E_C(t) = E_S(t)$	(5)

Equation (5) above is a first-order, linear ODE which can be solved using integrating factors for $E_C(t)$. This works even if the resistance is variable (i.e. $R = R(t)$).

Remark: If there is more than one current running at different spots in the circuit (for example, if you have a more complicated circuit), you may need a system of two or more ODEs (or a higher-order ODE). We will study these in Chapter 4.

Example 2: RL circuit



Given: $E_S(t)$ = voltage from source as function of time; R = resistance, L = inductance

Goal: Describe $I(t)$ = current at time t

Kirchoff's voltage law:	$E_R(t) + E_L(t) = E_S(t)$	(1)
Ohm's Law:	$E_R(t) = RI_R(t)$	(2)
Faraday's Law:	$E_L(t) = L\frac{dI}{dt}$	(3)
Only one current in circuit:	$I_R(t) = I_L(t) = I(t)$	(4)
Plug into equation (1):	$L\frac{dI}{dt} + RI(t) = E_S(t)$	(5)

Equation (5) above is a first-order, linear ODE which can be solved using integrating factors for $I(t)$. (If you need to know the voltage across the resistor, apply Ohm's Law to the answer: $E_R(t) = RI(t)$.)

2.7 Summary of Chapter 2

Theory of first-order linear ODEs

The general solution of any first-order, homogeneous linear equation

$$p_1(t)y'(t) + p_0(t)y(t) = 0$$

is a subspace spanned by one nonzero solution, i.e. is of the form $Cy_h(t)$ for any nonzero solution y_h. The general solution of any first-order (not necessarily homogeneous) linear equation

$$p_1(t)y'(t) + p_0(t)y(t) = q(t)$$

is the solution of the corresponding homogeneous equation plus any one particular solution of the original equation, i.e. has the form $y_p(t) + Cy_h(t)$.

Solution techniques

- Every linear, first-order, homogeneous, constant-coefficient ODE is of the form $y' = ry$; the solution of such an equation is $y = y_0 e^{rt}$ where $y_0 = y(0)$. This is the **exponential growth/decay** model.

- To solve a linear, first-order equation

$$p_1(t)y'(t) + p_0(t)y(t) = q(t),$$

first divide through by p_1, to rewrite the equation as

$$y' + p(t)y = q(t).$$

Then use one of these two methods:

Integrating factors: multiply through both sides of the equation by the integrating factor $\mu(t) = \exp\left(\int p(t)\,dt\right)$ and then identify the left-hand side as $\frac{d}{dt}\left(\mu(t)y(t)\right)$; the general solution is eventually

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)\,dt\right].$$

Undetermined coefficients: first, solve the corresponding homogeneous equation

$$y' + p(t)y = 0$$

and call the solution y_h. Then, to a particular solution of the original equation, "guess" what y_p is based on what the function q is. Substitute your guess for y_p into the ODE and use that equation to solve for the constant(s) in your guess. The general solution of the equation is then

$$y = y_p + Cy_h.$$

- To solve a first-order equation which is not linear, try one of two methods:

Separation of variables: a first-order ODE is called **separable** if it can be rewritten as $f(y)y' = h(t)$. To solve a separable ODE, separate the variables and integrate both sides.

Note: homogeneous linear equations are also separable and can be solved via separation of variables.

Exact equations: a first-order ODE is **exact** if it is of the form

$$M(t, y) + N(t, y)y' = 0.$$

where $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$. The solution of an exact equation is $\psi(t, y) = C$ where

$$\psi(t, y) = \int M(t, y)\, dt = \int N(t, y)\, dy.$$

Applications

Applications of first-order models include compartmental models, population dynamics, mixing problems, Newtonian mechanics, heating and cooling, and electrical circuits.

2.8 Exam 1 Review

What should you expect on the exam:

Solve first-order ODEs/IVPs: separable equations, exact equations, first-order linear equations

Solve second-order ODEs that either have no y or no t in them (HW # 79-82)

Verifying whether or not a given equation is or is not a solution of an ODE or IVP

Sketch and/or analyze pictures of slope fields and/or phase lines

Euler's method: perform a couple of steps for a simple equation

Applications: heating & cooling, population models, circuits, etc. You will be given formulas (the heating/cooling equation, the RL equation, the RC equation, etc.) if necessary.

Question(s) on vocabulary

Some practice questions:

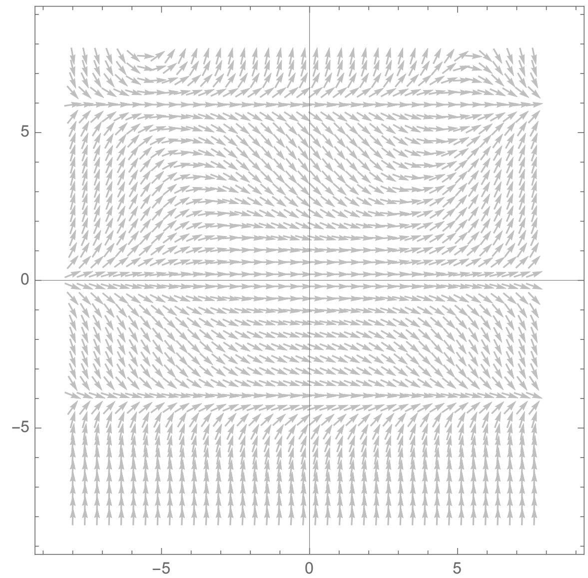
1. a) What is the difference between a "general solution" and a "particular solution"?
   b) What is meant by the "order" of an ODE?
   c) What does it mean for an ODE to be "linear"?
   d) What is meant by "existence/uniqueness" (in the context of ODEs)?
   e) What is the general form of the general solution of a first-order linear ODE?

2. Consider the initial value problem

$$\begin{cases} y' = 2y - t \\ y(0) = 1 \end{cases}$$

   Suppose you wanted to estimate $y(30)$ by performing Euler's method with 15 steps. Find the points (t_1, y_1) and (t_2, y_2) that would be obtained by this method.

3. Here is the picture of a slope field associated to an ODE of the form $y' = \phi(t, y)$:



a) Estimate the value of y' when $t = 5$ and $y = -2$.

b) Suppose $y(-2) = 4$. Estimate $y(2)$.

c) Suppose $y(-2) = 4$. Estimate $y'(2)$.

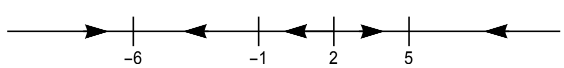
d) Suppose $y(0) = -2$. Find $\lim_{t\to\infty} y(t)$.

e) Sketch the solution of this ODE satisfying $y(0) = 3$ on the slope field.

f) Find the equation of one particular solution of this ODE.

g) Is the equation which generates this slope field autonomous? Why or why not?

4. Here is the picture of the phase line associated to an ODE of the form $y' = \phi(y)$:



a) Suppose $y(1) = 3$. Find $\lim_{t\to\infty} y(t)$.

b) Suppose $y(-4) = 3$. Find $\lim_{t\to-\infty} y(t)$.

c) Find the equilibria of this system, and classify them as stable, unstable or semistable.

d) Suppose $y(0) = y_0$. Find all values of y_0 for which $\lim_{t\to\infty} y(t) = 5$.

5. Sketch the phase lines for each ODE. Find the equilibria and classify them as stable, unstable or semistable:

a) $y' = y^2 - 7y + 12$

b) $y' = y^4 - 8y^3$

c) $y' = \sin y + 1$

6. Solve each given differential equation or initial value problem. In questions (a), (c) and (i), your answer must be written in the form $y = f(t)$.

a) $\frac{dy}{dt} = \frac{t}{y^2}$

b) $\begin{cases} \frac{dy}{dt} = \frac{t^2}{y^3} \\ y(0) = -1 \end{cases}$

c) $\begin{cases} \frac{dy}{dt} = (y^2+1)t \\ y(0) = 1 \end{cases}$

d) $y' = \frac{y}{7}$

e) $y' = \frac{t}{1+t^2}$

f) $y' = \frac{y}{1+y^2}$

g) $3t^3y^2\frac{dy}{dt} + 3t^2y^3 = 5t^4$

h) $t^2y^3 + t(1+y^2)y' = 0$

i) $y' = -2ty + 4e^{-t^2}$

j) $t\frac{dy}{dt} + 2y = t^2 - t$

k) $\begin{cases} \frac{dy}{dt} = \frac{-y}{t} + 2 \\ y(1) = 3 \end{cases}$

l) $y'' - 4y' = e^{2t}$

m) $y' - 2y = 4\sin t - 7\cos t$

n) $yy'' = (y')^3$

7. A 5% sulfuric acid solution is pumped into a 24 L tank at a rate of 3 L / min. The tank initially contains 10% sulfuric acid, and is kept well-stirred at all times. If the tank drains at a rate of 4 L / min, find the amount of sulfuric acid in the tank at time $t = 3$.

Solutions

WARNING: I did these by hand. There might be errors.

1. a) Given an ODE, the set of all solutions of that ODE is called the **general solution** of the ODE. This solution will have one or more arbitrary constants in it. If you are given an initial value, you can plug that initial value into the general solution, solving for the constant in the general solution. This produces a solution of the ODE with no arbitrary constants, which is called a **particular solution** of the ODE.

   b) The **order** of an ODE is the highest-order of derivative appearing in the equation; it is also equal to the number of arbitrary constants in the general solution.

   c) An ODE is **linear** if it is of the form $q(t) = p_0(t)y + p_1(t)y' + p_2(t)y'' + ... + p_n(t)y^{(n)}$ where $q, p_0, p_1, ..., p_n$ are functions of t. Equivalently, an ODE is linear if it is of the form $T(y) = q$ where T is a linear differential operator on $C^\infty(\mathbb{R}, \mathbb{R})$.

d) A first-order initial value problem of the form $y' = \phi(t, y); y(t_0) = y_0$ has one and only one solution, which is of the form $y = f(t)$, so long as the function ϕ is "nice" (i.e. ϕ and $\frac{\partial \phi}{\partial y}$ are both continuous).

e) The general solution of a first-order linear ODE always has the form $y = y_p + Cy_h$, where $y_p = y_p(t)$ is any one particular solution of the equation and $y_h = y_h(t)$ is any nonzero solution of the associated homogeneous equation.

2. To use Euler's method, first compute $\Delta t = \frac{t_n - t_0}{n} = \frac{30-0}{15} = 2$.

$(t_0, y_0) = (0, 1)$ so $\phi(t_0, y_0) = 2(1) - 0 = 2$. Therefore

$$t_1 = t_0 + \Delta t = 0 + 2 = 2 \quad \text{and} \quad y_1 = y_0 + \phi(t_0, y_0)\Delta t = 1 + 2(2) = 5$$

so $(t_1, y_1) = (2, 5)$. Now $\phi(t_1, y_1) = 2(5) - 2 = 8$. Therefore

$$t_2 = t_1 + \Delta t = 2 + 2 = 4 \quad \text{and} \quad y_2 = y_1 + \phi(t_1, y_1)\Delta t = 5 + 8(2) = 21$$

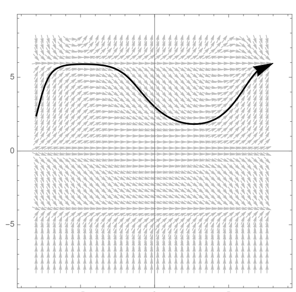
so $(t_2, y_2) = (4, 21)$.

3. a) At $(5, -2)$, the slope y' appears to be about -1

b) Follow the slope field from $(-2, 4)$ until $t = 2$ and read off the y value to get $y(2) \approx 1.5$.

c) Follow the slope field from $(-2, 4)$ until $t = 2$ and read off the slope at 2 to get $y'(2) \approx 0$.

d) As the solution through $(0, -2)$ goes to the right, its y-value approaches $\lim_{t\to\infty} y(t) = -4$.

e) 

f) $y = -4$, $y = 0$ and $y = 6$ all appear to be solutions.

g) Notice that the slope at $(0, 2)$ is different from the slope at $(5, 2)$. Therefore the slope y' cannot depend only on y, so this equation cannot be autonomous.

4. a) $\lim_{t\to\infty} y(t) = 5$ (follow the arrow forward from $y = 3$).

b) $\lim_{t\to -\infty} y(t) = 2$ (follow the arrow backward from $y = 3$).

c) $y = -6$ is stable; $y = -1$ is semistable; $y = 2$ is unstable; $y = 5$ is stable.

d) If $y(0) > 2$, then $\lim_{t\to\infty} y(t) = 5$. (Equivalent answer: $(2, \infty)$.)

5. a) Set $\phi(y) = y^2 - 7y + 12$ equal to zero and solve for y to get $y = 3$ and $y = 4$ (those are the equilibria). To classify them, note that $\phi'(y) = 2y - 7$. So $\phi'(3) = -1 < 0$ so 3 is stable; $\phi'(4) = 1 > 0$ so 4 is unstable. (So the phase line has arrows going from $-\infty$ toward 3, 4 towards 3, and 4 towards ∞.)

   b) Set $\phi(y) = y^4 - 8y^3 = y^3(y - 8)$ equal to zero to get $y = 0$ and $y = 8$ (those are the equilibria). To classify them, compute $\phi'(y) = 4y^3 - 24y^2$. $\phi'(8) = 512 > 0$ so 8 is unstable. As for $y = 0$, $\phi'(0) = 0$ so compute $\phi''(y) = 12y^2 - 48y$. That means $\phi''(0) = 0$. Now compute $\phi'''(y) = 24y - 48$; $\phi'''(0) = -24 < 0$. Since the first nonzero derivative at 0 is odd (it was the third derivative) and negative, it is as if the first derivative was negative, so the function ϕ is decreasing at zero, which makes $y = 0$ a stable equilibria. (So the phase line has arrows going from $-\infty$ toward 0, from 8 toward 0 and from 8 toward ∞).

   c) Set $\phi(y) = \sin y + 1 = 0$ and solve for y to get $y = \frac{3\pi}{2} + 2\pi N$ where N is an integer. To classify these, compute $\phi'(y) = \cos y$ and observe that $\cos(\frac{3\pi}{2} + 2\pi N) = 0$. Therefore, go to the second derivative: $\phi''(y) = -\sin y$ so $\phi''(\frac{3\pi}{2} + 2\pi N) = -1 < 0$. That makes all these equilibria semistable (all the arrows in the phase line go from left to right since $\phi(y) \geq 0$ for all y).

6. a) This is a separable equation. Write it as $y^2\, dy = t\, dt$ and integrate both sides to get $\frac{1}{3}y^3 = \frac{1}{2}t^2 + C$; then solve for y to get $y = \sqrt[3]{\frac{3}{2}t^2 + 3C}$ which could also be written as $y = \sqrt[3]{\frac{3}{2}t^2 + C}$.

   b) This equation is separable: write it as $y^3\, dy = t^2\, dt$ and integrate both sides to get $\frac{1}{4}y^4 = \frac{1}{3}t^3 + C$. Then plug in the initial condition $(0, -1)$ to get $\frac{1}{4} = 0 + C$, i.e. $C = \frac{1}{4}$. Therefore the particular solution is $\frac{1}{4}y^4 = \frac{1}{3}t^3 + \frac{1}{4}$.

   c) This equation is separable: write it as $\frac{1}{y^2+1}\, dy = t\, dt$ and integrate both sides to get $\arctan y = \frac{1}{2}t^2 + C$. Plug in the initial condition $(0, 1)$ to get $\frac{\pi}{4} = 0 + C$, i.e. $C = \frac{\pi}{4}$. Thus the particular solution is $\arctan y = \frac{1}{2}t^2 + \frac{\pi}{4}$, i.e. $y = \tan\left(\frac{1}{2}t^2 + \frac{\pi}{4}\right)$.

   d) This is an exponential growth model whose solution is $y = y_0 e^{(1/7)t}$.

   e) This equation is already separated: integrate both sides to get $y = \frac{1}{2}\ln(1 + t^2) + C$. (The t-integral needs the u-substitution $u = 1 + t^2$.)

   f) This equation is separable: write it as $\frac{y^2+1}{y}\, dy = 1\, dt$ and integrate both sides to get $\frac{1}{2}y^2 + \ln y = t + C$. (To integrate the left-hand side, rewrite $\frac{y^2+1}{y}$ as $\frac{y^2}{y} + \frac{1}{y} = y + \frac{1}{y}$.)

g) First, rewrite this equation as $3t^3y^2\frac{dy}{dt}+(3t^2y^3-5t^4)=0$. This equation is exact since $\frac{\partial}{\partial t}(3t^3y^2) = 9t^2y^2 = \frac{\partial}{\partial y}(3t^2y^3-5t^4)$. Therefore, the solution is $\psi(t,y)=C$ where ψ can be found by integrating:

$$\psi(t,y) = \int 3t^3y^2\,dy = t^3y^3 + C(t)$$
$$\psi(t,y) = \int (3t^2y^3-5t^4)\,dt = t^3y^3 - t^5 + D(y)$$

These integrals can be rectified by choosing $C(t)=-t^5$ and $D(y)=0$, so $\psi(t,y)=t^3y^3-t^5$ so the general solution is $t^3y^3-t^5=C$.

h) This equation is separable; rewrite using algebra to get $\frac{1+y^2}{y^3}\,dy = -t\,dt$, i.e. $\left(y^{-3}+\frac{1}{y}\right)dy = -t\,dt$. Integrate both sides to obtain $\frac{-1}{2}y^{-2}+\ln y = \frac{-1}{2}t^2+C$.

i) Write this as $y'+2ty = 4e^{-t^2}$. This is first-order linear, so multiply through by the integrating factor

$$\exp\left[\int_0^t p_0(s)\,ds\right] = \exp\left[\int_0^t 2s\,ds\right] = e^{t^2}.$$

This produces the equation written below, which is solved by usual methods:

$$\begin{aligned} e^{t^2}y' + 2te^{t^2}y &= 4e^{-t^2}e^{t^2} \\ \frac{d}{dt}\left(ye^{t^2}\right) &= 4 \\ ye^{t^2} &= 4t + C \\ y &= 4te^{-t^2} + Ce^{-t^2}. \end{aligned}$$

j) This is first-order linear, so divide through by t to write the equation in the standard form $\frac{dy}{dt}+\frac{2}{t}y = t-1$. Now multiply through by the integrating factor

$$\exp\left[\int_0^t p_0(s)\,ds\right] = \exp\left[\int_0^t \frac{2}{s}\,ds\right] = e^{2\ln t} = t^2.$$

This produces the equation written below, which is solved by usual methods:

$$\begin{aligned} t^2y' + 2ty &= t^3 - t^2 \\ \frac{d}{dt}\left(t^2y\right) &= t^3 - t^2 \\ t^2y &= \frac{1}{4}t^4 - \frac{1}{3}t^3 + C \\ y &= \frac{1}{4}t^2 - \frac{1}{3}t + Ct^{-2}. \end{aligned}$$

k) Rewrite this first-order linear equation as $\frac{dy}{dt} + \frac{1}{t}y = 2$. Then multiply through by the integrating factor

$$\exp\left[\int_0^t p_0(s)\,ds\right] = \exp\left[\int_0^t \frac{1}{s}\,ds\right] = e^{\ln t} = t.$$

This produces the equation written below, which is solved by usual methods:

$$\begin{aligned} t\frac{dy}{dt} + y &= 2t \\ \frac{d}{dt}(ty) &= 2t \\ ty &= t^2 + C \\ y &= t + \frac{C}{t}. \end{aligned}$$

Now plug in the initial condition $(1, 3)$ to get $3 = 1 + \frac{C}{1}$, so $C = 2$. Thus the particular solution is $y = t + \frac{2}{t}$.

l) This is a second-order equation with no y: start with the substitution $v = y'$ to rewrite the equation as $v' - 4v = e^{2t}$. This is first-order linear with constant coefficients. The corresponding homogeneous equation is $v' - 4v = 0$ which has nonzero solution $v_h = e^{4t}$. Find a particular solution by undetermined coefficients: guess $v_p = Ae^{2t}$ and plug into the left-hand side to get $v_p' - 4v_p = 2Ae^{2t} - 4e^{2t} = (2A - 4)e^{2t}$. Since the right-hand side should be e^{2t}, we have $2A - 4 = 1$, i.e. $a = \frac{5}{2}$ so $v_p = \frac{5}{2}e^{2t}$. That means the general solution is

$$v(t) = y_p + Cy_h = \frac{5}{2}e^{2t} + Ce^{4t}.$$

Last, integrate this to obtain $y = \int v(t)\,dt = \frac{5}{4}e^{2t} + Ce^{4t} + D$.

m) This is first-order linear with constant coefficients. The corresponding homogeneous equation is $y' - 2y = 0$ which has nonzero solution $y_h = e^{2t}$. Find a particular solution via undetermined coefficients: guess $y_p = A\sin t + B\cos t$ and plug into the left-hand side to get

$$y_p' - 2y_p = A\cos t - B\sin t - 2A\sin t - 2B\cos t = (-B - 2A)\sin t + (A - 2B)\cos t.$$

Therefore we know that $-B - 2A = 4$ and $A - 2B = 7$. Solve these equations to get $A = \frac{-1}{5}$, $B = \frac{-18}{5}$. Thus the particular solution is $y_p = \frac{-1}{5}\sin t - \frac{18}{5}\cos t$ so the general solution is

$$y(t) = y_p + Cy_h = \frac{-1}{5}\sin t - \frac{18}{5}\cos t + Ce^{2t}.$$

n) This is a second-order equation with no t: start with the substitution $v = y' = \frac{dy}{dt}$ and rewrite the equation as $yv\frac{dv}{dy} = v^3$. Divide through by v, then separate variables to get $v^{-2}\,dv = y^{-1}\,dy$. Integrate both sides to obtain $\frac{-1}{v} = \ln y + C$, i.e. $v = \frac{-1}{\ln y + C}$. Now, since $v = \frac{dy}{dt}$, we have the equation $\frac{dy}{dt} = \frac{-1}{\ln y + C}$. This equation is also separable: rewrite it as $(\ln y + C)\,dy = -dt$ and integrate both sides (you need parts on the left-hand side) to get $y \ln y - y + Cy = -t + D$.

7. Let $y(t)$ be the amount of sulfuric acid in the tank at time t; we have $y(0) = (10\%)(24) = 2.4$. Since 5% sulfuric acid is pumped into the tank at a rate of 3 L/min, the "rate in" is $(.05)(3) = .15$. Since the tank drains at 4 L/min, the "rate out" is $4y$, so we have the initial value problem

$$\begin{cases} y' = .15 - 4y \\ y(0) = 2.4 \end{cases}$$

Solving this either with integrating factors or undetermined coefficients, we obtain the general solution $y = \frac{.15}{4} + Ce^{-4t}$. The particular solution is obtained by setting $t = 0$ and $y = 2.4$ and solving for C to get $C = 2.4 - \frac{.15}{4}$, making the particular solution $y = \frac{.15}{4} + \left(2.4 - \frac{.15}{4}\right)e^{-4t}$. Therefore at time $t = 3$, the amount of sulfuric acid in the tank is $y(3) = \frac{.15}{4} + \left(2.4 - \frac{.15}{4}\right)e^{-12}$.

Chapter 3

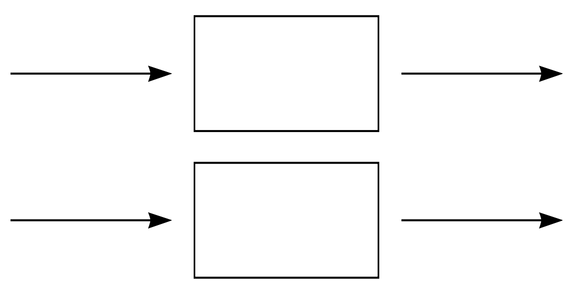
First-order linear systems

3.1 Language for systems of ODEs

Example: Consider an ecosystem with two species: predators (think wolves or lynx) and prey (think rabbits). Let

$$\begin{cases} x(t) = & \text{prey population at time } t \\ y(t) = & \text{predator population at time } t \end{cases}$$

Each of these quantities can be modeled by a compartmental diagram:



This chapter is about first-order systems of ODEs. This means that instead of having one function of t, we have several (say $x = x(t)$ and $y = y(t)$, or $y_1, y_2, ..., y_d$) which satisfy a collection of equations of the form

$$\begin{cases} x'(t) = \phi_1(t, x, y) \\ y'(t) = \phi_2(t, x, y) \end{cases} \qquad \text{Ex: } \begin{cases} x' = 2tx + y - 4t^2 \\ y' = xy + 4y\cos t - 2e^t x \end{cases}$$

or

$$\begin{cases} y_1'(t) = \phi_1(t, y_1, y_2, ..., y_d) \\ y_2'(t) = \phi_2(t, y_1, y_2, ..., y_d) \\ \qquad \vdots \\ y_d'(t) = \phi_d(t, y_1, y_2, ..., y_d) \end{cases} \qquad \text{Ex: } \begin{cases} y_1' = 2y_1 - y_2 + y_3 \\ y_2' = y_1 + 4y_3 + e^t \\ y_3' = 5y_1 + y_2 - 3y_3 - 6t - 1 \end{cases}$$

A **solution** of a system is a collection of functions $(x(t), y(t))$ or $(y_1(t), ..., y_d(t))$ which satisfy *all* the equations in the system. For example, for the system

$$\begin{cases} x' = -y \\ y' = x \end{cases}$$

one solution would be $x = \cos t$, $y = \sin t$:

To establish notation for systems, we borrow language from parametric equations (discussed in Math 230 and/or 320). Recall that if an object is moving in a plane so that the x- and y-coordinates of its position at time t are given by functions $x = x(t)$ and $y = y(t)$, then we combine these into a single function:

$$\mathbf{p}(t) = \overrightarrow{p}(t) = (x(t), y(t)) \qquad \text{(or just } \mathbf{p} = \overrightarrow{p} = (x, y)\text{)}$$

We do the same thing here: given two functions of t, say x and y, we combine these into a single object and write

$$\mathbf{y} = \mathbf{y}(t) = (x(t), y(t)) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

and given d functions of t, say $y_1, ..., y_d$, we combine these into a single object and write

$$\mathbf{y} = \mathbf{y}(t) = (y_1(t), ..., y_d(t)) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_d(t) \end{pmatrix}.$$

In either case $\mathbf{y}$ can be thought of in two (equivalent) ways:

1. as a vector, whose entries are functions of t; or

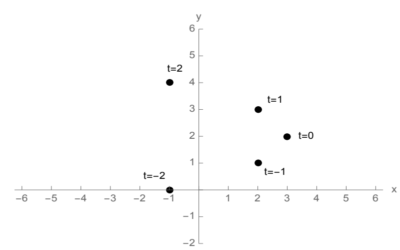
2. as a "$d \times 1$ column matrix", where the entries are listed one above the other.

The reason for writing the entries in a column matrix (as opposed to a row matrix) will become apparent later when we discuss matrix multiplication.

The graph of a set of parametric equations is a curve (which lives in a plane if $d = 2$ and in 3D space if $d = 3$):

Example: $\mathbf{y} = (3 - t^2, t + 2)$ i.e. $\begin{cases} x(t) = 3 - t^2 \\ y(t) = t + 2 \end{cases}$

t	x	y	(x, y)
-2	-1	0	$(-1, 0)$
-1	2	1	$(2, 1)$
0	3	2	$(3, 2)$
1	2	3	$(2, 3)$
2	-1	4	$(-1, 4)$



<u>**Notation:**</u> a first-order system of ODEs has the form

$$\begin{cases} y_1'(t) = \phi_1(t, y_1, y_2, ..., y_d) \\ y_2'(t) = \phi_2(t, y_1, y_2, ..., y_d) \\ \quad\vdots \\ y_d'(t) = \phi_d(t, y_1, y_2, ..., y_d) \end{cases};$$

we shorthand this entire system by writing

$$\mathbf{y}' = \Phi(t, \mathbf{y}) \qquad \text{or} \qquad \overrightarrow{y}' = \Phi(t, \overrightarrow{y}).$$

In this context,

$$\mathbf{y} \text{ means } \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_d(t) \end{pmatrix}; \ \mathbf{y}' \text{ means } \begin{pmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_d'(t) \end{pmatrix}; \text{ and } \Phi(t, \mathbf{y}) \text{ means } \begin{pmatrix} \phi_1(t, y_1, ..., y_d) \\ \phi_2(t, y_1, ..., y_d) \\ \vdots \\ \phi_d(t, y_1, ..., y_d) \end{pmatrix}.$$

In Math 330, the number of equations in the system (which we will denote by d) is always the same as the number of functions we are solving for (also d).

	FIRST-ORDER ODE $y' = \phi(t, y)$	FIRST-ORDER $d \times d$ SYSTEM $\mathbf{y}' = \Phi(t, \mathbf{y})$
METHOD OF APPROXIMATING SOLUTION	Euler's method: $t_{j+1} = t_j + \Delta t$ $y_{j+1} = y_j + \phi(t_j, y_j)\Delta t$	
EXISTENCE/ UNIQUENESS THEOREM	If ϕ and $\frac{\partial \phi}{\partial y}$ are cts in a rectangle containing (t_0, y_0), then the IVP has a unique solution	
WHAT DOES IT MEAN TO BE "LINEAR"?	$y' + p(t)y = q(t)$ (homogeneous if $q = 0$)	
STRUCTURE OF GENERAL SOLUTION OF LINEAR CLASS	**Solution of homogeneous:** Cy_h (subspace of $C^\infty(\mathbb{R})$ spanned by any one nonzero solution) **Solution of general eqn:** $y_p + Cy_h$ where y_p is any one solution	
SIMPLEST MODEL	exponential growth/decay $y' = ry \Rightarrow y(t) = y_0 e^{rt}$	
AUTONOMOUS EQUATIONS (SYSTEMS)	$y' = \phi(y)$ equilibrium solutions: $y = y_0$ where $\phi(y_0) = 0$ (stable if $\phi'(y_0) < 0$; unstable if $\phi'(y_0) > 0$)	
METHODS OF SOLUTION	separation of variables methods for exact equations integrating factors undetermined coefficients	
APPLICATIONS	compartmental models Newtonian mechanics heating and cooling electrical circuits	

3.2 Euler's method for systems

Recall from Chapter 1: Given a first-order IVP

$$\begin{cases} y' = \phi(t,y) \\ y_0 = y(t_0) \end{cases}$$

we can define (t_1, y_1), (t_2, y_2), ... recursively by the formula

$$\begin{cases} t_{j+1} = t_j + \Delta t \\ y_{j+1} = y_j + \phi(t_j, y_j)\Delta t \end{cases}$$

where $\Delta t \neq 0$. The sequence of points obtained by this process approximates the solution $y = y(t)$, and in particular the y_n obtained by this method is an approximation to $y(t_n)$. (The smaller Δt is, or the larger n is, the better the approximation.)

In Chapter 3 we have a system of ODEs, which together with an initial value $\mathbf{y}_0 = \mathbf{y}(t_0)$ gives:

$$\begin{cases} \mathbf{y}' = \Phi(t,\mathbf{y}) \\ \mathbf{y}_0 = \mathbf{y}(t_0) \end{cases}$$

Euler's method adapts in a straight-forward way:

Definition 3.1 *Given a first-order system of IVPs of the form*

$$\begin{cases} \mathbf{y}' = \Phi(t,y) \\ \mathbf{y}(t_0) = \mathbf{y}_0 \end{cases},$$

given a number $t_n \neq t_0$, and given a natural number n, set $\Delta t = \frac{t_n - t_0}{n}$. Define a sequence of points $(t_j, \mathbf{y}_j)$ recursively by setting

$$\begin{cases} t_{j+1} = t_j + \Delta t \\ \mathbf{y}_{j+1} = \mathbf{y}_j + \Phi(t_j, \mathbf{y}_j)\Delta t \end{cases}$$

The $\mathbf{y}_n$ obtained by this method is called **the approximation to $\mathbf{y}(t_n)$ obtained by Euler's method with n steps**. *n is called the* **number of steps** *and Δt is called the* **step size**.

As with single IVPs, the smaller Δt is, the better the approximation.

Example: Consider the system

$$\begin{cases} x' = y - x \\ y' = x + y \end{cases}$$

$$\begin{cases} x(0) = 2 \\ y(0) = 0 \end{cases}$$

Estimate (x, y) when $t = 3$ using 3 steps.

Solution: First, the step size is $\Delta t = \frac{t_n - t_0}{n} = \frac{3-0}{3} = 1$.

Now we use the formula of Euler's method to find ($\mathbf{y}_1$ and t_1) from ($\mathbf{y}_0$ and t_0):

Having found $\mathbf{y}_1 = (x_1, y_1) = (0, 2)$ and $t_1 = 1$, use the formula again:

$$\begin{cases} t_2 = t_1 + \Delta t = 1 + 1 = 2 \\ \mathbf{y}_2 = \mathbf{y}_1 + \Phi(t_1, \mathbf{y}_1)\Delta t = (0,2) + \Phi(1,(0,2)) \cdot 1 & = (0,2) + (2-0, 0+2) \\ & = (0,2) + (2,2) \\ & = (2,4). \end{cases}$$

Therefore $\mathbf{y}_2 = (2, 4)$ and $t_2 = 2$. Use the formula one more time:

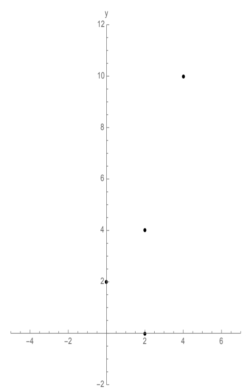
$$\begin{cases} t_3 = t_2 + \Delta t = 2 + 1 = 3 \\ \mathbf{y}_3 = \mathbf{y}_2 + \Phi(t_2, \mathbf{y}_2)\Delta t = (2,4) + \Phi(2,(2,4)) \cdot 1 & = (2,4) + (4-2, 2+4) \\ & = (2,4) + (2,6) \\ & = (4,10). \end{cases}$$

We conclude that $\mathbf{y}(3) \approx (4, 10)$.

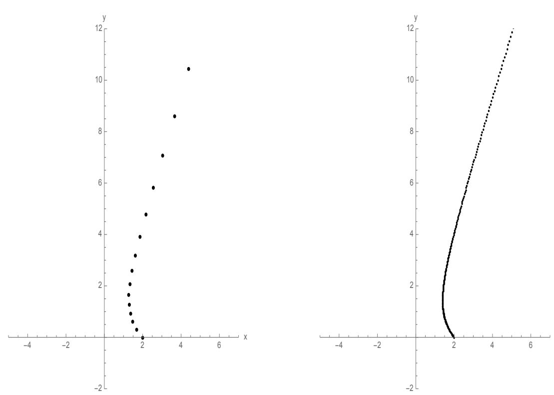
On the previous page, we got the following points from Euler's method with three steps:

$$(t_0, \mathbf{y}_0) = (0, (2, 0)) \quad (t_1, \mathbf{y}_1) = (1, (0, 2)) \quad (t_2, \mathbf{y}_2) = (2, (2, 4)) \quad (t_3, \mathbf{y}_3) = (3, (4, 10))$$

Let's plot these points as if they are coming from a set of parametric equations (i.e. you drop the t and just plot the $\mathbf{y} = (x, y)$ points, labeling them with the values of t):



Suppose you approached the same problem using 20 steps, or 300 steps, rather than 3. You get the following pictures:



It is apparent from these pictures that:

1. both x and y are approaching ∞ as $t \to \infty$;
2. x decreases initially but then starts increasing;
3. y increases at all times.

Euler's method for systems on *Mathematica*

As with Euler's method for single equations, we don't usually do Euler's method for systems by hand. The *Mathematica* notebook `eulermethodsystems.nb` (available on my web page) has the requisite code for implementing Euler's method for systems of 2 or 3 ODEs. Details of the commands are in this file (as with single equations, you need to run commands once to create modules called "`euler2D`" and "`euler3D`" when you start *Mathematica*), but the Euler's method command for systems of 2 ODEs is this:

```
euler2D[{y-x, x+y}, {t, 1, 7}, {x, 3}, {y, -1}, 100]
```

For systems of 3 ODEs, run something like this:

```
euler3D[{y, x t-2, 3x y^2 z}, {t, 1, 7}, {x, 3}, {y, -1}, {z, 2}, 300]
```

The file `eulermethodsystems.nb` also contains commands to produce plots coming from Euler's method.

3.3 Existence and uniqueness of solutions

Recall from Chapter 1: An IVP

$$\begin{cases} y' = \phi(t, y) \\ y_0 = y(t_0) \end{cases}$$

is guaranteed to have a unique solution $y = f(t)$ in some interval containing t_0 if the functions ϕ and $\frac{\partial \phi}{\partial y}$ are both continuous in some rectangle containing (t_0, y_0).

The proof of this statement comes from converting the IVP into an integral equation

$$f(t) = \int_0^t \phi(s, f(s))\, ds$$

and then solving this using Picard approximations: let f_0 be any function satisfying the initial condition and define

$$f_{j+1}(t) = \int_0^t \phi(s, f_j(s))\, ds$$

for each j; the functions f_j converge to a solution f as $t \to \infty$.

Picard's method for systems

All the material above translates directly to systems of ODEs. Consider the IVP

$$\begin{cases} \mathbf{y}' = \Phi(t, \mathbf{y}) \\ \mathbf{y}_0 = \mathbf{y}(t_0) \end{cases}$$

This can be converted into a system of integral equations by assuming $\mathbf{y} = \mathbf{f}(t)$:

$$\begin{cases} f_1(t) = \int_0^t \phi_1(s, \mathbf{f}(s))\, ds \\ \quad\vdots \qquad\qquad \vdots \\ f_d(t) = \int_0^t \phi_d(s, \mathbf{f}(s))\, ds \end{cases}$$

Choose any d functions satisfying the initial conditions; put them into a vector and call them $\mathbf{f}_0 = (f_{0,1}, f_{0,2}, ..., f_{0,d})$. Then, the Picard approximations are given recursively by defining

$$\mathbf{f}_{j+1} = (f_{j+1,1}, f_{j+1,2}, ..., f_{j+1,d})$$

in terms of $\mathbf{f}_j$ as follows:

$$\begin{cases} f_{j+1,1}(t) = \int_0^t \phi_1(s, \mathbf{f}_j(s))\, ds \\ f_{j+1,2}(t) = \int_0^t \phi_2(s, \mathbf{f}_j(s))\, ds \\ \quad\vdots \qquad\qquad \vdots \\ f_{j+1,d}(t) = \int_0^t \phi_d(s, \mathbf{f}_j(s))\, ds \end{cases}$$

Under certain reasonable hypotheses, you can show that each $f_{j,k}$ converges as $j \to \infty$ to a function f_k. The limit $\mathbf{f} = (f_1, f_2, ..., f_d)$ is a solution to the system, and this $\mathbf{f}$ is the only solution of the system. We won't actually do this procedure in this class, but you should be aware of the conclusion of all this due to its theoretical importance. That conclusion is:

Theorem 3.2 (Existence/uniqueness for first-order systems) *Suppose that* $\Phi = (\phi_1, \phi_2, ..., \phi_d)$ *is a function from* $\mathbb{R}^{d+1}$ *to* $\mathbb{R}^d$ *such that for all* i *and* j*, both* ϕ_j *and* $\frac{\partial \phi_j}{\partial y_i}$ *are continuous in some rectangular box in* $\mathbb{R}^{d+1}$ *containing* $(t_0, \mathbf{y}_0)$*. Then for some interval* I *of values of* t *containing* t_0*, the system of initial value problems*

$$\begin{cases} \mathbf{y}' = \Phi(t, \mathbf{y}) \\ \mathbf{y}_0 = \mathbf{y}(t_0) \end{cases}$$

has one and only one solution, which is of the form $\mathbf{y} = \mathbf{f}(t) = (f_1(t), f_2(t), ..., f_d(t))$ *for functions* $f_1, ..., f_d : \mathbb{R} \to \mathbb{R}$.

3.4 Matrices and matrix operations

In linear algebra, you consider *linear* systems of numerical equations. These look like

$$\begin{cases} 2x - y + 4z = 4 \\ x + 3y - 2z = -2 \\ 5x - 2y + z = 0 \end{cases}$$

(with maybe a different number of variables and/or equations).

To study these systems, you rely heavily on the theory of *matrices* and *linear transformations*. The same theory works to describe *linear* systems of ODEs, so we need to cover some linear algebra machinery here.

Definition 3.3 *Given positive integers m and n, an $m \times n$* **matrix** *is an array of numbers (or functions) a_{ij} where $1 \leq i \leq m$ and $1 \leq j \leq n$. We arrange the entries of the matrix in a rectangle as follows:*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{pmatrix}$$

The set of $m \times n$ matrices with entries in $\mathbb{R}$ is denoted $M_{mn}(\mathbb{R})$. The set of $m \times n$ matrices with entries that are functions of some variable t is denoted $M_{mn}(C^\infty)$. Two matrices are **equal** *if they are the same size and if all their entries coincide, i.e. $A = B$ if they are both $m \times n$ and if $a_{ij} = b_{ij}$ for all i, j.*

In particular, a_{ij} is the entry of A in the i^{th} row and j^{th} column. m is the number of rows of A; n is the number of columns of A.

In Math 330, for the most part we only care about matrices which have the same number of rows as columns:

Definition 3.4 *A matrix is called* **square** *if it has the same number of rows as columns. The set of square $n \times n$ matrices with entries in $\mathbb{R}$ is denoted $M_n(\mathbb{R})$ (as opposed to $M_{nn}(\mathbb{R})$).*

Example:

$$A = \begin{pmatrix} 1 & 6 \\ -4 & 5 \end{pmatrix} \text{ is square} \qquad B = \begin{pmatrix} 1 & 5 & 7 \end{pmatrix} \text{ is not square}$$

Definition 3.5 *Given a matrix A, the* **diagonal entries** *of A are the numbers a_{11}, a_{22}, a_{33}, ...a_{nn}. A matrix A is called* **diagonal** *if it is square and all of its nondiagonal entries are zero.*

Example:

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \text{ is diagonal; the diagonal entries are } 2, -1, 0 \text{ and } 4$$

Definition 3.6 *A square matrix is called* **upper triangular** *(abbreviated* **upper** Δ*) if all its entries below its diagonal are zero. A matrix is called* **lower triangular** *(abbreviated* **lower** Δ*) if all the entries above its diagonal are zero. A matrix is called* **triangular** *if it is either lower triangular or upper triangular.*

Example:

$$A = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix} \qquad B = \begin{pmatrix} -7 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & -3 & 4 \end{pmatrix}$$

Note: diagonal matrices are both upper and lower triangular.

Definition 3.7 *The* **trace** *of a matrix, denoted $tr(A)$, is the sum of the diagonal entries of A.*

Example:

$$A = \begin{pmatrix} 7e^t & \cos t & 4t \\ 1 & -2t^2 & \sin 3t \\ 0 & -t & -e^{-t} \end{pmatrix} \qquad tr(A) = 7e^t - 2t^2 - e^{-t}.$$

Definition 3.8 *The $n \times n$* **identity matrix***, denoted I or I_n, is the diagonal $n \times n$ matrix with all diagonal entries equal to 1.*

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

In general, a blank entry in a matrix means that entry is zero.

Example: For the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -6 \\ 0 & 0 & 4 \end{pmatrix}$$

1. Give the size of the matrix.
2. Write down the $(3, 2)-$entry of the matrix.
3. Write down the diagonal entries of the matrix.
4. What is the trace of the matrix?
5. Is the matrix diagonal? upper triangular? lower triangular?

Addition, scalar multiplication and differentiation of matrices

You can add two matrices of the same size and multiply a matrix by a real number, by performing these operations entry-by-entry.

Note: You can only add two matrices of the same size.

Example: Let

$$A = \begin{pmatrix} 2 & -1 & -3 \\ 3 & 0 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 2 & 2 \\ 3 & 1 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 0 \\ -1 & -5 \end{pmatrix}$$

You also differentiate matrices (whose entries are functions) term-by-term:

Definition 3.9 *Given $A \in M_{mn}(C^\infty)$, we define the* **derivative** *of A, denoted*

$$\frac{dA}{dt} = \frac{d}{dt}(A) = A'(t) = \dot{A} = A',$$

to be the $m \times n$ matrix obtained from A by differentiating its entries term-by-term.

Example: $A = \begin{pmatrix} e^t & \cos t \\ 4e^{2t} & \sin t \end{pmatrix}$

Example: $\mathbf{y} = \mathbf{y}(t) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$

Matrix multiplication

There is another operation one can perform on matrices. The importance of this operation will be seen later; for now we simply define it.

Definition 3.10 *Given matrices $A \in M_{mn}(F)$ and $B \in M_{pq}(F)$, if $n = p$ then we can define the* **product** *AB, which is an $m \times q$ matrix AB defined entrywise by setting*

$$(ab)_{ij} = \sum_{k=1}^{n(=p)} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}.$$

(If $n \neq p$, AB is undefined.)

Note:

- If A is a square matrix, we write A^2 for AA, A^3 for AAA, etc.
- If A isn't square, then A^2 is undefined.
- In general matrix multiplication is **not commutative**: $AB \neq BA$ most of the time, even if both products are defined.

Example: Let

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix}; \quad B = \begin{pmatrix} -2 & 1 \\ 1 & 3 \end{pmatrix}; \quad C = \begin{pmatrix} 5 & -1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}; \quad \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Compute each of the following quantities, or state that they are undefined (with justification).

(a) AB

(b) BA

(c) A^2

(d) B^2

(e) $B\mathbf{v}$

Theorem 3.11 (Properties of elementary matrix operations) *Let A, B, C be matrices, let I be the identity matrix of the appropriate size and let $k \in \mathbb{R}$. Then,* so long as everything is defined*, we have:*

1. $IA = A$ *and* $BI = B$*;*
2. $A(BC) = (AB)C$*;*
3. $k(AB) = (kA)B = A(kB)$*;*
4. $A(B + C) = AB + AC$ *and* $(A + B)C = AC + BC$*.*

A useful fact we will need later is the following:

Theorem 3.12 (Powers of a diagonal matrix) *If $A \in M_n(\mathbb{R})$ is a diagonal matrix, then A^r is also diagonal for each r, and the $(j, j)-$entry of A^r is $(a_{j,j})^r$.*

Example: Let $A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Then $A^2 = \begin{pmatrix} (-2)^2 & 0 & 0 \\ 0 & 3^2 & 0 \\ 0 & 0 & (-1)^2 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

WARNING: Computing the powers of a non-diagonal matrix is much harder (see the example of B^2 on the previous page).

Reasons that matrix multiplication is defined the way that it is (and why vectors are thought of as columns rather than rows): Consider this system of three equations in three variables:

$$\begin{cases} 4x - 2y + z &= 7 \\ 8x + 3y + 7z &= 12 \\ x + y - 4z &= -4 \end{cases}$$

More generally, given any system of d numerical equations in d variables, you can write down an $d \times d$ matrix A, a $d \times 1$ matrix $\mathbf{b}$, so that the equation, in matrix multiplication language, is

$$A\mathbf{x} = \mathbf{b}$$

(the goal being to solve for $\mathbf{x} = (x_1, x_2, ..., x_d) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}$).

Recall from Chapter 1: a function $T : V \to V$ is a *linear operator* if $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ and $T(r\mathbf{v}) = rT(\mathbf{v})$ for all $\mathbf{v}, \mathbf{w} \in V$, all $r \in \mathbb{R}$.

Another reason for the definition of matrix multiplication is that every linear operator on $\mathbb{R}^d$ is multiplication by a $d \times d$ square matrix:

Theorem 3.13 (Characterization of linear operators on $\mathbb{R}^d$) *For any matrix $A \in M_d(\mathbb{R})$, the function $T : \mathbb{R}^d \to \mathbb{R}^d$ described by*

$$T(\mathbf{v}) = A\mathbf{v}$$

is a linear operator on $\mathbb{R}^d$.

Conversely, given any linear operator $T : \mathbb{R}^d \to \mathbb{R}^d$, there is a matrix $A \in M_d(\mathbb{R})$ such that

$$T(\mathbf{v}) = A\mathbf{v}$$

for all $\mathbf{v} \in \mathbb{R}^d$.

PROOF To show that every function of the form $T(\mathbf{x}) = A\mathbf{x}$ is linear is pretty easy:

$$T(\mathbf{v}+\mathbf{w}) = A(\mathbf{v}+\mathbf{w}) = A\mathbf{v} + A\mathbf{w} = T(\mathbf{v}) + T(\mathbf{w}) \Rightarrow T \text{ preserves addition;}$$

$$T(r\mathbf{v}) = A(r\mathbf{v}) = rA\mathbf{v} = rT(\mathbf{v}) \Rightarrow T \text{ preserves scalar multiplication.}$$

Since T preserves addition and scalar multiplication, it is linear.

Conversely, to show that every linear operator on $\mathbb{R}^d$ is matrix multiplication is given by matrix multiplication is harder. See Theorem 5.5 of my Math 322 lecture notes. □

Recall that a "linear" equation is one of the form $T(\mathbf{x}) = \mathbf{b}$ where T is a linear operator. In light of the previous theorem, we now know:

Theorem 3.14 (Characterization of linear numerical systems) *Every linear system of d numerical equations in d variables is of the form*

$$A\mathbf{x} = \mathbf{b}$$

where A is an $d \times d$ matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^d$.

How would you solve such a system?

Inverses and determinants

We know what it means to add, subtract, and/or multiply two matrices. What does it mean to *divide* one matrix by another?

For real numbers, two things are true about division:

1.

2.

So to define "division" of matrices, we really only need to define the "reciprocal" of a matrix. However, with matrices we use the word "inverse" rather than "reciprocal":

Definition 3.15 *A matrix $A \in M_n(\mathbb{R})$ is called* **invertible** *if there is another matrix $A^{-1} \in M_n(\mathbb{R})$ and called an* **inverse (matrix)** *of A, such that*

$$AA^{-1} = I \text{ and } A^{-1}A = I.$$

Remark: Non-square matrices are never invertible.

Theorem 3.16 (Properties of inverses) *Let $A, B \in M_n(\mathbb{R})$ be invertible. Then*

1. *A has only one inverse.*
2. *$(A^{-1})^{-1} = A$.*
3. *$(AB)^{-1} = B^{-1}A^{-1}$.*

In linear algebra, you learn a lot about inverses (including how to compute them in general). In Math 330, you need to memorize these two formulas for inverses of 1×1 and 2×2 matrices:

Theorem 3.17 (1 × 1 inverses) *Let $A = (a)$. Then A is invertible if and only if $a \neq 0$, in which case*

$$A^{-1} = \left(\frac{1}{a}\right).$$

PROOF $AA^{-1} = (a)\left(\frac{1}{a}\right) = (1) = I$ and $A^{-1}A = \left(\frac{1}{a}\right)(a) = (1) = I$. □

Theorem 3.18 (2 × 2 inverses) *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Then* A *is invertible if and only if* $ad - bc \neq 0$, *in which case*

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}.$$

PROOF If you actually matrix multiply A and A^{-1} in either order, you will get I. Details are in Theorem 5.32 of my linear algebra notes. $\square$

Example: Find A^{-1} if $A = \begin{pmatrix} 3 & 7 \\ -2 & 1 \end{pmatrix}$.

To compute the inverse of a 3×3 or larger (square) matrix, use a computer (there is a way to do it by hand that you learn in linear algebra, but you don't need that in Math 330). See the next section for the appropriate *Mathematica* commands.

We saw above that in order for a 1×1 or 2×2 matrix to be invertible, there is a formula in terms of its entries that has to be nonzero:

Size of matrix	Matrix	what has to be nonzero for A to be invertible
1×1	$A = (a)$	a
2×2	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$	$ad - bc$
3×3	$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$	?
$n \times n$		?

In linear algebra, you learn that for any square matrix A, there is a magic number, defined as a formula of the terms of the entries of that matrix. This magic number is called the **determinant** of A, is denoted $\det A$ or $\det(A)$, and tells you whether or not A is invertible:

Theorem 3.19 *A square matrix is* A *is invertible if and only if* $\det A \neq 0$.

For a 1×1 matrix $A = (a)$, $\det A = a$.

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det A = ad - bc$.

Here is the formula for 3×3 matrices:

Theorem 3.20 *Let* $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in M_3(\mathbb{R})$. *Then*

$$\det A = aei + cdh + bfg - bdi - ceg - afh.$$

The proof of this theorem is beyond the scope of this course.

The formula in the preceding theorem is hard to remember, but there is a trick. Given a 3×3 matrix, copy the first two columns to the right of the matrix:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

Then multiply along the diagonals, and add the "upper" and "lower" products. The determinant is the bottom sum minus the top sum.

<u>**WARNING**</u>**: This technique does not work for 4×4 and larger matrices.** For large matrices, find the determinant using a computer (see the next section).

Example: $A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 1 \end{pmatrix}$. Find $\det A$.

A major reason why we care about determinants in Math 330 is that an appropriate determinant tells us whether or not a system of linear equations has a unique solution:

Theorem 3.21 *The system of d linear, numerical equations in d variables*

$$A\mathbf{x} = \mathbf{b} \quad \Leftrightarrow \quad \begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1d}x_d = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2d}x_d = b_2 \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{d1}x_1 + a_{d2}x_2 + ... + a_{dd}x_d = b_d \end{cases}$$

has a unique solution if and only if $\det A \neq 0$, *in which case the unique solution is* $\mathbf{x} = A^{-1}\mathbf{b}$.

PROOF If $\det A \neq 0$, then A is invertible, i.e. A^{-1} exists. Start with the matrix equation

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ \Leftrightarrow A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ \Leftrightarrow I\mathbf{x} &= A^{-1}\mathbf{b} \\ \Leftrightarrow \mathbf{x} &= A^{-1}\mathbf{b}. \ \square \end{aligned}$$

3.5 Matrix operations on *Mathematica*

Defining a matrix: To store a matrix whose entries are numbers as a variable, there are two methods.

1. Type in the matrix (carefully) using braces to separate the entries. For example, to save the matrix

   $$\begin{pmatrix} 2 & 4 & 7 \\ -5 & 3 & 1 \end{pmatrix}$$

   as A, execute

   ```
   A = {{2,4,7}, {-5,3,1}}
   ```

   Note that the entries are separated by commas, every row of the matrix needs braces around it, and the entire matrix needs braces around it.

To type in a column vector, you need only one set of braces, so if you execute `b = {1,2,3}` this defines the column vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

2. Use the Basic Math Assistant Pallette. Click "Pallettes" and "Basic Math Assistant", then on the Basic Math Assistant click the fourth tab under "Basic Commands" that looks like a matrix. In the *Mathematica* notebook, type `A=`, then click the large button that looks like a matrix, then click "AddRow" or "AddColumn" until the matrix is the appropriate size. Click in each box of the matrix and type in the appropriate numbers. For example, your command to define the A above would look like

   $$\texttt{A = } \begin{pmatrix} \texttt{2} & \texttt{4} & \texttt{7} \\ \texttt{-5} & \texttt{3} & \texttt{1} \end{pmatrix}$$

To define a matrix whose entries are functions, you must define the matrix as a function: type `A[t_] =` , then type the matrix and execute. For example:

```
A[t_] = {{Cos[t], Sin[t]}, {E^t, E^(-t)}}
```

defines the matrix $A = A(t) = \begin{pmatrix} \cos t & \sin t \\ e^t & e^{-t} \end{pmatrix}$. To call this matrix later, type `A[t]` (no underscore).

Matrix multiplication: To multiply two matrices in *Mathematica*, you need a period between the matrices. For example, after defining matrices A and B, you can compute the matrix product by

```
A.B
```

The output you will get won't look like a matrix; to make it look like a matrix you can type

```
A.B //MatrixForm
```

For matrix powers, you will need to type `A.A` rather than `A^2` (the command `A^2` just squares each entry of A). For a larger matrix power (say A^{100}), run the following: `MatrixPower[A,100] //MatrixForm`

Other matrix operations: Once you have saved a matrix as a letter or string, you can perform standard operations on it as follows (add `//MatrixForm` to the end of the command to make the output look like a matrix):

1. For the trace of A, execute `Tr[A]`.
2. To multiply A by a scalar (say 5), execute `5 A`.
3. To add two matrices (say A and B), execute `A + B`.

4. To get the i, j entry of a matrix, use double braces: execute `A[[2,3]]` (to get the $2, 3-$entry).
5. To call the $n \times n$ identity matrix, use a command like `IdentityMatrix[4]` (this generates the 4×4 identity matrix).
6. For the inverse of a matrix, type `Inverse[A]`.
7. For the determinant of a matrix, type `Det[A]`.
8. To take the derivative of a matrix of functions, execute `A'[t]`.

3.6 First-order linear systems of ODEs

We can adapt our definition of "linear" from Chapter 1 from this setting. Recall that an ODE is called **linear** if there are functions $p_0, p_1, ..., p_n$ (with $p_n \neq 0$) and q such that the equation has the form

Let's generalize this to systems. The key idea is that instead of multiplying each derivative of y by a single function (like p_0, p_1 or p_n), we have to multiply a vector of functions (like $\mathbf{y}' = (y_1', y_2', ..., y_d')$) by an object that will produce another vector of functions. The most general object that does this is a

This means we can describe linear systems of ODEs as follows:

Definition 3.22 *An n^{th}* **order, linear** *$d \times d$* **system** *of ODEs is any equation of the form*

$$A_n\mathbf{y}^{(n)} + A_{n-1}\mathbf{y}^{(n-1)} + ... + A_1\mathbf{y}' + A_0\mathbf{y} = \mathbf{q}$$

where $A_n, A_{n-1}, ..., A_1, A_0 \in M_d(C^\infty)$ are n $d \times d$ matrices of functions with A_n invertible for all t, where $\mathbf{q} = \mathbf{q}(t) = (q_1, q_2, ..., q_d)$ is a list of d functions of t, and where $\mathbf{y} = (y_1, ..., y_d)$ is a list of d functions of t (the goal is to find $\mathbf{y}$).

If all the entries of the A_j are constants, we say the system is **constant-coefficient**. *If every entry of $\mathbf{q}$ is 0 (i.e. $\mathbf{q} = \mathbf{0}$), we say the system is* **homogeneous**.

Example:

$$\begin{cases} t^3x' + (\sin t)y' & +3x - e^t y & = \cos 2t \\ 5tx' - 2y' & +4t^2 x & = e^{-t} \end{cases}$$

Example:

$$\begin{cases} 4x'' - 3y'' + z'' & +2x' + 3y' - 8z' & -x + y - 3z & = 0 \\ -2x'' + 5y'' - z'' & -3x' + 7y' & +y - 4z & = 0 \\ x'' & -6y' & +3x - y - 2z & = 0 \end{cases}$$

3.7 The structure of the solution set of a linear system

Definition 3.23 *Given the* $d \times d$ *linear system*

$$A_n \mathbf{y}^{(n)} + A_{n-1}\mathbf{y}^{(n-1)} + ... + A_1\mathbf{y}' + A_0\mathbf{y} = \mathbf{q},$$

the **corresponding homogeneous system** *is the system*

$$A_n \mathbf{y}^{(n)} + A_{n-1}\mathbf{y}^{(n-1)} + ... + A_1\mathbf{y}' + A_0\mathbf{y} = \mathbf{0}.$$

Theorem 3.24 *Suppose* $\mathbf{y}$ *and* $\widehat{\mathbf{y}}$ *are two solutions of the* $d \times d$ *linear system*

$$A_n \mathbf{y}^{(n)} + A_{n-1}\mathbf{y}^{(n-1)} + ... + A_1\mathbf{y}' + A_0\mathbf{y} = \mathbf{q}.$$

Then $\mathbf{y} - \widehat{\mathbf{y}}$ *is a solution of the corresponding homogeneous system*

$$A_n \mathbf{y}^{(n)} + A_{n-1}\mathbf{y}^{(n-1)} + ... + A_1\mathbf{y}' + A_0\mathbf{y} = \mathbf{0}.$$

(Compare this result with Theorem 2.10.)

PROOF This is a direct calculation:

$$\begin{aligned}
&A_n(\mathbf{y}-\widehat{\mathbf{y}})^{(n)} + A_{n-1}(\mathbf{y}-\widehat{\mathbf{y}})^{(n-1)} + ... + A_1(\mathbf{y}-\widehat{\mathbf{y}})' + A_0(\mathbf{y}-\widehat{\mathbf{y}})\\
&= \left[A_n\mathbf{y}^{(n)} + A_{n-1}\mathbf{y}^{(n-1)} + ... + A_1\mathbf{y}' + A_0\mathbf{y}\right]\\
&\qquad - \left[A_n\widehat{\mathbf{y}}^{(n)} + A_{n-1}\widehat{\mathbf{y}}^{(n-1)} + ... + A_1\widehat{\mathbf{y}}' + A_0\widehat{\mathbf{y}}\right]\\
&= \mathbf{q} - \mathbf{q}\\
&= \mathbf{0}. \ \square
\end{aligned}$$

Consequence: if $\mathbf{y}_p$ is any one solution of the $d \times d$ system, then for any solution $\mathbf{y}$ of the system, we have

Example: Suppose some 2×2 system of (non-homogeneous) equations has the following solution of its corresponding homogeneous system:

$$\begin{aligned}
\mathbf{y}_h(t) &= C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{4t} \sin t \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} C_1 e^t + C_2 e^{4t} \sin t \\ 2C_1 e^t + 3C_2 e^{4t} \sin t \end{pmatrix}
\end{aligned}$$

$$\text{i.e.} \quad \begin{cases} x_h(t) = C_1 e^t + C_2 e^{4t} \sin t \\ y_h(t) = 2C_1 e^t + 3C_2 e^{4t} \sin t \end{cases}.$$

If one particular solution of the system is

$$\mathbf{y}_p(t) = \begin{pmatrix} \sin 2t \\ 3\cos 2t \end{pmatrix} \quad \text{i.e.} \quad \begin{cases} x_p(t) = \sin 2t \\ y_p(t) = 3\cos 2t \end{cases},$$

describe all solutions of the system.

The solution set of the homogeneous equation

Recall from Chapter 2 the idea of a subspace:

Definition 3.25 *Let V be a vector space and let $W \subseteq V$. We say W is a* **subspace (of V)** *if*

1. *W is closed under addition, i.e. for any two vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ in W, $\mathbf{w}_1 + \mathbf{w}_2 \in W$; and*

2. *W is closed under scalar multiplication, i.e. for any vector $\mathbf{w} \in W$ and any scalar r, $r\mathbf{w} \in W$.*

In Chapter 2, we showed that the solution to any first-order linear, homogeneous ODE is a subspace of $C^\infty(\mathbb{R})$, spanned by a single nonzero solution:

Definition 3.26 *Let V be a vector space and let $\mathbf{v} \in V$ be a vector. The* **span** *of $\mathbf{v}$, denoted $Span(\mathbf{v})$, is the set of linear multiples of $\mathbf{v}$:*

$$Span(\mathbf{v}) = \{c\mathbf{v} : c \in \mathbb{R}\}$$

If $W \subseteq V$ is such that $W = Span(\mathbf{v})$, we say W is **spanned** *by $\mathbf{v}$.*

We now show the analogous property for first-order linear, homogeneous systems. (The difference is that the solution will be a subspace spanned by d solutions.) First, a definition:

Definition 3.27 *Let V be a vector space and let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_d \in V$ be vectors. The* **span** *of $\mathbf{v}_1, ..., \mathbf{v}_d$, denoted $Span(\mathbf{v}_1, ..., \mathbf{v}_d)$, is the set of linear combinations of the $\mathbf{v}_j$:*

$$Span(\mathbf{v}_1, ..., \mathbf{v}_d) = \left\{ \sum_{j=1}^{d} c_j \mathbf{v}_j : c_j \in \mathbb{R} \right\} = \{c_1\mathbf{v}_1 + ... + c_d\mathbf{v}_d : c_j \in \mathbb{R}\}$$

If $W \subseteq V$ is such that $W = Span(\mathbf{v}_1, ..., \mathbf{v}_d)$, we say W is **spanned** *by $\mathbf{v}_1, ..., \mathbf{v}_d$.*

In Math 330, we care about spans where the vectors are functions:

Example 1: $V = C^\infty(\mathbb{R})$; $f_1(t) = \sin t$; $f_2(t) = \cos t$.

Example 2: $V = C^\infty(\mathbb{R})$; $f_1(t) = t^2$; $f_2(t) = 3t^2$.

Example 3: $V = C^\infty(\mathbb{R})$; $f_1(t) = \sin^2 t$; $f_2(t) = \cos^2 t$; $f_3(t) = 1$.

Example 4: $V = (C^\infty(\mathbb{R}))^2$ (i.e. V is the set of ordered pairs of differentiable functions);

$$\mathbf{f}_1(t) = \begin{pmatrix} 2\sin t \\ \sin t + \cos t \end{pmatrix}; \mathbf{f}_2(t) = \begin{pmatrix} 2\sin t - \cos t \\ \cos t \end{pmatrix}$$

Notice that in Examples 2 and 3, you could drop one of the functions f_1, f_2, f_3 without changing the span. This is essentially because

Question: Under what circumstances can you do this in general? In other words, given a list of d functions $f_1, ..., f_d$, is there an identity relating the functions?

Definition 3.28 *Given vectors* $\mathbf{v}_1, ...\mathbf{v}_d$ *in vector space* V*, we say the set of vectors is* **linearly independent** *if the only constants* $c_1, ..., c_d$ *which make*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_d\mathbf{v}_d = 0$$

are $c_1 = c_2 = ... = c_d = 0$.

We say the set of vectors is **linearly dependent** *if there are constants* $c_1, ..., c_d$*, not all zero, which make*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_d\mathbf{v}_d = 0.$$

In linear algebra, you learn a lot about what makes vectors linearly independent or linearly dependent in general. Among other things, you learn:

Theorem 3.29 *Given a list of vectors* $\mathbf{v}_1, ...\mathbf{v}_d$ *in vector space* V*:*

1. *If the vectors are linearly independent, then you cannot remove any of the vectors from the list without changing the span of the list.*
2. *If the vectors are linearly dependent, then there is at least one vector which can be removed from the list without changing the span.*

So in some sense, a list of linearly independent vectors is a "minimal" list of vectors which span a particular subspace. With that in mind, we make the following definition:

Definition 3.30 *If* W *is a subspace of* V *spanned by* d *linearly independent vectors, then we say* W *is a subspace of* **dimension** d*, and we write* $\dim W = \dim(W) = d$*. In this situation, any collection of* d *linearly independent vectors in* d *is called a* **basis** *of* W*.*

A reasonable question is to ask whether a vector space or subspace can be of two different dimensions: more precisely, can a space W be spanned by 3 linearly independent vectors, and also be spanned by 4 linearly independent vectors? Linear algebra theory tells us that the answer is **NO**:

Theorem 3.31 *If* V *is a vector space or subspace with* $\dim V = d$*, then*

1. *Any collection of* d *linearly independent vectors must also span* V*, and hence be a basis of* V*.*
2. *Any collection of* d *vectors which span* V *must also be linearly independent, hence must be a basis of* V*.*

So if $\mathbf{v}_1, ..., \mathbf{v}_d$ are linearly independent vectors in a d-dimensional subspace W, then

$$W = Span(\mathbf{v}_1, ..., \mathbf{v}_d) = \{C_1\mathbf{v}_1 + ... + C_d\mathbf{v}_d : C_j \in \mathbb{R} \text{ for all } j\}.$$

The Wronskian

In Math 330, we only care about whether a set of functions (as opposed to a set of more general vectors) is linearly independent or not, and there is an easy test for that:

Definition 3.32 *Given d functions $f_1, ..., f_d \in C^\infty(\mathbb{R})$, the* **Wronskian** *of the functions is the function*

$$W = W(t) = W(f_1, ..., f_d)(t) = \det \begin{pmatrix} f_1(t) & f_2(t) & \cdots & f_d(t) \\ f_1'(t) & f_2'(t) & \cdots & f_d'(t) \\ f_1''(t) & f_2''(t) & \cdots & f_d''(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(d-1)}(t) & f_2^{(d-1)}(t) & \cdots & f_d^{(d-1)}(t) \end{pmatrix}.$$

Given d elements of $(C^\infty(\mathbb{R}))^d$, namely $\mathbf{f}_1 = (f_{11}, f_{12}, ..., f_{1d})$, $\mathbf{f}_2 = (f_{21}, ..., f_{d2})$, ..., $\mathbf{f}_d = (f_{d1}, ..., f_{dd})$, the **Wronskian** *of these functions is the function*

$$W = W(t) = W(\mathbf{f}_1, ..., \mathbf{f}_d)(t) = \det \begin{pmatrix} f_{11}(t) & f_{21}(t) & \cdots & f_{d1}(t) \\ f_{12}(t) & f_{22}(t) & \cdots & f_{d2}(t) \\ f_{13}(t) & f_{23}(t) & \cdots & f_{d3}(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_{1d}(t) & f_{2d}(t) & \cdots & f_{dd}(t) \end{pmatrix}.$$

Example: Find the Wronskian of $f_1(t) = \sin t$ and $f_2(t) = \cos t$.

Example: Find the Wronskian of $f_1(t) = \sin^2 t$, $f_2(t) = \cos^2 t$ and $f_3(t) = 1$.

Example: Find the Wronskian of $\mathbf{f}_1(t) = (e^t, e^{2t})$ and $\mathbf{f}_2(t) = (2e^t, 4e^{2t} + e^t)$.

Example: Find the Wronskian of $\mathbf{f}_1(t) = (2\sin t, \cos t)$ and $\mathbf{f}_2(t) = (4\sin t, 2\cos t)$.

The importance of the Wronskian is seen in the following theorem:

Theorem 3.33 *Given functions* $f_1, ..., f_d \in C^\infty(\mathbb{R})$, *if the Wronskian of the functions* $W(f_1, ..., f_d)(t) \neq 0$ *for at least one value of* t, *the functions are linearly independent.*

Given functions $\mathbf{f}_1, ..., \mathbf{f}_d \in (C^\infty(\mathbb{R}))^d$, *the Wronskian* $W(\mathbf{f}_1, ..., \mathbf{f}_d)(t) \neq 0$ *for at least one value of* t, *then* $\mathbf{f}_1, ..., \mathbf{f}_d$ *are linearly independent.*

PROOF Suppose that there are constants $c_1, ..., c_d$ such that

$$c_1 f_1(t) + c_2 f_2(t) + ... + c_d f_d(t) = 0.$$

Repeatedly differentiating both sides of this equation, we obtain the system

$$\begin{cases} c_1 f_1(t) + c_2 f_2(t) + ... + c_d f_d(t) = 0 \\ c_1 f_1'(t) + c_2 f_2'(t) + ... + c_d f_d'(t) = 0 \\ c_1 f_1''(t) + c_2 f_2''(t) + ... + c_d f_d''(t) = 0 \\ \vdots \\ c_1 f_1^{(d)}(t) + c_2 f_2^{(d)}(t) + ... + c_d f_d^{(d)}(t) = 0 \end{cases}$$

This system of linear equations has at least one solution: $(c_1, c_2 ..., c_d) = (0, 0, ..., 0)$. By Theorem 3.21, this is the only solution (i.e. the functions are linearly independent) if $W(t) \neq 0$. This proves the first statement; the second statement has a similar proof. □

Theorem 3.34 *Given any linear, homogeneous* $d \times d$ *system of ODEs*

$$A_n \mathbf{y}^{(n)} + A_{n-1}\mathbf{y}^{(n-1)} + ... + A_1 \mathbf{y}' + A_0 \mathbf{y} = \mathbf{0},$$

the set of solutions of this ODE is a subspace of $(C^\infty(\mathbb{R}))^d$.

PROOF First, we show the solution set is closed under addition. Let $\mathbf{y}$ and $\widehat{\mathbf{y}}$ be solutions. Then, plugging $\mathbf{y} + \widehat{\mathbf{y}}$ into the system, we get

$$\begin{aligned} & A_n(\mathbf{y} + \widehat{\mathbf{y}})^{(n)} + A_{n-1}(\mathbf{y} + \widehat{\mathbf{y}})^{(n-1)} + ... + A_1(\mathbf{y} + \widehat{\mathbf{y}})' + A_0(\mathbf{y} + \widehat{\mathbf{y}}) \\ & = \left[A_n \mathbf{y}^{(n)} + A_{n-1}\mathbf{y}^{(n-1)} + ... + A_1 \mathbf{y}' + A_0 \mathbf{y}\right] \\ & \qquad + \left[A_n \widehat{\mathbf{y}}^{(n)} + A_{n-1}\widehat{\mathbf{y}}^{(n-1)} + ... + A_1 \widehat{\mathbf{y}}' + A_0 \widehat{\mathbf{y}}\right] \\ & = \mathbf{0} + \mathbf{0} \\ & = \mathbf{0}. \ \square \end{aligned}$$

Therefore the solution set is closed under addition.

Next, we show the solution set is closed under scalar multiplication: let $\mathbf{y}$ be a solution and let $r \in \mathbb{R}$. Then, by plugging in $r\mathbf{y}$ to the system, we get

$$\begin{aligned}
&A_n(r\mathbf{y})^{(n)} + A_{n-1}(r\mathbf{y})^{(n-1)} + ... + A_1(r\mathbf{y})' + A_0(r\mathbf{y}) \\
&= r\left[A_n\mathbf{y}^{(n)} + A_{n-1}\mathbf{y}^{(n-1)} + ... + A_1\mathbf{y}' + A_0\mathbf{y}\right] \\
&= r\mathbf{0} \\
&= \mathbf{0}.
\end{aligned}$$

Therefore the solution set is closed under scalar multiplication, so it is a subspace. □

Knowing that the solution set of the homogeneous is a subspace, it is natural to ask what its dimension is:

Theorem 3.35 *Given any linear, first-order, homogeneous $d \times d$ system of ODEs*

$$A_1\mathbf{y}' + A_0\mathbf{y} = \mathbf{0},$$

the set of solutions of this system is a d-dimensional subspace of $(C^\infty(\mathbb{R}))^d$.

PROOF Let

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, ..., \mathbf{e}_d = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

For $j \in \{1, ..., d\}$, let $\mathbf{y}^{[j]}$ be the unique solution of the system satisfying the initial value $\mathbf{y}^{[j]}(0) = \mathbf{e}_j$.

Claim 1: $\mathbf{y}^{[1]}, \mathbf{y}^{[2]}, ..., \mathbf{y}^{[d]}$ are linearly independent.
Proof of Claim 1:

$$\begin{aligned}
W(\mathbf{y}^{[1]}, \mathbf{y}^{[2]}, ..., \mathbf{y}^{[d]})(0) &= \det \begin{pmatrix} y_1^{[1]}(0) & y_1^{[2]}(0) & \cdots & y_1^{[d]}(0) \\ y_2^{[1]}(0) & y_2^{[2]}(0) & \cdots & y_2^{[d]}(0) \\ \vdots & \vdots & \ddots & \vdots \\ y_d^{[1]}(0) & y_d^{[2]}(0) & \cdots & y_d^{[d]}(0) \end{pmatrix} \\
&= \det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1.
\end{aligned}$$

Since the Wronskian is nonzero at $t = 0$, it is everywhere nonzero, so the solutions $\mathbf{y}^{[1]}, \mathbf{y}^{[2]}, ..., \mathbf{y}^{[d]}$ are linearly independent, proving the claim.

Claim 2: $\mathbf{y}^{[1]}, \mathbf{y}^{[2]}, ..., \mathbf{y}^{[d]}$ span the solution set.

Proof of Claim 2: Let $\mathbf{y}$ be any solution of the system. To prove the claim, it is sufficient to prove that there are constants $c_1, ..., c_d$ so that

$$c_1\mathbf{y}^{[1]} + c_2\mathbf{y}^{[2]} + ... + c_d\mathbf{y}^{[d]} = \mathbf{y}.$$

Writing this coordinate-wise, this means we need to solve the system

$$(*)\ \begin{cases} c_1 y_1^{[1]} + c_2 y_1^{[2]} + ... + c_d y_1^{[d]} = y_1 \\ c_1 y_2^{[1]} + c_2 y_2^{[2]} + ... + c_d y_2^{[d]} = y_2 \\ \vdots \\ c_1 y_d^{[1]} + c_2 y_d^{[2]} + ... + c_d y_d^{[d]} = y_d \end{cases}$$

for $c_1, ..., c_d$. But since the Wronskian $W(\mathbf{y}^{[1]}, \mathbf{y}^{[2]}, ..., \mathbf{y}^{[d]})$ is everywhere nonzero, by Theorem 3.21 we see that the system $(*)$ can always be solved. This proves the claim.

Since we have a set of d functions (namely $\mathbf{y}^{[1]}, \mathbf{y}^{[2]}, ..., \mathbf{y}^{[d]}$) which are linearly independent and span the solution set, by Definition 3.30 the dimension of the solution set is d, and the set $\{\mathbf{y}^{[1]}, \mathbf{y}^{[2]}, ..., \mathbf{y}^{[d]}\}$ is a basis of the solution set. □

Putting this all together

Solution of first-order linear, homogeneous systems

Given the $d \times d$ linear, first-order homogeneous system

$$A_1\mathbf{y}' + A_0\mathbf{y} = \mathbf{0},$$

the solution set is a d-dimensional subspace. Therefore for any d linearly independent solutions $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_d$ of the system, the general solution of the homogeneous equation is

$$\boxed{\mathbf{y} = \mathbf{y}_h = C_1\mathbf{y}_1 + C_2\mathbf{y}_2 + ... + C_d\mathbf{y}_d}$$

where $C_1, C_2, ..., C_d$ are arbitrary constants.

(To verify that the functions $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_d$ are linearly independent, compute their Wronskian and show that the Wronskian is nonzero for at least one value of t.)

Solution of general linear systems

Given the $d \times d$ linear n^{th}-order system

$$A_n\mathbf{y}^{(n)} + A_{n-1}\mathbf{y}^{(n-1)} + ... + A_1\mathbf{y}' + A_0\mathbf{y} = \mathbf{q},$$

the general solution is any one particular solution $\mathbf{y}_p$ plus the solution of the corresponding homogeneous system, i.e. the general solution is

$$\boxed{\mathbf{y} = \mathbf{y}_p + \mathbf{y}_h}$$

If the equation is first-order, this means the general solution has the form

$$\boxed{\mathbf{y} = \mathbf{y}_p + C_1\mathbf{y}_1 + C_2\mathbf{y}_2 + ... + C_d\mathbf{y}_d}$$

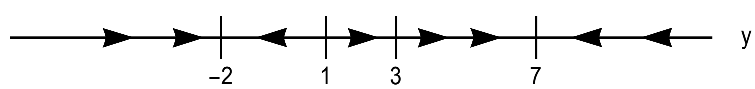
where $C_1, C_2, ..., C_d$ are arbitrary constants and $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_d$ are linearly independent solutions of the corresponding homogeneous system.

3.8 Autonomous systems; slope fields and phase planes

Recall from Chapter 1: A first-order ODE is called **autonomous** if it is of the form

$$y' = \phi(y);$$

to study such a system qualitatively we look for **equilibrium solutions** of such an ODE: $y = y_0$ is an equilibrium solution if and only if $\phi(y_0) = 0$. An equilibrium is **attracting** (a.k.a. **stable** a.k.a. a **sink**) if $\phi'(y_0) < 0$, it is **repelling** (a.k.a. **unstable** a.k.a. a **source**) if $\phi'(y_0) > 0$ and is **neutral** (a.k.a. **semistable**) if $\phi'(y_0) = 0$ and $\phi''(y_0) \neq 0$. We describe the behavior of a first-order ODE by drawing a **phase line** where the arrows indicate what happens to a solution $y(t)$ as $t \to \infty$:



In this section generalize these ideas to systems.

Definition 3.36 *A first-order $d \times d$ system of ODEs is called* **autonomous** *if it is of the form*

$$\mathbf{y}' = \Phi(\mathbf{y})$$

for some function $\Phi : \mathbb{R}^d \to \mathbb{R}^d$.

An IVP is called **autonomous** *if its differential equation is autonomous.*

Definition 3.37 *Let $\mathbf{y}' = \Phi(\mathbf{y})$ be an autonomous system. A constant function $\mathbf{y}$ which is a solution of this ODE is called an* **equilibrium (solution)** *of the ODE.*

Theorem 3.38 *The constant function $\mathbf{y} = \mathbf{y}_0$ is an equilibrium solution of the autonomous ODE $\mathbf{y}' = \Phi(\mathbf{y})$ if and only if $\Phi(\mathbf{y}_0) = \mathbf{0}$.*

Example: Find the equilibria of the system $x' = x + 2y, y' = y^2 + 4xy - 2x$.

Example: Find the equilibria of the system $\mathbf{y}' = (xy + x^2, x - 2y + 1)$.

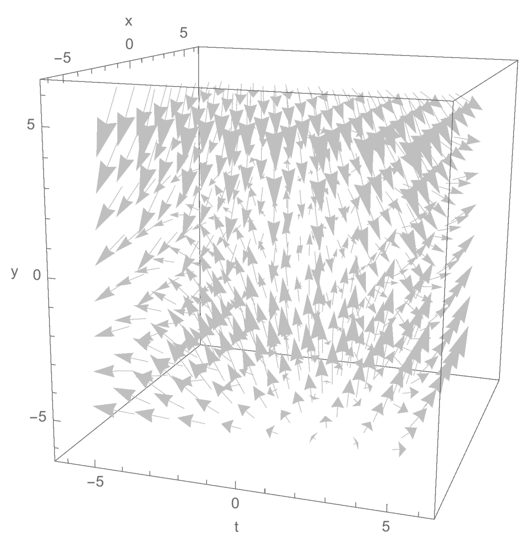
Slope fields for autonomous 2×2 systems

Recall that for a first-order ODE $y' = \phi(t, y)$, we drew pictures called **slope fields** which describe, at each point (t, y), a "mini-tangent" of slope $\phi(t, y)$.

Now, let's consider a first-order 2×2 system:

$$\mathbf{y}' = \Phi(t, \mathbf{y}) \;\Leftrightarrow\; \begin{cases} x' = \phi_1(t, x, y) \\ y' = \phi_2(t, x, y) \end{cases}$$

In order to draw a slope field that incorporates the t, you would need a 3-dimensional picture (because you need a dimension for each variable t, x and y). You can do this theoretically, but such a picture is usually really hard to read (because you can't interpret the perspective). Here is an example:



However, if the system is autonomous:

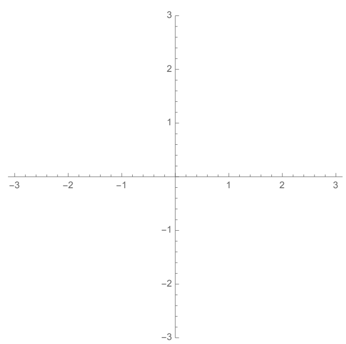
$$\mathbf{y}' = \Phi(\mathbf{y}) \;\Leftrightarrow\; \begin{cases} x' = \frac{dx}{dt} = \phi_1(x,y) \\ y' = \frac{dy}{dt} = \phi_2(x,y) \end{cases}$$

then you don't have to account for the t. You can draw a picture which includes "mini-tangents" at each point (x,y). The slope of the mini-tangent at (x,y) is

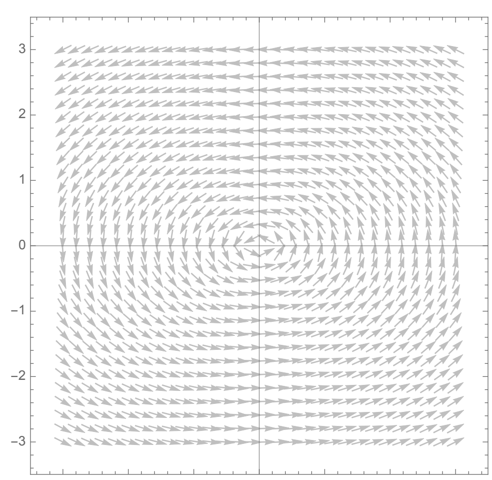
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\phi_2(x,y)}{\phi_1(x,y)}.$$

Example: $\begin{cases} x' = -2y \\ y' = x \end{cases}$

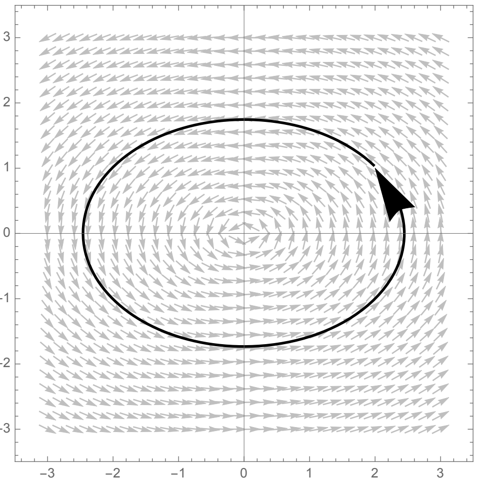
x	0	0	1	1	2	2	-1	-1	-1
y	1	2	1	-1	1	2	1	-1	0
(x,y)	$(0,1)$	$(0,2)$	$(1,1)$	$(1,-1)$	$(2,1)$	$(2,2)$	$(-1,1)$	$(-1,-1)$	$(-1,-3)$
$\frac{dx}{dt} = \phi_1(x,y)$	-2	-4	-2	2	-2	-4	-2	2	0
$\frac{dy}{dt} = \phi_2(x,y)$	0	0	1	1	2	2	-1	-1	-1
$\frac{dy}{dx}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	DNE

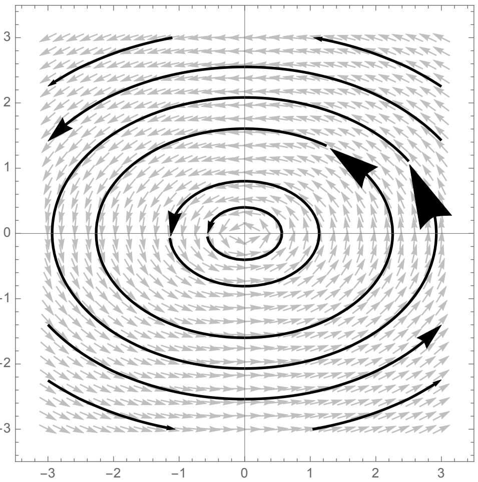


Here is a computer-generated picture of the slope field of the system;



As with single equations, the graph of a set of parametric equations which comprise a solution of the system must "flow with" the slope field. In this case, it appears that the solutions are ellipses centered at the origin:

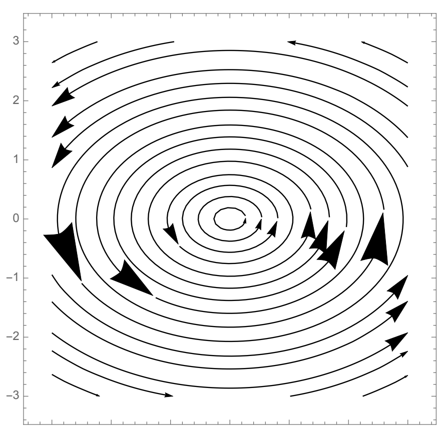




We can verify that the solutions are ellipses by actually solving the system; we'll learn how to do this in the forthcoming sections.

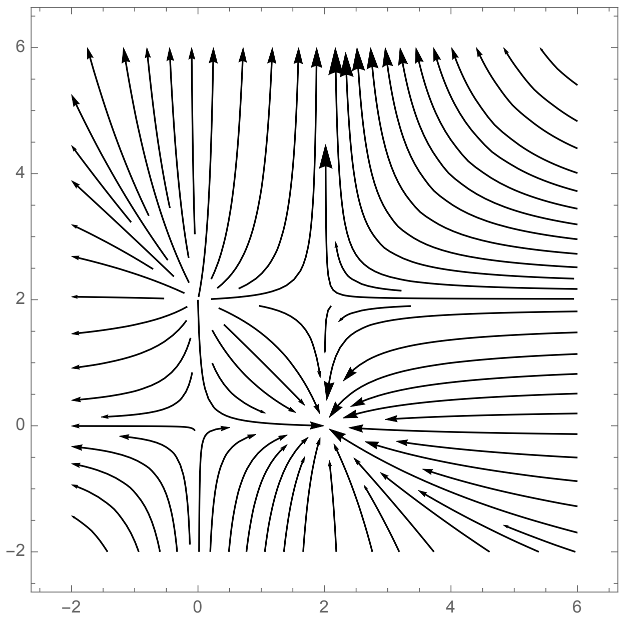
Phase plane analysis

If you take a slope field (as described above), and you sketch many solution curves on that slope field, you obtain a picture called a **phase plane** for the system. Often, you "erase" the mini-tangents from this picture and draw only the solution curves. Here is the phase plane for the above example ($x' = -2y, y' = -x$):



Phase planes are the analogue of phase lines for autonomous equations: they tell you how the solutions of a system behave as t changes.

Example: Below is the phase plane for some first-order 2×2 autonomous system $\mathbf{y}' = \phi(\mathbf{y})$:



1. Consider the solution to this system satisfying $x(0) = 1, y(0) = 4$.
   a) Sketch the graph of this solution on the phase plane.
   b) Find $\lim\limits_{t \to \infty} x(t)$ for this solution.
   c) Find $\lim\limits_{t \to \infty} y(t)$ for this solution.
   d) Find $\lim\limits_{t \to -\infty} x(t)$ for this solution.

2. Consider the solution to this system with initial condition $\mathbf{y}(0) = (5, 3)$. For this solution, does x increase or decrease as t increases? Does y increase or decrease as t increases?

3. Find all constant functions which solve the system.

Mathematica commands for slope fields and phase planes for systems

These commands can be found in the file `phaseplanes.nb`, available on my website.

Code to sketch the slope field

The following code will sketch a slope field and also sketch several solution curves (passing through randomly chosen points). Execute all this in a single *Mathematica* cell:

```
phi1[x_,y_] := x - y;
phi2[x_,y_] := x + 2y;
VectorPlot[{phi1[x,y], phi2[x,y]}, {x, -3, 3}, {y, -3, 3},
 VectorPoints -> 20, Axes -> True,
 VectorScale -> {Automatic, Automatic, None},
 VectorStyle -> Orange]
```

Code to sketch the slope field and several solution curves

The following code will sketch a slope field and also sketch several solution curves (passing through randomly chosen points). Execute all this in a single *Mathematica* cell:

```
phi1[x_,y_] := x - y;
phi2[x_,y_] := x + 2y;
VectorPlot[{phi1[x,y], phi2[x,y]}, {x, -3, 3}, {y, -3, 3},
 VectorPoints -> 20, Axes -> True,
 VectorScale -> {Automatic, Automatic, None},
 VectorStyle -> Orange,
 StreamPoints -> 35,
 StreamScale -> Full,
 StreamStyle -> Blue]
```

The first six lines are the same as the command above; the seventh line directs *Mathematica* to sketch 35 solution curves at random locations on the picture. The last line tells *Mathematica* what color to draw the solution curves.

Code to sketch the slope field and a solution curve passing through a specific point

The following code (executed in a single cell) will sketch a slope field and sketch a single solution curve passing through a given point (x_0, y_0). In this case the initial value is $(-1, 2)$:

```
phi1[x_,y_] := x - y;
phi2[x_,y_] := x + 2y;
VectorPlot[{phi1[x,y], phi2[x,y]}, {x, -3, 3}, {y, -3, 3},
 VectorPoints -> 20, Axes -> True,
 VectorScale -> {Automatic, Automatic, None},
 VectorStyle -> Orange,
 StreamPoints -> {{-1,2}},
 StreamScale -> Full,
 StreamStyle -> Blue]
```

Code to sketch phase planes (solution curves only; no mini-tangent lines)

```
phi1[x_,y_] := x - y;
phi2[x_,y_] := x + 2y;
StreamPlot[{phi1[x,y], phi2[x,y]}, {x, -4, 4}, {y, -4, 4},
 StreamPoints -> 100,
 StreamStyle -> Black,
 StreamScale -> Full]
```

Code to sketch a single solution curve (no mini-tangent lines)

```
phi1[x_,y_] := x - y;
phi2[x_,y_] := x + 2y;
StreamPlot[{phi1[x,y], phi2[x,y]}, {x, -4, 4}, {y, -4, 4},
 StreamPoints -> {{-1,2}},
 StreamStyle -> Black,
 StreamScale -> Full]
```

3.9 First-order, constant-coefficient homogeneous systems

Preliminaries

Since we assume A_1 is invertible, given any first-order, linear system

$$A_1\mathbf{y}' + A_0\mathbf{y} = \mathbf{q},$$

we can multiply through by A_1^{-1}, rearrange and redefine terms:

We have proven:

Theorem 3.39 *Every first-order, linear $d \times d$ system of ODEs can be rewritten as*

$$\mathbf{y}' = A\mathbf{y} + \mathbf{q}$$

for some $d \times d$ matrix of functions A and some list of functions $\mathbf{q} = (q_1, ..., q_d)$. The system is homogenous if and only if $\mathbf{q} = \mathbf{0}$, and constant-coefficient if and only if the entries of A are constants.

Example:

$$\begin{cases} x' = 3x - y + \sin t \\ y' = x + 2y - \cos t \end{cases}$$

Example:

$$\begin{cases} x' = e^t x + 2e^{3t} y - e^{-t} z \\ y' = e^{3t} x - e^t y + 4e^{-t} z \\ z' = e^{-t} x + e^t z \end{cases}$$

Recall from Chapters 1 & 2: the most simple first-order ODE is linear, homogeneous and constant-coefficient:

Now let's think about the most simple first-order system.

Since it is linear and first order, it must look like

Since it is homogeneous,

Since it is constant-coefficient,

Theorem 3.40 *Every first-order, linear, homogeneous, constant-coefficient of d ODEs can be written as*

$$\mathbf{y}' = A\mathbf{y}$$

where $\mathbf{y} = \mathbf{y}(t) = (y_1(t), y_2(t), ..., y_d(t))$ and A is an $d \times d$ matrix whose entries are numbers.

Note that any such system is autonomous, and its only equilibrium is $\mathbf{y} = \mathbf{0}$.

What should the solution of such a system be? In light of the answer for single equations

$$y' = ry \quad \Rightarrow$$

the solution of $\mathbf{y}' = A\mathbf{y}$ is probably something like

Matrix exponentials

Question: Given a square matrix A, what is $e^A = \exp(A)$? What is $e^{At} = \exp(At)$?

To answer this question, recall from Calculus II the Taylor series of e^t:

Lemma 3.41 *Let t be a real number. Then*

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{4!} + ...$$

(this series converges absolutely for all t).

Notice that the Taylor series formula for e^t involves only nonnegative integer powers of t, and addition. These are operations that are well-defined for square matrices, so we can define the exponential of a square matrix as follows:

Definition 3.42 *Let $A \in M_d(\mathbb{R})$. The* **(matrix) exponential** *of A, denoted e^A or $\exp(A)$, is*

$$e^A = \exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + ...$$

Remarks:

- If A is not square, then $A^2, A^3, ...$ do not exist, so e^A does not exist.
- If A is $d \times d$, then $A^2, A^3, A^4, ...$ are all $d \times d$ so e^A is also $d \times d$.
- It is not clear that this series converges (or what the word "converges" even means in this sense). **Fact:** this series converges for all square matrices A.
- You cannot compute matrix exponentials entry-by-entry:

$$\exp\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \begin{pmatrix} e^a & e^b \\ e^c & e^d \end{pmatrix}.$$

- For any constant t,

$$\begin{aligned} \exp(tA) = \exp(At) = \sum_{n=0}^{\infty} \frac{1}{n!}(At)^n &= \sum_{n=0}^{\infty} \frac{1}{n!} A^n t^n \\ &= I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + ... \end{aligned}$$

- If $A = \mathbf{0}$, the zero matrix, then $e^A = I + \mathbf{0} + \mathbf{0} + ... = I$. So $e^{\mathbf{0}} = I$.
- It is **not** always the case that

$$e^{A+B} = e^A e^B.$$

  This holds only if $AB = BA$ (which as we know is false, in general).

Theorem 3.43 *Let $A \in M_n(\mathbb{R})$ be a matrix whose entries are numbers. The only solution $\mathbf{y} = \mathbf{y}(t)$ to the first-order system*

$$\begin{cases} \mathbf{y}' = A\mathbf{y} \\ \mathbf{y}(0) = \mathbf{y}_0 \end{cases}$$

is

$$\mathbf{y}(t) = e^{At}\mathbf{y}_0.$$

PROOF First, let's check that the equation $\mathbf{y}' = A\mathbf{y}$ is satisfied. Let $\mathbf{y}(t) = e^{At}\mathbf{y}_0$.

Next, let's check the initial condition. If $\mathbf{y}(t) = e^{At}\mathbf{y}_0$, then when $t = 0$ we have

$$\mathbf{y}(0) = e^{A0}\mathbf{y}_0 = e^{\mathbf{0}}\mathbf{y}_0 = I\mathbf{y}_0 = \mathbf{y}_0$$

as desired.

Since both the ODE and initial value are satisfied by $\mathbf{y}$, $\mathbf{y}$ is a solution of the system. By the existence/uniqueness theorem, such a system has only one solution, so $\mathbf{y} = e^{At}\mathbf{y}_0$ is the only solution. □

Computing matrix exponentials

Example: Consider the first-order system

$$\begin{cases} x' = x + 3y \\ y' = 2x + 2y \end{cases}$$

with initial condition $x(0) = 1$, $y(0) = -3$. Find the solution of this system.

Theoretical solution:

Problem: How do you compute e^{At}?

Easier Example: Consider the first-order system

$$\begin{cases} x' = 7x \\ y' = -3y \end{cases}$$

with initial condition $x(0) = 1$, $y(0) = -3$. Find the solution of this system.

Theoretical solution 1: Write $A = \begin{pmatrix} 7 & 0 \\ 0 & -3 \end{pmatrix}$ and $\mathbf{y}_0 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. Then the solution is

$$\mathbf{y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{At}\mathbf{y}_0 = \exp\begin{pmatrix} 7t & 0 \\ 0 & -3t \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Theoretical solution 2: This system is "uncoupled" (the x and y have nothing to do with one another); from our study of exponential models in Chapter 1, the solution is

In fact, what we saw in the previous example is more general:

Theorem 3.44 (Exponentiation of diagonal matrices) *If Λ is diagonal with diagonal entries $\lambda_1, \lambda_2, ..., \lambda_n$, then*

$$\exp(\Lambda) = \begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix} \text{ and } \exp(\Lambda t) = \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix}.$$

PROOF Let Λ be as in the theorem. It is sufficient to prove the second statement, because by letting $t = 1$ in the second statement you get the first statement. By Theorem 3.12, for any k

$$(\Lambda t)^k = \begin{pmatrix} (\lambda_1 t)^k & & & \\ & (\lambda_2 t)^k & & \\ & & \ddots & \\ & & & (\lambda_n t)^k \end{pmatrix}$$

so

$$\begin{aligned} e^{\Lambda t} &= \sum_{k=0}^{\infty} \frac{1}{k!}(\Lambda t)^k = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!}(\lambda_1 t)^k & & & \\ & \sum_{k=0}^{\infty} \frac{1}{k!}(\lambda_2 t)^k & & \\ & & \ddots & \\ & & & \sum_{k=0}^{\infty} \frac{1}{k!}(\lambda_n t)^k \end{pmatrix} \\ &= \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix} \text{ as desired. } \square \end{aligned}$$

Example: If $\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$, then $e^{\Lambda t} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$.

Question: What do you do if the matrix is not diagonal?

Suppose $A \in M_n(\mathbb{R})$ can be "factored" into a product of three matrices as follows:

$$A = S\Lambda S^{-1}.$$

Then for any k,

$$A^k = (S\Lambda S^{-1})^k =$$

so

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k =$$

We have proven:

Theorem 3.45 *Let S be any invertible $n \times n$ matrix, and let Λ be any $n \times n$ matrix. Then if $A = S\Lambda S^{-1}$, we have*

$$e^A = \exp(S\Lambda S^{-1}) = Se^{\Lambda} S^{-1} \quad \textit{and} \quad e^{At} = \exp(S(\Lambda t)S^{-1}) = Se^{\Lambda t} S^{-1}.$$

These ideas lead to the following definition:

Definition 3.46 *A square $n \times n$ matrix A is called* **diagonalizable** *(a.k.a.* similar to a diagonal matrix*) if there is an invertible matrix $S \in M_n(\mathbb{R})$ and a* ***diagonal*** *matrix $\Lambda \in M_n(\mathbb{R})$ such that $A = S\Lambda S^{-1}$.*

Application: Suppose $A = \begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}^{-1}$.

At this point we have reduced the problem of computing the matrix exponentials of a diagonalizable matrix A to figuring out what the diagonal matrix Λ is and what the invertible matrix S is:

The following theory is derived in linear algebra (see Chapter 8 of my Math 322 notes):

> **Definition 3.47** *Let $A \in M_n(\mathbb{R})$. A number λ is called an* **eigenvalue** *of A if and only if one of the two equivalent conditions hold:*
>
> 1. *λ is a solution of the equation $\det(A - \lambda I) = 0$;*
> 2. *there is a nonzero vector $\mathbf{v}$ (called an* **eigenvector (of A corresponding to λ)***) such that $A\mathbf{v} = \lambda\mathbf{v}$.*

Note: eigenvectors, by definition, are **never** the zero vector.

Note: an eigenvector corresponding to an eigenvalue λ is **never** unique: if $\mathbf{v}$ is an eigenvector corresponding to λ, then so is $c\mathbf{v}$ for any nonzero constant c.

Recall our example from the beginning of this section:

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$$

Find the eigenvalues and eigenvectors of this matrix A.

Note that in this example, the eigenvalues sum to $4+(-1)=3=tr(A)$ and the eigenvalues multiply to $(4)(-1)=-4=1(2)-2(3)=\det A$. This is true in general:

Theorem 3.48 *Assume the eigenvalues of a matrix A are listed according to their multiplicities. Then:*

1. *The sum of the eigenvalues is $tr(A)$.*
2. *The product of the eigenvalues is $\det(A)$.*

The phrase "according to their multiplicities" means, for instance, that if $\det(A-\lambda I)$ ends up being $(\lambda-2)(\lambda+3)(\lambda+3)$, then $\lambda=-3$ should be listed twice (i.e. $\lambda=-3$ is an "eigenvalue of multiplicity 2").

The importance of eigenvalues and eigenvectors is seen in the following theorem:

Theorem 3.49 *Let $A \in M_n(\mathbb{R})$ be a square matrix. If A has n different real eigenvalues, then A is diagonalizable, in which case $A=S\Lambda S^{-1}$ where Λ is a diagonal matrix whose entries are the eigenvalues of A, and S is a matrix whose columns are the corresponding eigenvectors (written in the same order as the eigenvalues are written in Λ).*

Example: Diagonalize the matrix $A=\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$.

Using diagonalization to solve a system

Here was the example from the beginning of this section:

> Consider the first-order system
>
> $$\begin{cases} x' = x + 3y \\ y' = 2x + 2y \end{cases}$$
>
> with initial condition $x(0) = 1$, $y(0) = -3$. Find the solution of this system.

How to compute matrix exponentials

1. Given a $d \times d$ square matrix A, find the eigenvalues of A by solving the equation $\det(A - \lambda I) = 0$ for λ. Hopefully, you get d distinct λs, and hopefully all the λ are real numbers. This would guarantee that A is diagonalizable.

   (We'll talk about what to do if you don't get distinct, real λs later.)

2. For each λ, find an eigenvalue corresponding to λ by finding a nonzero $\mathbf{v}$ which satisfies the vector equation $A\mathbf{v} = \lambda\mathbf{v}$.

3. Let Λ be a diagonal matrices whose entries are the eigenvalues of A; let S be a matrix whose entries are the eigenvectors of A (written in the same order as the eigenvalues). Then

$$A = S\Lambda S^{-1}.$$

   (Find the inverse of S by methods from earlier in this chapter.)

4. $\exp(\Lambda t)$ is diagonal, whose entries are $e^{\lambda t}$s. Then

$$\exp(At) = S\exp(\Lambda t)S^{-1}$$

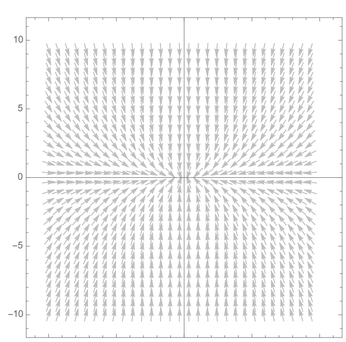
   (multiply these matrices back together to get the exponential).

What do the eigenvalues and eigenvectors "mean"?

Simple Example 1:

$$\begin{cases} x' = -2x \\ y' = -6y \end{cases} \quad \leftrightarrow \quad \mathbf{y}' = \begin{pmatrix} -2 & 0 \\ 0 & -6 \end{pmatrix} \mathbf{y}$$

Here is the vector field for this system:



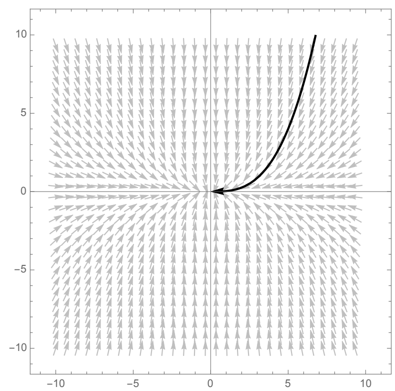
This isn't a very interesting system, because it is "uncoupled" (the x equation has all the xs and the y equation has all the ys. Since both equations represent exponential decay, we know the general solution is

$$\begin{cases} x = x_0 e^{-2t} \\ y = y_0 e^{-6t} \end{cases}$$

Suppose we were interested in the solution to this system with initial value $\mathbf{y}_0 = \mathbf{y}(0) = (5, 4)$. We know that its parametric equations are

$$\begin{cases} x = 5e^{-2t} \\ y = 4e^{-6t} \end{cases}$$

and its graph is

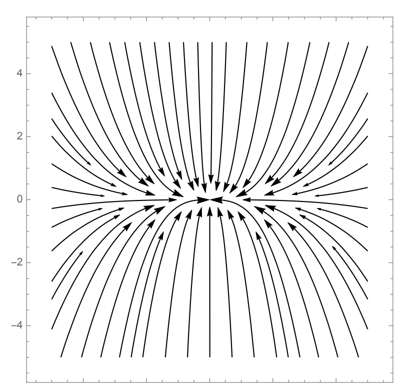


Question: What is the Cartesian equation of this curve?

More generally, given *any* initial value $\mathbf{y}_0 = (x_0, y_0)$ with $x_0 \neq 0$ and $y_0 \neq 0$, the solution of this system is

$$\begin{cases} x = x_0 e^{-2t} \\ y = y_0 e^{-6t} \end{cases} \quad \leftrightarrow y = Kx^3 \text{ for some constant } K.$$

So most of the solution curves are cubic. This matches the following picture of the phase plane for this system:



However: there are some "special" solutions of the system which are not cubic:

- Suppose $y_0 = 0$, $x_0 \neq 0$:

- Suppose $x_0 = 0$, $y_0 \neq 0$:

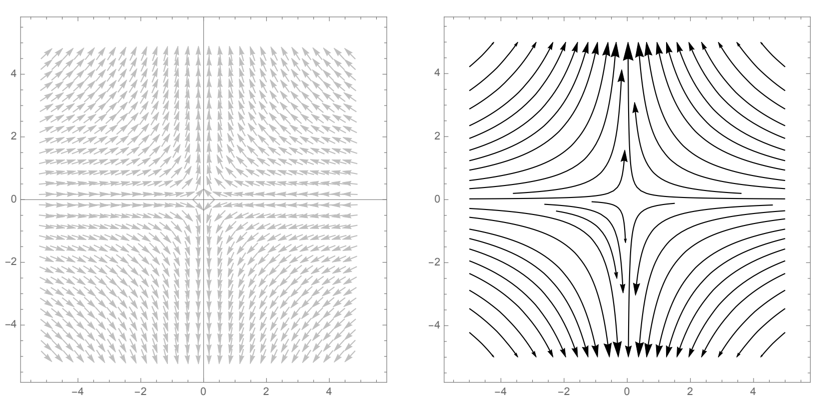
- Suppose $x_0 = y_0 = 0$: Then $x(t) = 0, y(t) = 0$ for all t. This is a constant solution $\mathbf{y} = \mathbf{0}$.

These straight-line solutions have a lot to do with the behavior of the other solutions. Think of the straight-line solutions as representing "forces" acting on a point in the xy-plane, where the coefficient on the exponential term of the corresponding variable is the "magnitude" of the force. The curve another point will travel reflects the sum effect of these "forces".

Simple Example 2:

$$\begin{cases} x' = -x \\ y' = y \end{cases} \quad \leftrightarrow \quad \mathbf{y}' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{y} \quad \Rightarrow \quad \begin{cases} x = x_0 e^{-t} \\ y = y_0 e^{t} \end{cases} \quad \Rightarrow \quad y = \frac{K}{x}$$

$\mathbf{y} = \mathbf{0}$ is again a constant solution; the straight-line solutions are again $x = 0$ and $y = 0$.



Harder Example (studied earlier in this section):

$$\begin{cases} x' = x + 3y \\ y' = 2x + 2y \end{cases} \qquad \leftrightarrow \qquad \mathbf{y}' = A\mathbf{y} \text{ where } A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$$

Eigenvalues and eigenvectors of A:

$$\lambda = -1 \leftrightarrow (3, -2) \qquad\qquad \lambda = 4 \leftrightarrow (1, 1)$$

General solution of the system:

$$\begin{aligned} \mathbf{y} = e^{At}\mathbf{y}_0 &= \begin{pmatrix} \frac{3}{5}e^{-t} + \frac{2}{5}e^{4t} & \frac{-3}{5}e^{-t} + \frac{3}{5}e^{4t} \\ \frac{-2}{5}e^{-t} + \frac{2}{5}e^{4t} & \frac{2}{5}e^{-t} + \frac{3}{5}e^{4t} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \\ &= \begin{pmatrix} \left(\frac{3}{5}x_0 - \frac{3}{5}y_0\right) e^{-t} + \left(\frac{2}{5}x_0 + \frac{3}{5}y_0\right) e^{4t} \\ \left(\frac{-2}{5}x_0 + \frac{2}{5}y_0\right) e^{-t} + \left(\frac{2}{5}x_0 + \frac{3}{5}y_0\right) e^{4t} \end{pmatrix} \end{aligned}$$

Let's suppose we wanted to see if there are any straight-line solutions which solve this system. To do this, let's try to choose the initial condition (x_0, y_0) so that either the e^{-t} terms or the e^{4t} terms drop out of the general solution:

- To make the e^{-t} terms drop out: we need

  So if $x_0 = y_0 = 1$, we get the solution

  $$\mathbf{y} = \begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix} = e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

  which has Cartesian equation $y = x$.

- To make the e^{4t} terms drop out: we need

  $$\frac{2}{5}x_0 + \frac{3}{5}y_0 = 0,$$

  in which case $y_0 = \frac{-2}{3}x_0$ and the solution when $x_0 = 3$, $y_0 = -2$ reduces to

  $$\mathbf{y} = \begin{pmatrix} 3e^{-t} \\ -2e^{-t} \end{pmatrix} = e^{-t} \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

  which has Cartesian equation $y = \frac{-2}{3}x$.

Summary: For the 2×2 linear, constant-coefficient system

$$\begin{cases} x' = x + 3y \\ y' = 2x + 2y \end{cases} \quad \leftrightarrow \quad \mathbf{y}' = A\mathbf{y} \text{ where } A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix},$$

we have found two solutions:

$$\mathbf{y}_1 = e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2 = e^{-t}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 3e^{-t} \\ -2e^{-t} \end{pmatrix}.$$

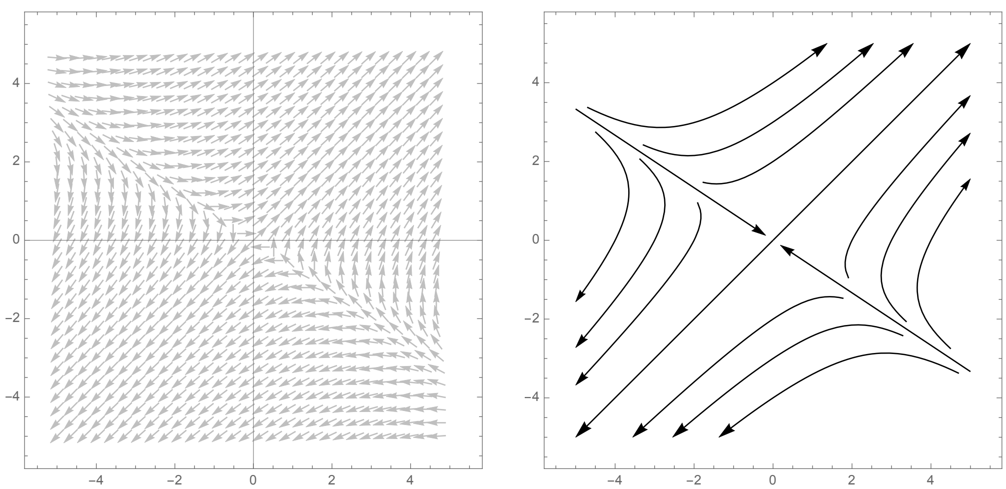
From the theory, we know that the solution set is 2 dimensional (since there are 2 equations), so any two linearly independent solutions will span the solution set. Let's verify that $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent by computing their Wronskian:

$$W(\mathbf{y}_1, \mathbf{y}_2) = \det \begin{pmatrix} e^{4t} & 3e^{-t} \\ e^{4t} & -2e^{-t} \end{pmatrix} = -5e^{3t} \neq 0$$

so $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent, so the general solution of the system is

$$\mathbf{y} = C_1\mathbf{y}_1 + C_2\mathbf{y}_2 = C_1 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

Slope field and phase plane of this system:



The behavior in this example is not a coincidence: suppose you have any straight-line solution of any $d \times d$ linear, constant-coefficient system $\mathbf{y}' = A\mathbf{y}$. Choose a vector $\mathbf{v}$ which goes along that straight line; then parameterize the straight line by the equations $\mathbf{y} = e^{\lambda t}\mathbf{v}$, where $\lambda \in \mathbb{R}$ is some constant. That means $\mathbf{y}' = \lambda e^{\lambda t}\mathbf{v}$ so by plugging in to the equation $\mathbf{y}' = A\mathbf{y}$, we get

In other words, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ.

Assuming there are d different λs, there will be d different straight-line solutions which can be shown to be linearly independent using the Wronskian. Therefore we can conclude:

Theorem 3.50 (General sol'n of const.-coeff. system with distinct eigenvalues) *Let $A \in M_d(\mathbb{R})$ be a matrix with d distinct, real eigenvalues $\lambda_1, \lambda_2, ..., \lambda_d$ whose corresponding eigenvectors are $\mathbf{v}_1, ..., \mathbf{v}_d$. Then if $\mathbf{y} = (y_1, y_2, ..., y_d)$, the general solution of the $d \times d$ system of ODEs*

$$\mathbf{y}' = A\mathbf{y}$$

is

$$\mathbf{y} = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 + ... + C_d e^{\lambda_d t}\mathbf{v}_d.$$

Example: Find the general solution of $\begin{cases} \frac{dx}{dt} = 8x - y \\ \frac{dy}{dt} = 11x - 4y \end{cases}$.

Example: (same system from previous pages)

$$\begin{cases} \frac{dx}{dt} = 8x - y \\ \frac{dy}{dt} = 11x - 4y \end{cases}$$

What is the particular solution if you have an initial value given to you, like $x(0) = 2$, $y(0) = 14$?

First, find the general solution (previous page):

Let $A = \begin{pmatrix} 8 & -1 \\ 11 & -4 \end{pmatrix}$; eigenvalues and eigenvectors of A are

$$\lambda = -3 \leftrightarrow (1, 11) \qquad \lambda = 7 \leftrightarrow (1, 1)$$

so the general solution is

$$\mathbf{y} = C_1 e^{-3t} \begin{pmatrix} 1 \\ 11 \end{pmatrix} + C_2 e^{7t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 e^{-3t} + C_2 e^{7t} \\ 11 C_1 e^{-3t} + C_2 e^{7t} \end{pmatrix}.$$

i.e.

$$\begin{cases} x(t) = C_1 e^{-3t} + C_2 e^{7t} \\ y(t) = 11 C_1 e^{-3t} + C_2 e^{7t} \end{cases}$$

Now, for the particular solution:

Plug in the known initial values and solve for C_1 and C_2.

Putting all this together

Example:

1. Find the general solution of the system

$$\begin{cases} x' = x + 3y \\ y' = -4x - 6y \end{cases}$$

2. Find the particular solution of this system satisfying $x(0) = 2$, $y(0) = -1$.

3. Find a basis of the solution set and verify that the functions in this basis are linearly independent.

4. For the particular solution found in # 2, find (x, y) when $t = 3$, and find $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.

Solution:

Mathematica code for eigenvalues, eigenvectors and matrix exponentials

It is unreasonable to solve (by hand) first-order linear, constant-coefficient systems of ODEs if there are three or more equations / variables in the system. Even computing the eigenvalues requires you to factor a cubic (or higher-degree) polynomial in λ, which is hard. **Therefore, we use computers.**

Example: Use *Mathematica* to find the general solution of the system of ODEs

$$\begin{cases} x' = 3x + 2z \\ y' = 3y - 2z \\ z' = 2x - 2y + z \end{cases}$$

Solution: First, define the matrix A:

```
A = {{3,0,2}, {0,3,-2}, {2,-2,1}}
```

Now, find the eigenvalues and eigenvectors of A: execute

```
Eigensystem[A] // MatrixForm
```

which produces output

```
(      5            3            -1       )
( {1, -1, 1}  {1, 1, 0}  {-1, 1, 2} )
```

This means the eigenvalues and eigenvectors of A are

$$\lambda = 5 \leftrightarrow \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \lambda = 3 \leftrightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \lambda = -1 \leftrightarrow \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

so the general solution is

$$\mathbf{y} = C_1 e^{5t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3 e^{-t} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

i.e.

$$\begin{cases} x(t) = C_1 e^{5t} + C_2 e^{3t} - C_3 e^{-t} \\ y(t) = -C_1 e^{5t} + C_2 e^{3t} + C_3 e^{-t} \\ z(t) = C_1 e^{5t} + 2C_3 e^{-t} \end{cases}$$

Example: Use *Mathematica* to find the particular solution of the preceding system with initial value $x(0) = 2$, $y(0) = -1$, $z(0) = 3$.

Solution # 1: Find the general solution as on the previous page. Then, plug in the initial condition (since $t = 0$, all the exponential terms are 1) and solve for the constants with a command like this (I'll call the constants `K`, `L`, `M` rather than C_1, C_2, C_3):

```
Solve[{2 == K + L - M, -1 == -K + L + M, 3 == K + 2M}, {K,L,M}]
```

You get output

$$\left\{\left\{\mathtt{K} \to 2, \mathtt{L} \to \tfrac{1}{2}, \mathtt{M} \to \tfrac{1}{2}\right\}\right\}$$

which means that $C_1 = 2, C_2 = \frac{1}{2}, C_3 = \frac{1}{2}$. Plug these into the general solution to get

$$\begin{cases} x(t) = 2e^{5t} + \frac{1}{2}e^{3t} - \frac{1}{2}e^{-t} \\ y(t) = -2e^{5t} + \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ z(t) = 2e^{5t} + 2 \cdot \frac{1}{2}e^{-t} \end{cases} \Rightarrow \begin{cases} x(t) = 2e^{5t} + \frac{1}{2}e^{3t} - \frac{1}{2}e^{-t} \\ y(t) = -2e^{5t} + \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ z(t) = 2e^{5t} + e^{-t} \end{cases}$$

Solution # 2: We know the theoretical solution is $\mathbf{y} = e^{At}\mathbf{y}_0$, so define A and $\mathbf{y}_0$ and compute the matrix exponential directly:

```
A =  {{3,0,2}, {0,3,-2}, {2,-2,1}}
y0 = {2,-1,3}
y[t_] = MatrixExp[A t].y0
```

This gives the answer which, after expanding with the `Expand[%]` command, in *Mathematica* output looks like

$$\{\frac{-e^{-t}}{2} + \frac{e^{3t}}{2} + 2e^{5t}, \frac{e^{-t}}{2} + \frac{e^{3t}}{2} - 2e^{5t}, e^{-t} + 2e^{5t}\}$$

and hand-written, this answer is

$$\begin{cases} x(t) = \frac{-1}{2}e^{-t} + \frac{1}{2}e^{3t} + 2e^{5t} \\ y(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{3t} - 2e^{5t} \\ z(t) = e^{-t} + 2e^{5t} \end{cases},$$

the same as what was found in Solution # 1.

Useful *Mathematica* commands in this context:

- `MatrixExp[A]` computes e^A;
- `Eigensystem[A]` gives the eigenvalues of A, together with their eigenvectors. If you get output that looks like

  ```
  {{-5, 0}, {{-1, 2}, {-3, 1}}}
  ```

  that means the eigenvalues and eigenvectors are

  $$\lambda = -5 \leftrightarrow (-1, 2) \qquad\qquad \lambda = 0 \leftrightarrow (-3, 1);$$

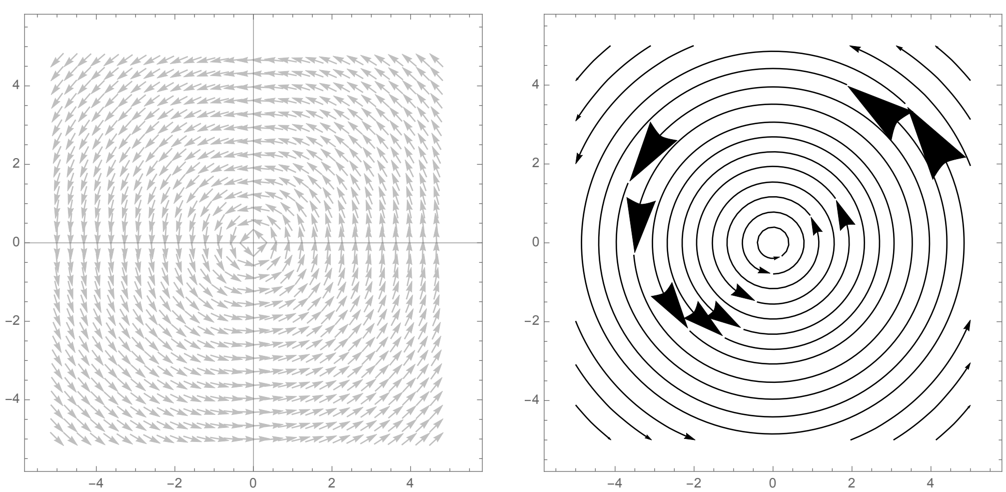
  the output looks better if you run `Eigensystem[A] // MatrixForm`
- `Eigenvalues[A]` gives just the eigenvalues of A;
- `Eigenvectors[A]` gives just the eigenvectors of A (in the same order as the eigenvalues were given with the preceding command);
- `CharacteristicPolynomial[A,x]` gives a formula for $\det(A - \lambda I)$ (with x instead of λ);
- `DiagonalizableMatrixQ[A]` asks whether or not A is a diagonalizable matrix (it returns `True` if it is, and `False` if it isn't).

3.10 A crash course in complex numbers

Example: Solve the system, with the given initial value:

$$\begin{cases} x' = -y \\ y' = x \end{cases} \qquad\qquad \begin{cases} x(0) = 1 \\ y(0) = 0 \end{cases}$$

Slope field and phase plane:



Notice that there are no straight line solutions (which makes sense, given that the matrix had no eigenvalues). The solutions appear to be circles.

Question: where do the circles come from?

A brief history of complex numbers

Recall: Quadratic formula (method of solution known 1600 BC, written down in 628 AD by Brahmagupta):

$$ax^2 + bx + c = 0 \qquad \Rightarrow \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Potential dilemma: what if $b^2 - 4ac < 0$?

History: up to the 1500s, people were happy to say that $ax^2 + bx + c = 0$ has no solution (because they believed you "can't" take the square root of a negative number).

What about a "cubic formula"?

$$ax^3 + bx^2 + cx + d = 0 \qquad \Rightarrow \qquad x = ?$$

Such a formula exists! (discovered by Cardano in 1545) Here it is:

$$x = -\frac{b}{3a} + \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}.$$

Bad news: there are two square roots in this formula, and often the number under these square roots is negative.

Cardano's fix: he "pretended" that square roots of these negative numbers existed and found that in the cubic formula, eventually these "pretend" numbers drop out and x ends up being a real number which solves the cubic equation.

Bombelli in 1572 called Cardano's pretend numbers "imaginary numbers". He decided to see what kinds of arithmetic one could do with these "imaginary" numbers, and showed that you could make sense of addition, multiplication, division, powers and roots of them.

Imaginary numbers were controversial until 1742, when Euler proved the Fundamental Theorem of Algebra, which states that any polynomial whose coefficients are real numbers has a root, if you allow imaginary numbers as solutions. In other words, if you write down the equation

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0,$$

then there is a solution x if you allow for the possibility that x is imaginary (the theorem is true even if the a_j are allowed to be imaginary).

Today: we have discovered that square roots of negative numbers are not really "imaginary". They describe physical quantities in fluid dynamics, electromagnetism, signal and image processing, quantum mechanics, and special and general relativity. As such, we don't call these numbers "imaginary"; we call them *complex numbers*.

The basics of complex numbers

Definition 3.51 *A* **complex number** *is any number of the form $z = x + iy$ where x and y are real numbers (for now, i is just a symbol... this symbol will be given meaning later). The set of complex numbers is denoted $\mathbb{C}$.*

Given a complex number $z = x + iy$, the **real part** *of z, denoted $\Re(z)$ or $Re(z)$, is x, and the* **imaginary part** *of z, denoted $\Im(z)$ or $Im(z)$, is y.*

A complex number z is called **pure imaginary** *if $\Re(z) = 0$; a complex number z is* **real** *if $\Im(z) = 0$.*

The set of $m \times n$ matrices with entries in $\mathbb{C}$ is denoted $M_{mn}(\mathbb{C})$, and the set of square $n \times n$ matrices with entries in $\mathbb{C}$ is denoted $M_n(\mathbb{C})$. Operations on these matrices are defined the same way they are for other types of matrices. The real part and imaginary part of a complex vector or matrix are computed term-by-term.

Note: for $z \in \mathbb{C}$, $\Re(z)$ and $\Im(z)$ are real numbers. For example:

$$\Re(2 + 5i) = 2 \qquad \Im(-1 - 4i) = -4$$

Comments on notation: Usually a complex number is denoted by z, w, or a Greek letter like ζ (zeta) or ξ (xi) or ω (omega). The letters s, t, u, v, x and y should not be used to denote complex numbers; they connote real numbers. In particular it is always understood with complex numbers that "z" means the complex number $z = x + iy$.

Remark: in general you want to avoid immediately thinking of a complex number as $x + iy$. Just think of it as z.

Definition 3.52 *The* **(complex) conjugate** *of $z = x + iy \in \mathbb{C}$ is $\overline{z} = x - iy$. The* **(complex) conjugate** *of a matrix or vector whose entries are complex numbers is obtained by taking the conjugate of each entry of the matrix/vector.*

Example: If $z = 2 - 7i$, then $\overline{z} = 2 + 7i$.

Example: Find $\Re(\mathbf{v})$ and $\overline{\mathbf{v}}$, if $\mathbf{v} = (3-2i, 7+i, -i)$.

Arithmetic in $\mathbb{C}$

Addition and subtraction in $\mathbb{C}$ are defined by combining like terms. For example,

$$(2-3i)+(1+i) = 3-2i \quad \text{and} \quad (-3+i)-(2+7i) = -5-6i.$$

Multiplication is defined by distributing terms, together with the law that $i^2 = -1$ (this is the first time we need the idea that $i = \sqrt{-1}$). For example:

$$(2+5i)(-1-2i) = -2-5i-4i-10i^2 = -2-9i+10 = 8-9i.$$

Division is trickier; to divide one complex number by a nonzero complex number, what you do is multiply through the numerator and denominator of the fraction by the conjugate of the denominator. An example:

$$(1+i) \div (3-4i) = \frac{1+i}{3-4i} = \frac{(1+i)(3+4i)}{(3-4i)(3+4i)} = \frac{-1+7i}{25} = \frac{-1}{25} + \frac{7}{25}i.$$

The operations thus defined satisfy all the elementary arithmetic properties: addition and multiplication are commutative and associative, addition and multiplication have identity elements ($0 = 0+0i$ and $1 = 1+0i$ respectively); every element has an additive inverse; every nonzero element has a reciprocal; the distributive property holds. Also:

Lemma 3.53 *Let $z_1, z_2 \in \mathbb{C}$. Then $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$.*

PROOF Write $z = x+iy$. If you worked out both sides of the equations in terms of x and y, you would see the left- and right- hand sides of the equations are equal. □

Lemma 3.54 *Let $z \in \mathbb{C}$. Then $z + \overline{z} = 2\Re(z)$ and $z - \overline{z} = 2i\Im(z)$.*

PROOF Write $z = x+iy$; then $z + \overline{z} = x+iy+(x-iy) = 2x = 2\Re(z)$. Also, $z - \overline{z} = (x+iy)-(x-iy) = 2iy = 2i\Im(z)$. □

Geometric interpretation of $\mathbb{C}$

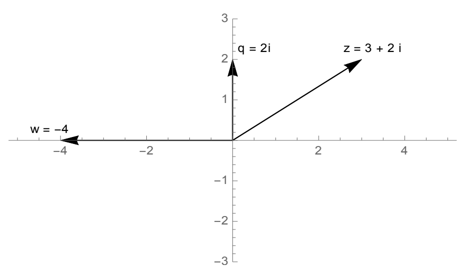
We think of the complex number $z = x + iy$ as if it is the vector (x, y) (which lies in a plane). Thus $\mathbb{C}$ is a plane (in the same way that $\mathbb{R}$ is a line). The "$x-$axis" of this plane is called the **real axis** and the "$y-$axis" of this plane is called the **imaginary axis**. Addition of complex numbers corresponds to "head-to-tail" or "parallelogram" addition of vectors, i.e.

$$(3 + 2i) + (1 - 3i) = 4 - i$$

is essentially the same as

$$(3, 2) + (1, -3) = (4, -1).$$

Observe that if we think of z as a vector, then $\overline{z}$ is the vector obtained by reflecting z through the real axis (note that $\overline{z} = z$ if and only if z is real):



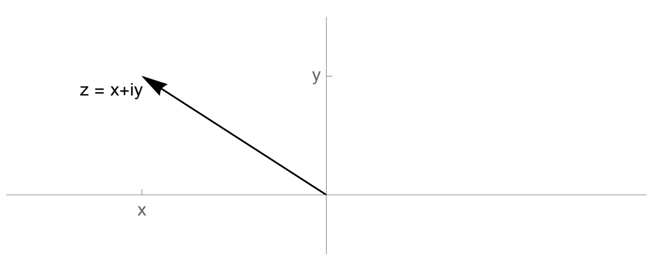
Question: How is multiplication of complex numbers interpreted geometrically?

Answer:

Lemma 3.55 *Let $z \in \mathbb{C}$. Then $z\overline{z}$ is real, and $z\overline{z} \geq 0$.*

PROOF Let $z = x+iy$, then $z\overline{z} = (x+iy)(x-iy) = x^2+iyx-iyx-i^2y^2 = x^2+y^2 \geq 0$. $\square$

Definition 3.56 *The* **absolute value** *a.k.a.* **norm** *a.k.a.* **modulus** *of a complex number $z = x + iy$ is $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$.*



The norm of a complex number is its distance from zero, so the "norm of a complex number" generalizes the notion of "absolute value of a real number".

Another view of division in $\mathbb{C}$: given $z_1, z_2 \in \mathbb{C}$, we have

$$z_1 \div z_2 = \frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2}.$$

Special case (reciprocals): If $z \neq 0$, then

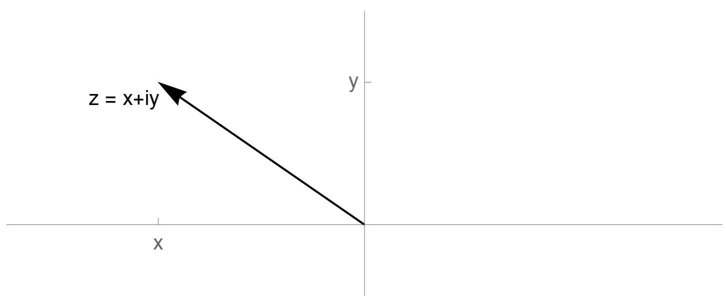
$$z^{-1} = \frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}.$$

In particular, if $|z| = 1$, then $z^{-1} = \overline{z}$ (useful special case: $\frac{1}{i} = i^{-1} = \overline{i} = -i$).

Lemma 3.57 *Let $z_1, z_2 \in \mathbb{C}$. Then $|z_1 z_2| = |z_1||z_2|$.*

PROOF Write $z_1 = x_1 + iy_1$ (same for z_2), if you work out both sides in terms of the xs and ys you would see that they are equal.

Definition 3.58 *Let $z = x + iy \in \mathbb{C}$. The* **argument** *of z, denoted $\arg(z)$, is any angle θ (in radians) such that $x = |z|\cos\theta$ and $y = |z|\sin\theta$.*

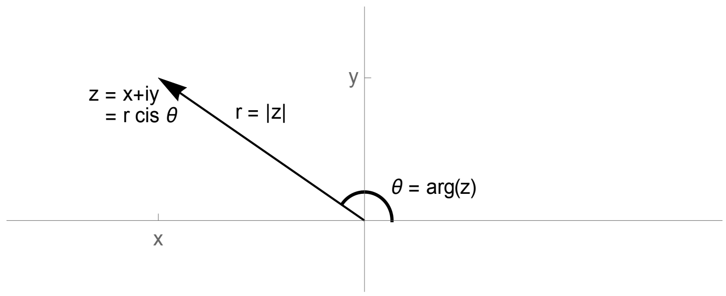


Given z, you can solve for $\theta = \arg z$ by setting $\theta = \arctan\left(\frac{y}{x}\right)$ if $x \neq 0$; if $x = 0$ then $\theta = \pi/2$ if $y > 0$ and $\theta = -\pi/2$ if $y < 0$. Notice that arguments are only defined up to multiples of 2π.

Definition 3.59 *The* **polar coordinates** *of a complex number z are (r, θ) where $r = |z|$ and $\theta = \arg z$. If the polar coordinates of z are (r, θ), we write*

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

or $z = r$ cis θ.



Euler's formula

To define functions like exponentials and trig functions of complex numbers, we use power series (because power series are made up only of addition, subtraction, multiplication and division, and all these operations are already defined for complex numbers).

There is an issue regarding what it means for a series of complex numbers to converge, but it turns out that any power series which converges for all real numbers also converges for all complex numbers.

Definition 3.60 *For any complex number $z \in \mathbb{C}$, define*

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$
$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.$$

From this, you can show that all the usual trigonometric and exponential identities that hold for real numbers also hold for complex numbers. Additionally, these operations preserve conjugation:

Lemma 3.61 *Let $z \in \mathbb{C}$. Then*

$$e^{\overline{z}} = \overline{e^z}; \quad \cos \overline{z} = \overline{\cos z}; \quad \sin \overline{z} = \overline{\sin z}.$$

PROOF Conjugation is preserved under multiplication and addition, and these three operations are made up of only multiplication and addition. □

More importantly, we have the following amazing identity which links exponential and trigonometric functions:

> **Theorem 3.62 (Euler's formula)** *For any* $z \in \mathbb{C}$, $e^{iz} = \cos z + i \sin z$.

PROOF

$$\begin{aligned}
e^{iz} &= \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \\
&= 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \ldots \\
&= 1 + iz + i^2\frac{z^2}{2!} + i^3\frac{z^3}{3!} + i^4\frac{z}{4!} + \ldots \\
&= 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - \frac{z^6}{6!} - i\frac{z^7}{7!} + \ldots \\
&= \left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots\right] + i\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right] \\
&= \cos z + i \sin z. \ \square
\end{aligned}$$

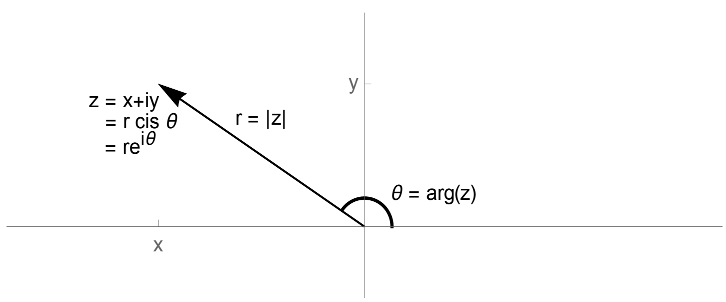
As an important consequence, we see that if z has polar coordinates (r, θ), then

$$z = r \text{ cis } \theta = r(\cos\theta + i\sin\theta) = r\cos\theta + ir\sin\theta = re^{i\theta}.$$

For such a z,

$$\overline{z} = r\cos\theta - ir\sin\theta = r\cos(-\theta) + i\sin(-\theta) = re^{-i\theta}.$$

In particular, if we write $z = re^{i\theta}$ where $r \geq 0$ and $\theta \in \mathbb{R}$, this means (r, θ) are the polar coordinates of z, so $r = |z|$ and $\theta = \arg z$.



> **Theorem 3.63** *Suppose* $z_1 = r_1$ *cis* θ_1 *and* $z_2 = r_2$ *cis* θ_2 *(this means* $r_1 = |z_1|, r_2 = |z_2|$*). Then* $z_1 z_2 = r_1 r_2$ *cis* $(\theta_1 + \theta_2)$.

PROOF Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then by elementary properties of exponentials, $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$. $\square$

This theorem tells us how to interpret multiplication geometrically in $\mathbb{C}$. Given two complex numbers, if those numbers are multiplied, then the "moduli multiply" (since $|z_1 z_2| = |z_1||z_2|$) and the "arguments add" (since this theorem implies $\arg(z_1 z_2) = \arg z_1 + \arg z_2$).

3.11 Complex eigenvalues

Recall the example at the start of the previous section that led us to complex numbers:

$$\begin{cases} x' = -y \\ y' = x \end{cases} \qquad \begin{cases} x(0) = 1 \\ y(0) = 0 \end{cases}$$

To solve this, remember that we let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and let $\mathbf{y} = \begin{pmatrix} x \\ y \end{pmatrix}$ so that the system became $\mathbf{y}' = A\mathbf{y}$. We then tried to find the eigenvalues of A:

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} &= 0 \\ \lambda^2 + 1 &= 0 \end{aligned}$$

From the previous page, we have

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = A = S\Lambda S^{-1} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \frac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$$

Therefore the solution of the system is

$$\begin{aligned}
e^{At}\mathbf{y}_0 = \quad & S \quad e^{\Lambda t} \quad S^{-1} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \frac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{2i} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \cos t + i \sin t & 0 \\ 0 & \cos(-t) + i \sin(-t) \end{pmatrix} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&\qquad \text{(using Euler's formula)} \\
&= \frac{1}{2i} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \cos t + i \sin t & 0 \\ 0 & \cos t - i \sin t \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&\qquad \text{(using identities } \cos(-t) = \cos t; \sin(-t) = -\sin t\text{)} \\
&= \frac{1}{2i} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \cos t + i \sin t \\ -\cos t + i \sin t \end{pmatrix} \\
&= \frac{1}{2i} \begin{pmatrix} i \cos t - \sin t + i \cos t + \sin t \\ \cos t + i \sin t - \cos t + i \sin t \end{pmatrix}
\end{aligned}$$

Just as in Cardano's cubic formula, the complex numbers are <u>a means to an end</u>: they drop out once Euler's formula is applied and everything is multiplied out.

To summarize, we have found that the solution to the system

$$\begin{cases} x' = -y \\ y' = x \end{cases} \qquad \begin{cases} x(0) = 1 \\ y(0) = 0 \end{cases}$$

is

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} .$$

The Cartesian equation of this set of parametric equations is $x^2 + y^2 = 1$, whose graph is a circle of radius 1 centered at the origin. This matches the slope field we started with.

The general situation

Theorem 3.64 (Complex Roots Theorem) *Suppose f is a polynomial with real coefficients, i.e. $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$. Then if z is any root of f (i.e. $f(z) = 0$), then the conjugate $\overline{z}$ is also a root of f.*

PROOF Suppose $f(z) = 0$. Then (using Lemma 2.39) we see $f(\overline{z}) = \overline{f(z)} = \overline{0} = 0$ as well. □

Recall that to find eigenvalues of a matrix with real entries, we find roots of the polynomial $\det(A - \lambda I)$. The Complex Roots Theorem implies, therefore, that for a real matrix, whenever a complex number is an eigenvalue of that matrix, so is its conjugate. More generally:

Theorem 3.65 *Let $A \in M_n(\mathbb{R})$ be a matrix whose entries are real. If $\lambda = a + ib \in \mathbb{C}$ is an eigenvalue of matrix A with eigenvector $\mathbf{v}$ ($\mathbf{v}$ may have complex entries), then the conjugate $\overline{\lambda} = a - ib$ is also an eigenvalue, whose eigenvector is $\overline{\mathbf{v}}$.*

PROOF We are given that $A\mathbf{v} = \lambda\mathbf{v}$. Take the conjugate of both sides:

To summarize:

Theorem 3.66 *If $A \in M_n(\mathbb{R})$ has n distinct (real or complex) eigenvalues, then A is diagonalizable over the complex numbers, i.e. $A = S\Lambda S^{-1}$ where $S, \Lambda \in M_n(\mathbb{C})$ are such that Λ is a diagonal matrix whose entries are the eigenvalues of A, and S is a matrix whose columns are the corresponding eigenvectors (written in the same order as the eigenvalues are written in Λ).*

Moreover, any non-real eigenvalues and eigenvectors of A come in complex conjugate pairs.

A shortcut in the complex case

Suppose you have a 2×2 system

$$\mathbf{y}' = A\mathbf{y}$$

where the eigenvalues of A are not real. In light of Theorem 3.50, the solution of the system is

$$\mathbf{y} = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

Since the eigenvalues and eigenvectors are complex-conjugate pairs, this solution can be rewritten as

$$\mathbf{y} = C_1 e^{\lambda t}\mathbf{v} + C_2 e^{\overline{\lambda} t}\overline{\mathbf{v}}.$$

In other words, a basis of the solution space is given by the two functions

$$\mathbf{y}_1 = e^{\lambda t}\mathbf{v}; \quad \mathbf{y}_2 = e^{\overline{\lambda} t}\overline{\mathbf{v}}.$$

Notice that since t is real, $t = \bar{t}$ so $\overline{\lambda} t = \overline{\lambda}\,\bar{t} = \overline{\lambda t}$. Since exponentiation preserves conjugation, we have

$$\mathbf{y}_2 = e^{\overline{\lambda} t}\overline{\mathbf{v}} = e^{\overline{\lambda t}}\overline{\mathbf{v}} = \overline{e^{\lambda t}}\,\overline{\mathbf{v}} = \overline{e^{\lambda t}\mathbf{v}} = \overline{\mathbf{y}_1}.$$

Therefore a basis of the solution space is given by $\{\mathbf{y}_1, \overline{\mathbf{y}_1}\}$. The problem is that these solutions have complex numbers in them.

Goal: Find a basis of the solution space which has only real numbers in it.

Idea: Since the solution space is a subspace, the sum of the two basis elements is also a solution:

$$\mathbf{y}_1 + \overline{\mathbf{y}_1} = 2\Re(\mathbf{y}_1)$$

Multiplying this sum by $\frac{1}{2}$ doesn't change whether or not it is a solution, so we now know $\Re(\mathbf{y}_1)$ is a solution. **This solution is real!**

Similarly, since the solution space is a subspace, the difference of the two basis elements is also a solution:

$$\mathbf{y}_1 - \overline{\mathbf{y}_1} = 2i\,\Im(\mathbf{y}_1)$$

Multiplying this sum by $\frac{1}{2i}$ doesn't change whether or not it is a solution, so we now know $\Im(\mathbf{y}_1)$ is a solution. **This solution is also real!**

What are these real and imaginary parts? Well, $\mathbf{y}_1 = e^{\lambda t}\mathbf{v}$. Write $\lambda = \alpha + i\beta$ and write $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ to get

$$\begin{aligned}
\mathbf{y}_1 &= e^{(\alpha+i\beta)t}(\mathbf{a}+i\mathbf{b}) \\
&= e^{\alpha t}e^{i\beta t}(\mathbf{a}+i\mathbf{b}) \\
&= e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(\mathbf{a}+i\mathbf{b}) \\
&= e^{\alpha t}\cos(\beta t)\,\mathbf{a} + ie^{\alpha t}\cos(\beta t)\,\mathbf{b} + ie^{\alpha t}\sin(\beta t)\,\mathbf{a} - e^{\alpha t}\sin(\beta t)\,\mathbf{b} \\
&= \Big[e^{\alpha t}\cos(\beta t)\,\mathbf{a} - e^{\alpha t}\sin(\beta t)\,\mathbf{b}\Big] + i\Big[e^{\alpha t}\cos(\beta t)\,\mathbf{b} + e^{\alpha t}\sin(\beta t)\,\mathbf{a}\Big]
\end{aligned}$$

Claim: The two solutions

$$e^{\alpha t}\cos vt\,\mathbf{a} - e^{\alpha t}\sin vt\,\mathbf{b} \quad \text{and } e^{\alpha t}\cos vt\,\mathbf{b} + e^{\alpha t}\sin vt\,\mathbf{a}$$

are linearly independent.

Proof of claim: Compute their Wronskian:

$$\begin{aligned}
W(t) &= \det\begin{pmatrix} a_1e^{\alpha t}\cos(\beta t) - b_1e^{\alpha t}\sin(\beta t) & b_1e^{\alpha t}\cos(\beta t) + a_1e^{\alpha t}\sin(\beta t) \\ a_2e^{\alpha t}\cos(\beta t) - b_2e^{\alpha t}\sin(\beta t) & b_2e^{\alpha t}\cos(\beta t) + a_2e^{\alpha t}\sin(\beta t) \end{pmatrix} \\
&\ \ \vdots \\
&\text{(lots of work with algebra and trig identities that I am omitting)} \\
&\ \ \vdots \\
&= e^{2\alpha t}(a_1b_2 - a_2b_1) \\
&= e^{2\alpha t}\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \\
&= e^{2\alpha t}\det\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}.
\end{aligned}$$

Because $\mathbf{a}$ and $\mathbf{b}$ are the real and imaginary parts coming from non-real eigenvector $\mathbf{v}$, they must be linearly independent (if they weren't, then the eigenvector could be taken to be real, in which case $\overline{\mathbf{v}} = \mathbf{v}$, in which case $\mathbf{v}$ would be an eigenvector corresponding to eigenvalues λ and $\overline{\lambda} \neq \lambda$. But this is impossible, since $A\mathbf{v}$ cannot be both $\lambda\mathbf{v}$ and $\overline{\lambda}\mathbf{v}$, since λ is not real.)

Therefore the determinant $\det\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}$ is never zero, so $W(t) \neq 0$, so the two solutions are linearly independent, proving the claim.

At this point, we have two linearly independent <u>real</u> solutions (namely $\Re(\mathbf{v}_1)$ and $\Im(\mathbf{v}_1)$), which must form a basis of the solution space since the solution space is two-dimensional. To summarize:

Theorem 3.67 (Solution of 2×2 system with complex eigenvalues) *Suppose*

$$\mathbf{y}' = A\mathbf{y}$$

is a 2×2 linear, constant-coefficient system of ODEs where A has non-real eigenvalues $\lambda = \alpha + i\beta$ and $\overline{\lambda} = \alpha - i\beta$. Let the corresponding eigenvectors be $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ and $\overline{\mathbf{v}} = \mathbf{a} - i\mathbf{b}$. Then the solution of the ODE is

$$\begin{aligned}\mathbf{y} &= C_1[e^{\alpha t}\cos(\beta t)\,\mathbf{a} - e^{\alpha t}\sin(\beta t)\,\mathbf{b}] + C_2[e^{\alpha t}\cos(\beta t)\,\mathbf{b} + e^{\alpha t}\sin(\beta t)\,\mathbf{a}].\\ &= C_1\Re(e^{\lambda t}\mathbf{v}) + C_2\Im(e^{\lambda t}\mathbf{v}).\end{aligned}$$

Example 1: Find the general solution of

$$\begin{cases} x' = 4x + y \\ y' = -5x + 2y \end{cases}$$

Solution: Let $A = \begin{pmatrix} 4 & 1 \\ -5 & 2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} x \\ y \end{pmatrix}$, so that the system becomes $\mathbf{y}' = A\mathbf{y}$.

First, find the eigenvalues:

$$0 = \det(A - \lambda I) = \det\begin{pmatrix} 4-\lambda & 1 \\ -5 & 2-\lambda \end{pmatrix} = (4-\lambda)(2-\lambda) + 5 = \lambda^2 - 6\lambda + 13$$

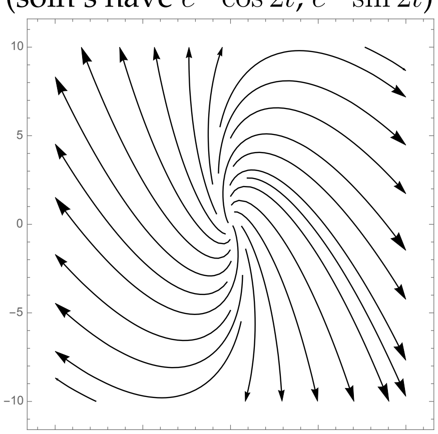
Next, find the corresponding eigenvectors:

Apply Theorem 3.67:

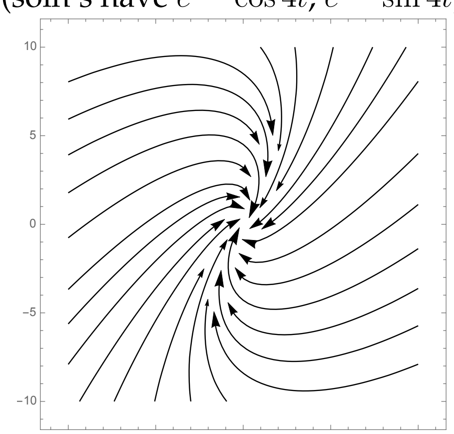
Example 2: Suppose $x'(t) = -11x(t) + 4y(t)$ and $y'(t) = -8x(t) - 3y(t)$. If $x(0) = 2$ and $y(0) = -3$, find $x(t)$ and $y(t)$.

Below are the phase planes for Examples 1 and 2:

Example 1: $\lambda = 3 \pm 2i$
(soln's have $e^{3t} \cos 2t$, $e^{3t} \sin 2t$)



Example 2: $\lambda = -7 \pm 4i$
(soln's have $e^{-7t} \cos 4t$, $e^{-7t} \sin 4t$)



3.12 Repeated eigenvalues

Example:

$$\begin{cases} x' = 3x + 2y \\ y' = 3y \end{cases}$$

Let's try this the same way as our previous examples: let $A = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$ and let $\mathbf{y} = (x, y)$ so that the system is $\mathbf{y}' = A\mathbf{y}$. Then the solution is $\mathbf{y} = e^{At}\mathbf{y}_0$. Let's compute this as usual:

Eigenvalues and eigenvectors of A:

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 2 \\ 0 & 3-\lambda \end{pmatrix} = (3-\lambda)^2 \quad \Rightarrow \quad \lambda = 3$$

This suggests that maybe the solution should look like

$$\mathbf{y} = C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{i.e.} \quad \begin{cases} x = C_1 e^{3t} \\ y = 0 \end{cases}$$

However: our theory tells us that the solution space should be two-dimensional, so it should be spanned by two linearly independent functions (and have two arbitrary constants). We only have one such function in the solution space: e^{3t} (and only one constant). **So this can't be right.**

Actually, this system is fairly easy to solve, because it is uncoupled ("uncoupled" means the y' equation has only y in it, so you can solve the equation for y while ignoring x).

$$\begin{cases} x' = 3x + 2y \\ y' = 3y \end{cases}$$

From the second equation, $y = C_1 e^{3t}$ (exponential growth). Substituting this into the first equation, we get

$$x' = 3x + 2C_1 e^{3t} \quad \Rightarrow \quad x' - 3x = 2C_1 e^{3t}$$

This equation is first-order, linear (but not homogeneous) and can therefore be solved with integrating factors:

Question: What does this solution have to do with eigenvalues and eigenvectors?

Suppose you have a 2×2 system $\mathbf{y}' = A\mathbf{y}$ where A has a repeated eigenvalue λ, whose eigenvector is $\mathbf{v}$. We know one nonzero solution is

$$\mathbf{y}_1 = e^{\lambda t}\mathbf{v};$$

we need to find a second linearly independent solution.

First try: $\mathbf{y}_2 = te^{\lambda t}\mathbf{w}$.
Plug this into the system $\mathbf{y}' = A\mathbf{y}$ and see what happens:

Second try: $\mathbf{y}_2 = te^{\lambda t}\mathbf{w}_1 + e^{\lambda t}\mathbf{w}_2$.
Plug this into the system $\mathbf{y}' = A\mathbf{y}$:

Theorem 3.68 *Suppose $A \in M_2(\mathbb{R})$ is not a scalar multiple of the identity. If A has a repeated eigenvalue λ of multiplicity 2, then the general solution of the system of two ODEs given by $\mathbf{y}' = A\mathbf{y}$ is*

$$\mathbf{y}(t) = C_1 e^{\lambda t}\mathbf{v} + C_2 \left[e^{\lambda t}\mathbf{w} + te^{\lambda t}\mathbf{v}\right]$$

where $\mathbf{v}$ is an eigenvector of A corresponding to eigenvalue λ, and $\mathbf{w}$ is some other nonzero vector satisfying $(A - \lambda I)\mathbf{w} = \mathbf{v}$; $\mathbf{w}$ is called a **generalized eigenvector** *of A corresponding to λ.*

Example: Find (x, y) when $t = 3$, if $x'(t) = 12x(t) + 4y(t)$ and $y'(t) = -x(t) + 16y(t)$, and if $(x, y) = (2, -1)$ when $t = 0$.

3.13 Summary of linear, const.-coeff., homogeneous linear systems

2×2 systems

Suppose $A \in M_2(\mathbb{R})$ and consider the system $\mathbf{y}' = A\mathbf{y}$ where $\mathbf{y} = (x, y)$. The unique solution to such a system with initial value $\mathbf{y}(0) = \mathbf{y}_0$ is always $\mathbf{y}(t) = e^{At}\mathbf{y}_0$; the general solution of the system is given by $\mathbf{y} = C_1\mathbf{y}_1 + C_2\mathbf{y}_2$, where $\mathbf{y}_1$ and $\mathbf{y}_2$ are any two linearly independent solutions of the system. There are four cases which describe all possible solutions:

Case 1: A has two distinct, real eigenvalues λ_1 and λ_2, whose eigenvectors are $\mathbf{v}_1$ and $\mathbf{v}_2$	$\Rightarrow$	The general solution is $\mathbf{y} = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$
Case 2: A has a pair of complex conjugate eigenvalues $\lambda = \alpha + i\beta$, $\overline{\lambda} = \alpha - i\beta$ whose eigenvectors are $\mathbf{v} = \mathbf{a} + i\mathbf{b}$, $\overline{\mathbf{v}} = \mathbf{a} - i\mathbf{b}$	$\Rightarrow$	The general solution is $\mathbf{y} = C_1\left[e^{\alpha t}\cos(\beta t)\,\mathbf{a} - e^{\alpha t}\sin(\beta t)\,\mathbf{b}\right] + C_2\left[e^{\alpha t}\cos(\beta t)\,\mathbf{b} + e^{\alpha t}\sin(\beta t)\,\mathbf{a}\right] = C_1\Re(e^{\lambda t}\mathbf{v}) + C_2\Im(e^{\lambda t}\mathbf{v})$
Case 3: A is not a scalar multiple of the identity, but A has a repeated eigenvalue λ with one eigenvector $\mathbf{v}$	$\Rightarrow$	The general solution is $\mathbf{y} = C_1 e^{\lambda t}\mathbf{v} + C_2\left[e^{\lambda t}\mathbf{w} + te^{\lambda t}\mathbf{v}\right]$ where $\mathbf{w}$ satisfies $(A - \lambda I)\mathbf{w} = \mathbf{v}$
Case 4: $A = \lambda I$ (i.e. A is a scalar multiple of the identity matrix)	$\Rightarrow$	The system is uncoupled; the equations can be solved separately to obtain the general solution $\begin{cases} x = C_1 e^{\lambda t} \\ y = C_2 e^{\lambda t} \end{cases}$

Systems of more than two ODEs

Suppose $A \in M_n(\mathbb{R})$ and consider the system $\mathbf{y}' = A\mathbf{y}$ where $\mathbf{y} = (y_1, ..., y_n)$. The unique solution to such a system with initial value $\mathbf{y}(0) = \mathbf{y}_0$ is always $\mathbf{y}(t) = e^{At}\mathbf{y}_0$; the general solution of the system is given by $\mathbf{y} = C_1\mathbf{y}_1 + C_2\mathbf{y}_2 + ...C_d\mathbf{y}_d$, where $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_d$ are any d linearly independent solutions of the system. To describe the solutions, you need to find the eigenvalues of A. In general:

- You might get some real eigenvalues of multiplicity one, say $\lambda_1, \lambda_2, ..., \lambda_k$, with respective eigenvectors $\mathbf{v}_1, ..., \mathbf{v}_k$.

  Each real eigenvalue λ_j of this type generates a solution of the form $e^{\lambda_j t}\mathbf{v}_j$.

- You might get some complex eigenvalues (coming in pairs of complex conjugates), say
$$\lambda_1, \overline{\lambda_1} = \alpha_1 \pm i\beta_1;\ \lambda_2, \overline{\lambda_2} = \alpha_2 \pm i\beta_2;\ ...;\ \lambda_k, \overline{\lambda_k} = \alpha_k \pm i\beta_k,$$
  whose corresponding eigenvectors are
$$\mathbf{v}_1, \overline{\mathbf{v}_1} = \mathbf{a}_1 \pm i\mathbf{b}_1;\ \mathbf{v}_2, \overline{\mathbf{v}_2} = \mathbf{a}_2 \pm i\mathbf{b}_2;\ ...;\ \mathbf{v}_k, \overline{\mathbf{v}_k} = \mathbf{a}_k \pm i\mathbf{b}_k.$$
  Each pair of eigenvalues $\alpha_j \pm i\beta_j$ of this type generates a pair of solutions
$$\Re(e^{\lambda_j t}\mathbf{v}_j) = e^{\alpha_j t}\cos\beta_j t\,\mathbf{a}_j - e^{\alpha_j t}\sin\beta_j t\,\mathbf{b}_j \text{ and } \Im(e^{\lambda_j t}\mathbf{v}_j) = e^{\alpha_j t}\cos\beta_j t\,\mathbf{b}_j + e^{\alpha_j t}\sin\beta_j t\,\mathbf{a}_j.$$

- You might get some repeated real eigenvalues of multiplicity greater than one, say
$$\lambda_1(\text{multiplicity } m_1), \lambda_2(\text{multiplicity } m_1), ..., \lambda_l(\text{multiplicity } m_l),$$
  where each eigenvalue λ_j has exactly one linearly independent eigenvector $\mathbf{v}_j$. **Each such eigenvalue λ_j of multiplicity m_j generates a list of m_j solutions of the form**
$$e^{\lambda_j t}\mathbf{v}_j, e^{\lambda_j t}\mathbf{w}_1 + te^{\lambda_j t}\mathbf{v}_j, e^{\lambda_j t}\mathbf{w}_2 + te^{\lambda_j t}\mathbf{w}_1 + t^2e^{\lambda_j t}\mathbf{v}_j, ...,$$
$$e^{\lambda t}\mathbf{w}_{m-1} + te^{\lambda_j t}\mathbf{w}_{m-2} + t^2e^{\lambda_j t}\mathbf{w}_{m-3} + ...t^{m_j-1}e^{\lambda_j t}\mathbf{w}_1 + t^{m_j-1}e^{\lambda_j t}\mathbf{v}_j.$$
  where $\mathbf{w}_i$ satisfies $(A - \lambda I)^i\mathbf{w}_i = \mathbf{v}_j$.

- You might get repeated real eigenvalues which have more than one linearly independent eigenvector. This is beyond the scope of Math 330, but for each such eigenvalue λ_j of multiplicity m_j, there are m_j solutions generated by this eigenvalue; each of these is a polynomial in t multiplied by $e^{\lambda_j t}$.

- You might get some repeated complex eigenvalues (coming in pairs of complex conjugates), say
$$\alpha_1 \pm i\beta_1(\text{multiplicity } n_1), \alpha_2 \pm i\beta_2(\text{multiplicity } n_1), ..., \alpha_m \pm i\beta_m(\text{multiplicity } n_m).$$
  This too is beyond the scope of Math 330, but for each pair of eigenvalues of multiplicity m_j, there are $2m_j$ solutions generated by this eigenvalue; each of these solutions is a polynomial in t multiplied by something like the solutions in the second bullet point above.

Each solution of $\mathbf{y}' = A\mathbf{y}$ is a linear combination of the n total functions generated by the eigenvalues of the $n \times n$ matrix A.

Example: Suppose A is some 9×9 matrix whose eigenvalues are $2, -1, 5, 3 \pm 2i$, and $\pm 6i$, where 5 has multiplicity 3 and all others have multiplicity 1. Assuming that there is one linearly independent eigenvector corresponding to each eigenvalue, this tells you that the general solution of $\mathbf{y}' = A\mathbf{y}$ is

$$
\begin{aligned}
\mathbf{y}(t) = C_1 e^{2t}\mathbf{v} \\
+ C_2 e^{-t}\mathbf{v} \\
+ C_3 e^{5t}\mathbf{v} + C_4(e^{5t}\mathbf{w}_1 + te^{5t}\mathbf{v}) + C_5(e^{5t}\mathbf{w}_2 + te^{5t}\mathbf{w}_1 + t^2 e^{5t}\mathbf{v}) \\
+ C_6(e^{3t}\cos 2t\,\mathbf{a} - e^{3t}\cos 2t\,\mathbf{b}) + C_7(e^{3t}\cos 2t\,\mathbf{b} + e^{3t}\cos 2t\,\mathbf{a}) \\
+ C_8(\cos 6t\,\mathbf{a} - \sin 6t\,\mathbf{b}) + C_9(\cos 6t\,\mathbf{b} + \sin 6t\,\mathbf{a}).
\end{aligned}
$$

Note: The $\mathbf{v}$s, $\mathbf{a}$s and $\mathbf{b}$s in different lines of this solution are not the same vectors. If the letters appear in the same horizontal line, they are the same vector.

An example with *Mathematica*

Solve $\mathbf{y} = A\mathbf{y}$, $\mathbf{y}(0) = \mathbf{y}_0$ where

$$A = \begin{pmatrix} 4 & -6 & 6 & 0 & -6 & 6 \\ 6 & -8 & 6 & 0 & -5 & 5 \\ 6 & -6 & 4 & 0 & -5 & 5 \\ 5 & -5 & 0 & 3 & -4 & 3 \\ 2 & -2 & 0 & 2 & -4 & 5 \\ 2 & -2 & 0 & 2 & -2 & 3 \end{pmatrix}; \quad \mathbf{y}_0 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -3 \\ 1 \\ 4 \end{pmatrix}.$$

Step 1: Type in the matrices and save them as something like `A` and `y0`.

Step 2: Find the eigenvalues of A:

In: `Eigenvalues[A]`

Out: `{4, 2+i, 2-i, -2, -2, -2}`

This tells you that the solutions y_j should each be of the form

$$C_1e^{4t} + C_2e^{2t}\cos t + C_3e^{2t}\sin t + C_4e^{-2t} + C_5te^{-2t} + C_6t^2e^{-2t}.$$

Step 3: Compute the solution $\mathbf{y}(t) = e^{At}\mathbf{y}_0$:

In: `y[t_] = MatrixExp[A t].y0`

Out: some stuff

In: `Expand[y[t]] // MatrixForm`

Out:

$$\begin{pmatrix} e^{4t} \\ e^{-2t} + e^{4t} - 3e^{-2t}t \\ -2e^{-2t} + e^{4t} - 3e^{-2t}t \\ -2e^{-2t} - \left(\frac{1}{2} + \frac{5i}{2}\right)e^{(2-i)t} - \left(\frac{1}{2} - \frac{5i}{2}\right)e^{(2+i)t} - 3e^{-2t}t \\ -3e^{-2t} + (2-3i)e^{(2-i)t} + (2+3i)e^{(2+i)t} \\ (2-3i)e^{(2-i)t} + (2+3i)e^{(2+i)}t \end{pmatrix}$$

To simplify this, we need a bit more theory of complex numbers:

Theorem 3.69 *For any real numbers a, b, u and v,*

$$(a+ib)e^{(u+iv)t} + (a-ib)e^{(u-iv)t} = 2ae^{ut}\cos vt - 2be^{ut}\sin vt.$$

PROOF First, notice that for any complex number $z = x + iy$,

$$z + \overline{z} = (x+iy) + (x-iy) = 2x = 2\Re(z).$$

Now, let $w = u + iv$ and $c = a + ib$. Then the left-hand side of the theorem is

$$ce^{wt} + \overline{c}e^{\overline{w}t} = ce^{wt} + \overline{ce^{wt}} = 2\Re(ce^{wt}).$$

Finally,

$$\Re(c^{wt}) = Re((a+ib)(e^{ut}\cos vt + ie^{ut}\sin vt)) = ae^{ut}\cos vt - be^{ut}\sin vt.$$

Put these equations together to prove the theorem. □

Using this theorem, the solution *Mathematica* produced on the previous page:

$$\mathbf{y}(t) = \begin{pmatrix} e^{4t} \\ e^{-2t} + e^{4t} - 3e^{-2t}t \\ -2e^{-2t} + e^{4t} - 3e^{-2t}t \\ -2e^{-2t} - \left(\frac{1}{2} + \frac{5i}{2}\right)e^{(2-i)t} - \left(\frac{1}{2} - \frac{5i}{2}\right)e^{(2+i)t} - 3e^{-2t}t \\ -3e^{-2t} + (2-3i)e^{(2-i)t} + (2+3i)e^{(2+i)t} \\ (2-3i)e^{(2-i)t} + (2+3i)e^{(2+i)}t \end{pmatrix}$$

can (and should) be rewritten as

$$\mathbf{y}(t) = \begin{pmatrix} e^{4t} \\ e^{-2t} + e^{4t} - 3te^{-2t} \\ -2e^{-2t} + e^{4t} - 3te^{-2t} \\ -2e^{-2t} - e^{2t}\cos t - 5e^{2t}\sin t - 3te^{-2t} \\ -3e^{-2t} + 4e^{2t}\cos t - 6e^{2t}\sin t \\ 4e^{2t}\cos t - 6e^{2t}\sin t \end{pmatrix}.$$

3.14 Non-homogeneous systems

Recall from Chapter 2 that to solve a first-order, linear, constant-coefficient ODE that is not homogeneous, you use the method of undetermined coefficients:

Example: $y' + 4y = e^t$

Solution: First, the corresponding homogeneous is $y' + 4y = 0$, i.e. $y' = -4y$ which has solution $y_h = Ce^{-4t}$.

Now, "guess" a particular solution. Let's try $y_p = Ae^t$. Plugging this into the left-hand side of the original equation, we have

$$\begin{aligned}(Ae^t)' + 4(Ae^t) &= e^t \\ \Rightarrow 5Ae^t &= e^t \\ \Rightarrow A &= \frac{1}{5}\end{aligned}$$

Therefore $y_p = \frac{1}{5}e^t$, so the general solution is $y(t) = y_p + y_h = \frac{1}{5}e^t + Ce^{-4t}$.

This same technique works for systems of the form $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$ where A has constant coefficients. First, solve the corresponding homogeneous system $\mathbf{y}' = A\mathbf{y}$ using the techniques of the previous sections; then find a particular solution $\mathbf{y}_p$ using the method of undetermined coefficients.

Example: Find the general solution of the system

$$\begin{cases} x' = x - 3y + e^{2t} \\ y' = 4x - 6y \end{cases}$$

Solution: First, establish notation: define

$$A = \begin{pmatrix} 1 & -3 \\ 4 & -6 \end{pmatrix} \quad \mathbf{q} = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

This means we are trying to solve the system

$$\mathbf{y}' = A\mathbf{y} + \mathbf{q}$$

so we start with the corresponding homogeneous system

$$\mathbf{y}' = A\mathbf{y}$$

Eigenvalues of A:

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & -3 \\ 4 & -6-\lambda \end{pmatrix} = (1-\lambda)(-6-\lambda) + 12$$
$$= \lambda^2 + 5\lambda + 6$$
$$= (\lambda+2)(\lambda+3)$$
$$\Rightarrow \lambda = -2, \lambda = -3$$

Eigenvectors:

$$\lambda = -2 : A\mathbf{x} = -2\mathbf{x} \Rightarrow \begin{cases} x - 3y = -2x \\ 4x - 6y = -2y \end{cases} \Rightarrow x = y \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = -3 : A\mathbf{x} = -3\mathbf{x} \Rightarrow \begin{cases} x - 3y = -3x \\ 4x - 6y = -3y \end{cases} \Rightarrow 4x = 3y \Rightarrow \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Therefore the solution of the homogeneous is

$$\mathbf{y}_h = C_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Undetermined coefficients: Now we find $\mathbf{y}_p$. Since

$$\mathbf{q} = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}, \text{ let's try } \mathbf{y}_p = \begin{pmatrix} Ae^{2t} \\ Be^{2t} \end{pmatrix}.$$

We have

$$\mathbf{y}_p' = \begin{pmatrix} 2Ae^{2t} \\ 2Be^{2t} \end{pmatrix};$$

this should equal

$$A\mathbf{y}_p + \mathbf{q} = \begin{pmatrix} 1 & -3 \\ 4 & -6 \end{pmatrix} \begin{pmatrix} Ae^{2t} \\ Be^{2t} \end{pmatrix} + \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} = \begin{pmatrix} (A - 3B + 1)e^{2t} \\ (4A - 6B)e^{2t} \end{pmatrix}.$$

Equating terms of $\mathbf{y}_p'$ with those of $A\mathbf{y}_p + \mathbf{q}$, we get

$$\begin{cases} 2A = A - 3B + 1 \\ 2B = 4A - 6B \end{cases}$$

From the second equation, $A = 2B$, and from the first equation, we get $5B = 1$ so $B = \frac{1}{5}$ (and therefore $A = 2B = \frac{2}{5}$).

Therefore

$$\mathbf{y}_p = \begin{pmatrix} \frac{2}{5}e^{2t} \\ \frac{1}{5}e^{2t} \end{pmatrix}$$

and the general solution of the original system is

$$\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p = \begin{pmatrix} \frac{2}{5}e^{2t} + C_1 e^{-2t} + 3C_2 e^{-3t} \\ \frac{1}{5}e^{2t} + C_1 e^{-2t} + 4C_2 e^{-3t} \end{pmatrix},$$

i.e.

$$\begin{cases} x(t) = \frac{2}{5}e^{2t} + C_1 e^{-2t} + 3C_2 e^{-3t} \\ y(t) = \frac{1}{5}e^{2t} + C_1 e^{-2t} + 4C_2 e^{-3t} \end{cases}.$$

3.15 Classification of equilibria

Recall from Chapter 1 that to classify an equilibrium y_0 of a single first-order autonomous equation $y' = \phi(y)$, you find the sign of the derivative $\phi'(y_0)$:

$$\phi'(y_0) > 0 \Leftrightarrow y_0 \text{ is unstable}$$
$$\phi'(y_0) < 0 \Leftrightarrow y_0 \text{ is stable}$$
$$\phi'(y_0) = 0 \text{ and } \phi''(y_0) \neq 0 \Leftrightarrow y_0 \text{ is semistable}$$

So classifying the equilibria of a system $\mathbf{y}' = \Phi(\mathbf{y})$ should have something to do with the "derivative" of Φ. But Φ is a function of several variables, which has several components. What is the "derivative" of Φ?

Definition 3.70 *Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a function of the form*

$$\Phi(x_1, ...x_d) = (\phi_1(x_1, ...x_d), \phi_2(x_1, ...x_d), ..., \phi_d(x_1, ...x_d)).$$

The **total derivative** *of Φ is the $d \times d$ matrix*

$$D\Phi = D\Phi(x_1, ..., x_d) = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_d} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \phi_d}{\partial x_1} & \frac{\partial \phi_d}{\partial x_2} & \cdots & \frac{\partial \phi_d}{\partial x_d} \end{pmatrix}.$$

Example: Find the total derivative of Φ, where $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\Phi(x, y) = (3x^2 + 2xy, 4xy^3 - 2x^2 y).$$

Example: Find $D\Phi(\pi, 0, \frac{\pi}{2})$ of $\Phi = (\phi_1, \phi_2, \phi_3)$, if

$$\phi_1(x, y, z) = 2\sin x + \cos y; \phi_2(x, y, z) = 3\cos x - 2\sin z;$$

$$\phi_3(x, y, z) = 3\cos x - 2\sin y + 5\cos z.$$

Example: Suppose $\Phi(\mathbf{y}) = A\mathbf{y}$, where $A \in M_d(\mathbb{R})$. Find $D\Phi$.

For a system, you classify an equilibrium $\mathbf{y}_0$ of a system of first-order autonomous systems by *finding the signs of the eigenvalues of the total derivative* $D\Phi(\mathbf{y}_0)$:

Definition 3.71 *An equilibrium solution* $\mathbf{y} = \mathbf{y}_0$ *of an autonomous system is called* **stable** *(or* **asymptotically stable** *or* **attracting** *or a* **sink***) if there is an open disk* E *(an* **open disk** *is the set of points inside a circle) of initial values, centered at* $\mathbf{y}_0$*, such that if* $\mathbf{y}(t_0) \in E$*,*

$$\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{y}_0.$$

An equilibrium solution $\mathbf{y} = \mathbf{y}_0$ *of an autonomous system is called* **semistable** *(or* **neutral***) if it is not stable, but there is an open disk* E *(an* **open disk** *is the set of points inside a circle) of initial values, centered at* $\mathbf{y}_0$*, such that if* $\mathbf{y}(t_0) \in E$*, then* $\mathbf{y}(t)$ *stays close to* $\mathbf{y}_0$ *for all* $t \geq 0$*.*

An equilibrium solution $\mathbf{y} = \mathbf{y}_0$ *of an autonomous system is called* **unstable** *(or* **asymptotically unstable***) if it is neither stable nor semistable.*

To classify equilibria, we look at the signs of the (real parts of the) eigenvalues of $D\Phi(\mathbf{y}_0)$ (these play the same role as the signs of $\phi'(y_0)$ for single equations):

Theorem 3.72 (Classification of equilibria) *Suppose $\mathbf{y} = \mathbf{y}_0$ is an equilibrium solution of autonomous ODE $\mathbf{y}' = \Phi(\mathbf{y})$. Then:*

1. $\Phi(\mathbf{y}_0) = \mathbf{0}$*;*
2. *If the real part of every eigenvalue of $D\Phi$ is negative, then $\mathbf{y}_0$ is stable.*
3. *If the real part of any eigenvalue of $D\Phi$ is positive, then $\mathbf{y}_0$ is unstable.*
4. *If $D\Phi$ has only eigenvalues with nonpositive real parts, and has at least one eigenvalue with zero real part, then $\mathbf{y}_0$ is semistable.*

Example: Consider the autonomous (but not linear) 2×2 system of ODEs

$$\begin{cases} x'(t) = x^2 - 2x \\ y'(t) = 4 - (x+2)(y-1) \end{cases}$$

Find the equilibria of this system and classify them as stable, unstable or semistable.

Note that in the previous problem, the eigenvalues of $D\Phi(\mathbf{y}_0)$ could be read off from the matrix. This is because the matrix is triangular:

Theorem 3.73 *If A is a triangular matrix, then the eigenvalues of A are its diagonal entries.*

PROOF See Chapter 8 of my Math 322 lecture notes.

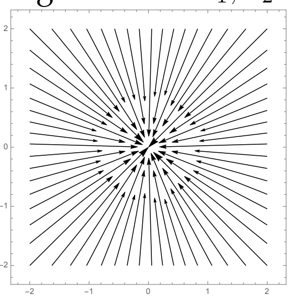
Example: Find and classify all the equilibria of the 2×2 system

$$\begin{cases} x' = x - y + 1 \\ y' = y(x-2) \end{cases}$$

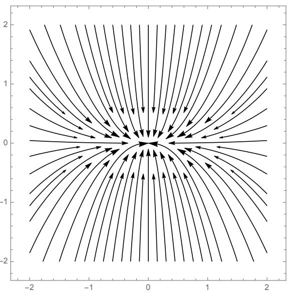
CLASSES OF STABLE EQUILIBRIA:

Stable node (two negative, real eigenvalues)

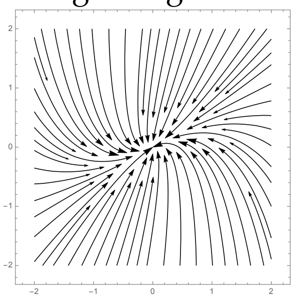
$\lambda_1 = \lambda_2 < 0$
w/ 2 lin. indep.
eigenvectors $\mathbf{v}_1, \mathbf{v}_2$



$\lambda_1 < \lambda_2 < 0$

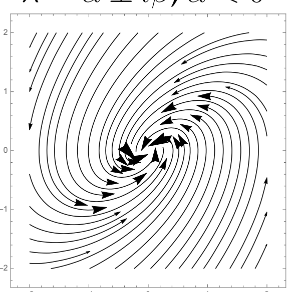


$\lambda_1 = \lambda_2 < 0$
w/ 1 eigenvector $\mathbf{v}$
and 1 gen. eigenvec. $\mathbf{w}$

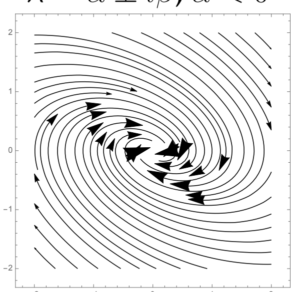


Stable spiral (two non-real eigenvalues with negative real part)

$\lambda = \alpha \pm i\beta, \alpha < 0$



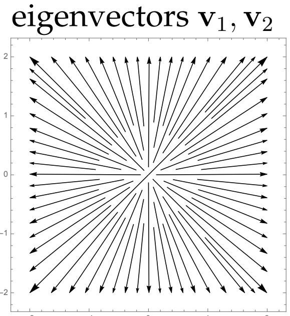
$\lambda = \alpha \pm i\beta, \alpha < 0$



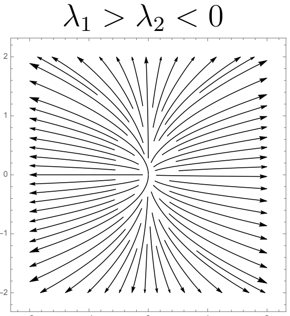
CLASSES OF UNSTABLE EQUILIBRIA:

Unstable node (two positive, real eigenvalues)

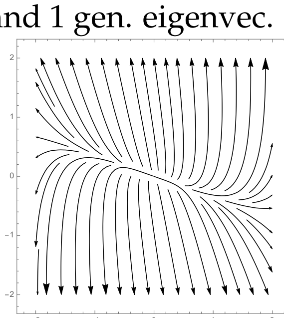
$\lambda_1 = \lambda_2 > 0$
w/ 2 lin. indep.
eigenvectors $\mathbf{v}_1, \mathbf{v}_2$



$\lambda_1 > \lambda_2 < 0$

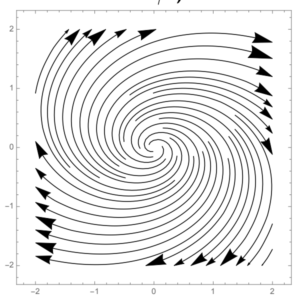


$\lambda_1 = \lambda_2 > 0$
w/ 1 eigenvector $\mathbf{v}$
and 1 gen. eigenvec. $\mathbf{w}$

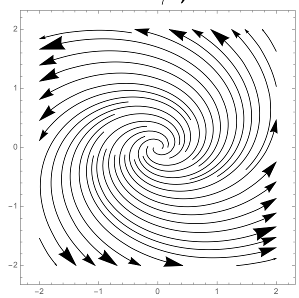


Unstable spiral (two non-real eigenvalues with positive real part)

$\lambda = \alpha \pm i\beta, \alpha > 0$

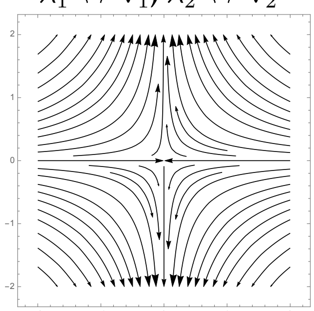


$\lambda = \alpha \pm i\beta, \alpha > 0$

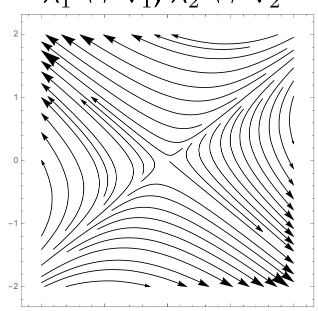


Saddle (one positive, real eigenvalue and one negative, real eigenvalue)

$\lambda_1 > 0, \lambda_2 < 0$
$\lambda_1 \leftrightarrow \mathbf{v}_1$, $\lambda_2 \leftrightarrow \mathbf{v}_2$



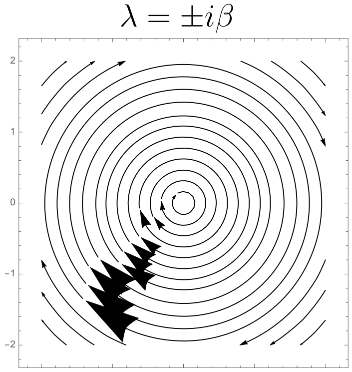
$\lambda_1 > 0, \lambda_2 < 0$
$\lambda_1 \leftrightarrow \mathbf{v}_1$, $\lambda_2 \leftrightarrow \mathbf{v}_2$



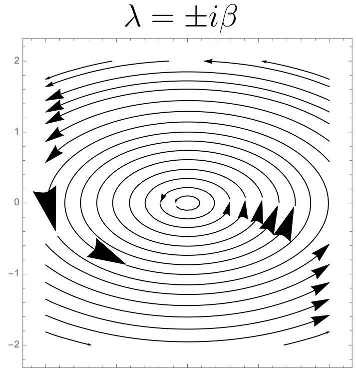
CLASSES OF SEMISTABLE EQUILIBRIA:

Center (two non-real eigenvalues with zero real part)

$\lambda = \pm i\beta$



$\lambda = \pm i\beta$



3.16 The trace-determinant plane

Recall that an equilibrium $\mathbf{y}_0$ of an autonomous equation $\mathbf{y}' = \Phi(\mathbf{y})$ can be classified by looking at the eigenvalues of its determinant.

Suppose $\mathbf{y}' = \Phi(\mathbf{y})$ is a 2×2 system and that $\mathbf{y}_0$ is an equilibrium of this system. It turns out that you can classify this equilibrium without finding the eigenvalues of $D\Phi(\mathbf{y}_0)$: you only need to compute the trace and determinant of $D\Phi(\mathbf{y}_0)$.

Suppose $D\Phi(\mathbf{y}_0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the eigenvalues of $D(\Phi(\mathbf{y}_0))$ can be computed as usual, in terms of a, b, c and d:

$$\begin{aligned} 0 = \det(D\Phi(\mathbf{y}_0) - \lambda I) &= \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= ad - a\lambda - d\lambda + \lambda^2 - bc \\ &= \lambda^2 - (a+d)\lambda + (ad - bc) \\ &= \lambda^2 - (trA)\lambda + \det A. \end{aligned}$$

Solve for λ using the quadratic formula to get

$$\lambda = \frac{trA \pm \sqrt{(trA)^2 - 4\det A}}{2} = \frac{t \pm \sqrt{t^2 - 4d}}{2}$$

where $t = trA$ and $d = \det A$.

Suppose the equilibrium is a saddle. That means one of the eigenvalues is positive and one is negative, i.e.

In this case, the product of the eigenvalues is

$$\left(\frac{t + \sqrt{t^2 - 4d}}{2}\right)\left(\frac{t - \sqrt{t^2 - 4d}}{2}\right) = \frac{t^2 - (t^2 - 4d)}{4} = d.$$

This shows that *the equilibrum is a saddle if and only if* $d < 0$.

From the previous page,

$$\lambda = \frac{trA \pm \sqrt{(trA)^2 - 4\det A}}{2} = \frac{t \pm \sqrt{t^2 - 4d}}{2}$$

where $t = trA$ and $d = \det A$.

Next, recall that the eigenvalues of a matrix sum to the trace t. So if the eigenvalues have the same sign, their sign will coincide with the sign of the trace.

Last, take a look at the expression under the square root:

1. if $d > \frac{1}{4}t^2$, then the number under the square root is negative. This means the eigenvalues are (non-real) complex numbers with real part $\frac{trA}{2}$, so the equilibrium is

2. if $d = \frac{1}{4}t^2$, then the number under the square root is zero. This means the eigenvalue $\frac{trA}{2}$ is real and repeated, so the equilibrium is a

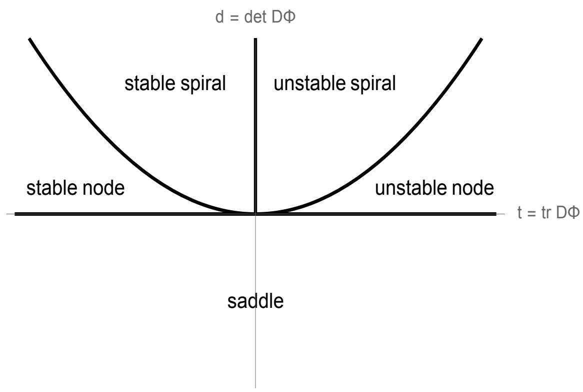
3. if $0 < d < \frac{1}{4}t^2$, then the number under the square root is positive. This means there are two real eigenvalues which must have the same sign since their product is d (which is positive). This makes the equilibrium a node.

To summarize, we have proven this statement:

Theorem 3.74 (Trace-determinant plane) *Suppose $\mathbf{y}_0$ is an equilibrium of a 2×2 autonomous system $\mathbf{y}' = \Phi(\mathbf{y})$. Let $t = trD\Phi(\mathbf{y}_0)$ and let $d = \det D\Phi(\mathbf{y}_0)$.*

1. *If $d < 0$, then $\mathbf{y}_0$ is a saddle.*
2. *If $0 < d \leq \frac{t^2}{4}$, then $\mathbf{y}_0$ is a node. The node is stable if $t < 0$ and unstable if $t > 0$.*
3. *If $d > \frac{t^2}{4}$, then $\mathbf{y}_0$ is a spiral or center. $\mathbf{y}_0$ is a stable spiral if $t < 0$, a center if $t = 0$, and an unstable spiral if $t > 0$.*

The previous theorem is sometimes described with the following picture, called the **trace-determinant plane**:



Example: Suppose $\mathbf{y}_0$ is an equilibrium of some 2×2 system $\mathbf{y}' = \Phi(\mathbf{y})$ and $D\Phi(\mathbf{y}_0) = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$. Classify this equilibrium.

Example: Suppose $\mathbf{y}_0$ is an equilibrium of some 2×2 system $\mathbf{y}' = \Phi(\mathbf{y})$ such that $tr D\Phi(\mathbf{y}_0) = 3$ and $\det D\Phi(\mathbf{y}_0) = 11$. Classify this equilibrium.

Example: Suppose $\mathbf{y}_0$ is an equilibrium of some 2×2 system $\mathbf{y}' = \Phi(\mathbf{y})$ such that $tr D\Phi(\mathbf{y}_0) = 0$ and $\det D\Phi(\mathbf{y}_0) = -2$. Classify this equilibrium.

3.17 Applications of first-order systems

Compartmental models

Example: Consider two tanks (say A and B) connected by two pipes which pump fluid between the tanks. Each tank holds 24 liters of a brine solution. Fresh water flows into tank A at a rate of 6 L/min, and fluid is drained out of tank B at the same rate. Suppose that 8 L of fluid per minute is being pumped from tank A to tank B, and 2 L/min of fluid per minute is pumped from tank B to tank A. If the tanks are kept well-stirred, and if tank A initially contains 3 kg of salt, and tank B initially contains 1 kg of salt, find the amount of salt in each tank at time $t > 0$.

This yields the following linear, constant-coefficient system of ODEs:

$$\begin{cases} A'(t) = \frac{-1}{3}A + \frac{1}{12}B \\ B'(t) = \frac{1}{3}A - \frac{1}{3}B \end{cases} \quad \Leftrightarrow \quad \mathbf{y}' = \begin{pmatrix} \frac{-1}{3} & \frac{1}{12} \\ \frac{1}{3} & \frac{-1}{3} \end{pmatrix} \mathbf{y} \qquad \text{initial value } \mathbf{y}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

From the previous page:

$$\begin{cases} A'(t) = \frac{-1}{3}A + \frac{1}{12}B \\ B'(t) = \frac{1}{3}A - \frac{1}{3}B \end{cases} \quad \Leftrightarrow \quad \mathbf{y}' = \begin{pmatrix} \frac{-1}{3} & \frac{1}{12} \\ \frac{1}{3} & \frac{-1}{3} \end{pmatrix}\mathbf{y}; \qquad \mathbf{y}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Eigenvalues:

$$0 = \det\begin{pmatrix} \frac{-1}{3} - \lambda & \frac{1}{12} \\ \frac{1}{3} & \frac{-1}{3} - \lambda \end{pmatrix} = \lambda^2 + \frac{2}{3}\lambda + \frac{1}{12} = \left(\lambda + \frac{1}{2}\right)\left(\lambda + \frac{1}{6}\right) \Rightarrow \lambda = \frac{-1}{2}, \lambda = \frac{-1}{6}$$

Eigenvectors:

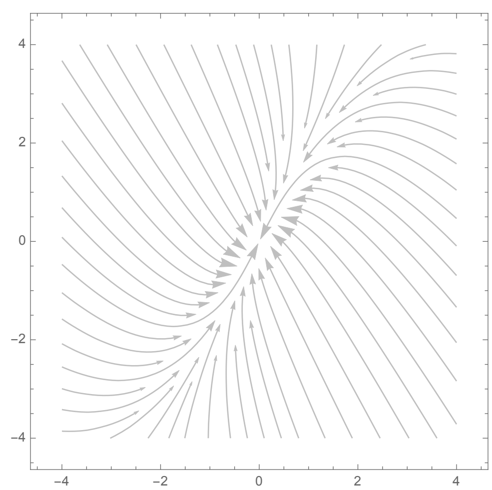
$$\lambda = \frac{-1}{2} \leftrightarrow \begin{pmatrix} -1 \\ 2 \end{pmatrix} \qquad\qquad \lambda = \frac{-1}{6} \leftrightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

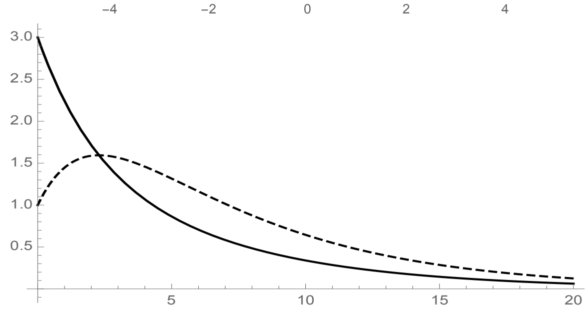
General solution:

$$\mathbf{y} = C_1 e^{(-1/2)t}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + C_2 e^{(-1/6)t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ i.e. } \begin{cases} A(t) = -C_1 e^{-t/2} + C_2 e^{-t/6} \\ B(t) = 2C_1 e^{-t/2} + 2C_2 e^{-t/6} \end{cases}$$

Find the particular solution using the initial condition:

$$\mathbf{y}_0 = \begin{pmatrix} -C_1 + C_2 \\ 2C_1 + 2C_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} C_1 = -1.25 \\ C_2 = 1.75 \end{matrix} \quad \Rightarrow \quad \begin{cases} A(t) = 1.25e^{-t/2} + 1.75e^{-t/6} \\ B(t) = -2.5e^{-t/2} + 3.5e^{-t/6} \end{cases}$$





Predator-prey population dynamics

We started this chapter with an example taken from the population dynamics of interacting species. Here is a more formal version of this setup:

Let

$$\begin{aligned} R(t) &= \text{ rabbit population at time } t \text{ (i.e. prey)} \\ L(t) &= \text{ lynx population at time } t \text{ (i.e. predators)} \end{aligned}$$

From the discussion at the beginning of the chapter, we saw that the compartments behaved as follows:

where a, b, c and d are positive constants. Thus we have the following system of ODEs, called the **Lotka-Volterra equations**:

$$\begin{cases} R' = aR - bRL \\ L' = cRL - dL \end{cases}$$

This system is not linear, but it is autonomous, so we can study it via phase planes and equilibria analysis.

Equilibria:

Classification of equilibria: First, find the total derivative:

$$\Phi(R, L) = (aR - bRL, cRL - dL) \Rightarrow D\Phi = \begin{pmatrix} \frac{\partial \phi_1}{\partial R} & \frac{\partial \phi_1}{\partial L} \\ \frac{\partial \phi_2}{\partial R} & \frac{\partial \phi_1}{\partial L} \end{pmatrix} = \begin{pmatrix} a - bL & -bR \\ cL & cR - d \end{pmatrix}.$$

Equilibrium $(0,0)$*:*

$$D\Phi(0,0) = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix} \Rightarrow \text{Eigensystem } \lambda = a \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \lambda = -d \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Therefore $(0,0)$ is a saddle (one positive eigenvalue, one negative eigenvalue).

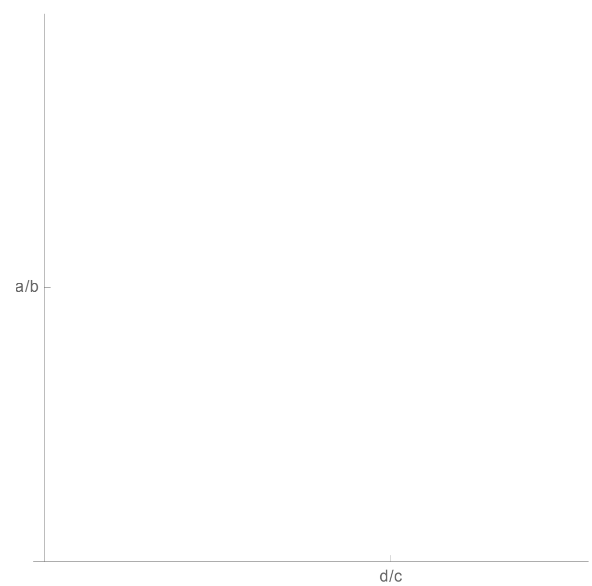
Equilibrium $\left(\frac{d}{c}, \frac{a}{b}\right)$*:*

$$D\Phi\left(\frac{d}{c}, \frac{a}{b}\right) = \begin{pmatrix} a - b(a/b) & -b(d/c) \\ c(a/b) & c(d/c) - d \end{pmatrix} = \begin{pmatrix} 0 & -bd/c \\ ca/b & 0 \end{pmatrix}$$

$$\Rightarrow \text{Eigenvalues } \lambda = \pm i\sqrt{ad}$$

Therefore $\left(\frac{d}{c}, \frac{a}{b}\right)$ is a center.

Phase plane analysis:



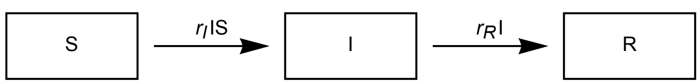
SIR model of disease spread

The study of how diseases spread across a population is called **epidemiology**. Let's consider an example which shows how systems of ODEs are used in this context.

Setup: Consider a contagious disease (think chicken pox) where once people get the disease, they recover; if they recover, they build up an immunity to the disease and can't get the disease again. This divides the population into three classes:

$$\begin{aligned} S(t) &= \text{ fraction of susceptible individuals at time } t \\ I(t) &= \text{ fraction of infective individuals at time } t \\ R(t) &= \text{ fraction of recovered (or dead) individuals at time } t \end{aligned}$$

We assume that the disease acts more quickly than the population can reproduce, so the total population $S(t)+I(t)+R(t)$ is constant, and always equal to 1 (therefore the birth rate is assumed to be zero). That means that there are really only two unknown functions ($S(t)$ and $I(t)$), because you can always find $R(t)$ by taking $1 - S(t) - I(t)$. This leads to the following compartmental model, called the **SIR model**:



In this model:

$$\begin{aligned} r_I &= \text{ rate of infection} \\ r_R &= \text{ rate at which infected individuals recover (or die)} \end{aligned}$$

This model translates into the following system of ODEs:

$$\begin{cases} \frac{dS}{dt} = -r_I IS \\ \frac{dI}{dt} = r_I IS - r_R I \end{cases}$$

(We also know that $\frac{dR}{dt} = r_R I$, but we don't need this since $S + I + R = 1$.)

From the previous page:

$$\begin{cases} \frac{dS}{dt} = -r_I I S \\ \frac{dI}{dt} = r_I I S - r_R I \end{cases}$$

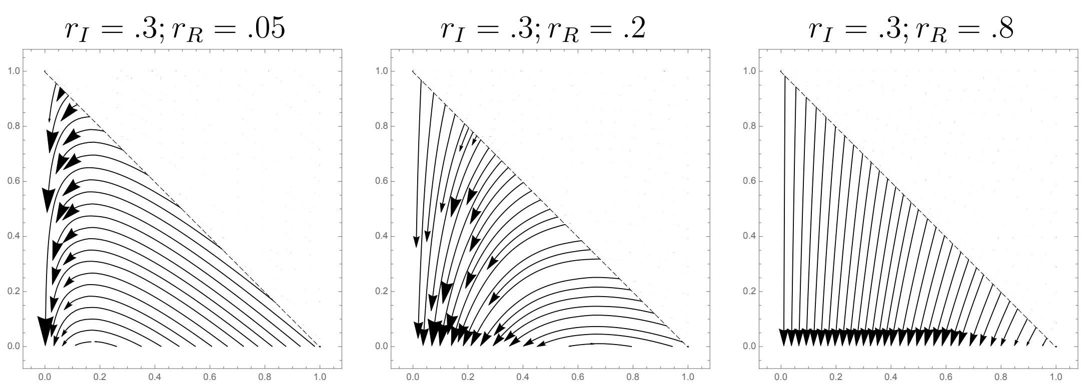
This system is not linear, but it is autonomous.

Equilibria:

$$\begin{cases} 0 = -r_I I S \\ 0 = I(r_I S - r_R) \quad \Rightarrow I = 0 \text{ or } S = \frac{r_R}{r_I} \end{cases}$$

So every point with $I = 0$ is an equilibrium of his system.

Example phase planes:



SIR models can be adapted to account for diseases that take longer to act (in which case there are births and deaths in the population). This is done in the homework exercises.

3.18 Exam 2 Review

On this exam, you may use: one 4″ × 6″ note card with whatever you want on both sides.

What should you expect to be asked on the exam:

- Solve 2×2 constant-coefficient, linear ODEs and/or IVPs
- Find and classify equilibria of $2{\times}2$ autonomous systems (using trace-determinant plane if necessary)
- Analyze pictures of phase planes
- Perform one or two steps of Euler's method for systems, given a simple equation
- Applications / story problems (interconnected tanks, population models, etc.)
- Answer question(s) on theory and/or vocabulary

NOT on the exam:

- Eigenvalues of 3×3 or larger matrices
- Computing matrix exponentials (unless you choose to use them)
- Matrix operations (unless you choose to use them)
- Spans / subspaces / linear independence / basis / Wronskian / other linear algebra issues
- Picard's method for systems / integral equation for systems
- Questions that require you to know any *Mathematica* code

Some practice questions:

1. a) What does it mean for a $d \times d$ system of ODEs to be "linear"?

   b) What is the form of the general solution of a $d \times d$ first-order, linear system of ODEs?

   c) What is Euler's formula? (*Note:* this is not Euler's method) What is the importance of Euler's formula in differential equations?

2. Consider the initial value problem

$$\begin{cases} \mathbf{y}' = (y + t, 2y - x) \\ \mathbf{y}(0) = (2, 1) \end{cases}$$

   where $\mathbf{y} = (x, y)$. Estimate $\mathbf{y}(4)$ by using Euler's method with two steps.

3. Find and classify the equilibria of the autonomous system

$$\begin{cases} x' = 2x - xy \\ y' = xy - 3y + 4x - 12 \end{cases}.$$

4. Suppose $\mathbf{y}_0$ and $\mathbf{y}_1$ are the two equilibria of some 2×2 system $\mathbf{y}' = \Phi(\mathbf{y})$. If $D\Phi(\mathbf{y}_0)$ has trace 8 and determinant 12, and if $D\Phi(\mathbf{y}_0)$ has trace 8 and determinant 18, classify the equilibria $\mathbf{y}_0$ and $\mathbf{y}_1$.

5. Find the general solution of each system. In addition:

   - In parts (a) and (c), write your answer coordinate-wise, i.e. as $\begin{cases} x(t) = \text{ something} \\ y(t) = \text{ something} \end{cases}$.
   - In part (b), give a rough sketch of the phase plane of the system.
   - In parts (b), (c) and (e), classify the equilibrium $\mathbf{0}$ as a node, spiral, saddle or center.

   a) $\begin{cases} x' = -4x + 9y \\ y' = -4x - 16y \end{cases}$

   b) $\begin{cases} x' = 7x + 8y \\ y' = 16x - y \end{cases}$

   c) $\begin{cases} x' = 3x - 8y \\ y' = 4x - 5y \end{cases}$

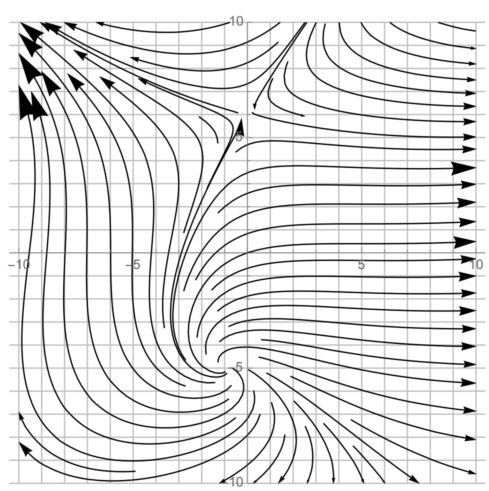
   d) $\begin{cases} x' = -x + 3y + 12e^{-t} \\ y' = 3x - y \end{cases}$

   e) $\begin{cases} x' = -3x + 13y \\ y' = -10x + 3y \end{cases}$

   f) $\mathbf{y}' = A\mathbf{y}$ where A is a 3×3 matrix with eigenvalues $\lambda = 2$, $\lambda = -3$ and $\lambda = 5$ and respective eigenvectors $(1, 2, -3)$, $(1, -1, 5)$ and $(0, -2, 1)$.

6. Find the particular solution of this initial value problem: $\begin{cases} \mathbf{y}' = (-13x + 2y, 3x - 8y) \\ \mathbf{y}(0) = (5, 2) \end{cases}$

7. Here is the phase plane corresponding to some 2×2 autonomous system of ODEs $\mathbf{y}' = \Phi(\mathbf{y})$:



   a) Sketch the graph of the solution of this system satisfying $\mathbf{y}(0) = (-6, -4)$.

b) Suppose $\mathbf{y} = (x, y)$ is the solution of this system satisfying $\mathbf{y}(0) = (6, 2)$. Find $\lim_{t\to-\infty} y(t)$.

c) Suppose $\mathbf{y} = \mathbf{f}(t)$ is the solution of this system satisfying $\mathbf{f}(0) = (4, 8)$. Find r such that $(8, r)$ is on the graph of $\mathbf{f}$.

d) Find all saddles of this system (if any). Classify each saddle as stable or unstable.

e) Find all nodes of this system (if any). Classify each node as stable or unstable.

f) Find all spirals of this system (if any). Classify each spiral as stable or unstable.

8. Two large tanks (call them X and Y) each hold 240 L of liquid. They are interconnected by a pipe which pumps liquid from tank X to tank Y at a rate of 8 L/min. A brine solution of concentration 0.1 kg/L of salt flows into tank X at a rate of 10 L/min; the solution flows out of the system of tanks via two pipes (one pipe allows flow out of tank X at 2 L/min and another pipe allows flow out of tank Y at 8 L/min). Suppose that initially, tank Y contains pure water but tank X contains 60 kg of salt; assume that at all times the liquids in each tank are kept mixed.

   a) Draw a compartmental diagram that models this situation.

   b) Write down the initial value problem which models this situation, clearly defining your variables.

   c) Solve the initial value problem you wrote down in part (a).

   d) Find the amount of salt in tank Y at time 240.

Solutions

WARNING: as always, these might have errors.

1. a) A $d \times d$ system of ODEs is linear if it can be written in the form

$$A_n\mathbf{y}^{(n)} + A_{n-1}\mathbf{y}^{(n-1)} + ... + A_1\mathbf{y}' + A_0\mathbf{y} = \mathbf{q}$$

   where the A_j are $d \times d$ matrices whose entries are functions of t, and $\mathbf{q}$ is a vector whose entries are functions of t.

   b) The general solution of a $d \times d$, first-order linear system $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$ is of the form

$$\begin{aligned}\mathbf{y} &= \mathbf{y}_p + \mathbf{y}_h \\ &= \mathbf{y}_p + (C_1\mathbf{y}_1 + C_2\mathbf{y}_2 + ...C_d\mathbf{y}_d)\end{aligned}$$

where $\mathbf{y}_p$ is any one particular solution of the system, and $\mathbf{y}_1, ..., \mathbf{y}_d$ are linearly independent solutions of the corresponding homogeneous system $\mathbf{y}' = A\mathbf{y}$.

c) Euler's formula says that for any complex number θ, $e^{i\theta} = \cos\theta + i\sin\theta$. This formula explains why a system of differential equations $\mathbf{y}' = A\mathbf{y}$ has solutions with cosines and sines in them whenever A has non-real eigenvalues.

2. Think of the system as $\mathbf{y}' = \Phi(t, \mathbf{y})$; $\mathbf{y}_0 = \mathbf{y}(t_0)$. First, $\Delta t = \frac{t_n - t_0}{n} = \frac{4-0}{2} = 2$. Now $\Phi(t_0, \mathbf{y}_0) = \Phi(0, (2,1)) = (1+0, 2(1)-2) = (1,0)$ so

$$\begin{cases} t_1 = t_0 + \Delta t = 0 + 2 = 2 \\ \mathbf{y}_1 = \mathbf{y}_0 + \Phi(\mathbf{y}_0)\Delta t = (2,1) + (1,0)2 = (4,1). \end{cases}$$

Next, $\Phi(t_1, \mathbf{y}_1) = \Phi(2, (4,1)) = (2+1, 2(1)-4) = (3,-2)$ so

$$\begin{cases} t_2 = t_1 + \Delta t = 2 + 2 = 4 \\ \mathbf{y}_2 = \mathbf{y}_1 + \Phi(\mathbf{y}_1)\Delta t = (4,1) + (3,-2)2 = (10,-3). \end{cases}$$

Thus $\mathbf{y}(4) \approx (10, -3)$.

3. Thinking of the system as $\mathbf{y}' = \Phi(\mathbf{y})$, we set $\Phi(\mathbf{y}) = \mathbf{0}$ and solve for $\mathbf{y}$. Start with the first equation:

$$0 = 2x - xy = x(2-y) \Rightarrow x = 0 \text{ or } y = 2.$$

If $x = 0$, plugging in the second equation gives $-3y - 12 = 0$, i.e. $y = -4$, so one equilibrium is $(0,-4)$. If $y = 2$, plugging in the second equation gives $2x - 6 + 4x - 12 = 0$, i.e. $x = 3$, so the other equilibrium is $(3,2)$. To classify the equilibria, find the eigenvalues of $D\Phi$:

$$D\Phi = \begin{pmatrix} 2-y & -x \\ y+4 & x-3 \end{pmatrix};$$

$D\Phi(0,-4) = \begin{pmatrix} 6 & 0 \\ 0 & -3 \end{pmatrix}$; this matrix has negative determinant, so $(0,-4)$ is an **unstable saddle**.

$D\Phi(3,2) = \begin{pmatrix} 0 & -3 \\ 6 & 0 \end{pmatrix}$; this matrix has trace 0 and positive determinant, so $(3,2)$ is a **center**.

4. Let $t = tr(D\Phi(\mathbf{y}_0))$ and $d = \det(D\Phi(\mathbf{y}_0))$; we have $t = 8$ and $d = 12$. Since $d < \frac{t^2}{4}$ and $t > 0$, $\mathbf{y}_0$ is an **unstable node**.

   Now, let $t = tr(D\Phi(\mathbf{y}_1))$ and $d = \det(D\Phi(\mathbf{y}_1))$; we have $t = 8$ and $d = 18$. Since $d > \frac{t^2}{4}$ and $t > 0$, $\mathbf{y}_0$ is an **unstable spiral**.

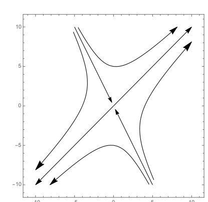
5. Answers may vary in this problem, due to different choices of eigenvectors. All of these problems are approached by first finding eigenvalues via the equation $\det(A-\lambda I) = 0$ and then finding eigenvectors by the equation $A\mathbf{v} = \lambda\mathbf{v}$.

   a) Eigenvalue $\lambda = -10$ (repeated twice).
   Eigenvector $\mathbf{v}$ (found by setting $A\mathbf{v} = \lambda\mathbf{v}$): $\mathbf{v} = (-3, 2)$.
   Then, solve $(A - \lambda I)\mathbf{w} = \mathbf{v}$ to find a generalized eigenvector: $\mathbf{w} = (1, -1)$.
   So the general solution is $\mathbf{y} = C_1 e^{-10t}\begin{pmatrix} -3 \\ 2 \end{pmatrix} + C_2\left[e^{-10t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + te^{-10t}\begin{pmatrix} -3 \\ 2 \end{pmatrix}\right]$.
   Written coordinate-wise: $\begin{cases} x(t) = (-3C_1 + C_2)e^{-10t} - 3C_2te^{-10t} \\ y(t) = (2C_1 - C_2)e^{-10t} + 2C_2te^{-10t} \end{cases}$.

   b) The general solution, coming from eigenvalues and eigenvectors, is

$$\mathbf{y} = C_1 e^{15t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-9t}\begin{pmatrix} 1 \\ -2 \end{pmatrix};$$

   0 is an unstable saddle since it has one positive and one negative eigenvalue (alternatively, because the matrix has negative determinant). The straight-line solutions are $y = x$ (which goes away from the origin since its eigenvalue is positive) and $y = -2x$ (which goes toward the origin (since its eigenvalue is negative), so the phase plane looks roughly like this:



   c) Eigenvalues: $\lambda = -1 \pm 4i$; eigenvectors: $(1 \pm i, 1)$.
   Since the eigenvalues are non-real with negative real part (alternatively, since the matrix has negative trace t and determinant d satisfying $d > \frac{t^2}{4}$), **0** is a stable spiral.
   The general solution is

$$\mathbf{y} = C_1\left[e^{-t}\cos 4t\begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^{-t}\sin 4t\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] + C_2\left[e^{-t}\cos 4t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-t}\sin 4t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right].$$

   Written coordinate-wise:

$$\begin{cases} x(t) = (C_1 + C_2)e^{-t}\cos 4t + (C_2 - C_1)e^{-t}\sin 4t \\ y(t) = C_1 e^{-t}\cos 4t + C_2 e^{-t}\sin 4t \end{cases}.$$

d) Start by solving the homogeneous system $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}$.

By usual methods, this solution is $\mathbf{y}_h = C_1 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Next, find a particular solution by guessing $\mathbf{y}_p = \begin{pmatrix} Ae^{-t} \\ Be^{-t} \end{pmatrix}$. Plugging in the original system, we get $\begin{cases} -A = -A + 3B + 12 \\ -B = 3A - B \end{cases}$ which has solution $A = 0, B = -4$. Therefore $\mathbf{y}_p = \begin{pmatrix} 0 \\ -4e^{-t} \end{pmatrix}$ so the solution of the original system is

$$\mathbf{y} = \mathbf{y}_p + \mathbf{y}_h = \begin{pmatrix} 0 \\ -4e^{-t} \end{pmatrix} + C_1 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

e) Eigenvalues: $\lambda = \pm 11i$; eigenvectors: $(3 \mp 11i, 10)$
Since the eigenvalues are pure imaginary (alternatively, since the matrix A has zero trace and positive determinant), $\mathbf{0}$ is a center.
The general solution is

$$\begin{aligned} \mathbf{y} &= C_1 \left[\cos 11t \begin{pmatrix} 3 \\ 10 \end{pmatrix} - \sin 11t \begin{pmatrix} -11 \\ 0 \end{pmatrix}\right] + C_2 \left[\cos 11t \begin{pmatrix} -11 \\ 0 \end{pmatrix} + \sin 11t \begin{pmatrix} 3 \\ 10 \end{pmatrix}\right] \\ &= \begin{pmatrix} (3C_1 - 11C_2)\cos 11t + (11C_1 + 3C_2)\sin 11t \\ 10C_1 \cos 11t + 10C_2 \sin 11t \end{pmatrix}. \end{aligned}$$

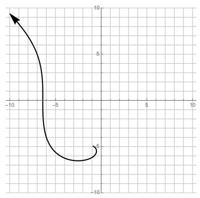
f) $\mathbf{y} = C_1 e^{2t} \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + C_3 e^{5t} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$.

6. Eigensystem: $\lambda = -14 \leftrightarrow \begin{pmatrix} 2 \\ -1 \end{pmatrix}$; $\lambda = -7 \leftrightarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

General solution: $\mathbf{y} = C_1 e^{-14t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + C_2 e^{-7t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Plug in $\mathbf{y}(0) = (5, 2)$ to get $\begin{cases} 5 = 2C_1 + C_2 \\ 2 = -C_1 + 3C_2 \end{cases}$ Solve for C_1 and C_2 to get $C_1 = \frac{13}{7}$, $C_2 = \frac{9}{7}$. Thus the particular solution is

$$\begin{aligned} \mathbf{y} &= \frac{13}{7} e^{-14t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \frac{9}{7} e^{-7t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{26}{7} e^{-14t} + \frac{9}{7} e^{-7t} \\ \frac{-13}{7} e^{-14t} + \frac{27}{7} e^{-7t} \end{pmatrix}. \end{aligned}$$

7. a) 

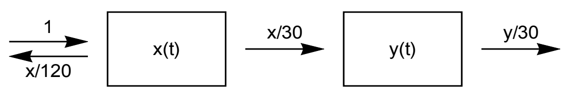
   b) $\lim_{t\to-\infty} y(t) = -5$.

   c) $r = 7$.

   d) This system has an unstable saddle at $(0, 6)$ (saddles are always unstable).

   e) This system has no nodes.

   f) This system has an unstable spiral at $(-1, -5)$.

8. a) 

   b) Let $x(t)$ = the amount of salt in tank X at time t, and let $y(t)$ = the amount of salt in tank Y at time t. Then we have the initial value problem

$$\begin{cases} x'(t) = -\frac{x}{30} - \frac{x}{120} + 1 \\ y'(t) = \frac{x}{30} - \frac{y}{30} \end{cases} \qquad \begin{cases} x(0) = 60 \\ y(0) = 0 \end{cases}$$

   c) We can rewrite the system as $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$ where $\mathbf{y} = (x, y)$, $\mathbf{q} = (1, 0)$ and $A = \begin{pmatrix} \frac{-1}{24} & 0 \\ \frac{1}{30} & \frac{-1}{30} \end{pmatrix}$. To solve this, first solve the homogeneous system using eigenvalues and eigenvectors (notice that since this matrix is triangular, you can read off the eigenvalues as the diagonal entries):

$$\det(A - \lambda I) = (\frac{-1}{24} - \lambda)(\frac{-1}{30} - \lambda) \Rightarrow \lambda = \frac{-1}{24}, \frac{-1}{30}$$

   The corresponding eigenvectors (letting $\mathbf{v} = (x, y)$ and solving $A\mathbf{v} = \lambda\mathbf{v}$) are:

$$\lambda = \frac{-1}{24} : \begin{cases} \frac{-1}{24}x = \frac{-1}{24}x \\ \frac{1}{30}x - \frac{1}{30}y = \frac{-1}{24}y \end{cases} \Rightarrow x - y = \frac{-5}{4}y \Rightarrow x = \frac{-1}{4}y \Rightarrow \mathbf{v} = (1, -4)$$

$$\lambda = \frac{-1}{30} : \begin{cases} \frac{-1}{24}x = \frac{-1}{30}x \\ \frac{1}{30}x - \frac{1}{30}y = \frac{-1}{30}y \end{cases} \Rightarrow x = 0 \Rightarrow \mathbf{v} = (0, 1)$$

   So the general solution of the homogeneous is

$$\mathbf{y}_h = C_1 e^{-t/24} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + C_2 e^{-t/30} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now for the particular solution. Since $\mathbf{q}$ has constant entries, guess $\mathbf{y}_p = \begin{pmatrix} A \\ B \end{pmatrix}$. Then, by plugging in the original system we have

$$\begin{cases} 0 = \frac{-A}{24} + 1 \\ 0 = \frac{A}{30} - \frac{B}{30} \end{cases}$$

Therefore $A = B = 24$, so $\mathbf{y}_p = (24, 24)$. Therefore

$$\mathbf{y} = \mathbf{y}_p + \mathbf{y}_h = \begin{pmatrix} 24 \\ 24 \end{pmatrix} + C_1 e^{-t/24} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + C_2 e^{-t/30} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

i.e.

$$\begin{cases} x(t) = C_1 e^{-t/24} + 24 \\ y(t) = -4C_1 e^{-t/24} + C_2 e^{-t/30} + 24 \end{cases}$$

Plugging in the initial conditions $x(0) = 60, y(0) = 0$, we get $60 = C_1 + 24$ (i.e. $C_1 = 26$) and $0 = -4C_1 + C_2 + 24$ (i.e. $C_2 = 80$). Therefore the particular solution is

$$\begin{cases} x(t) = 26e^{-t/24} + 24 \\ y(t) = -104e^{-t/24} + 80e^{-t/30} + 24 \end{cases}$$

d) $y(240) = -104e^{-240/24} + 80e^{-240/30} + 24 = -104e^{-10} + 80e^{-8} + 24.$

Chapter 4

Higher-order linear equations

4.1 Reduction of order

Recall that a n^{th} order, linear ODE has the form

We know from Chapters 1-3 how to approach first-order equations and systems. On the face of things, it seems like we would have to invent entirely new theory to study n^{th}-order equations and systems. However, there is a trick called **reduction of order**. Here's how this trick works for linear systems:

Start with the n^{th}-order linear equation

$$p_n y^{(n)} + p_{n-1} y^{(n-1)} + ... + p_2 y'' + p_1 y' + p_0 = q$$

and rewrite this equation (renaming the coefficient functions p_j if necessary) by solving for $y^{(n)}$ to get

$$y^{(n)} = p_0 y + p_1 y' + p_2 y'' + ... + p_{n-1} y^{(n-1)} + q \tag{4.1}$$

Here comes the trick. We turn the n^{th}-order *equation* (4.1) into a first-order *system* as follows: first, define for $j \in \{0, ..., n-1\}$,

$$y_j(t) = y^{(j)}(t)$$

and arrange these y_j into a vector:

$$\mathbf{y}(t) = \begin{pmatrix} y_0(t) \\ y_1(t) \\ y_2(t) \\ \vdots \\ y_{n-1}(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \\ y''(t) \\ \vdots \\ y^{(n-1)}(t) \end{pmatrix}.$$

Now,

$$\mathbf{y}' = \frac{d}{dt}\begin{pmatrix} y_0(t) \\ y_1(t) \\ y_2(t) \\ \vdots \\ y_{n-2}(t) \\ y_{n-1}(t) \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} y(t) \\ y'(t) \\ y''(t) \\ \vdots \\ y^{(n-2)}(t) \\ y^{(n-1)}(t) \end{pmatrix} = \begin{pmatrix} y'(t) \\ y''(t) \\ y'''(t) \\ \vdots \\ y^{(n-1)}(t) \\ y^{(n)}(t) \end{pmatrix} = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ \vdots \\ y_{n-1}(t) \\ y^{(n)}(t) \end{pmatrix}$$

Substituting in the formula from Equation (4.1) (previous page), we get

$$\begin{aligned}
\mathbf{y}' &= \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{n-1}(t) \\ p_0 y_0(t) + p_1 y_1(t) + \ldots + p_{n-2} y_{n-2}(t) + p_{n-1} y_{n-1}(t) + q(t) \end{pmatrix} \\
&= \begin{pmatrix} & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \\ p_0 & p_1 & p_2 & \cdots & \cdots & p_{n-1} \end{pmatrix} \begin{pmatrix} y_0(t) \\ y_1(t) \\ y_2(t) \\ \vdots \\ y_{n-2}(t) \\ y_{n-1}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ q(t) \end{pmatrix} \\
&= A\mathbf{y} + \mathbf{q}
\end{aligned}$$

where

$$A = \begin{pmatrix} & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \\ p_0 & p_1 & p_2 & \cdots & \cdots & p_{n-1} \end{pmatrix} \quad \text{and} \quad \mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ q(t) \end{pmatrix}.$$

Note that if we can solve the first-order system

$$\mathbf{y}' = A\mathbf{y} + \mathbf{q},$$

then by reading the first row off this solution, we have solved the original n^{th}-order equation!

In general, an n^{th}-order ODE can be turned into a linear $n \times n$ system of first-order ODEs using this procedure, and a $d \times d$ system of n^{th}-order ODEs can be converted into a $dn \times dn$ system of first-order ODEs using similar techniques.

Example: Convert the following second-order equation to a first-order system of the form $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$:

$$y'' + 4e^{-3t}y' - 3e^{t}y = 7e^{2t}$$

Example: Convert the following third-order system to a first-order system of the form $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$:

$$\begin{cases} x''' & -4x'' + 2y'' & -3x' + 5y' & -x + 7y & = 0 \\ x''' + y''' & +3y'' & -2x' & +3x - 4y & = e^{t} \end{cases}$$

Since solving a n^{th}-order system is equivalent to solving a first-order system with more variables and more equations, the existence/uniqueness theory we developed for first-order systems lifts to this setting. The only issue is what constitutes an initial value.

When solving a first-order system $\mathbf{y}' = \Phi(t, \mathbf{y})$, an initial value is some $\mathbf{y}(t_0) = \mathbf{y}_0$, which is really a list of values of all the components of $\mathbf{y}$ at $t = t_0$. In this setting, since the components of $\mathbf{y}$ are the zeroth to $(n-1)$th derivatives of y, to specify an initial value for an n^{th}-order equation means to specify the values of

at some value t_0.

Definition 4.1 *Given an n^{th}-order ODE, an* **initial value** *of the ODE is a number t_0, together with the values $y(t_0), y'(t_0), ..., y^{(n-1)}(t_0)$ of y and its first $(n-1)$ derivatives at t_0. We write such an initial value as*

$$(t_0, y(t_0), y'(t_0), ..., y^{(n-1)}(t_0)) \quad \textit{or} \quad (t_0, y_0, y_0', y_0'', ..., y_0^{(n-1)}).$$

An n^{th}-order ODE together with an initial value of this type is called a **initial value problem**.

Theorem 4.2 (Existence/uniqueness for n^{th}-order equations) *Suppose*

$$\phi = \phi(t, y, y', ..., y^{(n-1)}) : \mathbb{R}^{n+1} \to \mathbb{R}$$

is a function such that $\phi, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial y'}, ..., \frac{\partial \phi}{\partial y^{(n-1)}}$ are each continuous for all $(t, y, y', ..., y^{n-1})$ in some box in $\mathbb{R}^{n+1}$ containing $(t_0, y_0, y_0', ..., y_0^{(n-1)})$. Then for some interval of t values containing t_0, the initial value problem

$$\begin{cases} y^{(n)} = \phi(t, y, y', ..., y^{(n-1)}) \\ y(t_0) = y_0, y'(t_0) = y_0', ..., y^{(n-1)}(t_0) = y_0^{n-1} \end{cases}$$

has one and only one solution which is of the form $y = f(t)$ for some function $f : \mathbb{R} \to \mathbb{R}$.

Theorem 4.3 (Existence/uniqueness for n^{th}-order systems) *Suppose*

$$\Phi = (\phi_1, \phi_2, ..., \phi_n) : \mathbb{R}^{n+1} \to \mathbb{R}^n$$

is a function whose components are

$$\phi_j = \phi_j(t, y, y', ..., y^{(n-1)}) : \mathbb{R}^{n+1} \to \mathbb{R}.$$

If for every j, $\phi_j, \frac{\partial \phi_j}{\partial y}, \frac{\partial \phi_j}{\partial y'}, ..., \frac{\partial \phi_j}{\partial y^{(n-1)}}$ *are continuous at all* $(t, y, y', ..., y^{n-1})$ *in some box in* $\mathbb{R}^{n+1}$ *containing* $(t_0, y_0, y_0', ..., y_0^{(n-1)})$*, then for some interval of* t *values containing* t_0*, the initial value problem*

$$\begin{cases} \mathbf{y}^{(n)} = \Phi(t, \mathbf{y}, \mathbf{y}', ..., \mathbf{y}^{(n-1)}) \\ \mathbf{y}(t_0) = \mathbf{y}_0, \mathbf{y}'(t_0) = \mathbf{y}_0', ..., \mathbf{y}^{(n-1)}(t_0) = \mathbf{y}_0^{n-1} \end{cases}$$

has one and only one solution which is of the form $\mathbf{y} = \mathbf{f}(t)$ *for some function* $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$.

4.2 n^{th}-order, linear, constant-coefficient equations

The homogeneous case

Recall from Chapter 1 that an n^{th}-order, linear, homogeneous constant-coefficient ODE looks like

$$p_n y^{(n)} + p_{n-1} y^{(n-1)} + ... + p_2 y'' + p_1 y' + p_0 y = 0$$

where $p_0, ..., p_n$ are constants, and $p_n \neq 0$.

Motivating Example: $2y''' - 3y'' - 9y' + 10y = 0$

Method of Solution: Convert this to a first-order system, and solve the system. From the preceding discussion, we start by solving the equation for y''':

Then let

$$\mathbf{y} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$$

so that the equation is converted to the first order system

$$\mathbf{y}' = A\mathbf{y} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & \frac{9}{2} & \frac{3}{2} \end{pmatrix} \mathbf{y}.$$

The solution of the original equation, which was

$$2y''' - 3y'' - 9y' + 10y = 0,$$

is the top row of $\mathbf{y}(t) = e^{At}\mathbf{y}_0$.

To compute this solution, we'd start by finding eigenvalues:

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -5 & \frac{9}{2} & \frac{3}{2} - \lambda \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -5 & \frac{9}{2} & \frac{3}{2} - \lambda \end{pmatrix} \begin{matrix} -\lambda & 1 \\ 0 & -\lambda \\ -5 & \frac{9}{2} \end{matrix}$$

This generalizes: starting with an n^{th}-order, constant-coefficient ODE, you can always read off the formula for $\det(A - \lambda I)$ by looking at the constants in the original equation.

Definition 4.4 *Given an n^{th}-order linear, constant-coefficient ODE*

$$p_n y^{(n)} + p_{n-1} y^{(n-1)} + ... + p_2 y'' + p_1 y' + p_0 y = q,$$

the n^{th} degree polynomial equation

$$p_n \lambda^n + p_{n-1} \lambda^{n-1} + ... + p_2 \lambda^2 + p_1 \lambda + p_0 = 0$$

is called the **characteristic equation** *of the ODE.*

When you perform reduction of order on a constant-coefficient, n^{th}-order equation to obtain a first-order system

$$\mathbf{y}' = A\mathbf{y},$$

it turns out that the eigenvalues of A are exactly the roots of the characteristic equation (to prove this in general, you would need to know how to compute determinants of matrices via "minors"; you learn this technique in linear algebra).

Therefore, if the characteristic equation has distinct real roots, then the solution of the system is

$$\mathbf{y} = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2 + ... + C_n e^{\lambda_n t} \mathbf{v}_n.$$

If we choose eigenvectors where the first component of each eigenvector is 1, and then read off the top row of this solution to find y, we can conclude:

Theorem 4.5 *Suppose that the characteristic equation of an n^{th}-order linear, homogeneous constant-coefficient ODE has n distinct, real roots $\lambda_1, ..., \lambda_n$. Then, the general solution of the ODE is*

$$y = y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + ... + C_n e^{\lambda_n t}.$$

Example: Find the particular solution of the ODE

$$y'' - 7y' + 10y = 0$$

satisfying $y(0) = 2$ and $y'(0) = 17$.

What about complex roots of the characteristic equation?

Suppose the characteristic equation has complex eigenvalues $\alpha \pm i\beta$ with corresponding eigenvectors $\mathbf{a} \pm i\mathbf{b}$. That means (in the 2×2 case) that the solution of the system is

$$\mathbf{y} = C_1 \left[e^{\alpha t} \cos(\beta t)\, \mathbf{a} - e^{\alpha t} \sin(\beta t)\, \mathbf{b} \right] + C_2 \left[e^{\alpha t} \cos(\beta t)\, \mathbf{b} + e^{\alpha t} \sin(\beta t)\, \mathbf{a} \right].$$

As before, you can choose an eigenvector whose top entry is 1; reading off the top row of this solution and combining the like terms, we get (after renaming the constants)

$$y = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t).$$

This generalizes:

Theorem 4.6 *Whenever the characteristic equation of an n^{th}-order linear, homogeneous constant-coefficient ODE has a pair of complex conjugate roots $\alpha \pm i\beta$, the general solution of the ODE contains terms of the form*

$$C_1 e^{\alpha t} \cos(\beta t) \quad \textit{and} \quad C_2 e^{\alpha t} \sin(\beta t).$$

Example: Find the general solution of the ODE $y''' + 2y'' + 5y' = 0$.

What about repeated real roots?

Recall that if you have a repeated eigenvalue in a system $\mathbf{y}' = A\mathbf{y}$, you get solutions which contain $e^{\lambda t}$, $te^{\lambda t}$, etc. This translates into our setting as follows:

Theorem 4.7 *Whenever the characteristic equation of an n^{th}-order linear, homogeneous constant-coefficient ODE has a repeated root λ of multiplicity m, the general solution of the ODE contains terms of the form*

$$C_1e^{\lambda t}, C_2te^{\lambda t}, C_3t^2e^{\lambda t}, ..., \text{ and } C_mt^{m-1}e^{\lambda t}.$$

Example: Find the general solution of the ODE $y'' - 6y' + 9y = 0$.

Summary

To solve a constant-coefficient homogeneous ODE, find the roots of the characteristic equation:

- each real root λ of multiplicity 1 generates a solution of the form $e^{\lambda t}$;
- each pair of non-real roots $\alpha \pm i\beta$ of multiplicity 1 generates two solutions of the form $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$;
- each repeated real root λ of multiplicity $m > 1$ generates m solutions of the form $e^{\lambda t}, te^{\lambda t}, t^2e^{\lambda t}, ..., t^{m-1}e^{\lambda t}$;
- (not assessed in Math 330) each repeated pair of non-real roots $\alpha \pm i\beta$ of multiplicity $m > 1$ generates $2m$ solutions of the form

  $$e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t), te^{\alpha t}\cos(\beta t), te^{\alpha t}\sin(\beta t), ..., t^{m-1}e^{\alpha t}\cos(\beta t), t^{m-1}e^{\alpha t}\sin(\beta t).$$

Multiply each solution generated by a root of the characteristic equation by an arbitrary constant, and add them up to get the general solution of the homogeneous n^{th}-order equation.

What about non-homogeneous equations?

We already know from Chapters 1 and 2 that for linear n^{th}-order equations, if y_h is the solution of the corresponding homogeneous equation, then for any particular solution y_p of the original equation, the solution set of the original equation is

$$y = y_p + y_h.$$

So to solve an n^{th}-order, constant-coefficient non-homogeneous system, we first solve the corresponding homogeneous and then find a particular solution y_p using undetermined coefficients.

Example: Solve the ODE $y''' - 3y'' + 3y' - y = 4e^{2t}$.

Solution: The characteristic equation is

$$\begin{aligned}\lambda^3 - 3\lambda^2 + 3\lambda - 1 &= 0 \\ (\lambda - 1)^3 &= 0 \text{ (use } \textit{Mathematica} \text{ to factor if necessary)} \\ \Rightarrow \lambda &= 1 \text{ (repeated three times)}\end{aligned}$$

(The *Mathematica* command to factor is `Factor[x^3 - 3x^2 + 3x - 1]`.)

Therefore the solution of the homogeneous is

$$y_h = C_1e^t + C_2te^t + C_3t^2e^t.$$

Example: Solve the ODE $y''' - 3y'' + 3y' - y = e^t$.

Solution: This has the same homogeneous equation as the example on the previous page, so

$$y_h = C_1 e^t + C_2 t e^t + C_2 t^2 e^t.$$

We might try $y_p = Ae^t$, but this is already part of the homogeneous solution y_h, so we need to try something else.

4.3 Variation of parameters

Recall that to solve a second-order, linear, non-homogeneous ODE like

$$y'' + p_1(t)y' + p_0(t)y = q(t),$$

you first solve the corresponding homogeneous equation

$$y'' + p_1(t)y' + p_0(t)y = 0.$$

The solution to the homogeneous is the span of two linearly independent solutions (say y_1 and y_2):

Then the solution to the original non-homogeneous equation is

Question: How do you find y_p?

Answer # 1: *Undetermined coefficients:* guess y_p (with some unknown constants), and try to figure out what the constants have to be.

Upside of this method: It's pretty easy to implement, if your guessed y_p is correct.

Drawback of this method: You have to "guess" the class of y_p correctly to get started, and if the $q(t)$ in the ODE is weird, you have no way to guess y_p correctly, so this method only works if $q(t)$ is "nice".

Answer # 2:

How the method works

Goal: Find particular solution y_p of

$$y'' + p_1(t)y' + p_0(t)y = q(t) \tag{4.2}$$

given that the solution of the corresponding homogeneous is

$$y_h(t) = C_1 y_1(t) + C_2 y_2(t).$$

Idea: replace the constants C_1 and C_2 in the above expression with *functions* $c_1(t)$ and $c_2(t)$, and try to find functions that work in the original equation.

Suppose $y_p = c_1(t)y_1(t) + c_2(t)y_2(t)$. Differentiate both sides of this equation twice (using the Product Rule) to get

$$y_p' = c_1'(t)y_1(t) + c_1(t)y_1'(t) + c_2'(t)y_2(t) + c_2(t)y_2'(t)$$

$$y_p'' = [c_1'(t)y_1(t) + c_2'(t)y_2(t)]' + c_1'(t)y_1'(t) + c_1(t)y_1''(t) + c_2'(t)y_2'(t) + c_2(t)y_2''(t)$$

Trick: to simplify this, let's assume that $c_1'(t)y_1(t) + c_2'(t)y_2(t) = 0$. Then the derivatives become

$$y_p = c_1(t)y_1(t) + c_2(t)y_2(t) \tag{4.3}$$

$$y_p' = c_1(t)y_1'(t) + c_2(t)y_2'(t) \tag{4.4}$$

$$y_p'' = c_1'(t)y_1'(t) + c_1(t)y_1''(t) + c_2'(t)y_2'(t) + c_2(t)y_2''(t) \tag{4.5}$$

and by plugging in to the left-hand side of the original equation (4.2), we get

$$y_p'' + p_1 y_p' + p_0 y_p = [c_1'y_1' + c_1 y_1'' + c_2'y_2' + c_2 y_2''] + p_1 [c_1 y_1' + c_2 y_2'] + p_0 [c_1 y_1 + c_2 y_2]$$

$$= c_1(y_1' + p_1 y_1' + p_0 y_1) + c_2(y_2'' + p_1 y_2' + p_0 y_2) + c_1'y_1' + c_2'y_2'$$

$$= c_1'y_1' + c_2'y_2'.$$

To summarize, we have shown:

Theorem 4.8 (Variation of parameters for second-order equations) *If y_1 and y_2 are two linearly independent solutions of the second-order, linear, homogeneous equation $y''(t) + p_1(t)y'(t) + p_0(t)y(t) = 0$, then for any functions c_1 and c_2 which solve the system of equations*

$$\begin{cases} c_1'y_1 + c_2'y_2 &= 0 \\ c_1'y_1' + c_2y_2' &= q(t) \end{cases},$$

the function $y_p = c_1(t)y_1(t) + c_2(t)y_2(t)$ is a particular solution of the second-order linear equation $y''(t) + p_1(t)y'(t) + p_0(t)y(t) = q(t)$.

Good news: The system in the above theorem can always be solved (uniquely) for c_1' and c_2' by addition-elimination.

This is the advantage of variation of parameters (as opposed to undetermined coefficients): you can always do it, no matter the function $q(t)$, and you don't need to make any guesses.

Bad news: The functions you get for c_1' and c_2' need to be integrated to get back to c_1 and c_2; this integration is not always "doable".

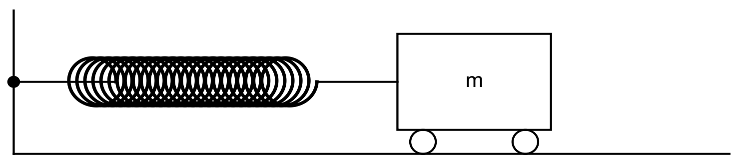
Example: $y'' - 2y' + y = \frac{e^t}{t}$.

Example: $y'' + 4y = 6\csc t$.

4.4 Applications of higher-order equations and systems

Mass-spring systems

Consider a mass attached to a fixed point by a spring:



Let $x(t)$ be the horizontal position of the mass at time t, and scale the x-axis so that $x = 0$ is the position of the mass when the spring is at rest.

Our goal is to determine the position of the mass at time t (i.e. find $x(t)$). To do this, we will determine the forces acting on the mass:

1. the spring exerts a force on the mass, which by Hooke's Law is proportional to the distance the spring is stretched/compressed. Since the force acts in the opposite direction to where the object moves, we have

$$F_{spring}(x) = -kx$$

   The larger k is, the stiffer the spring.

2. as the object moves, it encounters friction from the surface it rolls/slides across. This force is proportional to the velocity of the object, and acts in the opposite direction that the object is moving:

$$F_{friction}(x) = -bx'$$

   The larger b is, the greater the friction.

3. there may be other external forces (magnetic or electrical fields, gravity, etc.) Call these $F_{external}$.

From Newton's laws of motion, we know that the sum of these forces on the object must equal $ma = mx''(t)$. This gives the equation

which can be rewritten as follows:

Definition 4.9 *The second-order ODE*

$$mx''(t) + bx'(t) + kx(t) = F_{ext}(t)$$

where m and k are positive constants and b is a nonnegative constant, is called the **oscillator equation** *(or just an* **oscillator***). b is called the* **damping coefficient***; if $b > 0$ the oscillator is called* **damped***; if $b = 0$ the oscillator is called* **undamped***. If F_{ext} is nonzero, the oscillator is called* **driven***. An oscillator that is undamped, but not driven is called a* **simple oscillator***.*

Solution of the simple oscillator

The simple oscillator is the second-order, constant-coefficient homogeneous equation

$$mx''(t) + kx(t) = 0.$$

To solve it, consider the characteristic equation

$$m\lambda^2 + k = 0$$

which has solution $\lambda = \pm i\sqrt{\frac{k}{m}}$. Thus the general solution is

$$x(t) = C_1 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right).$$

This can be rewritten. Set

$$\psi = -\arctan\frac{C_2}{C_1} \quad \text{and} \quad A = \sqrt{C_1^2 + C_2^2}$$

Then notice that

$$\begin{aligned} A\cos\left(\sqrt{\frac{k}{m}}\,t + \psi\right) &= A\cos\left(\sqrt{\frac{k}{m}}\,t\right)\cos\psi - A\sin\left(\sqrt{\frac{k}{m}}\,t\right)\sin\psi \\ &= (A\cos\psi)\cos\left(\sqrt{\frac{k}{m}}\,t\right) - (A\sin\psi)\sin\left(\sqrt{\frac{k}{m}}\,t\right) \\ &= C_1\cos\left(\sqrt{\frac{k}{m}}\,t\right) - (-C_2)\sin\left(\sqrt{\frac{k}{m}}\,t\right). \end{aligned}$$

which is the same as the original solution. We have proven:

Theorem 4.10 (Solution of the simple oscillator) *The simple oscillator*

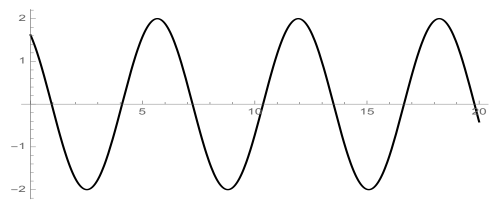
$$mx''(t) + kx(t) = 0$$

has as its general solution

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}\,t + \psi\right)$$

where A and ψ are the two arbitrary constants. A is called the **amplitude** *and ψ is called the* **phase shift**.

The graph of a simple oscillator is something like this:



Solution of the damped, but undriven, oscillator

The damped, undriven oscillator is the second-order, constant-coefficient homogeneous equation

$$mx''(t) + bx'(t) + kx(t) = 0.$$

To solve it, consider the characteristic equation

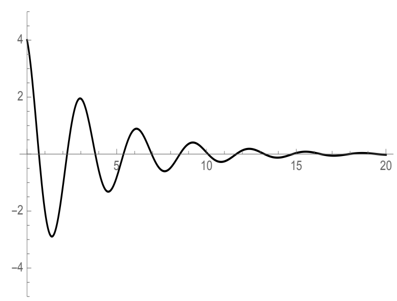
$$m\lambda^2 + b\lambda + k = 0$$

which by the quadratic formula, has solution $\lambda = \frac{-b\pm\sqrt{b^2-4mk}}{2m}$. Thus there are three cases:

- If b is small (more precisely, if $b^2 - 4mk < 0$), the oscillator is called **underdamped** (the effects of the spring outweigh the friction). In this case, there are two complex conjugate solutions λ_1 and λ_2 of the characteristic equation, so the general solution is

  $$x(t) = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t).$$

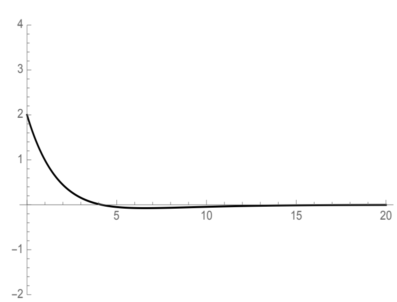
  In particular, you can show α *must be negative* (HW), so the solutions have graphs which look like



- If b is large (more precisely, if $b^2 - 4mk > 0$), the oscillator is called **overdamped** (the frictional force outweighs the harmonics coming from the spring). In this case, there are two real solutions λ_1 and λ_2 of the characteristic equation, *both of which must be negative* (HW). Thus the general solution is

  $$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

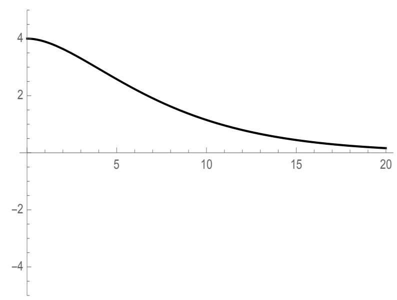
  and the graph of x looks something like



- If b is such that $b^2 - 4mk = 0$, then you get a repeated root in the characteristic equation $\lambda = \frac{-b}{2m}$. Thus the general solution is

$$x(t) = C_1 e^{(-\frac{b}{2m})t} + C_2 t e^{(-\frac{b}{2m})t}$$

and the graph of x looks like

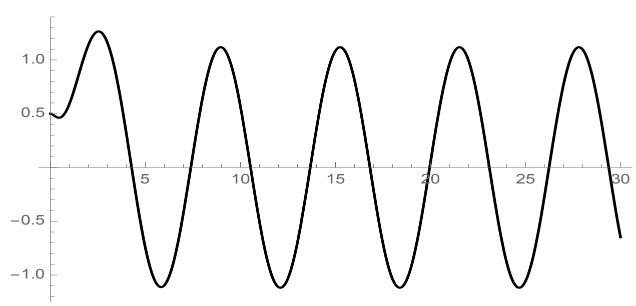


so we again consider the oscillator to be **overdamped** (but the damping takes longer to occur).

An example of a driven oscillator

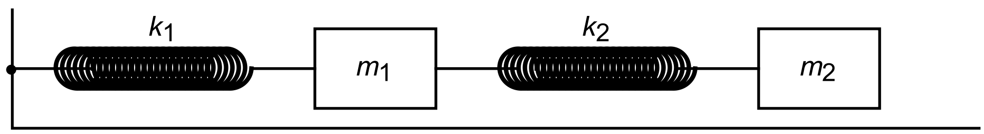
Example: A mass-spring system is driven by an external force of $5 \sin t$. The mass equals 1, the spring constant equals 3, and the damping coefficient is 4. If the mass is initially located at $x(0) = \frac{1}{2}$ and at rest, find its equation of motion.

Here is the graph of $x(t) = \frac{-1}{2}e^{-3t} + 2e^{-t} + \frac{1}{2}\sin t - \cos t$:

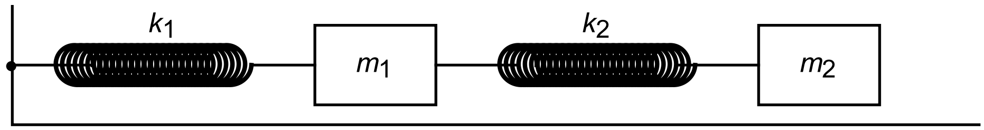


Coupled mass-spring systems

(A fancy way of saying "more than one mass hooked up by springs")



Recall: For the coupled mass-spring system



we obtained the second-order system of ODEs

$$\begin{cases} m_1 x_1'' + (k_1 + k_2)x_1 - k_2 x_2 = 0 \\ m_2 x_2'' + k_2 x_2 - k_2 x_1 = 0 \end{cases}$$

which is reduced in order to the first-order system

$$\mathbf{x}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} & 0 & 0 \end{pmatrix} \mathbf{x}$$

The eigenvalues of this matrix can be computed with *Mathematica*. Suppose we have an example with $m_1 = 2$, $m_2 = 1$, $k_1 = 4$ and $k_2 = 2$. If both objects are displaced 4 units to the right of their equilibrium positions and then released, then the initial condition of the system is

$$x(0) = 4; y(0) = 4; x'(0) = 0; y'(0) = 0 \quad \text{i.e.} \quad \mathbf{x}_0 = \begin{pmatrix} 4 \\ 4 \\ 0 \\ 0 \end{pmatrix}.$$

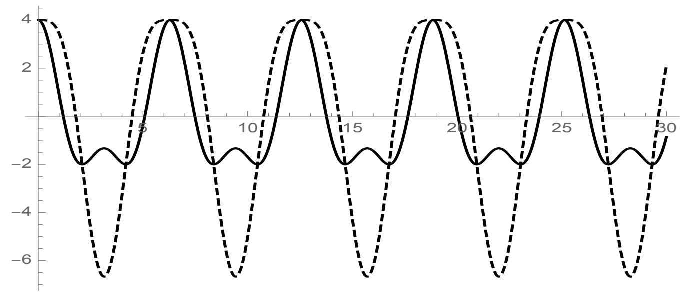
So the solution is

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = \begin{pmatrix} \frac{8}{3}\cos t + \frac{4}{3}\cos 2t \\ \frac{16}{3}\cos t - \frac{4}{3}\cos 2t \\ \frac{-8}{3}\sin t - \frac{8}{3}\sin 2t \\ \frac{-16}{3}\sin t + \frac{8}{3}\sin 2t \end{pmatrix}$$

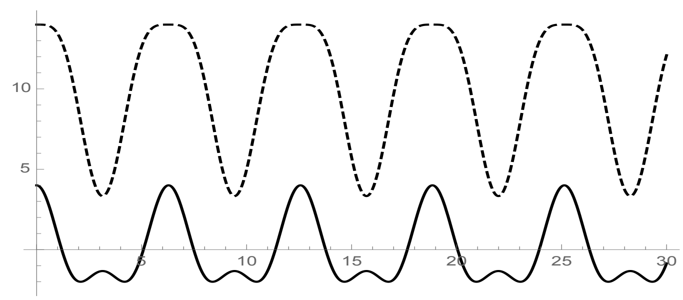
(computed with *Mathematica*). We need only the top two rows of this solution, so

$$\begin{cases} x_1(t) = \frac{8}{3}\cos t + \frac{4}{3}\cos 2t \\ x_2(t) = \frac{16}{3}\cos t - \frac{4}{3}\cos 2t \end{cases}$$

Here are the graphs of $x_1(t)$ and $x_2(t)$ (x_1 is the solid line):

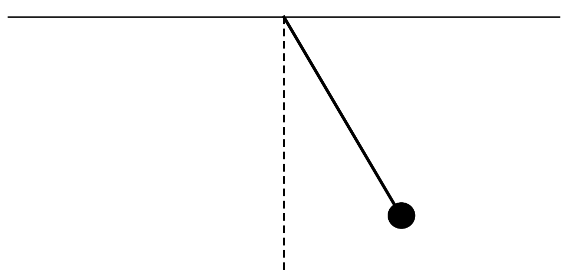


Keep in mind, however, that the vertical axis on the above picture measures each object's displacement *from when the respective springs are at rest*. If you want a picture that tracks the position of the two objects on the same scale, you need to add the length of spring # 2 when it is at rest to x_2, the position of mass # 2. This yields the following picture, if that length is, for example, 10 units:



Pendulums

We start by considering a single pendulum where a mass m is at the end of a beam of length l; the other end of the beam has fixed position.

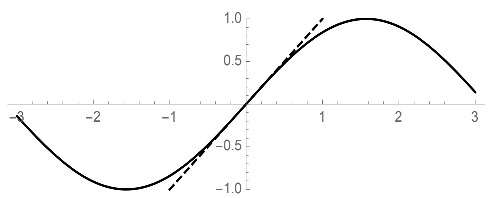


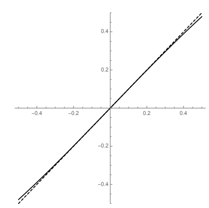
We assume for simplicity that the pendulum is contained in a plane (so the mass only moves back and forth, not in a circle or ellipse). We also suppose that the only force acting on the pendulum is gravity (no friction or air resistance). By Newton's Second Law (in the context of rotational forces),

Definition 4.11 *The* **undamped pendulum equation** *which gives the angle θ at which the mass hangs as a function of t is*

$$\theta'' + \frac{g}{l}\sin\theta = 0.$$

This is **<u>NOT</u>** a linear equation. However, if θ is small, then we can use a Taylor polynomial (of order 1 to approximate $\sin\theta$:





Thus the pendulum equation, when θ is small, can be rewritten as

$$\theta'' + \frac{g}{l}\theta = 0.$$

This **<u>IS</u>** linear (and homogeneous and constant-coefficient).

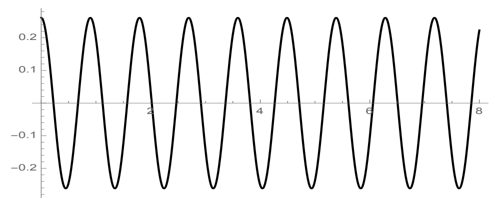
Remark: This equation is called the **linearization** of the pendulum equation. In general, to study a nonlinear ODE one starts by constructing a "linearization" of that differential equation, then applying the methods of this course. If only there was a Math 331: Differential Equations 2.....

In fact, we've seen the linearized equation before:

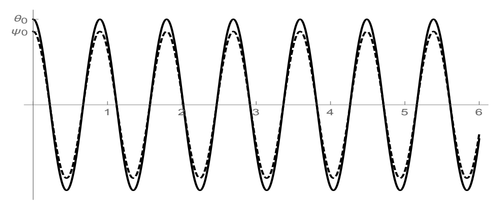
The solution is

$$\theta(t) = A\cos\left(\sqrt{\frac{g}{l}}t + \psi\right)$$

where A and ψ are the arbitrary constants. As with the simple oscillator, you get you get **simple harmonic** (i.e. periodic) **motion**.



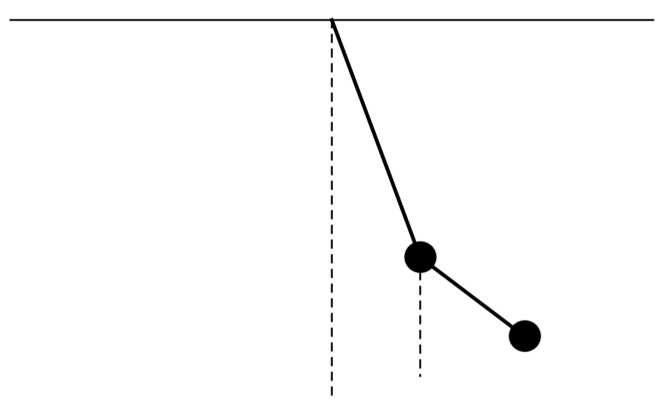
Moreover, so as long as the angle θ_0 where the pendulum starts is small, notice that if you started two identical pendulums at two very slightly different angles (say θ_0 and ψ_0), the solution curves would stay very close together (their graphs would be cosine functions with slightly different amplitude and phase shift):



In other words the pendulums would be very close to moving in harmony (and they have the exactly the same frequency $\sqrt{\frac{g}{l}}$).

Double pendulums

Now let's consider a double pendulum, which looks like this:



The double pendulum satisfies the second-order (non-linear) system of ODEs (trust me on this):

$$\begin{cases} (m_1+m_2)l_1\theta_1'' + m_2 l_2 \cos(\theta_1-\theta_2)\theta_2'' - m_2 l_2 \sin(\theta_1-\theta_2)(\theta_2')^2 + g(m_1+m_2)\sin\theta_1 = 0 \\ m_2 l_2 \theta_2'' + m_2 l_1 \cos(\theta_1-\theta_2)\theta_1'' - m_2 l_1 \sin(\theta_1-\theta_2)\theta_1^2 + m_2 g \sin\theta_2 = 0 \end{cases}$$

In this system, $g \approx 9.8$ is the gravitational constant, m_1 and m_2 are the masses at the ends of the pendulums, and l_1 and l_2 are the lengths of the pendulums.

Again, this is non-linear, but assuming θ_1 and θ_2 are small, we can assume $\sin\theta_j \approx \theta_j$, $\cos(\theta_1-\theta_2) \approx 1$ and $\sin(\theta_1-\theta_2) \approx 0$. This yields the linearized equations

$$\begin{cases} (m_1+m_2)l_1\theta_1'' + m_2 l_2 \theta_2'' + (m_1+m_2)g\theta_1 = 0 & (1) \\ m_2 l_2 \theta_2'' + m_2 l_1 \theta_1'' + m_2 g\theta_2 = 0 & (2) \end{cases}$$

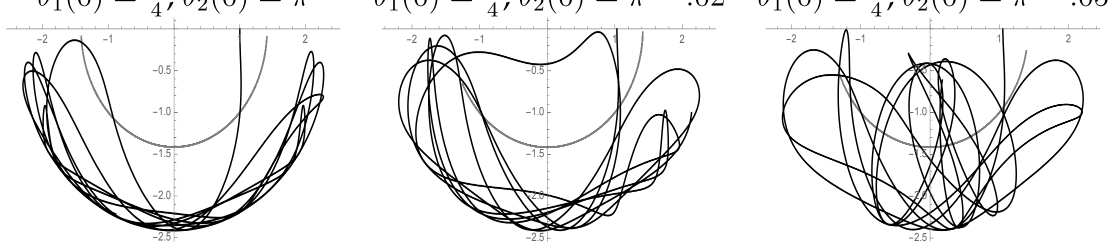
which can be solved via reduction of order, followed by eigentheory/etc. (HW)

It is hopeless to solve the double pendulum equation analytically, so we approximate solutions numerically (using Euler's method or another method) on a computer. See the file `doublependulum.nb` on my webpage for *Mathematica* code that will do this.

What's significant about the double pendulum is that very similar initial values lead to wildly different behavior.

Trajectories of the double pendulum with $m_1 = m_2 = 1$, $l_1 = \sqrt{2}$, $l_2 = 1$:

$\theta_1(0) = \frac{\pi}{4}$, $\theta_2(0) = \pi$ $\qquad$ $\theta_1(0) = \frac{\pi}{4}$, $\theta_2(0) = \pi - .02$ $\qquad$ $\theta_1(0) = \frac{\pi}{4}$, $\theta_2(0) = \pi - .05$

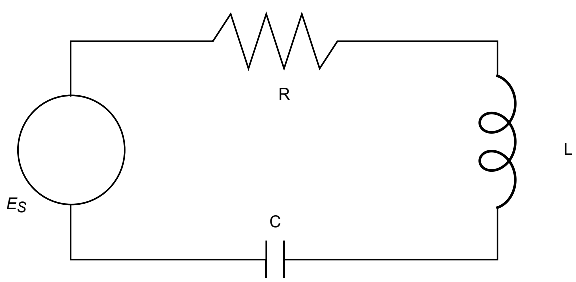


The idea of similar initial values leading to wildly different future behavior is the subject of a branch of mathematics called **chaos theory**.

Electrical circuits

In Chapter 2 we examined very simple (RC and RL) circuits. Using ideas from systems and higher-order equations, we can examine more complicated circuits.

Example: RLC circuit



From the physics equations in Chapter 2, we can derive an ODE for the charge in the circuit at time t:

(q = charge; $I = q'$ = current; E = voltage; R = resistance; C = capacitance; L = inductance)

Kirchoff's voltage law:	$E_L(t) + E_R(t) + E_C(t) = E_S(t)$	(1)
Ohm's Law:	$E_R(t) = RI_R(t)$	(2)
Faraday's Law:	$E_L(t) = L\frac{dI}{dt}$	(3)
Only one current in circuit:	$I_R(t) = I_L(t) = I_C(t) = I(t)$	(4)
Capacitor formula:	$E_C(t) = \frac{1}{C}q(t)$	(5)
Plug into equation (1):	$L\frac{dI}{dt} + RI(t) + \frac{1}{C}q(t) = E_S(t)$	(6)
Definition of current: ($I(t) = q'(t)$)	$Lq''(t) + Rq'(t) + \frac{1}{C}q(t) = E_S(t)$	(7)

Equation (7) is a second-order, linear ODE called the **RLC circuit equation** which can be solved using the usual methods.

Note the analogies between this equation and the damped oscillator. These connections reveal some initial parallels between gravitational mechanics and electromagnetism, two of the four fundamental mechanisms in physics (the other two being the weak and strong nuclear forces).

	Mass-spring system with damping	**RLC electrical series circuit**
EQUATION	$mx'' + bx' + kx = F_{ext}(t)$	$Lq'' + Rq' + \frac{1}{C}q = E_S(t)$
SOLVING FOR	displacement x	charge q
OTHER ANALOGOUS QUANTITIES	velocity x'	current $I = q'$ ($I = q'$)
	mass m	inductance L
	damping constant b (from friction)	resistance R
	spring constant k	(capacitance)$^{-1}$ $\frac{1}{C}$
	external force $F_{ext}(t)$	voltage source $E_S(t)$

The analogies between these systems are really only seen when you build the mathematical models for them. This exemplifies what (higher-level) math is ***really*** about: math is really about placing physical (or economic or biological) models into a sufficiently abstract framework so that we can see the similarities (and/or distinctions) between seemingly different things.

THE END

4.5 Final Exam Review

On the final exam, you may use: one sheet of paper (max size 8.5″ × 11″) with whatever you want on both sides.

What you should expect to be asked on the final exam: Essentially everything:

Theory: Answer questions on course vocabulary and/or concepts (existence/uniqueness, general vs. particular solution, linear vs. nonlinear, etc.)

Qualitiative: Create and/or analyze pictures of vector fields, phase lines, phase planes, etc.; compute and classify equilibria for equations and/or systems

Numerical: Euler's method for single equations and/or systems

Analytic: Solve ODEs and/or systems of ODEs, using the various methods discussed during the semester

Applications: Set up and/or solve story problems involving ODEs and/or systems of ODEs
Including: population models, heating and cooling problems, tank problems, predator-prey models, SIR models, mass-spring systems, pendulums, RC, RL and RLC electrical circuits

Practice questions on the material of Chapter 4:

1. a) Why did I say "there is no such thing as a higher-order linear ODE" in class?
   b) What is meant by an "initial value" of a n^{th}-order ODE (or system of ODEs)?
   c) What is meant by the phrase "characteristic equation" (in the context of ODEs)?

2. Convert each equation or system to a first-order system of the form $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$. Be sure to clearly define what $\mathbf{y}$, A and $\mathbf{q}$ are.
   a) $y''' + 5ty'' - 2t^2y' = 7e^t$
   b) $\begin{cases} x'' + x' - 2y' + 5x - 6y = 0 \\ x'' - y'' - x' + 3x - y = 0 \end{cases}$

3. Find the general solution of each system:
   a) $y'' - 3y' - 40y = 0$
   b) $y^{(4)} + 9y^{(3)} + 18y'' = 0$
   c) $y'' - 4y' + 10 = 0$
   d) $2y'' + 32y' + 128y = 5e^{-2t}$
   e) $y'' - 7y' - 44y = 1037\sin t$

f) $y'' + 5y' - 6y = 2e^t$

g) $y'' + 4y = 8e^{2t}$

4. Suppose the characteristic equation of a n^{th}-order, linear, constant-coefficient ODE is
$$(\lambda - 3)^3(\lambda + 2)\lambda^2.$$
   a) What is the order of this equation?
   b) Write the general solution of this equation.

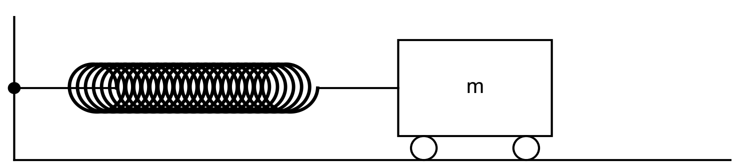
5. Find the particular solution of each system:

   a) $\begin{cases} y'' + 12y' + 32y = 0 \\ y(0) = 14 \\ y'(0) = -4 \end{cases}$

   b) $\begin{cases} y'' + 2y' + 10y = 0 \\ y(0) = 3 \\ y'(0) = 1 \end{cases}$

6. Two large tanks (call them X and Y) each hold 240 L of liquid. They are interconnected by a pipe which pumps liquid from tank X to tank Y at a rate of 8 L/min. A brine solution of concentration 0.1 kg/L of salt flows into tank X at a rate of 10 L/min; the solution flows out of the system of tanks via two pipes (one pipe allows flow out of tank X at 2 L/min and another pipe allows flow out of tank Y at 8 L/min). Suppose that initially, tank Y contains pure water but tank X contains 60 kg of salt; assume that at all times the liquids in each tank are kept mixed.
   a) Draw a compartmental diagram that models this situation.
   b) Write down the initial value problem which models this situation, clearly defining your variables.
   c) Solve the initial value problem you wrote down in part (a).
   d) Find the amount of salt in tank Y at time 240.

7. Consider an object attached to a fixed point by a spring, as in this picture:



Suppose that the mass of the object is 5 kg, the spring constant is 15 N/m and the damping coefficient (i.e. coefficient of friction) is 20 N sec/m, and the entire system (at time t) is subject to no external force. If at time 0, the mass has initial velocity 1 m/sec to the right but is 2 m to the left of where it would be at rest, find the equation of motion of the object. Is this system overdamped or underdamped (explain your answer)? Which is the more significant part of the system, the frictional force or the harmonics from the spring?

8. Consider an undamped pendulum of length 4.9 m and mass 15 kg.

    a) Write the (nonlinear) undamped pendulum equation which governs this motion.

    b) Write the linearization of this undamped pendulum equation.

    c) Find the general solution of the linearized equation you wrote in part (b).

9. A series RLC circuit has a constant voltage source of $E_S(t) = 20$ volts, a resistor of $80\,\Omega$, an inductor of 4 H, and a capacitor of $\frac{1}{1000}$ F. If the initial current is zero and the initial charge on the capacitor is 4 coulombs:

    a) Find the charge in the circuit as a function of time t.

    b) Find the current in the circuit as a function of time t.

Solutions

WARNING: as always, these might have errors.

1. a) Every higher-order linear ODE can be converted to a first-order system of ODEs, so we really only need to know how to solve first-order linear ODEs and systems to solve higher-order ODEs.

    b) An initial value is a list of values of $y, y', y'', y''', ..., y^{(n-1)}$ at the same t_0.

    c) The characteristic equation of the higher-order, constant-coefficient ODE $p_n y^{(n)} + p_{n-1} y^{(n-1)} + ... + p_2 y'' + p_1 y' + p_0 y = q(t)$ is the polynomial equation $p_n \lambda^n + ... + p_2 \lambda^2 + p_1 \lambda + p_0 = 0$.

2. a) First, rewrite the equation as $y''' = 2t^2 y' - 5ty''$. Then we can let $\mathbf{y} = (y, y', y'')$; let $\mathbf{q} = (0, 0, 7e^t)$ and let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2t^2 & -5t \end{pmatrix}$. Then $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$ as wanted.

    b) From the first equation, we know $x'' = -5x + 6y - x' + 2y'$. Next, by multiplying the second equation by -1 and then adding it to the first equation, we get $y'' + 2x' - 2y' + 2x - 5y = 0$, i.e. $y'' = -2x + 5y -$

$2x' + 2y'$. Therefore, we can let $\mathbf{y} = (x, y, x', y')$; let $\mathbf{q} = \mathbf{0}$ and let $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 6 & -1 & 2 \\ -2 & 5 & -2 & 2 \end{pmatrix}$. Then $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$ as wanted.

3. a) The characteristic equation is $\lambda^2 - 3\lambda - 40 = (\lambda - 8)(\lambda + 5)$ so $\lambda = 8$ and $\lambda = -5$. Thus the general solution is $y = C_1 e^{8t} + C_2 e^{-5t}$.

   b) The characteristic equation is $\lambda^4 + 9\lambda^3 + 18\lambda^2 = \lambda^2(\lambda + 6)(\lambda + 3)$ so $\lambda = 0$ (repeated twice), $\lambda = -6$ and $\lambda = -3$. Thus the general solution is $y = C_1 + C_2 t + C_3 e^{-6t} + C_4 e^{-3t}$.

   c) The characteristic equation is $\lambda^2 - 4\lambda + 10$ so $\lambda = \frac{4 \pm \sqrt{16-40}}{2} = \frac{4 \pm i 2\sqrt{6}}{2} = 2 \pm \sqrt{6}\, i$. Thus the general solution is $y = C_1 e^{2t} \cos(\sqrt{6}\, t) + C_2 e^{2t} \sin(\sqrt{6}\, t)$.

   d) First, solve the homogeneous equation; the characteristic equation is $2\lambda^2 + 32\lambda + 128 = 2(\lambda + 8)^2$ so $\lambda = -8$ (repeated twice). Therefore $y_h = C_1 e^{-8t} + C_2 t e^{-8t}$.

   Next, find a particular solution using undetermined coefficients: guess $y_p = Ae^{-2t}$ and plug in to obtain $8Ae^{-2t} - 64Ae^{-2t} + 128Ae^{-2t} = 5e^{-2t}$, i.e. $72A = 5$, i.e. $A = \frac{5}{72}$ so $y_p = \frac{5}{72} e^{-2t}$.
   Finally, $y = y_p + y_h = \frac{5}{72} e^{-2t} + C_1 e^{-8t} + C_2 t e^{-8t}$.

   e) First, solve the homogeneous equation; the characteristic equation is $\lambda^2 - 7\lambda - 44 = (\lambda - 11)(\lambda + 4)$ so $\lambda = -11$ and $\lambda = 4$ Therefore $y_h = C_1 e^{-11t} + C_2 e^{4t}$.

   Next, find a particular solution using undetermined coefficients: guess $y_p = A \sin t + B \cos t$ and plug in to obtain $(-A \sin t - B \cos t) - 7(A \cos t - B \sin t) - 44(A \sin t + B \cos t) = 13 \sin t$, i.e.

$$\begin{cases} -A + 7B - 44A = 1037 \\ -B - 7A - 44B = 0 \end{cases}$$

   so $A = \frac{-45}{2}$, $B = \frac{7}{2}$. Thus $y_p = \frac{-45}{2} \sin t + \frac{7}{2} \cos t$.
   Finally, $y = y_p + y_h = \frac{-45}{2} \sin t + \frac{7}{2} \cos t + C_1 e^{-11t} + C_2 e^{4t}$.

   f) First, solve the homogeneous equation; the characteristic equation is $\lambda^2 + 5\lambda - 6 = (\lambda + 6)(\lambda - 1)$ so $\lambda = -6$ and $l = 1$. Therefore $y_h = C_1 e^{-6t} + C_2 e^t$.

   Next, find a particular solution using undetermined coefficients: guess $y_p = Ate^t$ and plug in to obtain $(2Ae^t + Ate^t) + 5(Ae^t + Ate^t) - 6Ate^t = 2e^t$, i.e. $7A = 2$, i.e. $A = \frac{2}{7}$. Therefore $y_p = \frac{2}{7} te^t$.
   Finally, $y = y_p + y_h = \frac{2}{7} te^t + C_1 e^{-6t} + C_2 e^t$.

g) First, solve the homogeneous equation; the characteristic equation is $\lambda^2 + 4 = (\lambda - 2i)(\lambda + 2i)$ so $\lambda = \pm 2i$. Therefore $y_h = C_1 \cos 2t + C_2 \sin 2t$. Next, find a particular solution using undetermined coefficients: guess $y_p = Ae^{2t}$ and plug in to obtain $4Ae^{2t} + 4Ae^{2t} = 8e^{2t}$, i.e. $8A = 8$ so $A = 1$. Therefore $y_p = e^{2t}$.
Finally, $y = y_p + y_h = e^{2t} + C_1 \cos 2t + C_2 \sin 2t$.

4. a) Since the characteristic equation has degree $3 + 1 + 2 = 6$, this equation has order 6.
   b) $y = C_1 e^{3t} + C_2 t e^{3t} + C_3 t^2 e^{3t} + C_4 e^{-2t} + C_5 + C_6 t$.

5. a) First, solve the system; the characteristic equation is $\lambda^2 + 12\lambda + 32 = (\lambda + 8)(\lambda + 4)$ so the general solution is $y = C_1 e^{-8t} + C_2 e^{-4t}$. Now, plug in the initial values to get

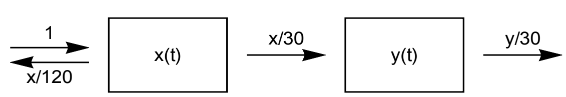
$$\begin{cases} 14 = C_1 + C_2 \\ -4 = -8C_1 - 4C_2 \end{cases}$$

and solve to get $C_1 = -13$, $C_2 = 27$. Therefore the particular solution is $y = -13e^{-8t} + 27e^{-4t}$.

   b) First, solve the system; the characteristic equation is $\lambda^2 + 2\lambda + 10 = 0$ which has solution $\lambda = \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$. Therefore the general solution is $y = C_1 e^{-t} \cos 3t + C_2 e^{-t} \sin 3t$. Plug in the initial values to get

$$\begin{cases} 3 = C_1 \\ 1 = -C_1 + 3C_2 \end{cases}$$

so $C_1 = 3$, $C_2 = \frac{4}{3}$. Therefore the particular solution is $y = 3e^{-t} \cos 3t + \frac{4}{3} e^{-t} \sin 3t$.

6. a) 

   b) Let $x(t)$ = the amount of salt in tank X at time t, and let $y(t)$ = the amount of salt in tank Y at time t. Then we have the initial value problem

$$\begin{cases} x'(t) = -\frac{x}{30} - \frac{x}{120} + 1 \\ y'(t) = \frac{x}{30} - \frac{y}{30} \end{cases} \qquad \begin{cases} x(0) = 60 \\ y(0) = 0 \end{cases}$$

   c) We can rewrite the system as $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$ where $\mathbf{y} = (x, y)$, $\mathbf{q} = (1, 0)$ and $A = \begin{pmatrix} \frac{-1}{24} & 0 \\ \frac{1}{30} & \frac{-1}{30} \end{pmatrix}$. To solve this, first solve the homogeneous system using eigenvalues and eigenvectors:

$$\det(A - \lambda I) = (\frac{-1}{24} - \lambda)(\frac{-1}{30} - \lambda) \Rightarrow \lambda = \frac{-1}{24}, \frac{-1}{30}$$

The corresponding eigenvectors (letting $\mathbf{v} = (x, y)$ and solving $A\mathbf{v} = \lambda\mathbf{v}$) are:

$$\lambda = \frac{-1}{24} : \begin{cases} \frac{-1}{24}x = \frac{-1}{24}x \\ \frac{1}{30}x - \frac{1}{30}y = \frac{-1}{24}y \end{cases} \Rightarrow x - y = \frac{-5}{4}y \Rightarrow x = \frac{-1}{4}y\mathbf{v} = (1, -4)$$
$$\lambda = \frac{-1}{30} : \begin{cases} \frac{-1}{24}x = \frac{-1}{30}x \\ \frac{1}{30}x - \frac{1}{30}y = \frac{-1}{30}y \end{cases} \Rightarrow x = 0 \Rightarrow \mathbf{v} = (0, 1)$$

So the general solution of the homogeneous is

$$\mathbf{y}_h = C_1 e^{-t/24} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + C_2 e^{-t/30} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now for the particular solution. Since $\mathbf{q}$ has constant entries, guess $\mathbf{y}_p = \begin{pmatrix} A \\ B \end{pmatrix}$. Then, by plugging in the original system we have

$$\begin{cases} 0 = \frac{-A}{24} + 1 \\ 0 = \frac{A}{30} - \frac{B}{30} \end{cases}$$

Therefore $A = B = 24$, so $\mathbf{y}_p = (24, 24)$. Therefore

$$\mathbf{y} = \mathbf{y}_p + \mathbf{y}_h = \begin{pmatrix} 24 \\ 24 \end{pmatrix} + C_1 e^{-t/24} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + C_2 e^{-t/30} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

i.e.

$$\begin{cases} x(t) = C_1 e^{-t/24} + 24 \\ y(t) = -4C_1 e^{-t/24} + C_2 e^{-t/30} + 24 \end{cases}$$

Plugging in the initial conditions $x(0) = 60, y(0) = 0$, we get $60 = C_1 + 24$ (i.e. $C_1 = 26$) and $0 = -4C_1 + C_2 + 24$ (i.e. $C_2 = 80$). Therefore the particular solution is

$$\begin{cases} x(t) = 26e^{-t/24} + 24 \\ y(t) = -104e^{-t/24} + 80e^{-t/30} + 24 \end{cases}$$

d) $y(240) = -104e^{-240/24} + 80e^{-240/30} + 24 = -104e^{-10} + 80e^{-8} + 24.$

7. Use the oscillator equation:

$$\begin{aligned} mx''(t) + bx'(t) + kx(t) &= F_{ext}(t) \\ \Rightarrow 5x'' + 20x' + 15x &= 0 \\ \Rightarrow x'' + 4x' + 3x &= 0 \end{aligned}$$

Now solve this; the characteristic equation is $\lambda^2 + 4\lambda + 3 = (\lambda + 3)(\lambda + 1)$ so the general solution is $y = C_1 e^{-3t} + C_2 e^{-t}$. Plugging in the initial values $x(0) = -2$, $x'(0) = 1$, we get

$$\left\{ -2 = C_1 + C_2 1 = -3C_1 - C_2 \right.$$

Therefore $C_1 = \frac{1}{2}$ and $C_2 = \frac{-5}{2}$ so the particular solution is $y = \frac{1}{2}e^{-3t} - \frac{5}{2}e^{-t}$.

Last, this system is overdamped since the eigenvalues coming from the characteristic equation are real. Since the system is overdamped, the frictional force outweighs the harmonics coming from the spring.

8. a) $\theta'' + \frac{9.8}{4.9}\sin\theta = 0$

   b) $\theta'' + \frac{9.8}{4.9}\theta = 0$, i.e. $\theta'' + 2\theta = 0$.

   c) The characteristic equation is $\lambda^2 + 2 = 0$, which has roots $\lambda = \pm i\sqrt{2}$. Therefore the general solution is $\theta = C_1\cos(t\sqrt{2}) + C_2\sin(t\sqrt{2})$.

9. a) Start with the RLC circuit equation (q is the charge, and $I = q'$ is the current):

$$\begin{aligned} Lq'' + Rq' + \frac{1}{C}q &= E_S(t) \\ 4q'' + 80q' + 1000q &= 20 \\ q'' + 20q' + 250q &= 5 \end{aligned}$$

To solve this, start with the characteristic equation $\lambda^2 + 20\lambda + 250 = 0$, which has solutions $\lambda = -10 \pm i5\sqrt{6}$. Thus the general solution of the homogeneous is

$$q_h = C_1 e^{-10t}\cos(5\sqrt{6}\,t) + C_2 e^{-10t}\sin(5\sqrt{6}\,t).$$

Now we can find the particular solution by guessing $q_p = A$: we get $0 + 0 + 250A = 5$ so $A = \frac{1}{50}$. That makes the general solution

$$q(t) = q_p + q_h = \frac{1}{50} + C_1 e^{-10t}\cos(5\sqrt{6}\,t) + C_2 e^{-10t}\sin(5\sqrt{6}\,t).$$

To find the particular solution, plug in the given values $q'(0) = 0$ and $q(0) = 4$ to get

$$\begin{cases} 4 = \frac{1}{50} + C_1 \\ 0 = -10C_1 + 5\sqrt{6}C_2 \end{cases} \Rightarrow C_1 = \frac{199}{50}, C_2 = \frac{199}{25\sqrt{6}}.$$

Thus the particular solution, which gives the charge at time t, is

$$q(t) = \frac{1}{50} + \frac{199}{50}e^{-10t}\cos(5\sqrt{6}\,t) + \frac{199}{25\sqrt{6}}\sin(5\sqrt{6}\,t).$$

b) The current is the derivative of the charge:

$$I(t) = q'(t) = \frac{-199\sqrt{6}}{10}e^{-10t}\cos(5\sqrt{6}\,t) + \frac{199\sqrt{6}}{15}e^{-10t}\sin(5\sqrt{6}\,t).$$

Appendix A

Homework

Algebra and calculus review

1. Solve for y in terms of t:

   a) $\frac{1}{\sqrt{y}+1} = 3t^2 + 2$

   b) $\frac{1}{y} = 2t - 3$

   c) $\frac{1}{y^2} = \frac{1}{t} + 2$

   d) $\arctan y = 5t - 4$

2. Solve for y in terms of t:

   a) $y^2 - 2y = t + 4$

   b) $\sin 3y = 4 \cos t$

   c) $t^2 + 2y^2 = 8$

   d) $\ln(y + 1) = \ln(t + 1) - 2$

3. Solve for y in terms of t; simplifying your answer as much as possible.

   a) $\sqrt{ty} = t$

   b) $\ln y = 4 \ln t + 3$

   c) $e^{y-2} = 3e^{t-1}$

   d) $e^{3y} = e^t + 1$

4. Compute the following limits:

   a) $\lim\limits_{t\to\infty} e^t$

   b) $\lim\limits_{t\to\infty} e^{-2t}$

   c) $\lim\limits_{t\to\infty} \frac{3}{2+e^t}$

   d) $\lim\limits_{t\to\infty} \frac{3}{2+e^{-t}}$

5. Compute the following limits:

   a) $\lim\limits_{t\to\infty} (e^{-3t} + 2e^{-t})$

   b) $\lim\limits_{t\to\infty} \sin 2t$

   c) $\lim\limits_{t\to\infty} e^t \sin t$

   d) $\lim\limits_{t\to\infty} e^{-2t} \cos t$

6. Find the derivative of each function:

a) $f(t) = 3t^2 - 7t + 5$

b) $f(t) = (t^2 + 3)^8$

c) $f(t) = t^2 e^{2t} + 4te^{2t}$

d) $f(t) = 4te^{-t}$

7. Find the derivative of each function:

a) $f(t) = \sin 4t$

b) $f(t) = e^{-t} \sin 2t$

8. Find the derivative of each function:

a) $f(t) = \sin^3(4t^2)$

b) $f(t) = e^{3t} \cos 4t - 2e^{3t} \sin 4t$

9. Find the first, second and third derivative of each function:

a) $f(t) = e^{-2t}$

b) $f(t) = \sin 4t + \cos 4t$

c) $f(t) = t \cos t$

d) $f(t) = -te^{3t} + 2e^{3t}$

10. Compute the following integrals:

a) $\int t \, dt$

b) $\int (3t^4 - 2t^7) \, dt$

c) $\int t^{-1} \, dt$

d) $\int t^{-2} \, dt$

11. Compute the following integrals:

a) $\int \frac{4}{t} \, dt$

b) $\int \frac{1}{t} \, dt$

c) $\int \frac{6}{t^2} \, dt$

d) $\int \frac{1}{\sqrt{t}} \, dt$

12. Compute the following integrals:

a) $\int 12\sqrt{t} \, dt$

b) $\int \sqrt[3]{t} \, dt$

c) $\int (\sqrt{t} + 1) \, dt$

d) $\int \sqrt{t+1} \, dt$

13. Compute the following integrals:

a) $\int 2 \sin t \, dt$

b) $\int (\sin 3t + \cos t) \, dt$

c) $\int \frac{1}{t^2+1} \, dt$

d) $\int \left(11e^{2t} - 18e^{-3t} + 7e^{-t}\right) \, dt$

14. Compute the following integrals. Your answer will have a t in it.

a) $\int_0^t (2s+1)\,ds$

b) $\int_1^t 2s^{-1/2}\,ds$

c) $\int_0^t e^{-s}\,ds$

d) $\int_1^t \frac{4}{s}\,ds$

15. Compute the following integrals (using integration by parts):

a) $\int te^{-3t}\,dt$

b) $\int t\ln t\,dt$

16. Compute the following integrals:

a) $\int \frac{2}{t^2-4}\,dt$

b) $\int \sin t\cos t\,dt$

17. Compute the following integrals:

a) $\int \frac{e^t}{e^t+1}\,dt$

b) $\int \frac{e^t+1}{e^t}\,dt$

18. Compute the following integrals:

a) $\int \frac{t^2+2}{(t-1)^2}\,dt$

b) $\int \frac{(t-2)^2}{t}\,dt$

19. Some sequences in mathematics are defined by a method called **recursion**: this means that each term in the sequence is given by a formula which depends on previous terms. For example, suppose you are given

$$x_0 = 3 \text{ and } x_{n+1} = 2x_n.$$

Then you can find $x_1, x_2, x_3, \ldots$ by repeatedly plugging in to the second equation as follows:

$$\begin{aligned} x_1 &= 2x_0 = 2(3) = 6 \\ x_2 &= 2x_1 = 2(6) = 12 \\ x_3 &= 2x_2 = 2(12) = 24 \\ &\vdots \qquad \vdots \end{aligned}$$

In each part of this problem, you are given a recursive formula and a value of x_0. Based on this formula, write down x_1, x_2, x_3, and x_4.

a) $x_0 = 1; x_{n+1} = -x_n$

b) $x_0 = 18; x_{n+1} = \frac{1}{3}x_n + 3$

c) $x_0 = 1; x_{n+1} = 2^{x_n}$

d) $x_0 = 3; x_{n+1} = 2 - \frac{1}{2}x_n$

20. Suppose $x_0 = 0$, $x_1 = 1$ and $x_{n+2} = x_{n+1} + 2x_n$. Find x_7.

A.1 Problems from Chapter 1

Problems from Section 1.1

21. In each part, you are given an ODE and a possible solution of that ODE. Determine whether or not the possible solution is actually a solution of the ODE.

    Hint: Plug the possible solution in to both sides of the ODE and see if they are equal.

    a) *ODE:* $\frac{dy}{dt} = \frac{y^2-1}{t^2+2t}$
    Possible solution: $y = 2t + 1$
    b) *ODE:* $y'' + y = t^2 + 2$
    Possible solution: $y = \sin t + t^2$
    c) *ODE:* $\frac{d^2y}{dt^2} - \frac{dy}{dt} = 2y$
    Possible solution: $y = e^{2t} - 3e^{-t}$

22. In each part, you are given an ODE and a possible solution of that ODE. Determine whether or not the possible solution is actually a solution of the ODE

    Hint: To compute derivatives of the possible solutions, differentiate implicitly.

    a) *ODE:* $\frac{dy}{dt} = \frac{t}{y}$
    Possible solution: $t^2 + y^2 = 9$
    b) *ODE:* $\frac{dy}{dt} = \frac{2ty}{y-1}$
    Possible solution: $y - \ln y - 1 = t^2$

23. In each part, you are given an initial value problem and a possible solution of that IVP. Determine whether or not the possible solution is actually a solution of the IVP:

    a) *IVP:* $\begin{cases} y' = 6y \\ y(0) = 2 \end{cases}$
    Possible solution: $y = 2e^{6t}$
    b) *IVP:* $\begin{cases} y' = 6y \\ y(0) = -1 \end{cases}$
    Possible solution: $y = 2e^{6t} + -3$

24. In each part, you are given an initial value problem and a possible solution of that IVP. Determine whether or not the possible solution is actually a solution of the IVP:

    a) *IVP:* $\begin{cases} y' = \frac{-t}{y} \\ y(4) = 3 \end{cases}$
    Possible solution: $t^2 + y^2 = 20$
    b) *IVP:* $\begin{cases} y' = \frac{-t}{y} \\ y(4) = 3 \end{cases}$
    Possible solution: $t^2 + y^2 = 25$

Problems from Section 1.2

25. Determine whether each of the formulas T given below defines a linear operator on $C^\infty(\mathbb{R},\mathbb{R})$ (no proof is required, just write "linear" or "not linear"):

    a) $T(y) = y + t$
    b) $T(y) = 4y^{(7)}$
    c) $T(y) = y'' + ty' - (3\sin t)y$
    d) $T(y) = (y')^2 + 2y - 4y''$

26. Suppose T is the linear differential operator defined by
$$T(y) = e^t y''' - e^{-t}y'' + e^{4t}y' - 2y.$$

    a) What is the order of this operator?
    b) Find $T(2e^t)$.
    c) Find $T(e^{2t})$.
    d) Find $T(4)$.

27. Suppose T is the fourth-order linear differential operator defined by setting $p_0(t) = 2t^2$, $p_1(t) = t^3$, $p_2(t) = 8t^4$, $p_3(t) = 0$ and $p_4(t) = 2t^6$ and using the formula given in Definition 1.12 of the lecture notes.

    a) Write the formula for $T(y)$.
    b) Find $T(2t^6)$.

In each part of Problems 28-31 you are given an ODE. For each equation:
(i) give the order of the equation;
(ii) give the number of arbitrary constants you would expect in the general solution;
(iii) classify the equation as linear or nonlinear;
(iv) if the equation is linear, determine whether or not it is homogeneous;
(v) if the equation is linear, determine whether or not it is constant-coefficient.

28. a) $\frac{dy}{dt} + 5t\frac{dy}{dt} - 3y^2 = y^7 \sin t$
    b) $4y - \cos y^3 + y'''' - t^2y'' = e^t y'' - y^2$
    c) $e^{t+y}\frac{d^2y}{dt^2} + \cos t\frac{dy}{dt} = 0$

29. a) $t^2\frac{dy}{dt} + t^4 y = 3t\frac{dy}{dt} - 2t$
    b) $y^8 + y' = e^{3t}$
    c) $y^{(8)} + y' = e^{3t}$

30. a) $2y\frac{d^2y}{dt^2} - y^2\frac{dy}{dt} = 0$
    b) $e^t\frac{d^2y}{dt^2} + \cos t\frac{dy}{dt} = 0$
    c) $2\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 6y = 0$

31. a) $8y'' - 5y' + 3y = 6ty''$
    b) $y^{(4)} + y''' + y'' + y' + y = e^t$
    c) $\frac{y'}{y} = 2$

Problems from Section 1.3

32. Suppose that some quantity is changing as time passes so that the rate of increase of the quantity is equal to 4.7 times the size of the quantity.

    a) Write the ODE represented by this model.
    b) Write the general solution of this ODE.
    c) Write the particular solution of this ODE corresponding to the situation where you start with 5 units of the quantity.
    d) Write the particular solution of this ODE corresponding to the situation where at time 3, you have 12 units of the quantity.

33. Suppose that $y(t)$ is some quantity which grows at a rate proportional to its size.

    a) Suppose that initially, I have exactly twice as much of the quantity as you do. After 50 years, will I have more than twice as much as you, exactly twice as much as you, or less than twice as much as you? Explain.
    b) Suppose that initially, I have exactly one more unit of the quantity than you do (we both have a positive amount). After 50 years, will the difference between our holdings be greater than one, equal to one, or less than one? Explain.

34. The amount of money in a retirement account grows proportionally to the amount of money in the account. Suppose initially that there is $\$100$ in the account, and that five years later, there is $\$118$ in the account.

    a) Find a formula for the amount of money $y = y(t)$ in the account t years after the account is opened.
    b) Find the (exponential) rate of growth of the account, expressed as an annual percentage rate.
    c) Find the amount of money in the account thirty-five years after it is opened.

35. The **half-life** of a substance which decays exponentially is the amount of time it takes for the substance to decay to half of its original amount. Carbon-14 has a half-life of 5730 years.

    a) Find the (exponential) rate of decay of carbon-14.

b) If you have 25 grams of carbon-14 now, how much will you have 20000 years from now? (A decimal approximation is OK.)

c) If you wanted to have 5 g of carbon-14 10000 years from now, how much should you start with now? (A decimal approximation is OK.)

Problems from Section 1.4

36. Consider the ODE $y' = t - 2y$.

    a) Find four points (t, y) where $y' = 0$.

    b) Find y' at the points $(2, 3)$, $(0, 3)$, $(0, -2)$, $(2, 2)$, $(-2, 0)$, $(-1, 0)$, and $(-2, 1)$.

    c) Sketch the mini-tangent lines corresponding to the points used in parts (a) and (b). (There should be one picture with all the mini-tangents on it.)

37. Consider the ODE $y' = \frac{1}{35}(y^4 + y^3 - 20y^2)$.

    a) Use *Mathematica* to draw a picture of the slope field associated to this ODE, in the viewing window $[-8, 8] \times [-8, 8]$. Attach a printout of this picture; please label the picture "37".

    b) Find the equation of three explicit solutions to this ODE.

    c) Let $g(t)$ be the solution to this ODE passing through $(-2, -3)$.

       i. Find $\lim_{t\to\infty} g(t)$.

       ii. Find $\lim_{t\to-\infty} g(t)$.

    d) Sketch (by hand) the graph of the solution to this ODE satisfying $y(2) = 4$ on the picture you printed in part (a).

38. Consider the ODE $y' = \frac{1}{10}(y^3 - ty^2 + 2y^2 - 9y + 9t - 18)$, and let $h(t)$ be the solution to this ODE satisfying $h(0) = 1$.

    a) Use *Mathematica* to draw a picture of the slope field associated to this ODE, in the viewing window $[-6, 10] \times [-8, 8]$, with the stream line for h shown. Attach a printout of this picture; please label the picture "38".

    b) Estimate $h(3)$ and $h(5)$.

    c) Estimate $h'(2)$.

    d) Estimate all $t > 0$ (if any) for which $h(t) = 2$.

    e) Estimate all $t > 0$ (if any) for which $h(t) = 6$.

    f) Find $\lim_{t\to\infty} h(t)$.

    g) Find $\lim_{t\to-\infty} h(t)$.

39. Consider the initial value problem

$$\begin{cases} y' = y - \arctan t \\ y(0) = y_0 \end{cases}$$

where y_0 is a constant.

a) Use *Mathematica* to draw a picture of the slope field associated to this ODE, in the viewing window $[0, 10] \times [-5, 5]$, with several stream lines shown. Attach a printout of this picture; please label the picture "39".

b) Suppose you knew that y_0 was between 1.5 and 2.5. Is this information sufficient to describe the qualitative behavior of y for large t (i.e. to find $\lim_{t\to\infty} y(t)$)? Explain.

c) Suppose you knew that y_0 was between 0 and 1. Is this information sufficient to describe the qualitative behavior of y for large t (i.e. to find $\lim_{t\to\infty} y(t)$)? Explain.

Problems from Section 1.5

40. Consider the initial value problem

$$\begin{cases} y' = 5 + 2t - 3y \\ y(0) = 5 \end{cases}.$$

Perform Euler's method by hand (show your work) to compute (t_1, y_1), (t_2, y_2), (t_3, y_3) and (t_4, y_4) for $\Delta t = 1$.

41. Consider the initial value problem

$$\begin{cases} y' = \frac{1}{2} - t + 2y \\ y(0) = 2 \end{cases}.$$

Perform Euler's method by hand (show your work) to estimate $y(2)$, using four steps.

42. Consider the initial value problem

$$\begin{cases} y' = 5 - 3\sqrt{y} \\ y(0) = 2 \end{cases}.$$

a) Use *Mathematica* to implement Euler's method to estimate $y(3)$, using $\Delta t = .01$.

b) Attach a printout of the graph of the points obtained from Euler's method in part (a). Label the picture "42".

43. Consider the initial value problem

$$\begin{cases} y' = \frac{4-ty}{1+y^2} \\ y(0) = 3 \end{cases}.$$

a) Use *Mathematica* to implement Euler's method to estimate $y(60)$ using 300 steps.

b) Attach a printout of the graph of the points obtained from Euler's method in part (a). Please label the picture "43 (b)".

c) Based on the picture you get in part (b), do you trust your answer from part (a)? Why or why not?

d) Use *Mathematica* to implement Euler's method to estimate $y(60)$ using 3000 steps.

e) Attach a printout of the graph of the points obtained from Euler's method in part (d). Please label the picture "43 (e)".

f) Based on the picture you get in part (e), do you trust your answer from part (d)? Why or why not?

g) **(Optional; extra credit)** Explain thoroughly why you got the picture you did in part (b). (Saying "I didn't use enough steps" is insufficient; you need to describe the particular properties of the vector field of this ODE that lead to the specific picture you get in part (b).)

Problems from Section 1.6

44. Consider the initial value problem

$$\begin{cases} y' = 1 - y^3 \\ y(0) = 0 \end{cases}.$$

a) Use Picard's method of successive approximations, with $f_0(t) = 0$, to find $f_1(t)$, $f_2(t)$, and $f_3(t)$.

b) Graph f_3 (for $t \geq 0$ only) as obtained in part (a). If you use a *Mathematica* graph, label it as "44 (b)".

c) Plot the vector field associated to this ODE (viewing window $[0,4] \times [0,4]$) with the stream line passing through the given initial value; label the picture "44 (c)".

d) Based on your pictures, would you say that the function f_3 obtained in part (a) is, or is not, a good approximation of the solution $y = f(t)$ of the IVP?

e) Would it be reasonable to estimate $y(4)$ by computing $f_3(4)$? Why or why not?

45. Consider the initial value problem

$$\begin{cases} y' = -y - 1 \\ y(0) = 0 \end{cases} .$$

a) Use Picard's method of successive approximations, with $f_0(t) = 0$, to find $f_1(t)$, $f_2(t)$, $f_3(t)$ and $f_4(t)$.

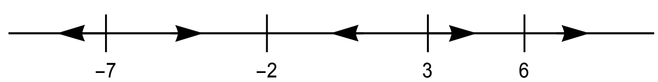
b) Based on your answers to part (a), write $f(t) = \lim_{j \to \infty} f_j(t)$ as an infinite series.

c) Recognize the infinite series from part (b) as being related to a "common" one, and find a formula for $f(t)$.

Hint: the series you get will probably have initial index 1, not 0. Take this into account.

Problems from Section 1.7

In problems 46-50, assume that this is the phase line of some autonomous ODE $y' = \phi(y)$:



46. a) Write the equation of one explicit solution of this ODE.

b) Find all sinks of this ODE (if there aren't any, say so).

c) Find all unstable equilibria of this ODE (if there aren't any, say so).

d) Find all semistable equilibria of this ODE (if there aren't any, say so).

47. a) Suppose $y(0) = 0$. Find $\lim_{t \to \infty} y(t)$.

b) Suppose $y(0) = -10$. Find $\lim_{t \to \infty} y(t)$.

c) Suppose $y(2) = 5$. Find $\lim_{t \to -\infty} y(t)$.

d) Suppose $y(-1) = 6$. Find $\lim_{t \to -\infty} y(t)$.

48. a) Suppose $y(0) = 4$. Is $y(t)$ an increasing function or decreasing function?

b) Suppose $y(0) = y_0$. For what values of y_0 is $\lim_{t \to \infty} y(t) = -2$?

c) Suppose $y(0) = y_0$. For what values of y_0 is $\lim_{t \to \infty} y(t) > 0$?

49. a) Sketch a (possible) graph of the function ϕ.

b) Sketch a (possible) slope field for this ODE.

c) Sketch a graph of the particular solution of this ODE whose initial condition is $y(1) = 0$.

d) **(Optional; extra credit)** Write down a formula for ϕ which would produce the phase line shown before Problem 46.

50. For each quantity, determine whether the quantity is positive, negative or zero:

    Hint: Your answer to Problem 49 (a) may be helpful.

    a) $\phi(-7)$ c) $\phi(4)$ e) $\phi'(3)$
    b) $\phi(0)$ d) $\phi'(-2)$ f) $\phi'(6)$

In each part of Problems 51-54, you are given an autonomous equation. For each equation:
(i) find all equilibria of the equation;
(ii) classify each equilibrium as stable, unstable or semistable;
(iii) sketch the phase line of the equation.

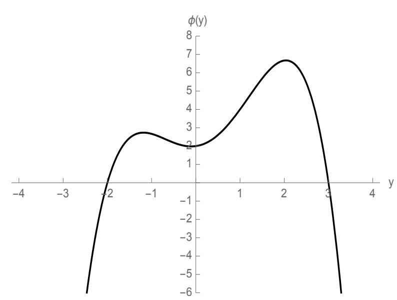
51. a) $y' = y^2 - 9$ b) $\frac{dy}{dt} = 2\cos y$

52. a) $y' = e^y - 1$ b) $y' = e^{-2y} - 1$

53. a) $\frac{dy}{dt} = 2y\ln(6/y)$ b) $y' = y^3(y-2)^2(y+3)(y-5)$

    Hint: In part (b) of Problem 53, graph the function, and estimate the values of its derivative at the equilibria by looking at the graph.

54. $y' = \phi(y)$, where ϕ is a function whose graph is as follows:



55. Suppose that the population y of a certain species of fish in a given area of the ocean (i.e. in a "fishery") would be described by a logistic equation, if there was no fishing. Since fish are delicious, we want to catch some of these fish so we can eat them. But if we catch too many fish, the population may be

driven to extinction. That would be bad. This question deals with a model for managing the fishery (which is actually used in the real world, by the way).

Let E be a constant which denotes the level of **effort** being put into fishing (the greater E is, the harder people work at catching fish). It is reasonable to assume that the rate at which they catch fish depends on the fish population y (because the more fish there are, the easier to catch them). So a simple expression which can be used to describe the rate at which fish are caught is E times y. To model the fish population including this effect, you adapt the logistic equation by subtracting the rate at which fish are caught, leading to what is called the **Schaefer model**, used in biology and environmental science. Here is that model:

$$\frac{dy}{dt} = ry(L-y) - Ey.$$

a) Find the two equilibria of this equation (in terms of the constants r, L and E).

b) Suppose $E < Lr$. What will the fish population be in the long run? Explain.

   Hint: This has something to do with whether or not the equilibria you found in part (a) are stable or unstable.

c) Suppose $E > Lr$. What will the fish population be in the long run? Explain.

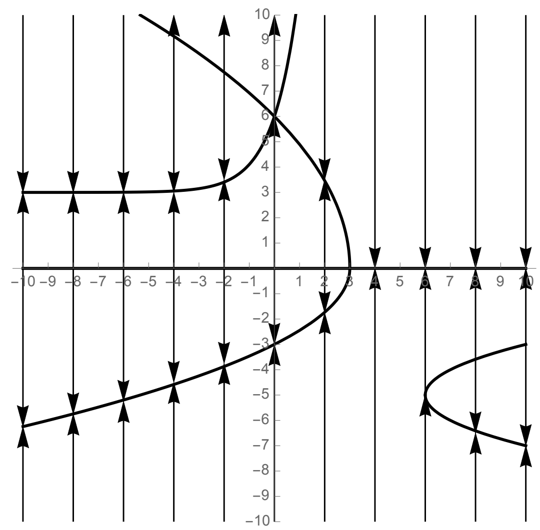
d) A **sustainable yield** Y of the fishery is a quantity of fish that can be caught indefinitely. It is the product of the effort E and the stable equilibrium fish population corresponding to effort E. Find Y as a function of E (this is called the **yield-effort** curve).

e) Determine the value of E which maximizes Y (and thereby produces the maximum sustainable yield).

   Hint: Maximize the function $Y(E)$ using the method you learn in Calculus 1.

Problems from Section 1.8

In Questions 56 and 57, use the following bifurcation diagram for a parameterized family of ODEs $y' = \phi(y; r)$:



56. a) Find all values of r (if any) at which the family has a saddle-node bifurcation.
    b) Find all values of r (if any) at which the family has a transcritical bifurcation.
    c) Find all values of r (if any) at which the family has a pitchfork bifurcation.
    d) Sketch the phase line corresponding to the equation where $r = 2$.
    e) Find all equilibria corresponding to the situation where $r = 6$, and classify them as stable, unstable or semistable.
    f) Suppose $r = -2$ and $y(0) = 1$. Find $\lim_{t\to\infty} y(t)$.

57. a) Suppose you know that r is somewhere between 5 and 7 and that $y(0) > 0$. Can you accurately predict the long-term behavior of $y(t)$? If so, what is this behavior? If not, why not?
    b) Suppose you know that r is somewhere between -7 and -6 and that $y(0)$ is somewhere between 1 and 5. Can you accurately predict the long-term behavior of $y(t)$? If so, what is this behavior? If not, why not?
    c) Suppose you know that r is somewhere between 5 and 7 and that $y(0) = -5$. Can you accurately predict the long-term behavior of $y(t)$? If so, what is this behavior? If not, why not?

In each part of Problems 58-61 you are given a parameterized family of ODEs (r is the parameter). For each parameterized family of ODEs:

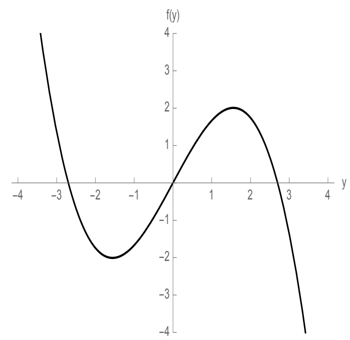
(i) find all value(s) of r at which bifurcations occur;
(ii) classify each bifurcation as saddle-node, pitchfork, transcritical or degenerate;
(iii) sketch a bifurcation diagram for the family.

58. a) $y' = y^2 + 6y + r$ b) $y' = y^2 - r^2$

59. a) $y' = y + \frac{ry}{1+y^2}$ b) $y' = (y^2 - r)(y^2 - 16)$

60. **(Optional; extra credit)** $y' = ry + y^3 - y^5$

61. $y' = f(y) + r$, where f has the graph given below:



Hint: As r changes, how does the graph of $f(y) + r$ change? What changes in this graph "cause" a bifurcation?

A.2 Problems from Chapter 2

Problems from Section 2.1

62. Find the general solution of each ODE:

a) $y' - 2y = 0$ b) $7y' = y$

63. Find the general solution of each ODE:

a) $y' = y\sqrt{t}$ b) $\frac{dy}{dt} = y \cos t$

64. Find the particular solution of each IVP:

a) $\begin{cases} y' \ln t = \frac{y}{t} \\ y(e) = 3 \end{cases}$ b) $\begin{cases} y'e^t + y' = ye^t \\ y(0) = 4 \end{cases}$

65. Suppose you know that $y = 6e^{4t} \sin 2t$ is a solution to some homogeneous, linear, first-order ODE.

a) Is $y = 9e^{4t} \sin 2t$ also a solution of this ODE? Why or why not?

b) Is $y = 6e^{3t} \sin 2t$ also a solution of this ODE? Why or why not?

Problems from Section 2.2

66. Solve each given differential equation or initial value problem, using the method of integrating factors:

a) $\begin{cases} y' - 2y = 3e^t \\ y(0) = 4 \end{cases}$

b) $\frac{dy}{dt} = te^{-t} + 1 - y$

67. Solve each given differential equation or initial value problem, using the method of integrating factors:

a) $ty' - y = t^2 e^{-t}$

b) $\begin{cases} t^3 y' + 4t^2 y = e^{-t} \\ y(-1) = 0 \end{cases}$

68. Solve each given differential equation or initial value problem:

a) $\begin{cases} y' = \frac{\cos t - 2ty}{t^2} \\ y(\pi) = 0 \end{cases}$

b) $t\frac{dy}{dt} + 2y = \sin t$

Problems from Section 2.3

69. Suppose you have a first-order linear ODE, where $y = 3e^{2t} \sin 2t$ is a solution of the equation and $y = 2e^{4t} \sin 2t$ is a solution of the corresponding homogeneous equation. Write the general solution of the ODE.

70. Solve each given differential equation or initial value problem, using the method of undetermined coefficients:

a) $y' + 3y = e^{-2t}$

b) $\begin{cases} y' - 4y = 2\cos 2t \\ y(0) = -3 \end{cases}$

71. Solve each given differential equation or initial value problem, using the method of undetermined coefficients:

a) $\begin{cases} y' + y = te^{2t} \\ y(0) = 24 \end{cases}$

b) $y' - 4y = 2t^2 + 8t$

72. Consider the ODE $y' + 7y = 20e^{-7t}$.

a) Find a nonzero solution of the corresponding homogeneous equation.

b) Try the method of undetermined coefficients with guess $y_p = Ae^{-7t}$. Does this work? Explain.

c) Try the method of undetermined coefficients with guess $y_p = Ate^{-7t}$. Does this work? Explain.

d) Solve the ODE $y' - 3y = 9e^{3t}$, by first solving the corresponding homogeneous equation and then using the method of undetermined coefficients. To find the appropriate formula to guess for y_p, use the previous example in this question as a guide.

e) For each given ODE:

(i) find the solution y_h of the corresponding homogeneous equation; and

(ii) write down what you would need to guess for y_p to utilize the method of undetermined coefficients.

You do not need to solve the equation.

i. $y' + 6y = e^{4t}$

ii. $y' + 6y = e^{-6t}$

iii. $y' + 6y = e^{2t}$

iv. $y' - 2y = e^{2t}$

f) Write down a general rule which tells you when your "normal" guess won't work (when attempting the method of undetermined coefficients), and how to adapt your guess so that it will work.

You are responsible for implementing this rule on Exam 1.

Problems from Section 2.4

73. Find the general solution of each ODE:

a) $y' = \sqrt{y(t+1)}$

b) $\frac{dy}{dt} = \sec y \sin t$

74. Find the general solution of each ODE:

a) $y' = \frac{t^2}{y(4+t^3)}$

b) $ty' = 1 - y^2$

75. Find the general solution of each ODE; write your answer as a function $y = f(t)$.

a) $e^y y' = 4$

b) $\frac{dy}{dt} = (y \sec t)^2$

76. Find the particular solution of each initial value problem; write your answer as a function $y = f(t)$:

a) $\begin{cases} \frac{dy}{dt} = \frac{1-2t}{y} \\ y(1) = -2 \end{cases}$ b) $\begin{cases} yy' = 4t \\ y(1) = -3 \end{cases}$

77. a) Solve the following initial value problem: $\begin{cases} 2(y-1)y' = e^t \\ y(0) = 2 \end{cases}$

    b) **(Optional; extra credit)** Write your answer to part (a) as a function $y = f(t)$, simplified as much as possible.

78. Let $y = f(t)$ be the solution of the initial value problem $\begin{cases} y' = ty^2 - 2y^2 \\ y(0) = 1 \end{cases}$. Find $f(4)$.

Problems 79-82 introduce some methods to solve second-order ODEs. A second-order ODE always has the form

$$\phi(t, y, y', y'') = 0 \quad \text{a.k.a. } \phi\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right) = 0.$$

I haven't said anything in class about how to solve second-order ODEs, but in two special cases you can use a trick, together with our techniques for solving first-order ODEs, to solve a second-order equation. **You should be able to solve these types of second-order equations on Exam 1.**

- *Case 1:* The equation has no y in it, i.e. the equation is of the form

  $$\phi(t, y', y'') = 0.$$

  To solve this, let $v(t) = y'$. Then $y'' = v'(t)$ so by substituting v for y' and v' for y'', the equation can be rewritten as $\phi(t, v, v') = 0$. This rewritten equation is first-order! Solve it for v, then integrate v to get the solution $y(t) = \int v(t)\,dt$ of the original second-order equation.

- *Case 2:* The equation has no t in it, i.e. the equation is of the form

  $$\phi(y, y', y'') = 0 \quad \text{a.k.a. } \phi\left(y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right) = 0.$$

  To solve this, pretend y is an independent variable and let $v = v(y) = \frac{dy}{dt}$. Then

  $$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}(v(y)) = \frac{dv}{dy} \cdot \frac{dy}{dt} = \frac{dv}{dy}v.$$

  This means you can replace $\frac{dy}{dt}$ with v and $\frac{d^2y}{dt^2}$ with $\frac{dv}{dy}v$ in the original equation to get

  $$\phi\left(y, v, \frac{dv}{dy}v\right) = 0.$$

This is a first-order equation in v, which can be solved by usual methods to find $v = v(y)$. Then solve the equation $\frac{dy}{dt} = v(y)$ (this equation is usually separable) to find y in terms of t.

In Problems 79-82, solve the given second-order ODE or IVP. Write your final answer in the form $y = f(t)$.

79. a) $t\frac{d^2y}{dt^2} = 2\frac{dy}{dt} + 2$ b) $ty''e^{y'} = e^{y'} - 1$

80. a) $\begin{cases} \frac{d^2y}{dt^2} + 2\left(\frac{dy}{dt}\right)^2 \tan y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$ b) $y'' = -2t(y')^2$

81. a) $y'' = -y$ b) $(y^2+1)y'' = 2y(y')^2$

Hint: In part (a), you will need the following integration rule, which you can use without proof:

$$\int \frac{1}{\sqrt{C-y^2}}\,dy = \arcsin\left(\frac{y}{\sqrt{C}}\right) + D.$$

82. a) $y'' - 2y' = 12\sin t$ b) $y'' + 4y' = 10e^{3t}$

Hint: Combine this new technique with the method of undetermined coefficients.

Problems from Section 2.5

83. Determine whether or not each of the given differential equations is exact. If the equation is exact, find its solution.

a) $(2t+3) + (2y-2)\frac{dy}{dt} = 0$
b) $(2t+4y) + (2t-2y)y' = 0$
c) $(e^t\cos y + 2\cos t)\frac{dy}{dt} + e^t\sin y - 2y\sin t = 0$
d) $y' = \frac{e^t\sin y + 3y}{3t - e^t\sin y}$

84. Find the solution of each initial value problem:

a) $\begin{cases} 2t - y + 2yy' - ty' = 0 \\ y(1) = 3 \end{cases}$

b) $\begin{cases} \frac{y}{t} + 6t + (\ln t)y' = 0 \\ y(1) = 7 \end{cases}$

Note: this IVP could also be written as $\frac{y}{t} + 6t + (\ln t)y' = 0$, $y(1) = 7$.

85. Consider the differential equation $y + (2t - e^y)y' = 0$.

   a) Show this equation is not exact.

   b) Show that after you multiply through the entire equation by y, the equation becomes exact.

   c) Solve the equation.

86. **(Optional; extra credit)** Find the general solution of the differential equation

$$(t+2)\sin y + t\cos y \frac{dy}{dt} = 0.$$

Hint: You have to figure out something to multiply the equation through by to make the equation become exact.

Problems from Section 2.6

In Problems 87-92, use the following setup: A nitric acid solution flows at a constant rate of 6 L/min into a large tank that initially holds 200 L of a 0.5% nitric acid solution. The solution inside the tank is kept well stirred, and flows out of the tank at a rate of 8 L/min. The solution entering the tank is 20% nitric acid. Let $y(t)$ be the amount (i.e. volume) of nitric acid in the tank at time t (where t is in minutes).

87. a) Determine the volume of solution in the tank as a function of t.

   Hint: The tank starts with 200 L of solution; 6 L of solution flows in per minute and 8 L of solution flows out per minute.

   b) Draw a compartmental diagram (with boxes and arrows) representing this situation (the compartment should be $y(t)$).

   WARNING: The "rate out" is tricky. To get this, you have to multiply the rate at which fluid flows out of the tank by the concentration of fluid in the tank; the concentration of fluid in the tank has to take into account the fact that the volume of fluid in the tank is not constant.

   c) Write an initial value problem which models this setup.

   d) Using the initial value problem you wrote in part (c), use Euler's method with $\Delta t = 1$ to estimate the amount of fluid in the tank at time 60.

   e) Using the initial value problem you wrote in part (c), use Euler's method with $\Delta t = .001$ to estimate the amount of fluid in the tank at time 60.

88. a) Use *Mathematica* to sketch the slope field for this differential equation, with the stream line corresponding to $y(t)$ included. Use the viewing window $[-5, 100] \times [-5, 50]$. Attach a printout of your slope field, labelled as "88 (a)".

b) Based on your slope field, estimate the amount of nitric acid in the tank at time 20.

c) Based on your slope field, estimate all times t where there is exactly 10 L of nitric acid in the tank.

d) Based on your slope field, estimate the maximum amount of nitric acid that is in the tank at any one instant.

e) Based on your slope field, estimate the time when the amount of nitric acid in the tank is maximized.

89. Solve the IVP of Problem 87 (c), writing your answer in the form $y = f(t)$.

    Hint: Use integrating factors.

90. a) Have *Mathematica* sketch the graph of the solution you obtain in Problem 89 (where $0 \leq t \leq 100$). Attach a printout of this graph, labelled as "90 (a)". (Check that the graph you get is consistent with the picture you obtained in Problem 88 (a); if it isn't, something is wrong.)

    b) Use your answer to Problem 89 to find the exact value of $y(60)$.

    c) Find a decimal value of the value of $y(60)$, and compare it to what you got in parts (d) and (e) of Problem 87. Were those estimates accurate?

    d) **(Optional; extra credit)** Find the exact answers (not decimal approximations) to the questions asked in parts (d) and (e) of Problem 87.

91. a) Find a formula for the concentration of nitric acid at time t.

    Hint: This is tricky, because you have to divide the amount of nitric acid in the tank by the volume, and unlike the example in class, the volume of solution in the tank is not a constant.

    b) Have *Mathematica* sketch the graph of the function you obtain in part (a) (where $0 \leq t \leq 100$). Attach a printout of this graph, labelled as "91 (b)".

    c) From the graph produced in part (b), estimate the first time t when the solution in the tank is 10% nitric acid.

    d) **(Optional; extra credit)** Find the exact value of the first time at which the solution in the tank is 10% nitric acid.

92. Suppose you wanted to know how much nitric acid was in the tank at time 150. I claim that this is not computable by figuring $y(150)$. Why not? What is the amount of nitric acid in the tank at time 150? What is the concentration of nitric acid in the tank at time 150? Explain your answers.

In Problems 93-96, use the following setup: A parachutist has mass 75 kg. She jumps out of a helicopter (assume her initial velocity is 0) which is 2000 m above the ground, and falls toward the ground under the influence of two forces: gravity (which is 9.8 m/sec^2) and air resistance (drag coefficient 30 N sec/m).

93. a) Draw a free body diagram representing this situation.
    b) Write an initial value problem which models this situation. Set the problem up so that $v > 0$ corresponds to downward motion.
    c) Sketch the phase line for the equation you wrote in part (b).
    d) What is the terminal velocity of the parachutist (assuming she never pulls her rip cord)?
    e) Use *Mathematica* to sketch the slope field for this differential equation, with the stream line corresponding to $v(t)$ included (use the viewing window $[-5, 50] \times [-5, 50]$). Attach a printout of your slope field, labelled as "93 (d)".

94. a) Solve the IVP you wrote down in Question 93 (b).
    b) Use *Mathematica* to sketch the graph of $v(t)$ (use the viewing window $[-5, 50] \times [-5, 50]$). Attach a printout of this graph, labelled as "94 (b)". (Check that this graph is consistent with the slope field obtained in Problem 93 (e); if it isn't, something is wrong.)

95. a) Find a formula which gives the parachutist's height above the ground at time t.

    Hint: In Problem 94 (a), you found the parachutist's velocity at time t. You know what the parachutist's height at time 0 is (that is given in the setup). How do you get from an object's velocity back to its position (the parachutists' height is like her position)?

    WARNING: Keep in mind that $v > 0$ corresponds to the parachutist falling, given how we have set this problem up.
    b) Her parachute is set to open automatically when her velocity reaches 20 m/sec. How many seconds after she jumps out of the helicopter will her parachute open? (A decimal answer is OK; it might be useful to have *Mathematica* solve an equation here.)
    c) How high above the ground will she be when her parachute opens? (Again, a decimal answer is OK.)

96. **(Optional; extra credit)**

    a) After the parachutist's chute opens, assume that the forces affecting her velocity are gravity (still 9.8 m/sec^2) and air resistance (but now, the drag coefficient is 90 N sec/m rather than 30 because the chute provides greater air resistance).

    How many seconds after her parachute opens will she hit the ground? (A decimal answer is OK.)

    Hint: To solve this question, you need to write an appropriate initial value problem, and then solve it.

b) What will her velocity be when she hits the ground? (A decimal answer is OK.)

c) Graph the parachutist's velocity against elapsed time, starting at $t = 0$ (when she jumps out of the helicopter) and ending when she hits the ground.

97. A bottle of champagne is originally at room temperature ($70°$). It is chilled in ice ($32°$). Suppose that it takes 15 minutes for the champagne to chill to $60°$. How long (including the original 15 minutes) will it take for the wine to reach $46°$? (A decimal answer is OK, but you need to actually solve the equation... don't rely on pictures or Euler's method.)

    Hint: This is a heating and cooling problem with $U(t) = H(t) = 0$, so this is closely related to Example 1 on page 92 of the lecture notes.

98. Choose one of problems (a), (b):

    a) An RC electrical circuit (see page 96 of the lecture notes) with a $1\,\Omega$ resistor and a 10^{-4} F capacitor is driven by a voltage $E_S(t) = \sin 2t$ V. If the initial capacitor voltage is zero, find:
        i. the voltage across the capacitor at time t;
        ii. the voltage across the resistor at time t (to get this, apply Kirchoff's voltage law to the answer to (i); and
        iii. the current at time t (to get this, apply Ohm's law to the answer to part (ii).

    b) An RL electrical circuit (see page 97 of the lecture notes) with a $1\,\Omega$ resistor and a $.01$ H inductor is driven by a generator whose voltage at time t is $E_S(t) = \sin 2t$ V. If the initial current across the inductor is 0, find:
        i. the current at time t;
        ii. the voltage across the resistor at time t (to get this, apply Ohm's Law to the answer to (i); and
        iii. the voltage across the inductor at time t.

A.3 Problems from Chapter 3

Problems from Section 3.1

99. In each part, you are given a system of first-order ODEs and a possible solution of that system. Determine whether or not the possible solution is actually

a solution of the system.

a) $\begin{cases} x'(t) = 3x - 2y \\ y'(t) = 2x - 2y \end{cases}$ POSSIBLE SOL'N: $\begin{cases} x(t) = 8e^{2t} + e^{-t} \\ y(t) = 4e^{2t} + 2e^{-t} \end{cases}$

b) $\begin{cases} x'(t) = x + 3y \\ y'(t) = -4x - y \end{cases}$ POSSIBLE SOL'N: $\begin{cases} x(t) = 2\cos 11t + 2\sin 11t \\ y(t) = -8\sin 11t \end{cases}$

100. In each part, you are given an IVP, together with a possible solution. Determine whether or not the possible solution is actually a solution of the IVP.

a) IVP: $\begin{cases} \begin{cases} x'(t) = 2y \\ y'(t) = x - y \end{cases} \\ \mathbf{y}(0) = (2, -3) \end{cases}$ POSSIBLE SOL'N: $\begin{cases} x(t) = \frac{2}{3}e^{-2t} - \frac{2}{3}e^{t} \\ y(t) = \frac{-2}{3}e^{-2t} - \frac{1}{3}e^{t} \end{cases}$

b) IVP: $\begin{cases} \begin{cases} x'(t) = 2y \\ y'(t) = x - y \end{cases} \\ \mathbf{y}(0) = (2, -3) \end{cases}$ POSSIBLE SOL'N: $\begin{cases} x(t) = \frac{8}{3}e^{-2t} - \frac{2}{3}e^{t} \\ y(t) = \frac{-8}{3}e^{-2t} - \frac{1}{3}e^{t} \end{cases}$

101. In each part, you are given a set of parametric equations $\mathbf{y}(t)$. For each set of parametric equations:
   (i) Write the set of parametric equations out, coordinate by coordinate (see the answers if you don't understand what this means).
   (ii) Find the values of $\mathbf{y}(0)$, $\mathbf{y}(1)$ and $\mathbf{y}(2)$.

   a) $\mathbf{y}(t) = (t^2 - 2, t + 3)$ b) $\mathbf{y}(t) = (2t, 5t, -7t)$

In each part of Problems 102-104, you are given a set of two parametric equations of the form $\mathbf{y}(t) = (x(t), y(t))$. For each set of parametric equations:

(i) Use the following command in *Mathematica* to sketch the graph of these parametric equations in the xy-plane:

```
ParametricPlot[{formula for x(t), formula for y(t)},
 {t, -100, 100}, PlotRange -> {{xmin, xmax}, {ymin, ymax}}]
```

This command plots the graph of the parametric equations in the viewing window $[xmin, xmax] \times [ymin, ymax]$. You are responsible for choosing an appropriate viewing window for each set of equations (I'd start with $[-10, 10] \times [-10, 10]$ and then zoom in or out as necessary).

(ii) Print the graph you get in *Mathematica* (labelling it with the problem number/letter).

(iii) On the graph you print, draw (by hand) an arrow on the curve indicating the direction of motion.

(iv) Indicate (by hand) the point on the graph corresponding to $t = 0$ by drawing a thick point and labelling that point "$t = 0$".

102. a) $\mathbf{y}(t) = (\cos t, \sin t)$ b) $\mathbf{y}(t) = (e^t, 3e^{-t})$

Hint: In *Mathematica*, e is `E`, not `e`.

103. a) $\mathbf{y}(t) = (e^t \cos 2t, e^t \sin 2t)$ b) $\mathbf{y}(t) = (-2\sin 3t, \cos 3t)$

104. a) $\mathbf{y}(t) = (3e^t, e^{2t})$ b) $\mathbf{y}(t) = \left(\frac{4e^t+e^{-t}}{10}, \frac{2e^t-5e^{-t}}{10}\right)$

Problems from Section 3.2

105. Consider the initial value problem

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} x' = x - y \\ y' = 2x + y \end{array} \right. \\ \mathbf{y}(0) = (2,1) \end{array} \right. .$$

Perform Euler's method by hand (show your work), to compute the points $(t_1, \mathbf{y}_1)$, $(t_2, \mathbf{y}_2)$ and $(t_3, \mathbf{y}_3)$ when $\Delta t = 1$.

106. Consider the initial value problem

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} x' = -y \\ y' = x \end{array} \right. \\ \mathbf{y}(0) = (3,0) \end{array} \right. .$$

Perform Euler's method by hand (show your work) to estimate $\mathbf{y}(2)$, using four steps.

107. Consider the initial value problem

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} x' = x + 2y \\ y' = -3x + y \end{array} \right. \\ \mathbf{y}(0) = (-2,1) \end{array} \right. .$$

Use *Mathematica* to implement Euler's method to estimate $\mathbf{y}(4)$, using 1000 steps.

108. Consider the initial value problem

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} x' = -4x - 8y \\ y' = 8x + 4y \end{array} \right. \\ \mathbf{y}(0) = (1,1) \end{array} \right. .$$

a) Use *Mathematica* to implement Euler's method to estimate $\mathbf{y}(30)$, using 10000 steps.

b) Attach a printout of the graph of the points obtained from Euler's method in part (a) (use the viewing window $[-10, 10] \times [-10, 10]$). Label the graph "108 (b)".

c) Based on the picture you get in part (b), describe (in your own words) the qualitative behavior of the solutions as t increases.

109. Consider the initial value problem

$$\begin{cases} \begin{cases} x' = -3x - 6y \\ y' = 6x - 3y \end{cases} \\ \mathbf{y}(0) = (-8, 7) \end{cases}.$$

a) Use *Mathematica* to implement Euler's method to estimate $\mathbf{y}(50)$, using 1000 steps.

b) Attach a printout of the graph of the points obtained from Euler's method in part (a) (use the viewing window $[-10, 10] \times [-10, 10]$). Label the graph "109 (b)".

c) Based on the picture you get in part (b), describe (in your own words) the qualitative behavior of the solutions as t increases.

110. Consider the initial value problem

$$\begin{cases} \begin{cases} x' = -y - z \\ y' = x \\ z' = x - z \end{cases} \\ \mathbf{y}(0) = (1, 2, -1) \end{cases}.$$

a) Use *Mathematica* to implement Euler's method to estimate $\mathbf{y}(5)$, using 5 steps.

b) Use *Mathematica* to implement Euler's method to estimate $\mathbf{y}(5)$, using 100 steps.

c) Based on what you get in part (b), would you say your answer to (a) is accurate? Why or why not?

d) Use *Mathematica* to implement Euler's method to estimate $\mathbf{y}(5)$, using 1000 steps.

e) Based on what you get in part (d), would you say your answer to (b) is accurate? Why or why not?

f) Use *Mathematica* to implement Euler's method to estimate $\mathbf{y}(5)$, using 10000 steps.

g) Use *Mathematica* to implement Euler's method to estimate $\mathbf{y}(5)$, using 100000 steps. (This took my computer 34 seconds to run.)

h) Suppose you needed an approximation of $\mathbf{y}(5)$ correct to two decimal places. Based on what you have done in this problem, is it sufficient to use Euler's method with 10000 steps? Explain.

i) If you needed an approximation of $\mathbf{y}(5)$ correct to three decimal places. Based on what you have done in this problem, is it sufficient to use Euler's method with 10000 steps? Explain.

Problems from Section 3.4

In Problems 111-117, use the following matrices and vectors to compute the indicated quantities. If the quantity does not exist, just write "does not exist".

$$A = \begin{pmatrix} 5 & -2 \\ 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} -3 & -1 \\ 0 & 4 \end{pmatrix} \qquad C = \begin{pmatrix} 4 & 2 \\ -8 & -4 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$G = \begin{pmatrix} 1 & -3 & 0 \\ 2 & 2 & -5 \\ 4 & 1 & -2 \end{pmatrix} \qquad H = \begin{pmatrix} 4 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$K = \begin{pmatrix} \cos 2t & \sin 3t \\ 4\sin 2t & -\sin 3t \end{pmatrix} \qquad \mathbf{x} = \mathbf{x}(t) = (e^t \cos t, e^t \sin t) \qquad \mathbf{z} = \mathbf{z}(t) = (\sin t, 2\cos 3t, 4e^{7t})$$

111.
a) the trace of C
b) the trace of G
c) $3A$
d) $2A - B + C$
e) $A - 2I$ (*Hint:* I is not given above. You should know what I is.)
f) $C - \lambda I$ (your answer should be in terms of λ, the Greek letter "lambda".)

112.
a) K'
b) $\mathbf{z}'(t)$
c) AB
d) G^2
e) $A\mathbf{x}$
f) $BC\mathbf{x}$

113.
a) AK
b) D^8
c) J^{20}
d) H^3

114.
a) H^n (your answer should be in terms of n).
b) $(Dt)^n$ (your answer should be in terms of n and t).

115. a) A^{-1} b) C^{-1} c) $AB^{-1}A$

116. a) $\det A$ b) $\det D$ c) $\det G$

117. a) $\det(A - \lambda I)$ b) $\det(B - \lambda I)$

Hint: Your answers should be in terms of λ.

118. a) Is the matrix A given before Problem 111 a diagonal matrix?
b) Is the matrix H given before Problem 111 a diagonal matrix?

Problems from Section 3.7

119. For each given set of functions, compute the Wronskian. Then, determine whether or not the functions are linearly independent.

a) $f_1(t) = t^2 + 5t$, $f_2(t) = t^2 - 5t$
b) $f_1(t) = e^{3t}$, $f_2(t) = e^{3(t-1)}$
c) $f_1(t) = 1$, $f_2(t) = t$, $f_3(t) = t^2$
d) $f_1(t) = \sin x$, $f_2(x) = \cos x$, $f_3(x) = \sin 2x$, $f_4(x) = \cos 2x$
Hint: To compute a 4×4 determinant, use *Mathematica* (there's also a command which computes the Wronskian of a set of functions directly).

120. Show that the Wronskian of the functions $\mathbf{f}_1(t) = (e^t, 2e^t - e^{-t}, 3e^t)$, $\mathbf{f}_2(t) = (4e^t, e^{-t}, 0)$ and $\mathbf{f}_3(t) = (-2e^t, 4e^t - 3e^{-t}, 6e^t)$ is zero. Then show these functions are linearly dependent by finding constants c_1, c_2 and c_3 (not all zero) such that $c_1\mathbf{f}_1 + c_2\mathbf{f}_2 + c_3\mathbf{f}_3 = \mathbf{0}$.

121. a) Prove that if a and b are different constants, then e^{at} and e^{bt} are linearly independent.
Hint: Compute the Wronskian of the two functions in terms of a and b.
b) Prove that if a, b and c are three different constants, then $\{e^{at}, e^{bt}, e^{ct}\}$ is a linearly independent set.
Hint: Compute the Wronskian of the functions, and then use *Mathematica* to factor the Wronskian using the `Factor[ ]` command. This will explain why the Wronskian is not equal to zero.

Note: the facts proved in this problem generalize: if $a_1, ..., a_d$ are <u>different</u> constants, then $\{e^{a_1 t}, e^{a_2 t}, ..., e^{a_d t}\}$ is a linearly independent set. (You should remember this fact, so you don't have to re-verify it over and over.)

122. **(Optional; extra credit)** Prove that for any constants a and b (with $b \neq 0$), the functions $e^{at} \sin bt$ and $e^{at} \cos bt$ are linearly independent.

123. Suppose that $\mathbf{y}_p(t) = (2e^{5t}, -3e^{5t}, e^{5t})$ is a solution of some linear system $A_1\mathbf{y}' + A_0\mathbf{y} = \mathbf{q}$, and suppose that the solution of the corresponding homogeneous system $A_1\mathbf{y}' + A_0\mathbf{y} = \mathbf{0}$ is

$$\mathbf{y}_h(t) = C_1 e^{4t}\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + C_3 e^{t}\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}.$$

a) How many equations comprise this system?

b) Write the general solution of this system in vector form.

c) Write the general solution coordinate-wise.

d) Write the particular solution corresponding to $C_1 = 3$, $C_2 = -2$ and $C_3 = 2$.

e) Find the value of $\mathbf{y}(3)$ for the solution found in part (d).

f) Find the value of $x(2)$ for the solution found in part (d).

g) Find the particular solution satisfying $\mathbf{y}(0) = (-1, 4, 2)$.

Problems from Section 3.8

124. Consider the system of ODEs

$$\begin{cases} x' = x + 4y \\ y' = x - 2y \end{cases}.$$

For each point (x, y) below, compute the value of $\frac{dy}{dx}$ at that point. Then sketch the mini-tangents corresponding to those points (there should be one picture, with all the vectors on it; each mini-tangent should have an arrowhead on it, drawn in the manner of the example on page 110 of the Fall 2017 lecture notes).

$$\begin{array}{cccc} (1,0) & (0,2) & (0,-2) & (-3,0) \\ (1,3) & (-2,-3) & (3,-1) & (-3,2) \\ (-2,2) & (-1,1) & (1,-1) & (2,-2) \end{array}$$

125. For each given initial value problem, use *Mathematica* to sketch the slope field of the system and the graph of the solution of the IVP (the intent here is for you to use Command 3 from the file `phaseplanes.nb`). Label your pictures with the problem number/letter:

a) $\begin{cases} x' = 3y \\ y' = x \\ x(0) = -2 \\ y(0) = 3 \end{cases}$

Note: this system of IVPs could also be written $\mathbf{y}' = (3y, x)$; $\mathbf{y}(0) = (-2, 3)$.

b) $\mathbf{y}' = (x - xy, x + 2y^2 - x^2 y)$; $\mathbf{y}(0) = (1, 0)$

126. Same directions as the previous problem:

a) $\mathbf{y}' = (y \sin(x + y), xe^{-x-y})$; $\mathbf{y}(0) = (1, -1)$

b) $\mathbf{y}' = (2xy, x - 2y + x^2 y^3)$; $\mathbf{y}(2) = \left(-1, \frac{5}{4}\right)$

127. Find all equilibria of each autonomous system of ODEs:

a) $\mathbf{y}' = (y^2 - 3y - 4, 2x + 3y)$

b) $\mathbf{y}' = (y - x, y^2 + xy - 4)$

128. Find all equilibria of each autonomous system of ODEs:

a) $\begin{cases} x' = 4x^2 - y \\ y' = 1 - y^2 \end{cases}$

b) $\begin{cases} x' = (x - 2)(y + 3) \\ y' = (y - 1)(x + 4) \end{cases}$

129. Consider the system of two ODEs:

$$\begin{cases} x' = x - 3 \\ y' = 2 - 2y \end{cases}$$

a) Use *Mathematica* to draw the phase plane (not the slope field) for this system (use the viewing window $[-8, 8] \times [-8, 8]$ and ask *Mathematica* to draw at least 200 solution curves). Print this picture, labelling it as "129".

b) On your picture, sketch the graph of the solution to this system satisfying $\mathbf{y}(0) = (0, 4)$.

c) Either by looking at your picture or by doing some algebra, find all constant functions which solve the system.

d) Suppose $\mathbf{y}(0) = (0, -5)$. In this situation:

i. Is $x(t)$ an increasing function, a decreasing function, or a constant function of t?

ii. Is $y(t)$ an increasing function, a decreasing function, or a constant function of t?

iii. What is $\lim\limits_{t \to \infty} x(t)$?

iv. What is $\lim\limits_{t \to \infty} y(t)$?

e) Suppose $\mathbf{y}(1) = (5, 5)$. Answer the same questions (i)-(iv) as in part (d).

f) Suppose $\mathbf{y}(0) = (3, -2)$. Answer the same questions (i)-(iv) as in part (d).

130. Consider the system of two ODEs:

$$\begin{cases} x' = 4y - xy - x + 4 \\ y' = 2x - xy - y + 2 \end{cases}$$

a) Use *Mathematica* to draw the phase plane (not the slope field) for this system (use the viewing window $[-10, 10] \times [-8, 8]$ and ask *Mathematica* to draw at least 200 solution curves). Print this picture, labelling it as "130".

b) Either by looking at your picture or by doing some (nontrivial) algebra, find all constant functions which solve the system.

c) Suppose $\mathbf{y}(0) = (7, -7)$.

i. Which of these statements best describes the behavior of $x(t)$ in this situation?
A. $x(t)$ increases for all t
B. $x(t)$ decreases for all t
C. initially, $x(t)$ is increasing, but then it becomes decreasing
D. initially, $x(t)$ is decreasing, but then it becomes increasing

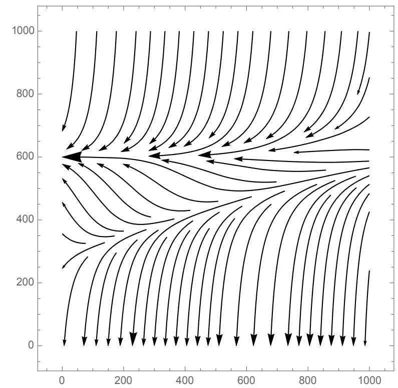
ii. Which of these statements best describes the behavior of $y(t)$ in this situation?
A. $y(t)$ increases for all t
B. $y(t)$ decreases for all t
C. initially, $y(t)$ is increasing, but then it becomes decreasing
D. initially, $y(t)$ is decreasing, but then it becomes increasing

d) Suppose $\mathbf{y}(1) = (1, -6)$. Answer the same questions (i) and (ii) as in part (c).

e) Suppose $\mathbf{y}(0) = (8, 9)$. Answer the same questions (i) and (ii) as in part (c).

f) Let E be the set of points (x_0, y_0) such that $\lim\limits_{t\to\infty} x(t) = 4$ if $\mathbf{y}(0) \in E$. On the picture you obtained in part (a), shade the set of points which belong to E.

131. A biologist is studying the population of two species, X and Y. He lets $x(t)$ and $y(t)$ represent the population of these species at time t, and based on his biology research, he comes up with a system of ODEs modeling this situation. He asks *Mathematica* to sketch a picture of the phase plane for this system, and *Mathematica* produces this:



a) Suppose that the current population of species X is somewhere between 700 and 800, and that the current population of species Y is somewhere between 800 and 900. Based on this model, can the biologist say what will happen to the population of these species in the long run? If so, what will happen? If not, why can't he tell?

b) Suppose that the current population of species X is somewhere between 400 and 500, and that the current population of species Y is somewhere between 200 and 250. Based on this model, can the biologist say what will happen to the population of these species in the long run? If so, what will happen? If not, why can't he tell?

c) Suppose that the current population of species X is somewhere between 600 and 700, and that the current population of species Y is somewhere between 400 and 600. Based on this model, can the biologist say what will happen to the population of these species in the long run? If so, what will happen? If not, why can't he tell?

132. **(Optional; extra credit)** The observed growth of tumors can be explained by the following mathematical model. Let $N(t)$ be the number of cells in the tumor at time t (so the bigger N is, the worse the tumor is). Some of the cells in the tumor "proliferate" (i.e. they split to make more cancerous cells, growing the tumor); let $P(t)$ be the number of proliferating cells in the tumor at time t. The functions P and N are modeled by the following system of ODEs:

$$\begin{cases} P' = cP - r(N)P \\ N' = cP \end{cases}$$

where c is a positive constant and $r : \mathbb{R} \to \mathbb{R}$ is an increasing function which represents the rate at which proliferating cells become non-proliferating (c and r depend on things like the type of cancer, the patient's age, weight, height, sex, etc.). Suppose that a patient starts with a tumor consisting of one proliferating cancerous cell (so that the initial condition is $P(0) = 1, N(0) = 1$) and suppose that $c = 30$.

For each of the following functions $r(N)$, determine the long-term size of the tumor, i.e. the number of cancerous cells the patient will end up with in the long run.

a) $r(N) = N^2$

b) $r(N) = N$

c) $r(N) = \sqrt{N} \ln N$

d) $r(N) = \ln N$

Based on what you observe in parts (a)-(d), describe (in English) some property (or properties) of the function r and the long-term size of the tumor.

Hint: in parts (a)-(d), have *Mathematica* sketch the phase line starting at the point $(1, 1)$ for the appropriate system(s), and figure out where each solution curve ends.

Problems from Section 3.9

Note: In Problems 133-136, you can (and probably should) use *Mathematica* to check your answers, but I want to see all the steps worked out by hand.

133. Compute the exponential of the following matrix:

$$\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$$

134. Find the general solution of this system of ODEs:

$$\begin{cases} x' = x + y \\ y' = 4x + y \end{cases}$$

135. Find the particular solution of the system

$$\begin{cases} \mathbf{y}' = (2x - y, 3x - 2y) \\ \mathbf{y}(0) = (-6, 4) \end{cases}$$

136. Find the general solution of this system of ODEs:

$$\begin{cases} x' = 4x - 3y \\ y' = 8x - 6y \end{cases}$$

137. Consider the 2×2 system of ODEs $\mathbf{y}' = A\mathbf{y}$ where

$$A = \begin{pmatrix} -7 & 1 \\ -6 & -2 \end{pmatrix}.$$

a) Find the eigenvalues and eigenvectors of A.

b) Find a basis of the solution set, and verify that the functions in this basis are linearly independent by computing their Wronskian.

c) Find the general solution of the system $\mathbf{y}' = A\mathbf{y}$.

d) If $\mathbf{y} = (x, y)$ is any solution of the system $\mathbf{y}' = A\mathbf{y}$, find $\lim\limits_{t\to\infty} x(t)$ and $\lim\limits_{t\to\infty} y(t)$.

e) Find the Cartesian equation(s) of all straight-line solutions of the system.

f) Find the particular solution of $\mathbf{y}' = A\mathbf{y}$ satisfying $\mathbf{y}(0) = (2, -1)$.

g) For the solution found in part (f), find (the exact values of) $\mathbf{y}(2)$ and $\mathbf{y}(-2)$.

138. Consider the 3×3 initial value problem $\begin{cases} \mathbf{y}' = A\mathbf{y} \\ \mathbf{y}(0) = (1, 3, -2) \end{cases}$ where

$$A = \begin{pmatrix} 3 & 0 & -1 \\ -2 & 2 & 1 \\ 8 & 0 & -3 \end{pmatrix}.$$

a) Use *Mathematica* to find the particular solution of this system (recall that the command `MatrixExp[ ]` computes matrix exponentials).

b) Based on the solution you got in part (a), what are the eigenvalues of the matrix A? Explain how you answered this question by looking at the solution to part (a).

139. Use *Mathematica* to find the general solution of each following system of ODEs:

a)
$$\begin{cases} x' = 3x + 2y + 4z \\ y' = 2x \qquad\quad + 2z \\ z' = 4x + 2y + 3z \end{cases}$$

b)
$$\begin{cases} x' = x + y + z \\ y' = 2x + y - z \\ z' = -8x - 5y - 3z \end{cases}$$

c)
$$\begin{cases} w' = 20w - 101x - 80y - 5z \\ x' = 7w - 7x - 28y - 28z \\ y' = -14w - 49x + 56y - 7z \\ z' = 3w - 75x - 12y + 36z \end{cases}$$

Problems from Sections 3.11 to 3.14

140. Find the general solution of this system of ODEs:

$$\begin{cases} x' = 3x - 2y \\ y' = 4x - y \end{cases}$$

141. Find the particular solution of this initial value problem:

$$\begin{cases} \mathbf{y}' = (x - 5y, x - 3y) \\ \mathbf{y}(0) = (1, 1) \end{cases}$$

142. a) Find the general solution of this system of ODEs:

$$\begin{cases} x' = 3x - 4y \\ y' = x - y \end{cases}$$

b) Write the Cartesian equation(s) of any straight-line solutions of this system.

143. Find the particular solution of this initial value problem:

$$\begin{cases} \mathbf{y}' = (x - 4y, 4x - 7y) \\ \mathbf{y}(0) = (3, 2) \end{cases}$$

144. Find the general solution of this system of ODEs:

$$\begin{cases} x' = 2x - y + 8e^{2t} \\ y' = 3x - 2y + 20e^{2t} \end{cases}$$

Write your answer coordinate-wise, i.e. as $\begin{cases} x(t) = \text{ something} \\ y(t) = \text{ something} \end{cases}$.

145. Find the particular solution of this initial value problem:

$$\begin{cases} \mathbf{y}' = (-3x + 2y, x - 2y + 3) \\ \mathbf{y}(0) = (1, 2) \end{cases}$$

146. Find the general solution of this system of ODEs:

$$\begin{cases} x' = 2x - 5y - \cos 3t \\ y' = x - 2y + \sin 3t \end{cases}$$

Write your answer coordinate-wise.

147. Suppose you are given a system of ODEs $\mathbf{y}' = A\mathbf{y}$ where A has the indicated properties. Write the general solution of the system (write each answer coordinate-wise):

a) A is a 4×4 matrix with the following eigenvalues and eigenvectors:

$$\lambda = -3 \leftrightarrow \begin{pmatrix} 3 \\ 0 \\ 1 \\ 4 \end{pmatrix} \quad \lambda = 2 \leftrightarrow \begin{pmatrix} 2 \\ 1 \\ -3 \\ 1 \end{pmatrix} \quad \lambda = 4 \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \lambda = -6 \leftrightarrow \begin{pmatrix} 0 \\ 3 \\ 5 \\ 1 \end{pmatrix}.$$

b) A is a 3×3 matrix with eigenvalues $\lambda = 2$ and $\lambda = 4$ ($\lambda = 4$ is repeated twice). An eigenvector corresponding to $\lambda = 2$ is $(1, 5, 0)$ and an eigenvector corresponding to $\lambda = 4$ is $(1, 2, -1)$. A generalized eigenvector for $\lambda = 4$ is $(0, 3, -4)$.

c) A is a 4×4 matrix with eigenvalues $\lambda = 3 \pm 2i$ and $\lambda = 0$ ($\lambda = 0$ is repeated twice). An eigenvector corresponding to $\lambda = 3 + 2i$ is $(1 - i, 1 + 2i, i, 0)$ and an eigenvector corresponding to $\lambda = 0$ is $(1, 3, -2, 4)$. A generalized eigenvector corresponding to $\lambda = 0$ is $(0, 1, 4, -2)$.

148. Find the solution of each system (using *Mathematica* to compute quantities as necessary). Write the answers coordinate-wise, and simplify them as much as possible. In particular, your answers should not contain i.

a) $\begin{cases} x' = y + z \\ y' = x + z \\ z' = x + y \end{cases}$

b) $\begin{cases} \mathbf{y}' = (-45x - 90y - 45z, 16x - 116y - 4z, 76x - 191y - 244z) \\ \mathbf{y}(0) = (1, 3, -2) \end{cases}$

c) $\begin{cases} \mathbf{y}' = (y_1 - y_2 + y_3, 2y_1 - 2y_2 + 3y_3, \frac{4}{3}y_1 - y_2 + \frac{2}{3}y_4, 3y_1 - y_2 - 2y_3 + y_4) \\ \mathbf{y}(0) = (2, 0, 1, -1) \end{cases}$

Hint: use the `ExpToTrig[ ]` command, then the `Simplify[ ]` command to get rid of the imaginary numbers in part (c).

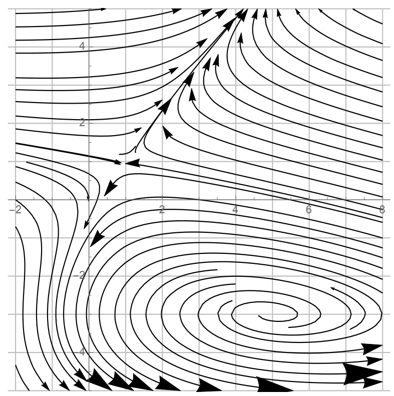
Problems from Section 3.15

In Problems 149-151, find and classify the equilibria of each autonomous system. Your classification should be specific (i.e. your answer should be "saddle", "stable node", "unstable node", "stable spiral", "unstable spiral" or "center").

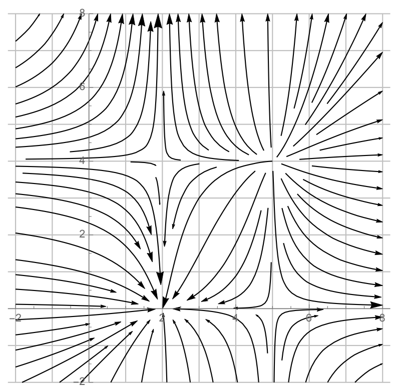
149. a) $\begin{cases} x' = 2x + y - 3 \\ y' = x + 4y + 1 \end{cases}$ b) $\begin{cases} x' = -2y - x - 1 \\ y' = x + 3 \end{cases}$

150. a) $\begin{cases} x' = (x-1)(4+y) \\ y' = -(x+2)(y-3) \end{cases}$ b) $\begin{cases} x' = e^x - e^{-y} \\ y' = e^{2x} - e^y \end{cases}$

151. a) The system whose phase plane is



b) The system whose phase plane is



Problems from Section 3.16

152. Consider a 2×2 autonomous system $\mathbf{y}' = \Phi(\mathbf{y})$; let $\mathbf{y}_0$ be an equilibrium of this system. Classify the equilibrium $\mathbf{y}_0$ based on the given information:

a) $\mathrm{tr} D\Phi(\mathbf{y}_0) = 8$ and $\det D\Phi(\mathbf{y}_0) = 1$.

b) $\mathrm{tr} D\Phi(\mathbf{y}_0) = -3$ and $\det D\Phi(\mathbf{y}_0) = -2$.

c) $\mathrm{tr} D\Phi(\mathbf{y}_0) = -6$ and $\det D\Phi(\mathbf{y}_0) = 3$.

d) $\mathrm{tr} D\Phi(\mathbf{y}_0) = 0$ and $\det D\Phi(\mathbf{y}_0) = 4$.

153. Consider the 2×2 system $\mathbf{y}' = A\mathbf{y}$ where A is a 2×2 matrix with constant entries. Classify the equilibrium at the origin based on the given information:

a) tr$A = -1$ and $\det A = 7$.

b) tr$A = 5$ and $\det A = -1$.

c) tr$A = 2$ and $\det A = 3$.

d) tr$A = 0$ and $\det A = -6$.

Problems from Section 3.17

154. Two large tanks (call them X and Y) each hold 100 L of liquid. They are interconnected by pipes which pump liquid from tank X to tank Y at 3 L/min and from tank Y to tank X at 1 L/min. A brine solution of concentration 0.2 kg/L of salt flows into tank X at a rate of 5 L/min; the solution flows out of the system of tanks via two pipes (one pipe allows flow out of tank X at 2 L/min and another pipe allows flow out of tank Y at 2 L/min). Suppose that initially, tank Y contains pure water but tank X contains 40 kg of salt; assume that at all times the liquids in each tank are kept mixed.

a) Draw a compartmental diagram which models this situation.

b) Write an initial value problem modeling the situation, where $x(t)$ and $y(t)$ represent the amount of salt in tanks X and Y respectively, at time t, and $\mathbf{y} = (x, y)$.

c) Use *Mathematica* to draw a picture of the vector field for your system, with the solution curve to your initial value problem indicated. Use the viewing window $[0, 60] \times [0, 60]$; attach a printout of this picture, labelled as "154 (c)".

d) Based on your picture from part (c), estimate the amount of salt in tank X at the instant when tank Y has 10 kg of salt in it.

e) Estimate the amount of salt in each tank 8 minutes after the initial situation by having *Mathematica* implement Euler's method with 10000 steps.

f) Solve the initial value problem you wrote down in part (b). Write your answer coordinate-wise.

g) Have *Mathematica* graph the functions $x(t)$ and $y(t)$ on the same axes (in the viewing window $[0, 100] \times [0, 60]$). The appropriate *Mathematica* commands are

`x[t_] =` *whatever formula you get for* $x(t)$
`y[t_] =` *whatever formula you get for* $y(t)$
`Plot[{x[t],y[t]}, {t, 0, 100}, PlotRange -> {0,60}]`

Attach a printout of these graphs (labelled as "154 (g)"). Make sure you indicate which graph is $x(t)$ and which is $y(t)$.

h) Find the amount of salt in each tank at time 8 (I want both the exact amount and a decimal approximation). Compare your answer to part (e); how accurate was your estimate in part (e)?

i) Find the concentration of salt in tank X at time 8.

155. In this problem, we explore a model (called the **Gause competitive exclusion model**) which describes how the populations of two species behave, when the species are in competition for the same resource. Let X and Y be two species whose populations at time t are $x(t)$ and $y(t)$, respectively, where x and y satisfy the system of differential equations

$$\mathbf{y}' = \Phi(\mathbf{y}) \text{ a.k.a. } \begin{cases} x' = r_X x(L_X - x - \alpha y) \\ y' = r_Y y(L_Y - y - \beta x) \end{cases} .$$

In this setting, α is a nonnegative number called the **competition coefficient of Y on X**; it measures the degree to which increased numbers of species Y negatively impact the ability of species X to survive. Similarly, $\beta \geq 0$ is the competition coefficient of X on Y.

a) If $\alpha = \beta = 0$ (i.e. the species are not in competition), the populations of X and Y change according to what model? In light of this, what do the constants r_X, L_x, r_Y and L_Y represent?

b) This system of ODEs has four equilibria: three of them are $(0,0)$ (i.e. both species are extinct), $(0, L_Y)$ (species X is extinct), and $(L_X, 0)$ (species Y is extinct).

   i. Find the fourth equilibrium.

   ii. Explain why this fourth equilibrium is called the **coexistence equilibrium**.

c) Find $D\Phi(0,0)$. Compute the eigenvalues of $D\Phi(0,0)$ and show that $(0,0)$ is an unstable equilibrium.

   Hint: a matrix is called **triangular** if all the entries below its diagonal are zero, or if all the entries above its diagonal are zero. The eigenvalues of a triangular matrix are always its diagonal entries.

d) Find $D\Phi(0, L_Y)$. Compute the eigenvalues of $D\Phi(0, L_Y)$ (the hint of part (c) is useful).

e) Based on your computations in part (d), show that $(0, L_Y)$ is a stable equilibrium if and only if $\alpha > \frac{L_X}{L_Y}$.

f) Find $D\Phi(L_X, 0)$. Compute the eigenvalues of $D\Phi(L_X, 0)$.

g) Based on your computations in part (f), show that $(L_X, 0)$ is a stable equilibrium if and only if $\beta > \frac{L_Y}{L_X}$.

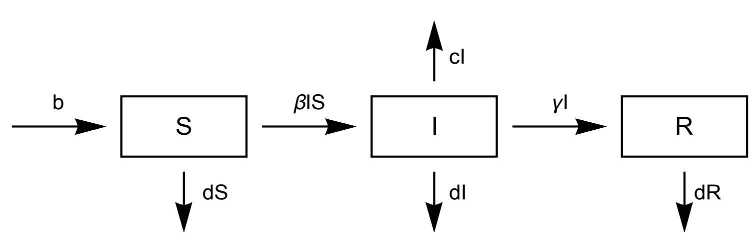
h) Suppose that neither $(0, L_Y)$ nor $(L_X, 0)$ are stable.
   i. Show that the following three inequalities hold:
   $$L_X - \alpha L_Y > 0 \qquad \beta L_X - L_Y < 0 \qquad \alpha\beta - 1 < 0$$
   ii. Compute $D\Phi$ at the coexistence equilibrium.
   iii. Compute and simplify $\det D\Phi$ and tr $D\Phi$ at the coexistence equilibrium using *Mathematica*.
   iv. Use the inequalities described in part (i) to show that the coexistence equilibrium must be stable.

i) If the two species occupy the same ecological niche (i.e. they eat the same food, live in the same places, etc.), then it is reasonable to assume that $\beta = \frac{1}{\alpha}$. For example, if X and Y are species of beetles where each Y beetle eats twice as much grain as each X beetle, then α would be 2 and β would be $\frac{1}{2}$. What happens to the coexistence equilibrium in this case? What does that mean about the two species?

156. In the SIR model we studied in class, we assumed that the disease acted quickly (before any births or deaths could occur). Now let's assume that the disease acts more slowly, so that we can account for births and deaths in our model. Let:

$$\begin{aligned}
S &= \text{ the population of the susceptible class (as before)} \\
I &= \text{ the population of the infective class (as before)} \\
R &= \text{ the population of the recovered class (as before)} \\
b &= \text{ the birth rate (assume this is constant)} \\
d &= \text{ the death rate due to factors other than the disease} \\
c &= \text{ the death rate due to the disease} \\
\beta &= \text{ the rate at which infections take place} \\
\gamma &= \text{ the rate at which infected patients recover}
\end{aligned}$$

This leads to the following compartmental diagram:



Note that we can no longer assume that $S + I + R = 1$, because the total population may change as time passes.

a) Write the system of differential equations for S, I and R. (In the rest of this problem, we will call this system $\mathbf{y}' = \Phi(\mathbf{y})$ where $\mathbf{y} = (S, I, R)$.)

b) There is one equilibrium of the system which has the form $(S^{\#}, 0, 0)$, where $S^{\#} \neq 0$ (this is called the **disease-free equilibrium**; it represents the situation where no one has the disease). Find $S^{\#}$ in terms of the variables listed above.

c) Find $D\Phi(S^{\#}, 0, 0)$, and have *Mathematica* compute the eigenvalues of this matrix.

d) Show that the disease-free equilibrium is stable if and only if $\beta b < d(c + d + \gamma)$.

e) There is a second equilibrium called the **endemic equilibrium** (because in this situation the disease persists), which has the form (S^*, I^*, R^*). Find S^*, I^* and R^* in terms of the variables listed above.

f) Find $D\Phi(S^*, I^*, R^*)$ and have *Mathematica* compute the eigenvalues of this matrix.

g) Show that the endemic equilibrium is stable if and only if $\beta b > d(\gamma + c + d)$.

h) Explain why your answers to parts (d) and (g) make sense, given what the variables in the problem mean.

A.4 Problems from Chapter 4

Problems from Section 4.1

157. Rewrite this third-order ODE as a first-order system of the form $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$. Be sure to carefully identify what $\mathbf{y}$, A and $\mathbf{q}$ are.

$$y''' - 4y'' + 7y' - 8y = \cos t$$

158. Rewrite this second-order system of equations as a first-order system of the form $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$. Be sure to carefully identify what $\mathbf{y}$, A and $\mathbf{q}$ are.

$$\begin{array}{llllll} x'' & +2x' & -3y' & +4x & & = 0 \\ y'' & -x' & +4y' & -7x & +3y & = 0 \end{array}$$

Problems from Section 4.2

159. Find the general solution of each differential equation:

a) $y'' - 8y' + 12y = 0$
b) $y'' = -7y' + 18y$
c) $y''' + 11y'' + 30y' = 0$

160. Find the general solution of each differential equation:

a) $y'' + 4y' + 11y = 0$
b) $y''' + 9y' = 0$
c) $y'' - 12y' + 36y = 0$

161. Find the general solution of each differential equation:

a) $y'' - 4y' - 21y = e^{4t}$
b) $y'' + y = \sin 2t$
c) $y'' - 25y = t$

162. Find the particular solution of each initial value problem:

a) $\begin{cases} y'' - 8y' + 15y = 0 \\ y(0) = 3 \\ y'(0) = 2 \end{cases}$ b) $\begin{cases} y'' + 4y' + 4y = 0 \\ y(0) = -6 \\ y'(0) = -1 \end{cases}$ c) $\begin{cases} y'' + 2y' + 5y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

163. Find the particular solution of this initial value problem:

$$\begin{cases} y'' - 16y' + 63y = e^t \\ y(0) = 11 \\ y'(0) = -5 \end{cases}$$

164. Find the general solution of this differential equation:

$$y^{(5)} + y^{(4)} - 4y''' - 8y'' - 32y' - 48y = 0.$$

Hint: use *Mathematica* to factor the characteristic equation.

Problems from Section 4.4

165. a) Explain why the α on page 239 of the lecture notes must be negative.
   b) Explain why the λ_1 and λ_2 on page 239 of the lecture notes must both be negative.

166. Consider a 12 kg mass attached to a fixed point by a spring (like the picture on the top of page 236 of the notes). If the spring constant is $\frac{1}{4}$ N/m and the damping coefficient (i.e. coefficient of friction) is $\frac{7}{2}$ N sec/m, and the entire system (at time t) is subject to an external force of $257 \cos 2t$ N, find the position of the mass at time t. Assume that at time 0, the mass has no initial velocity but is 8 m to the right of where it would be at rest.

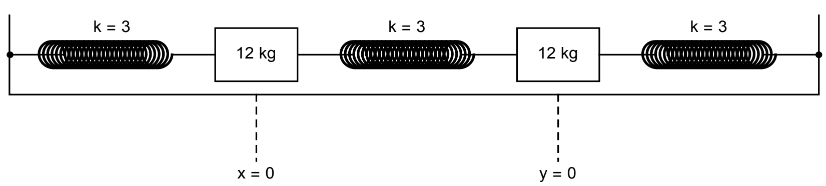
167. Consider a coupled mass-spring system like the one pictured at the top of page 243 in the notes. Assume that $m_1 = 6$ kg and $m_2 = 2$ kg, and the spring constants are $k_1 = 3$ N/m and $k_2 = 2$ N/m. If the first mass is displaced 4 m left of its equilibrium position and the second mass is displaced 2 m right of its equilibrium position, and then the masses are released with no initial velocity:

   a) Find the position of each mass at time t.

   b) Assuming that the spring between the two masses is 10 m long at rest, have *Mathematica* sketch a graph showing the positions of the masses at time t in the viewing window $[0, 50] \times [-5, 20]$ (attach a printout of this graph, labelled as "168"). The vertical axis should be scaled so that height 0 corresponds to the first mass being at equilibrium, as in the second picture on page 244 in the lecture notes.

      To graph two functions at once; see Problem 154 (g) for the appropriate commands.

168. Suppose that three identical springs, each with spring constant $k = 3$ and two identical masses, each of mass 12, are attached in a straight line, with the ends of the outside springs fixed. Here is a picture of the system:



Suppose that the masses move along a frictionless surface; let $x(t)$ and $y(t)$ be the displacement of the masses at time t, where $x = 0$ and $y = 0$ correspond to the system being at rest, and positive values of x and y indicate displacement to the right.

   a) Write down a second-order system of differential equations describing the system (coming from Newton's laws).

   b) Convert the system from part (a) to a first-order system of equations.

   c) Find the general solution of the system you wrote down in part (a).

      Hint: Use *Mathematica* to find eigenvalues and eigenvectors for the system you obtained in part (b).

d) Suppose that initially, the left-hand mass starts at $x = 2$ and the right-hand mass starts at $y = -1$, and that both masses have an initial rightward velocity of 1.
   i. Find $x(t)$ and $y(t)$ in this case.
   ii. Graph x and y on the same axes (see Problem 154 (g) for the appropriate *Mathematica* commands); attach a printout of this graph, labelled as "168 (d)". You are responsible for choosing a reasonable viewing window.
   iii. Find the positions of the masses (i.e. x and y) when $t = 3$.

169. Consider an undamped pendulum of length 2 m and mass 45 kg.
   a) Write the undamped pendulum equation in this case.
   b) Write the linearization of the equation you wrote in part (a).
   c) Find the general solution of the linearized equation you wrote in part (b).
   d) Suppose that at time 0, the pendulum is at angle $\theta = .05$ radians and has angular velocity $-.3$ radians/second. What is the largest angle obtained by the pendulum as it swings?
   Hint: Find the particular solution corresponding to these initial conditions; one aspect of this particular solution gives the answer to the question.

170. **(Optional; extra credit)** Consider the linearized double pendulum equation (coming from page 247 of the lecture notes; repeated here for convenience):
$$\begin{cases} (m_1+m_2)l_1\theta_1'' + m_2 l_2 \theta_2'' + (m_1+m_2)g\theta_1 = 0 \\ m_2 l_2 \theta_2'' + m_2 l_1 \theta_1'' + m_2 g \theta_2 = 0 \end{cases}$$
   a) Rewrite this system of two second-order equations as a first-order system of four equations $\mathbf{y}' = A\mathbf{y}$.
   Hint: To get started, pretend this is a system of two equations in the two variables θ_1'' and θ_2'' and solve for those two quantities in terms of the other variables.
   b) Use *Mathematica* to find the eigenvalues and eigenvectors of the matrix A, and write the general solution of the equation $\mathbf{y}' = A\mathbf{y}$ you wrote in part (a).
   c) Write the formulas for θ_1 and θ_2.

171. A series RLC circuit (as in page 248 of these notes) has a voltage source given by $E_S(t) = 10\cos 2t$ volts, a resistor of $2\,\Omega$, an inductor of $\frac{1}{4}$ H, and a capacitor of $\frac{1}{13}$ F. If the initial current is zero and the initial charge on the capacitor is 3.5 coulombs, find the current in the circuit as a function of time t.

A.5 Selected answers to the homework problems

1. (b) $y = \frac{1}{2t-3}$

2. a) $y = \sqrt{t+3} + 1$
   (c) $y = \pm\frac{1}{2}\sqrt{8-t^2}$

3. (c) $y = 2 + \ln 3 + t - 1$

4. a) ∞
   b) 0

6. a) $f'(t) = 6t - 7$

9. a) $f'(t) = -2e^{-2t}$
   $f''(t) = 4e^{-2t}$
   $f'''(t) = -8e^{-2t}$

10. a) $\frac{1}{2}t^2 + C$

12. (b) $\frac{3}{4}t^{4/3} + C$
    (c) $\frac{2}{3}t^{3/2} + t + C$

14. a) $t^2 + t$
    b) $4\sqrt{t} - 4$

15. (b) $\frac{-1}{4}t^2 + \frac{1}{2}t^2 \ln t + C$

16. (a) $\frac{1}{2}\ln(2-x) - \frac{1}{2}\ln(2+x) + C$
    (b) $\frac{1}{2}\sin^2 t + C$

18. a) $\frac{-3}{t-1} + (t-1) + 2\ln(t-1) + C$

19. a) $x_1 = -1$; $x_2 = 1$; $x_3 = -1$; $x_4 = 1$

20. 43

21. a) not a solution

22. a) solution

24. a) not a solution

25. a) not linear
    b) linear

28. (b) order 4;
    4 constants in gen. sol'n;
    nonlinear

29. a) order 1;
    1 constant in gen. sol'n;
    linear;
    not homogeneous;
    not constant-coefficient

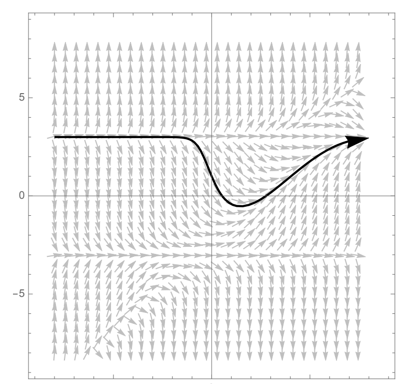
32. (d) $y = \frac{12}{e^{14.1}}e^{4.7t}$

34. (c) $\$318.55$.

35. (c) 16.7621 grams.

37. (b) $y = 4$, $y = 0$, $y = -5$

38. (a)



40. $(t_4, y_4) = (4, 61)$.

41. $y(2) \approx 33$.

46. (a) There are four possible answers: $y = -7$, $y = -2$, $y = 3$ or $y = 6$.
    (b) $y = -2$ is the only sink.

47. a) -2

50. (a) $\phi(-7)$ is negative
    (c) $\phi(4)$ is positive

(d) $\phi'(-2)$ is negative

51. (a) $y = -3$ is stable; $y = 3$ is unstable

52. (a) $y = 0$ is unstable

53. (a) $y = 0$ is unstable; $y = 6$ is stable

55. (a) $y = 0$ and $y = \frac{rL-e}{r}$.
    (e) $E = \frac{1}{2}rL$

57. (a) Yes; if $5 \le r \le 7$ and $y(0) > 0$ we know that $\lim_{t\to\infty} y(t) = 0$, so the long-term value of y is zero.

58. (a) There is only one bifurcation at $r = 9$; it is a saddle-node bifurcation.

62. (a) $y = Ce^{2t}$

63. (b) $y = Ce^{\sin t}$

64. (a) $y = 3\ln t$

66. (a) $y = 7e^{2t} - 3e^{t}$

67. (a) $y = -te^{-t} + Ct$

70. (a) $y = Ce^{-3t} + e^{-2t}$

71. (b) $y = Ce^{4t} - \frac{1}{2}t^2 - \frac{9}{4}t - \frac{9}{16}$

72. (a) $y_h = e^{-7t}$
    (b) No; on the left-side you get 0 which cannot equal $20e^{-7t}$.
    (d) $y = Ce^{3t} + 9te^{3t}$

74. a) $\frac{1}{2}y^2 = \frac{1}{3}\ln(4+t^3) + C$
    b) $y = \frac{Ct^2-1}{Ct^2+1}$

75. (a) $y = \ln(4t + C)$

79. (a) $y = Ct^3 - t + D$

80. (a) $y = \arctan(Ct + D)$

81. (a) $y = C\cos t + D\sin t$

82. (a) $y = Ce^{2t} - 2t^3 - 3t^2 - t + D$

83. a) Exact; solution is $y^2 - 2y + t^2 + 3t = C$
    b) Not exact

84. a) $y^2 - ty + t^2 = 7$

85. (c) $y^2t - ye^y + e^y = C$

87. (a) $200 - 2t$
    (c) $\begin{cases} y' = \frac{6}{5} - \frac{4y}{100-t} \\ y(0) = 1 \end{cases}$
    (e) 15.0017

88. (d) $y_{max} \approx 20$
    (e) $t \approx 35$

89. $y = \frac{-2}{5}(t-100) - \frac{39}{10^8}(t-100)^4$.

90. (b) $f(60) = \frac{9376}{625}$.

91. (a) $\frac{\frac{-2}{5}(100-t) - \frac{39}{10^8}(100-t)^4}{200-2t}$.
    (c) $t \approx 20$.

93. (b) $\begin{cases} \frac{dv}{dt} = 9.8 - \frac{30}{75}v \\ v(0) = 0 \end{cases}$
    (d) 24.5

94. $v = \frac{49}{2} - \frac{49}{2}e^{-(2/5)t}$.

95. (a) $h(t) = \frac{8245}{4} - \frac{49}{2}t - \frac{245}{4}e^{-(2/5)t}$.
    (b) $t = \frac{2}{5}\ln\frac{49}{9} \approx 4.2365$.

96. $t = \frac{15\ln\frac{7}{19}}{\ln\frac{14}{19}} \approx 49.0466$

99. a) solution
    b) not a solution

101. a) (i) $\begin{cases} x(t) = t^2 - 2 \\ y(t) = t + 3 \end{cases}$

(ii) $\mathbf{y}(0) = (-2, 3)$; $\mathbf{y}(1) = (-1, 4)$; $\mathbf{y}(2) = (2, 5)$.

102. (a) Use this code (all executed in one cell):

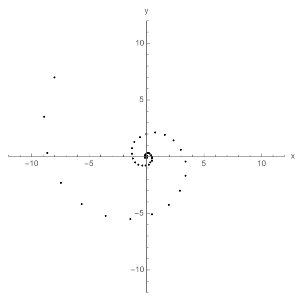
```
ParametricPlot[{Cos[t], Sin[t]}, {t, -100,100},
 PlotRange -> {{-2,2},{-2,2}}]
```

You will find that the graph is a circle.

105. $(t_1, \mathbf{y}_1) = (1, (3, 6))$; $(t_2, \mathbf{y}_2) = (2, (0, 18))$; $(t_3, \mathbf{y}_3) = (3, (-18, 36))$.

107. The x-coordinate when $t = 4$ is 92.1414.

109. (a) $\mathbf{y}(50) \approx (-6.531 \cdot 10^{-45}, 5.708 \cdot 10^{-45}) \approx (0, 0)$.

(b) 

110. (d) $\mathbf{y}(5) \approx (.515676, .362851, .326229)$.

(e) No, when more steps are used, the coordinates of $\mathbf{y}(5)$ change significantly.

111. a) 0

(c) $\begin{pmatrix} 15 & -6 \\ 3 & 9 \end{pmatrix}$

(e) $\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 4-\lambda & 2 \\ -8 & -4-\lambda \end{pmatrix}$

112. a) $\begin{pmatrix} -2\sin 2t & 3\cos 3t \\ 8\cos 2t & -3\cos 3t \end{pmatrix}$

b) $(\cos t, -6\sin 3t, 28e^{7t})$

c) $\begin{pmatrix} -15 & -13 \\ -3 & 11 \end{pmatrix}$

113. (b) $\begin{pmatrix} 2^8 & 0 \\ 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

115. a) $\begin{pmatrix} \frac{3}{17} & \frac{2}{17} \\ \frac{-1}{17} & \frac{5}{17} \end{pmatrix}$

116. (a) 17

(b) -2

117. (a) $\lambda^2 - 8\lambda + 17$

118. a) No

b) Yes

119. a) $W(t) = 20t^2$; the functions are linearly independent (since $W(1) \neq 0$).

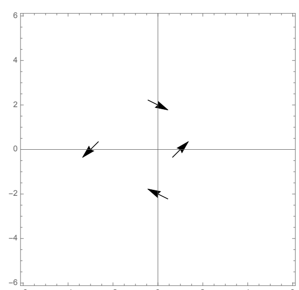
(c) $W(t) = 2$; the functions are linearly independent (since $W(0) \neq 0$).

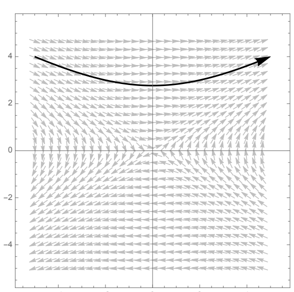
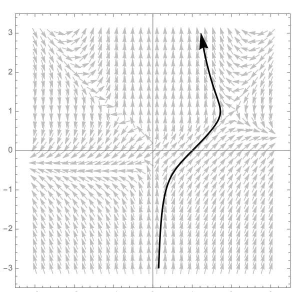
121. a) First, compute the Wronskian:

$$\begin{aligned} W(t) = \det \begin{pmatrix} e^{at} & e^{bt} \\ (e^{at})' & (e^{bt})' \end{pmatrix} &= \det \begin{pmatrix} e^{at} & e^{bt} \\ ae^{at} & be^{bt} \end{pmatrix} \\ &= be^{(a+b)t} - ae^{(a+b)t} \\ &= (b-a)e^{(a+b)t}. \end{aligned}$$

Since $a \neq b$, $W(t) \neq 0$, so the functions e^{at} and e^{bt} are linearly independent, as desired.

124. $\left.\frac{dy}{dx}\right|_{(1,0)} = 1$; $\left.\frac{dy}{dx}\right|_{(0,2)} = \frac{-1}{2}$; $\left.\frac{dy}{dx}\right|_{(0,-2)} = \frac{-1}{2}$; $\left.\frac{dy}{dx}\right|_{(-3,0)} = 1$. The slope field with these four mini-tangents is:



125. (a)  (b) 

127. (a) $(-6, 4)$ and $\left(\frac{3}{2}, -1\right)$

128. (a) $\left(\frac{1}{2}, 1\right)$ and $\left(\frac{-1}{2}, 1\right)$

129. (c) $\mathbf{y}(t) = (3, 1)$.

(d) i. $x(t)$ is decreasing

ii. $y(t)$ is increasing

iii. $\lim\limits_{t \to \infty} x(t) = -\infty$

iv. $\lim_{t\to\infty} y(t) = 1$

135. $\mathbf{y} = C_1e^t\binom{1}{1} + C_2e^{-t}\binom{1}{3}$

137. (c) $\mathbf{y} = C_1e^{-4t}\binom{1}{3} + C_2e^{-5t}\binom{1}{2}$.

141. $\mathbf{y} = \binom{e^{-t}\cos t - 3e^{-t}\sin t}{e^{-t}\cos t - e^{-t}\sin t}$

143. $\mathbf{y} = \binom{3e^{-3t}+4te^{-3t}}{2e^{-3t}+4te^{-3t}}$

144. $\begin{cases} x(t) = 4e^{2t} + C_1e^t + C_2e^{-t} \\ y(t) = 8e^{2t} + C_1e^t + 3C_2e^{-t} \end{cases}$

147. a) $\begin{cases} y_1(t) = 3C_1e^{-3t} + 2C_2e^{2t} \\ y_2(t) = 2C_2e^{2t} + 3C_4e^{-6t} \\ y_3(t) = C_1e^{-3t} - 3C_2e^{2t} + 5C_4e^{-6t} \\ y_4(t) = 4C_1e^{-3t} + C_2e^{2t} + C_3e^{4t} + C_4e^{-6t} \end{cases}$

148. a) $\begin{cases} x(t) = C_1e^{2t} - C_2e^{-t} - C_3e^{-t} \\ y(t) = C_1e^{2t} + C_3e^{-t} \\ z(t) = C_1e^{2t} + C_2e^{-t} \end{cases}$

149. a) The only equilibrium is $\left(\frac{13}{7}, \frac{-5}{7}\right)$, which is an unstable node.

151. (b) There are four equilibria: $(2,0)$ (stable node), $(5,0)$ (unstable saddle), $(5,4)$ (unstable node) and $(2,4)$ (unstable saddle).

152. a) Unstable node
   b) Unstable saddle
   c) Stable node

154. (b) $\begin{cases} \begin{cases} x' = \frac{-1}{20}x + \frac{1}{100}y + 1 \\ y' = \frac{3}{100}x - \frac{3}{100}y \end{cases} \\ \mathbf{y}(0) = (40,0) \end{cases}$

155. a) When $\alpha = \beta = 0$, the populations of X and Y behave according to logistic models.
   b) i. The fourth equilibrium is $\left(\frac{\alpha L_Y - L_X}{\alpha\beta - 1}, \frac{\beta L_X - L_Y}{\alpha\beta - 1}\right)$.
   c) $D\Phi(0,0) = \begin{pmatrix} r_XL_X & 0 \\ 0 & r_YL_Y \end{pmatrix}$. The eigenvalues of this matrix are r_XL_X and r_YL_Y, both of which are positive, so $(0,0)$ is unstable.

(d) $D\Phi(0, L_Y) = \begin{pmatrix} r_X L_X - \alpha r_X L_Y & 0 \\ -\beta r_Y L_Y & -r_Y L_Y \end{pmatrix}$. One eigenvalue of this matrix is $-r_Y L_Y < 0$; the other is $r_X(L_X - \alpha L_Y)$.

(e) If $\alpha > \frac{L_X}{L_Y}$, then $L_X - \alpha L_Y < 0$ so both eigenvalues of $D\Phi(0, L_Y)$ are negative, in which case $(0, L_Y)$ is stable.

(i) In this case, there is no coexistence equilibrium (since the denominators in the coexistence equilibrium would have to be zero).

156. (b) $S^\# = \frac{b}{d}$

(c) $D\Phi(\frac{b}{d}, 0, 0) = \begin{pmatrix} -d & -\beta\frac{b}{d} & 0 \\ 0 & & \\ 0 & \beta\frac{b}{d} - c - d - \gamma & 0 \\ 0 & \gamma & -d \end{pmatrix}$;

the eigenvalues are $-d$ (repeated twice) and $\frac{1}{d}(b\beta - d(c + d + \gamma))$.

(e) $S^* = \frac{c+d+\gamma}{\beta}$; $I^* = \frac{b\beta - d(c+d+\gamma)}{\beta(c+d+\gamma)}$; $R^* = \frac{\gamma(b\beta - d(c+d+\gamma))}{\beta d(c+d+\gamma)}$.

157. $\mathbf{y} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$; $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -7 & 4 \end{pmatrix}$; $\mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ \cos t \end{pmatrix}$.

159. a) $y = C_1 e^{2t} + C_2 e^{6t}$.

160. a) $y = C_1 e^{2t} \cos\left(\sqrt{7}t\right) + C_2 e^{2t} \sin\left(\sqrt{7}t\right)$.

161. a) $y = \frac{-1}{21} e^{4t} + C_1 e^{7t} + C_2 e^{-3t}$.

162. a) $y = \frac{13}{2} e^{3t} - \frac{7}{2} e^{5t}$.

165. a) If you convert the second-order equation $mx''(t) + bx'(t) + kx(t) = 0$ to a first-order system via reduction of order, the system becomes

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-b}{m} \end{pmatrix} \mathbf{x} = A\mathbf{x}.$$

We have $tr(A) = \frac{-b}{m} < 0$ and $\det A = \frac{k}{m} > 0$. Since the trace is negative and the determinant is positive, the equilibrium **0** is either a stable spiral or stable node, so both of its eigenvalues must have negative real part.

166. $x(t) = \frac{-764}{145} \cos 2t + \frac{112}{145} \sin 2t - \frac{396}{145} e^{-t/6} + 16e^{-t/8}$.

167. a) $\begin{cases} x(t) = \frac{-16}{7} \cos\left(t\sqrt{\frac{3}{2}}\right) - \frac{12}{7} \cos\left(\frac{t}{\sqrt{3}}\right) \\ y(t) = \frac{32}{7} \cos\left(t\sqrt{\frac{3}{2}}\right) - \frac{18}{7} \cos\left(\frac{t}{\sqrt{3}}\right) \end{cases}$

168. a) $\begin{cases} 12x'' = -3x + 3(y-x) \\ 12y'' = -3(y-x) - 3y \end{cases}$ (or something equivalent to this)

(d) i. $\begin{cases} x(t) = \frac{1}{2}\cos\frac{t}{2} + \frac{3}{2}\cos\frac{t\sqrt{3}}{2} + 2\sin\frac{t}{2} \\ y(t) = \frac{1}{2}\cos\frac{t}{2} - \frac{3}{2}\cos\frac{t\sqrt{3}}{2} + 2\sin\frac{t}{2} \end{cases}$

169. a) $\theta'' + \frac{9.8}{2}\sin\theta = 0.$

b) $\theta'' + \frac{9.8}{2}\theta = 0.$

(d) $\frac{\sqrt{409}}{140}$ radians (which is about 0.1444).

171. $I(t) = \frac{1}{2}\cos 2t - \frac{3}{2}\sin 2t - \frac{1}{2}e^{-4t}\cos 6t - \frac{47}{2}e^{-4t}\sin 6t.$

A.6 Extra credit problems

1. **The Snowplow Problem:** Suppose that one morning it starts snowing hard, but at a constant rate. A snowplow sets out at 9 AM to clear a road. At 11 AM, it has cleared 15 miles, and at 1 PM, it has cleared an additional 10 miles. When did it start snowing?

   To solve this problem, carry out the following steps:

   a) Let t be the time, measured in hours, with $t = 0$ corresponding to 9 AM. Let $h = h(t)$ be the height of the snow at time t. We will assume the rate of snowfall is constant, and equal to r. Write down a differential equation which could be used to solve for $h(t)$, if there was no plowing.

   b) Solve the equation from part (a) for h in terms of t (there will be an arbitrary constant; let's call this constant B so everyone is using the same notation).

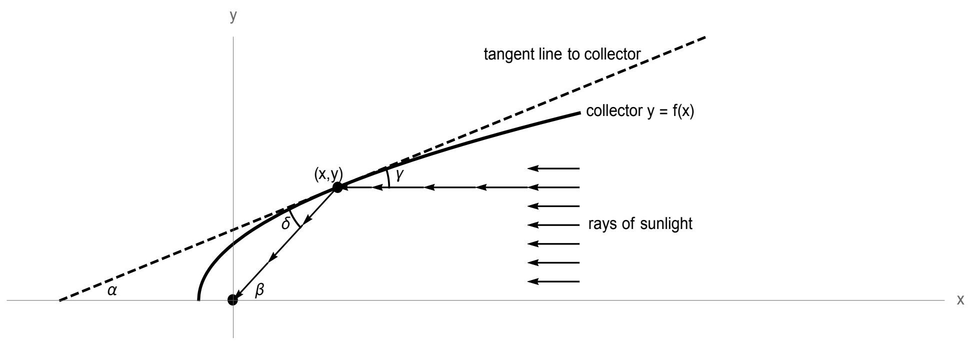
   c) Let $x(t)$ be the distance the snowplow has travelled at time t. It is reasonable to assume that the deeper the snow is, the slower the snowplow has to travel. A simple mathematical model for this is to assume that the velocity of the snowplow at time t is inversely proportional to the height $h(t)$. Use this to write a differential equation satisfied the function $x(t)$ (let's agree to use the letter k for the proportionality constant).

   d) Solve the differential equation of part (c) to obtain a formula for $x(t)$ (let's agree to call the arbitrary constant that appears here D).

   e) Use the initial conditions of this problem (given before part (a)) to solve for B, D and k.

   f) Answer the original question: when did it start snowing? Round your answer to the nearest minute.

2. **The Solar Collector Problem:** Suppose you want to design a solar collector which will concentrate the sun's rays at a point. The collector will be in the shape of a curve $y = f(x)$, and the point we want to concentrate the rays at will be $(0, 0)$. Suppose that the sun is located at the extreme positive x-axis, and that the light coming from the sun hits the collector, travelling in a horizontal path (see the picture below).



a) From physics, the law of reflection for rays of light says that angles γ and δ are equal. Use this, together with some facts from high-school geometry, to explain why $\beta = 2\alpha$.

b) From calculus, the slope of the tangent line at (x, y) is $\frac{dy}{dx}$. Use this to explain why $\frac{dy}{dx} = \tan\alpha$. Using similar logic, find $\tan\beta$ in terms of x and y.

c) Use the double-angle identity for tangent (look this up via Google if you don't know it) to show that

$$\frac{y}{x} = \frac{2\frac{dy}{dx}}{1 - \left(\frac{dy}{dx}\right)^2}.$$

d) Solve for $\frac{dy}{dx}$ in the above equation to show that the curve must satisfy

$$\frac{dy}{dx} = \frac{-x + \sqrt{x^2 + y^2}}{y}.$$

e) Solve the equation obtained in part (e) of this problem.
Hint: To solve this equation, you need a trick. Start by letting $u = x^2 + y^2$ and differentiate implicitly to obtain a formula for $\frac{du}{dx}$ in terms of x, y and $\frac{dy}{dx}$. Use this formula, together with the equation from part (d), to write a separable differential equation of the form $\frac{du}{dx} = $ *something*. Then solve this equation for $u = u(x)$, and use that to recover y in terms of x.

f) What shape are the solutions?

Appendix B

Mathematica information

B.1 General *Mathematica* principles

Mathematica is an extremely useful and powerful software package / programming language invented by a mathematician named Stephen Wolfram. Early versions of *Mathematica* came out in the late 1980s and early 1990s; the most recent version (which is loaded onto machines at FSU as of 2016) is *Mathematica* 10.

Mathematica does symbolic manipulation of mathematical expressions; it solves all kinds of equations; it has a library of important functions from mathematics which it recognizes while doing computations; it does 2- and 3-dimensional graphics; it has a built-in word processor tool; it works well with Java and C++; etc. One thing it doesn't do is prove theorems, so it is less useful for a theoretical mathematician than it is for an engineer or college student.

A bit about how *Mathematica* works: When you use the *Mathematica* program, you are actually running *two* programs. The "front end" of *Mathematica* is the part that you type on and the part you see. This part actually resides on the machine at which you are seated. The "kernel" is the part of *Mathematica* that actually does the calculations. If you type in `2 + 2` and hit [SHIFT]+[ENTER], the front end "sends" that information to the kernel which actually does the computation. The kernel then "sends" the result back to the front end, which displays the output `4` on the screen. Essentially, the way one uses *Mathematica* is by typing some "stuff" in, hitting [SHIFT]+[ENTER] to execute that stuff, and getting some output back from the program.

About *Mathematica* notebooks and cells: The actual files that *Mathematica* produces that you can edit and save are called *notebooks* and carry the file designation `*.nb`; they take up little space and can easily be saved to Google docs or on a flash drive, or emailed to yourself if you want them somewhere you can retrieve them. **Suggestion:** when saving any file, include the date in the file name (so it is easier to remember which file you are supposed to be open).

A *Mathematica* notebook is broken into *cells*. A cell can contain text, input, or output. A cell is indicated by a dark blue, right bracket (a "]") on the right-hand side of the notebook. To select a cell, click that bracket. This highlights the "]" in blue. Once selected, you can cut/copy/paste/delete cells as you would highlighted blocks of text in a Word document.

To change the formatting of a cell, select the cell, then click"Format / Style" and select the style you want. You may want to play around with this to see what the various styles look like. There are three particularly important styles:

- **input:** this is the default style for new cells you type
- **output:** this is the default style for cells the kernel produces from your commands
- **text:** changing a cell to text style allows you to make comments in between the calculations

Executing mathematical commands: To execute an input cell, put the cursor anywhere in the cell and hit [ENTER]. Well, not any [ENTER]; you have to use the [ENTER] on the numeric keypad at the far-right edge of the keyboard. The [ENTER] next to the apostrophe key (a.k.a. [RETURN]) gives you only a carriage return. You can also hold down the [SHIFT] key and hit either [ENTER] or [RETURN] to execute a command.

Important general concepts re: *Mathematica* syntax

1. **Multiplication:** use a star or a space: `2 * 3` or `2 3` will multiply numbers; `a x` means a times x; `ax` means the variable ax (in *Mathematica*, variables do not have to be named after one letter; they can be named by words or other strings of characters as well).

2. **Parentheses:** used for grouping and multiplication only. Parentheses mean "times" in *Mathematica*, and always mean that you intend to **multiply** what is in front of the parenthesis by what is inside the parenthesis.

3. **Brackets:** must be used to surround the input of any function or built-in *Mathematica* command. For example, to evaluate a function $f(x)$, you would type `f[x]`, not `f(x)`. Essentially, square brackets mean "**of**" in *Mathematica*.

4. **Capitalization:** All *Mathematica* commands and built-in functions begin with capital letters. For example, to find the sine of π, typing `sin(pi)` or `sin[pi]` does you no good (the first version would be the variable "sin" times the variable "pi", for instance). The correct syntax is `Sin[Pi]`. Similarly, e is `E` and i is `I` in *Mathematica*.

5. **Spaces:** *Mathematica* commands do not have spaces in them; for example, the inverse function of sine is `ArcSin`, not `Arc Sin` or `Arcsin`.

6. **Pallettes:** Lots of useful commands are available on the Basic Math Assistant Pallette, which can be brought up by clicking "Pallettes / Basic Math Assistant" on the toolbar. If you click on a button in the pallette, what you see appears in the cell. The tab halfway down this palette marked $d \int \Sigma$ has calculus commands, and the tab to the right of the $d \int \Sigma$ has matrix commands.

7. **Logarithms:** *Mathematica* does not know what `Ln` is. For natural logarithms (base e), type "Log[ ]". For common logarithms (base 10), type "Log10[ ]".

8. $\%$ refers to the last output (like "Ans" on a TI-calculator).

9. **Help:** To get help on a command, type "?" followed by the command you don't understand. If necessary, click the $\gg$ you get at the end of the help blurb to open a help browser. You can also find out how to do lots of stuff in *Mathematica* by using Google: search for what you want help on.

10. *Mathematica* gives exact answers (i.e. not decimals) for everything if possible. If you need a decimal approximation, use the command `N[ ]`. For example, `N[Pi]` spits out 3.14159...

11. If *Mathematica* freezes up in the middle of a calculation, click "Evaluation / Abort Evaluation" on the toolbar.

B.2 *Mathematica* quick reference guides

Basic operations

	Expression	*Mathematica* syntax
SPECIAL SYMBOLS	e	`E`
	π	`Pi`
	i (i.e. $\sqrt{-1}$)	`I`
	∞	`Infinity` (or use Basic Math Assistant palette)
ARITHMETIC	$3+4x$	`3 + 4x`
	$5-7$	`5 - 7`
	$8z$	`8z` or `8 z` or `8 * z`
	xy	`x y` (don't forget the space)
	$\frac{7}{3}$	`7/3`
	$\frac{x-7+2y}{a-7b}$	To get the fraction bar, type [CONTROL]+/ then use [TAB] to move between the top and bottom
	$\sqrt{32}$	`Sqrt[32]` (or type [CONTROL]+2 to get a $\sqrt{\ }$ sign) (or use Basic Math Assistant palette)
	$\sqrt[4]{40}$	`40^(1/4)` (or use Basic Math Assistant palette)
	$\lvert x-3\rvert$	`Abs[x-3]`
	$30!$ (factorial)	`30!`
EXPS AND LOGS	$\ln 3$	`Log[3]`
	$\log_6 63$	`Log[6,63]`
	$\log 18$	`Log10[18]` or `Log[10,18]`
	2^{7y}	`2^(7y)` (or type `2`, then [CONTROL]+6, then `7y`) (or use Basic Math Assistant palette)
	e^{x-5+x^2}	`E^(x-5+x^2)` or `Exp[x-5+x^2]` (or use Basic Math Assistant palette)
TRIG	$\sin\pi$	`Sin[Pi]`
	$\cos(x(y+1))$	`Cos[x(y+1)]`
	$\cot\left(\frac{2\pi}{3}+\frac{3\pi}{4}\right)$	`Cot[2 Pi/3 + 3 Pi/4]`
	$\arctan 1$	`ArcTan[1]`

Objective	*Mathematica* syntax
To call the preceding output	`%`
To get a decimal approximation to the preceding output	`N[%]` (or click **numerical value**)

Defining functions

Objective	*Mathematica* syntax
Define a function $f(x) = formula$	`f[x_] =` $formula$ (one equals sign, underscore after `x`)
Define parametric function $x = f(t), y = g(t)$	`f[t_] = {`$f(t)$`,`$g(t)$`}`
Define function of multiple variables $z = f(x, y)$	`f[x_,y_] =` $formula$

Tables and graphs (see also Section B.3)

Objective	*Mathematica* syntax
Generate table of values for f	`Table[{x,f[x]}, {x,`$xmin$`,`$xmax$`,`$step$`}]` (put `//TableForm` at end of command to arrange output in a table)
Plot the graph of $f(x) = formula$	`Plot[`$formula$`, {x,`$xmin$`,`$xmax$`}]`
Plot multiple graphs at once	`Plot[{`$formula$`,`$formula$, ..., $formula$`},` `{x,`$xmin$`,`$xmax$`}]`
Plot the graph of $f(x) = formula$ with range of $y-$values specified	`Plot[`$formula$`, {x,`$xmin$`,`$xmax$`},` `PlotRange -> {`$ymin$`,`$ymax$`}]`

Plot the graph of $f(x) = formula$ with x- and y-axes on same scale	`Plot[`$formula$`, {x,`$xmin$`,`$xmax$`},`
	`PlotRange -> {`$ymin$`,`$ymax$`},`
	`AspectRatio -> Automatic]`

Plot graph of a set of parametric equations (after defining them as $f(t)$)	`ParametricPlot[f[t], {t, -20,20},`
	`PlotRange -> {{`$xmin$`,`$xmax$`},`
	`{`$ymin$`,`$ymax$`}}]`

Function operations and calculus

Expression	*Mathematica* syntax
$f(x+3)$ (if f is a function)	`f[x+3]`
$xf(2x) - x^2 f(x)$	`x f[2x] - x^2 f[x]` (spaces important)
$(f \circ g)(x)$	`f[g[x]]`
$\lim_{x \to 4} f(x)$	`Limit[f[x], x -> 4]`
$f'(3)$	`f'[3]`
$g'''(x)$	`g'''[x]` or `D[g[x],{x,3}]`
partial derivative f_x	`D[f[x,y], x]`
partial derivative f_{yy}	`D[f[x,y], {y,2}]`
partial derivative f_{yxy}	`D[f[x,y], {y,2}, {x,1}]`
$\int x^2\,dx$	`Integrate[x^2,x]` (or use $\int$ sign on Basic Math Assistant palette)
$\int_2^5 \cos x\,dx$	`Integrate[Cos[x], {x, 2, 5}]` (or use $\int_\square^\square$ sign on Basic Math Assistant palette) (for a decimal approximation, use `NIntegrate`)
$\sum_{k=1}^{12} f(k)$	`Sum[f[k], {k, 1, 12}]` (or use Basic Math Assistant palette)
$\sum_{k=1}^{\infty} blah$	`Sum[`$blah$`, {k, 1, Infinity}]` (or use Basic Math Assistant palette)

To find the Taylor polynomial of order N for function f, centered at point a, execute

`Normal[Series[`f`, {`*variable*`, `a`, `N`}]]`

For example, to find the eighth Taylor polynomial of $f(x) = \cos 2x$ centered at 0, execute

```
Normal[Series[Cos[2x], {x, 0, 8}]]
```

Solving equations (see also section B.4)

Objective	*Mathematica* syntax
Find exact solution(s) to equation of form $lhs = rhs$	`Solve[`lhs `==` rhs`, x]` (two equals signs) (works only with polynomials or other relatively "easy"
Find decimal approximation(s) to solution(s) of equation $lhs = rhs$	`NSolve[`lhs `==` rhs`, x]` (two equals signs) (works only with "easy" equations) equations)
Find decimal approximation(s) to to solution(s) of equation $lhs = rhs$	`FindRoot[`lhs `==` rhs`, {x,`$guess$`}]` (two equals signs)

Other

Objective	*Mathematica* syntax
Find partial fraction decomposition	`Apart[ ]`
Combine rational terms (i.e. "undo" partial fractions)	`Together[ ]`
Factor a polynomial	`Factor[ ]`
Multiply an expression out (i.e. "FOIL" an expression) (i.e. "undo" factoring)	`Expand[ ]`
Simplify an expression	`Simplify[ ]`

B.3 Graphing functions with *Mathematica*

Defining a function in *Mathematica*

To graph a function $y = f(x)$ on *Mathematica*, you usually start by defining the function. For example, to define a function like $f(x) = 3\cos 4x - x$, execute

```
f[x_] = 3 Cos[4x] - x
```

You could just as well use a different letter for the independent variable. For example, typing

```
f[t_] = 3 Cos[4t] - t
```

would accomplish the same thing as above. However, don't mix and match! Typing

```
f[x_] = 3 Cos[4t] - t
```

doesn't accomplish anything, because there is a `x` on the left-hand side, and a `t` on the right-hand side.

The general syntax for defining a function is

$$function\,name\texttt{[}variable\texttt{\_] = }formula$$

it is important to include the underscore after the variable to tell *Mathematica* you are defining a function.

The basic Plot command

Immediately after defining a function as above, you will get (underneath your output) a list of suggested follow-up commands. One of these is plot. If you click the word plot, you will get a graph of the function you just defined. Here, *Mathematica* picks a range of x- and y-values it thinks will work well for the function you defined. It is useful to remember the syntax of this Plot command:

$$\texttt{Plot[}formula\texttt{, \{}variable\texttt{, }xmin\texttt{, }xmax\texttt{\}]}$$

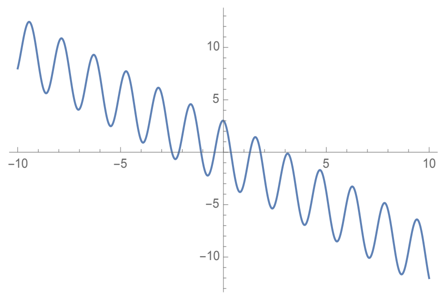
In this command:

- $formula$ is the function you want the graph of. It could be an expression like `f[x]` or `f[t]`, or a typed-out formula like `3 Cos[4x] - x`.
- $variable$ is the name of the independent variable (usually `x` or `t`); this must match the variable in the formula.

- $xmin$ and $xmax$ are numbers which represent, respectively, the left-most and right-most values of the independent variable shown on the graph. For example, if your Plot command has {x,-3,5} in it, then the graph will go from $x = -3$ to $x = 5$.

Here is an example, which plots $f(x) = 3\cos 4x - x$ from $x = -10$ to $x = 10$:

```
Plot[3 Cos[4x] - x, {x, -10,10}]
```



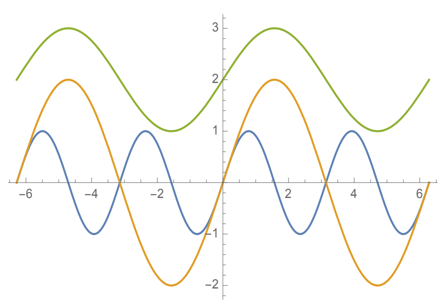
Plotting multiple functions at once

Suppose you want to plot more than one function on the same set of axes. To do this, you tweak the earlier Plot command by replacing the formula with a list of formulas inside squiggly braces, separated by commas. Thus the command you execute looks something like this:

Plot[{$formula1$, $formula2$, ...}, {$variable$, $xmin$, $xmax$}]

For example, the following command plots $\sin 2x$, $2\sin x$ and $\sin x + 2$ on the same set of axes:

```
Plot[{Sin[2x], 2 Sin[x], Sin[x] + 2}, {x, -2 Pi, 2 Pi}]
```



In *Mathematica* 10, the first graph you type will be blue; the second graph you type will be orange; the third graph you type is green; other graphs are in other colors. To change the way the graphs look, consult the end of this section.

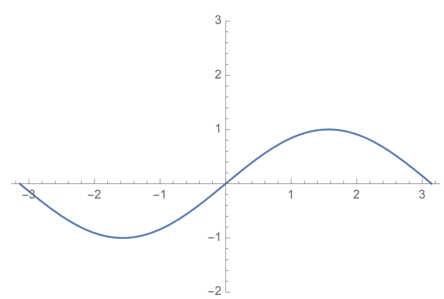
Specifying a range of y-values

By default, *Mathematica* just chooses a range of y-values it thinks will make the graph look good. If you want to force *Mathematica* to use a particular range of y-values, then you have to insert a phrase in the Plot command called `PlotRange`. This goes after the `{x,`$xmin$`,`$xmax$`}` and after another comma, but before the closing square bracket. The general command is

`Plot[{`$formulas$`}, {`var`,`$xmin$`,`$xmax$`}, PlotRange -> {`$ymin$`,`$ymax$`}]`

and an example of the code, which plots $\sin x$ on the viewing window $[-\pi, \pi] \times [-2, 3]$ is

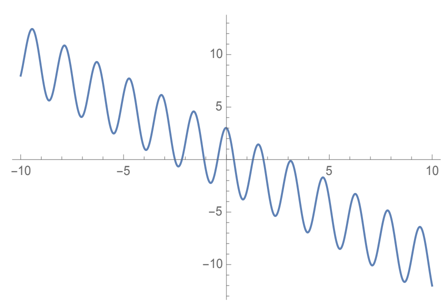
```
Plot[Sin[x], {x, -Pi, Pi}, PlotRange -> {-2,3}]
```



Making the x- and y-axes have the same scale on the screen

Here is the graph of $f(x) = 3\cos 4x - x$ that *Mathematica* produces with the command

```
Plot[3 Cos[4x] - x, {x, -10,10}]
```



If you look at this graph, the distance from the origin to $(5, 0)$ looks a lot longer than the distance from the origin to $(0, 5)$. But in actuality, both these distances are 5 units. The graph is distorted so that it fits nicely on your screen. To fix the distortion (you might want to do this if you needed to estimate the slope of a graph accurately), insert the command `AspectRatio -> Automatic` into the Plot command (similar to how you would insert a `PlotRange` command). This forces the number of pixels on your screen representing one unit in the x direction to be equal to the number of pixels on your screen representing one unit in the y direction. Here is the general syntax:

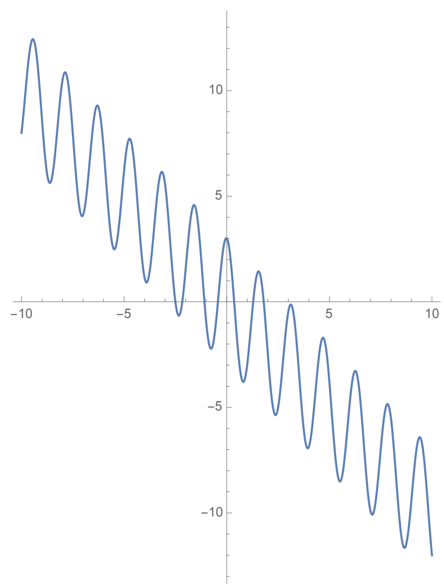
Plot[{*formulas*}, {*var*,*xmin*,*xmax*}, AspectRatio -> Automatic]

This command can also be used with the `PlotRange` command:

Plot[{*formulas*}, {*var*,*xmin*,*xmax*}, PlotRange -> {*ymin*,*ymax*}, AspectRatio -> Automatic]

Here is an example command:

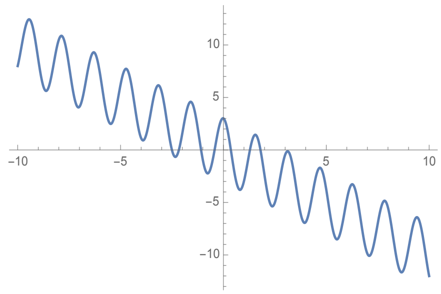
```
Plot[3 Cos[4x] - x, {x, -10,10}, AspectRatio -> Automatic]
```



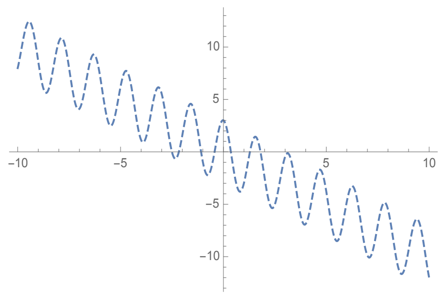
Changing the appearance of the curves

As mentioned earlier, by default *Mathematica* graphs all the functions with solid lines, using different colors for different formulas on the same picture. To change this, insert various directives into the Plot command using `PlotStyle`. Here are some examples:

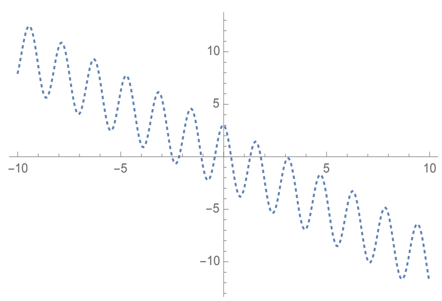
```
Plot[3 Cos[4x] - x, {x, -10,10}, PlotStyle -> Thick]
```




```
Plot[3 Cos[4x] - x, {x, -10,10}, PlotStyle -> Dashed]
```




```
Plot[3 Cos[4x] - x, {x, -10,10}, PlotStyle -> Dotted]
```



If you are plotting more than one function at once, then after the `PlotStyle ->`, you can type a list of graphics directives, separated by commas, enclosed by a set of squiggly braces. The directives will be applied to each function you are graphing, in the same order as they are typed after the `PlotStyle ->`. For example, this command plots $x, 2x$ and $3x$, where x is thick and black, $2x$ is red and dotted, and $3x$ is blue and dashed:

```
Plot[{x,2x,3x}, {x, -3,3},
 PlotStyle -> {{Thick, Black}, {Dotted, Red}, {Blue, Dashed}}]
```

Parametric equations

To define a set of parametric equations like $x = t\cos t$, $y = \sin 2t$ in *Mathematica*, type the following and execute:

```
f[t_] = {t Cos[t], Sin[2t]}
```

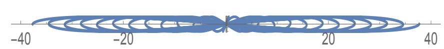
In general, the syntax is

name of function `[t_] =` {*formula for* x ,*formula for* y}

(You don't have to call the function f, of course.) You don't need to have only two coordinates x and y here; you could type more formulas, separated by commas, if necessary.

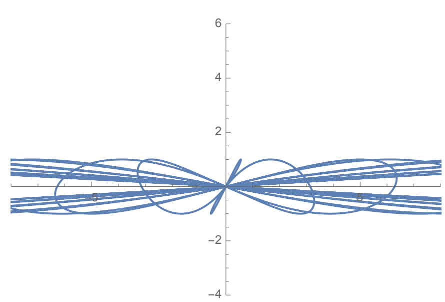
After defining the parametric equations as above, one follow-up command you get in the suggestions bar is parametric plot. Click that, and you will get a graph of the parametric equations in the xy-plane. The syntax looks like this:

```
ParametricPlot[f[t], {t, -40, 40}]
```



(The -40 and 40 are the range of t values considered; they don't matter except that they should be far apart for the purposes of getting a good graph.) Often the viewing window for such a picture is not very useful. To fix this, tweak the command by using `PlotRange`:

```
ParametricPlot[f[t], {t, -40, 40}, PlotRange -> {{-8,8},{-4,6}}]
```



The `PlotRange` part of the command specifies the range of $x-$ and $y-$values seen on the graph. For example, in this picture, x will go from -8 to 8 and y will run from -4 to 6.

You can also use the `PlotStyle` directives described in earlier labs if you want to change the appearance of the graph.

B.4 Solving equations with *Mathematica*

There are three methods to solve an equation using *Mathematica*. They have something in common: to solve an equation, the equation **must be typed with two equals signs** where the $=$ is. (A single equal sign is used in *Mathematica* to assign values to variables, which doesn't apply in the context of solving equations.)

The Solve command

To solve an equation of the form $lhs = rhs$, execute

`Solve[`lhs `==` rhs`,` $variable$`]`

where $variable$ is the name of the variable you want to solve for. For example, to solve $x^2 - 2x - 7 = 0$ for x, execute `Solve[x^2 - 2x - 7 == 0, x]`.

You can solve an equation for one variable in terms of others: for example, `Solve[a x + b == c, x]` solves for x in terms of a, b and c.

WARNING: The advantage of the `Solve` command is that it gives exact answers (no decimals); this can be a pro or con (as sometimes the exact answers are horrible to write down). The disadvantage is that it only works on polynomial, rational and other "easy" equations. It won't work on equations that mix-and-match trigonometry and powers of x like $x^2 = \cos x$.

The NSolve command

`NSolve` works exactly like `Solve`, except that it gives decimal approximations to the solutions. It has the same drawback as `Solve` in that it only works on reasonably "easy" equations. The syntax is

`NSolve[`lhs `==` rhs`,` $variable$`]`

The FindRoot command

To find decimal approximations to equations that are too hard for the `Solve` and `NSolve` commands, use `FindRoot`. This executes a numerical algorithm to estimate a solution to an equation. The good news is that this command always works; the bad news is that it requires an initial "guess" as to what the solution is (usually you determine the initial guess by graphing both sides of the equation and seeing roughly where the graphs cross). For example, to find a solution to $x^2 = \cos x$ near $x = 1$, execute

```
FindRoot[x^2 == Cos[x], {x, 1}]
```

and to find a solution to the same equation near $x = -1$, execute

```
FindRoot[x^2 == Cos[x], {x, -1}]
```

(these probably won't give the same solution). The general syntax for this command is

FindRoot[*lhs*==*rhs*, {*variable*,*guess*}]

B.5 Matrix operations on *Mathematica*

To define a matrix, there are two methods:

1. Use squiggly braces and commas to separate the entries. Each row should be surrounded by a squiggly brace, and the entire matrix should be surrounded by a set of squiggly braces, and everything should be separated by commas. For example, to define A as
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$
one could execute
```
A = {{1, 2}, {3, 4}}
```
Note that if you have a column matrix like $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, this matrix can be defined by just typing something like `B = {1, 2, 3}` (instead of having to type `B = {{1}, {2}, {3}}`).

2. On the Basic Math Assistant Pallette, under Basic Commands, click the matrix. Then type `A =` , then click the matrix in the palette. To add rows and columns, click `AddRow` or `AddColumn` until the matrix is the appropriate size. Then go into the matrix and type in each entry, moving between the locations using the [TAB] key or clicking on the location you want.

If the entries of the matrix are functions, then define the matrix as a function by executing `A[t_] =` ... instead of `A =` ...

Once you have defined all necessary matrices, *Mathematica* commands for operations on those matrices are given on the next page:

In this chart, if A is a matrix of functions, replace the `A` with `A[t]`.

Note: To make *Mathematica* display an answer as a matrix:

1. follow your command with `// MatrixForm`, or
2. once you've executed the command, choose Display as... matrix from the suggestions bar.

	Expression	Mathematica syntax
Basic matrix operations	Matrix addition / subtraction	
	$A+B$	`A + B`
	$A-B$	`A - B`
	Scalar multiplication	
	$3A$	`3A`
	nA	`n A` (space important)
	$-5A+\frac{1}{2}B$	`-5A + (1/2)B`
	Matrix product AB	`A.B` (the period is important)
	A^2	`A.A` or `MatrixPower[A,2]` (**not** `A^2`)
	A^7	`MatrixPower[A,7]` (**not** `A^7`)
	Trace $tr(A)$	`Tr[A]`
	Determinant $\det A$	`Det[A]`
	Transpose A^T	`Transpose[A]`
	To get the entry of matrix A in the i^{th} row and j^{th} column	`A[[i,j]]`
	To call the $n\times n$ identity matrix I	`IdentityMatrix[n]`
	Find derivative of a matrix of functions term-by-term	`A'[t]`
Linear systems	Matrix inverse A^{-1}	`Inverse[A]` (**not** `A^-1`)
	Find the rank of A (i.e. # of lin. indep. columns)	`MatrixRank[A]`
	Reduced row-echelon form of A	`RowReduce[A]`
	Find particular solution of $A\mathbf{x}=\mathbf{b}$	`LinearSolve[A,b]`
	Find basis of null space $N(A)$	`NullSpace[A]`
	Find least-squares solution $\widehat{\mathbf{x}}$ of $A\mathbf{x}=\mathbf{b}$	`LeastSquares[A,b]`
Eigentheory	Matrix exponential $e^A=\exp(A)$	`MatrixExp[A]`
	Eigenvalues and eigenvectors of A	`Eigensystem[A]`
	Just the eigenvalues of A	`Eigenvalues[A]`
	Just the eigenvectors of A	`Eigenvectors[A]`
	Find $\det(A-xI)$	`CharacteristicPolynomial[A,x]`
	Determine if A is diagonalizable	`DiagonalizableMatrixQ[A]`
	Determine if A is positive definite	`PositiveDefiniteMatrixQ[A]`
	Determine if A is negative definite	`NegativeDefiniteMatrixQ[A]`

B.6 Complex numbers in *Mathematica*

To type in a complex number in *Mathematica*, type `I` (capital I) for the imaginary number i. When *Mathematica* gets an answer with an i in it, it displays 𝕚 for the i.

Basically, commands with complex numbers are the same as they are with real numbers. To multiply $(7-5i)(3+2i)$, for example, just execute `(7 - 5I)(3 + 2I)`; to find e^i, execute `E^I`. There are some commands that are unique to complex numbers; those are given below:

Objective	*Mathematica* syntax
Find real part of complex number	`Re[ ]`
Find imaginary part of complex number	`Im[ ]`
Find norm of complex number	`Abs[ ]`
Find argument of complex number	`Arg[ ]`
Find conjugate of complex number	`Conjugate[ ]`

B.7 Slope fields for first-order equations

All this code is available in the file `slopefields.nb`, available on my web page.

Code to sketch the slope field of $y' = \phi(t,y)$:

Execute all this in a single *Mathematica* cell:

```
phi[t_,y_] := formula;"
VectorPlot[{1,phi[t,y]}, {t, -10,10}, {y, -8, 8},
 VectorPoints -> 20, Axes -> True,
 VectorScale -> {Automatic, Automatic, None},
 VectorStyle -> Orange]
```

Some of the things you can change as necessary:

- *formula* should be whatever $\phi(t,y)$ is equal to; for example, given the ODE $y' = 2t + y^2$, the first line of the code would be `phi[t_,y_] = 2t + y^2;`
- the numbers in the second line set the viewing windows; for example, in the above code the viewing window is $[-10,10] \times [-8,8]$;
- the `20` is the number of arrows drawn in each direction.

Code to sketch the slope field and several solution curves

Execute all this in a single *Mathematica* cell:

```
phi[t_,y_] := formula;
VectorPlot[{1,phi[t,y]}, {t, -10, 10}, {y, -8, 8},
 VectorPoints -> 20, Axes -> True,
 VectorScale -> {Automatic, Automatic, None},
 VectorStyle -> Orange,
 StreamPoints -> 35,
 StreamScale -> Full,
 StreamStyle -> {Black, Thick}]
```

The first five lines are the same as the command described earlier; the sixth line directs *Mathematica* to sketch 35 solution curves at random locations on the picture. The last line tells *Mathematica* what color to draw the solution curves.

Code to sketch the slope field and a solution curve passing through a specific point

The following code (executed in a single cell) will sketch a slope field and sketch a single solution curve passing through a given point (t_0, y_0), which is specified in the sixth line of the code. In this case the initial value is $(-1, 2)$:

```
phi[t_,y_] := formula;
VectorPlot[{1,phi[t,y]}, {t, -10, 10}, {y, -8, 8},
 VectorPoints -> 20, Axes -> True,
 VectorScale -> {Automatic, Automatic, None},
 VectorStyle -> Orange,
 StreamPoints -> {{-1,2}},
 StreamScale -> Full,
 StreamStyle -> {Black, Thick}]
```

B.8 Euler's method for first-order equations

All this code is available in the file `eulermethod.nb`, available on my web page.

Code to generate the program "euler"

Run this block of code **once** each time you start *Mathematica* to define a program called `euler`:

```
euler[f_, {t_, t0_, tn_}, {y_, y0_}, steps_] :=
 Block[{told = t0, yold = y0, thelist = {{t0, y0}}, t, y, h},
  h = N[(tn - t0)/steps];
  Do[tnew = told + h;
   ynew = yold + h *(f /.{t -> told, y -> yold});
   thelist = Append[thelist, {tnew, ynew}];
   told = tnew;
   yold = ynew, {steps}];
  Return[thelist];]
```

Implementing Euler's method

Once the above command is executed, you can then implement Euler's method with:

euler[*formula*, {t,t_0, t_n}, {y,y_0}, n]

Here,
- (t_0, y_0) is the initial value;
- t_n is the value of t where you want to estimate y (i.e. the ending value of t);
- n is the number of steps.

To get only the last point in the list (which is usually what you are most interested in), tweak this command as follows:

```
euler[3y, {t,1,3}, {y,-1}, 400][[401]]
```

The number in the double brackets should always be one more than the number of steps.

Plotting the points coming from Euler's method

Surround the `euler` command with `ListPlot[ ]`:

```
ListPlot[euler[3y, {t, 1,3}, {y, -1}, 400]]
```

B.9 Picard's method for first-order equations

You can also implement Picard's method using *Mathematica* (although this was not discussed in Chapter 1). To do this, run the following code in one cell:

```
f0[t_] = 0;
phi[t_,y_] = 2(1+y);
n = 12;
Do[f0[t_] = Integrate[phi[s,f0[s]], {s,0,t}], n];
f0[t]
```

In the command above:

- the first line contains the initial guess (which is in this case $f_0(t) = 0$);
- the second line contains the formula for $\phi(t, y)$ (which is in this case $\phi(t, y) = 2(1 + y)$;
- the third line is the number of steps (i.e. this command will compute f_{12});
- the last two lines should not be changed.

B.10 Euler's method for first-order 2×2 and 3×3 systems

All this code can be found in the file `eulermethodsystems.nb`, available on my website.

Code to generate the program "euler2D"

This creates a program called `euler2D` which can be executed to implement Euler's method for a 2×2 system:

```
euler2D[{f_, g_}, {t_, t0_, tn_}, {x_, x0_}, {y_, y0_}, steps_] :=
 Block[{told = t0, xold = x0, yold = y0, thelist = {{t0, {x0, y0}}},
   t, x, y, h}, h = N[(tn - t0)/steps];
  Do[tnew = told + h;
   xnew = xold + h*(f /. {t -> told, x -> xold, y -> yold});
   ynew = yold + h*(g /. {t -> told, x -> xold, y -> yold});
   thelist = Append[thelist, {tnew, {xnew, ynew}}];
   told = tnew;
   xold = xnew;
```

```
  yold = ynew, {steps}];
 Return[thelist];]
```

Implementing Euler's method for 2×2 systems

Once the above command is executed, you can then implement Euler's method with:

`euler2D[{`*formula for* x'`, ` *formula for* y'`}, {t,`t_0`, `t_n`}, {x,`x_0`}, {y,`y_0`}, `n`]`

To get only the last point in the list, add `[[`*number*`]]` to the end of the command, where *number* is one more than n, the number of steps used.

To plot the points obtained, use a command like this:

```
ListPlot[Transpose[
   euler2D[{y - x, x + y}, {t, 0, 3}, {x, 2}, {y, 0}, 3]][[2]],
  PlotRange -> {{-5, 7}, {-2, 12}},
 AspectRatio -> Automatic]
```

Code to generate the program "euler3D"

This creates a program called `euler3D` which can be executed to implement Euler's method for a 3×3 system:

```
euler3D[{f_, g_, k_}, {t_, t0_, tn_}, {x_, x0_}, {y_, y0_}, {z_, z0_},
   steps_] :=
 Block[{told = t0, xold = x0, yold = y0, zold = z0,
   thelist = {{t0, {x0, y0, z0}}}, t, x, y, z, h},
  h = N[(tn - t0)/steps];
  Do[tnew = told + h;
   xnew = xold + h*(f /. {t -> told, x -> xold, y -> yold, z -> zold});
   ynew = yold + h*(g /. {t -> told, x -> xold, y -> yold, z -> zold});
   znew = zold + h*(k /. {t -> told, x -> xold, y -> yold, z -> zold});
   thelist = Append[thelist, {tnew, {xnew, ynew, znew}}];
   told = tnew;
   xold = xnew;
   zold = znew;
   yold = ynew, {steps}];
  Return[thelist];]
```

Implementing Euler's method for 3×3 systems

Once the above command is executed, you can then implement Euler's method with:

euler3D[{*formula for* x', *formula for* y', *formula for* z'},
  {t,t_0, t_n}, {x,x_0}, {y,y_0}, {z,z_0}, n]

To get only the last point in the list, add [[*number*]] to the end of the command, where *number* is one more than n, the number of steps used.

To plot the points obtained, use a command like this:

```
ListPointPlot3D[Transpose[
   euler3D[{x - z, y + x, y - x}, {t, 0, 3},
     {x, 2}, {y, -2}, {z, 1},40]][[2]],
  PlotRange -> {{-6, 6}, {-16, 2}, {-10, 4}},
 AspectRatio -> Automatic]
```

B.11 Slope fields for first-order 2×2 systems

All this code can be found in the file `phaseplanes.nb`, available on my website.

Throughout this section, the assumption is that we are dealing with an autonomous 2×2 system of the form

$$\begin{cases} x'(t) = \phi_1(x, y) \\ y'(t) = \phi_2(x, y) \end{cases}$$

Otherwise, these commands are edited in a manner similar to how you would edit the commands from Section A.7.

Code to sketch the slope field

Execute in a single *Mathematica* cell:

```
phi1[x_,y_] := x - y;
phi2[x_,y_] := x + 2y;
VectorPlot[{phi1[x,y], phi2[x,y]}, {x, -3, 3}, {y, -3, 3},
 VectorPoints -> 20, Axes -> True,
 VectorScale -> {Automatic, Automatic, None},
 VectorStyle -> Orange]
```

Code to sketch the slope field and several solution curves

Execute in a single *Mathematica* cell:

```
phi1[x_,y_] := x - y;
phi2[x_,y_] := x + 2y;
VectorPlot[{phi1[x,y], phi2[x,y]}, {x, -3, 3}, {y, -3, 3},
 VectorPoints -> 20, Axes -> True,
 VectorScale -> {Automatic, Automatic, None},
 VectorStyle -> Orange,
 StreamPoints -> 35,
 StreamScale -> Full,
 StreamStyle -> Black]
```

Code to sketch the slope field and a solution curve passing through a specific point

Execute in a single *Mathematica* cell:

```
phi1[x_,y_] := x - y;
phi2[x_,y_] := x + 2y;
VectorPlot[{phi1[x,y], phi2[x,y]}, {x, -3, 3}, {y, -3, 3},
 VectorPoints -> 20, Axes -> True,
 VectorScale -> {Automatic, Automatic, None},
 VectorStyle -> Orange,
 StreamPoints -> {{-1,2}},
 StreamScale -> Full,
 StreamStyle -> Black]
```

Code to sketch phase planes (solution curves only; no mini-tangent lines)

Execute in a single *Mathematica* cell:

```
phi1[x_,y_] := x - y;
phi2[x_,y_] := x + 2y;
StreamPlot[{phi1[x,y], phi2[x,y]}, {x, -4, 4}, {y, -4, 4},
 StreamPoints -> 100,
 StreamStyle -> Black,
 StreamScale -> Full]
```

Code to sketch a single solution curve (no mini-tangent lines)

Execute in a single *Mathematica* cell:

```
phi1[x_,y_] := x - y;
phi2[x_,y_] := x + 2y;
StreamPlot[{phi1[x,y], phi2[x,y]}, {x, -4, 4}, {y, -4, 4},
 StreamPoints -> {{-1,2}},
 StreamStyle -> Black,
 StreamScale -> Full]
```

Index

$C^\infty(\mathbb{R}, \mathbb{R})$, 12
I, 119
$M_n(\mathbb{R})$, 118
$M_{mn}(C^\infty)$, 118
$M_{mn}(\mathbb{R})$, 118
$M_{mn}(\mathbb{C})$, 174
$\Im(z)$, 174
$\Re(z)$, 174
$\overline{z}$, 174
$\exp(A)$, 153
$\mathbb{C}$, 174
$d \times d$ linear system of ODEs, 131
e^t, Taylor series of, 153
i, 174
n^{th}-order system of ODEs, 131
Mathematica cells, 311
Mathematica code for Euler's method, 35, 328
Mathematica code for Euler's method (systems), 115, 330
Mathematica code for Picard's method, 330
Mathematica code for eigenvalues and eigenvectors, 169, 324
Mathematica code for matrix exponentials, 169, 324
Mathematica code for matrix operations, 129, 324
Mathematica code for phase planes, 149, 333
Mathematica code for slope fields, 26, 28, 149, 327, 333
Mathematica notebook, 311
Mathematica, calculus commands, 315
Mathematica, commands for complex numbers, 326
Mathematica, defining functions in, 314, 317
Mathematica, finding Taylor polynomials, 315
Mathematica, graphing with, 314, 317
Mathematica, solving equations with, 316

absolute value (of a complex number), 176
addition (in a vector space), 11
addition (of matrices), 120
addition in $\mathbb{C}$, 175
affine subspace, 71
amplitude, 238
argument (of a complex number), 177
arithmetic in $\mathbb{C}$, 175
asymptotically stable (equilibrium), 45, 199
asymptotically unstable (equilibrium), 45, 199
attracting, 199
attracting (equilibrium), 45
autonomous (system), 144

autonomous ODE, 41

basis, 137
bifurcation diagram, 53
bifurcation, pitchfork, 54
bifurcation, saddle-node, 53
bifurcation, transcritical, 55
bifurcations, 52
Bombelli, 173
Brahmagupta, 173

calculus with *Mathematica*, 315
capacitor, 96
Cardano, 173
carrying capacity, 50
cell (*Mathematica*), 311
center, 203
Chain Rule, 83
chaos theory, 247
characteristic equation, 227
classification of equilibria, 46
classification of equilibria (systems), 198, 200
coil, 96
compartmental models, 87, 207
complex conjugate, 174
complex eigenvalues, 180
complex number, 174
complex numbers in *Mathematica*, 326
complex numbers, arithmetic, 175
complex numbers, division, 177
complex numbers, geometry of, 176
complex numbers, history of, 173
complex numbers, multiplication of, 179
complex numbers, reciprocals of, 177
complex plane, 176
complex roots (of characteristic equation), 228
Complex Roots Theorem, 182
computing eigenvalues, 158
computing matrix exponentials, 155, 161
conjugate (of complex number), 174
constant-coefficient (system of ODEs), 131
constant-coefficient ODE, 18
cooling and heating models, 91, 92
corresponding homogeneous equation, 70
corresponding homogeneous system, 133
cosine (of a complex number), 178
coupled mass-spring systems, 242
cubic formula, 173
current (electrical circuit), 96
current law (electrical circuits), 95

damped (oscillator), 237
damped oscillator, example, 239
damping coefficient, 237
derivative of a matrix, 121
derivative, partial, 82
derivative, total, 198
determinant, 127
determinant of 1×1 matrix, 128
determinant of 2×2 matrix, 128
determinant of $3 times 3$ matrix, 128
diagonal entries (of a matrix), 118
diagonal matrix, 118
diagonal matrix, exponential of, 156
diagonal matrix, powers of, 123
diagonalizable (matrix), 157
diagonalizing a matrix, 157
diagram, bifurcation, 53
differential operator, 17
differentiation of matrices, 121
dimension (of vector space), 137
disease spread, model, 211
division (in $\mathbb{C}$), 177
division in $\mathbb{C}$, 175
double pendulum, 246
driven (oscillator), 237
driven oscillator, example, 240

eigenvalue, 158

eigenvalues and eigenvectors, *Mathematica* code for, 169, 324
eigenvalues, complex, 180
eigenvalues, repeated, 187
eigenvector, 158
eigenvector, generalized, 190
electrical circuits, 95, 248
epidemiology, 211
equality (of matrices), 118
equation, characteristic, 227
equilibria (systems), classification of, 200
equilibria, classification of, 46
equilibria, classification of (systems), 198
equilibrium (of ODE), 42
equilibrium (systems), 144
Euler, 173
Euler's formula, 179
Euler's method (equations), 31, 32
Euler's method (systems), 112
Euler's method, *Mathematica* code for, 35, 115, 328, 330
Euler's method, potential pitfall, 37
exact equation, 83
exact equation, solution of, 84
examples of ODEs, 5
Existence/Uniqueness Theorem for n^{th}-order linear equations, 224
Existence/Uniqueness Theorem for n^{th}-order linear systems, 225
Existence/Uniqueness Theorem for first-order ODEs, 38
Existence/Uniqueness Theorem for first-order systems, 117
exponential (of a complex number), 178
exponential (of a matrix), 153
exponential decay, 20, 21
exponential growth, 20, 21

Faraday's Law, 96, 248
first-order linear equation, solution of, 71
first-order linear system, solution of, 67
functions in *Mathematica*, 314, 317
Fundamental Theorem of Algebra, 173

general solution (of an ODE), 8
general solution (of ODE), 8
generalized eigenvector, 190
graph of parametric equations, 110
graphing with *Mathematica*, 314, 317

heating and cooling models, 91, 92
history of complex numbers, 173
homogeneous (system of ODEs), 131
homogeneous ODE, 18
homogeneous, first-order linear equation, solution of, 62
Hooke's Law, 236

identity matrix, 119
imaginary axis, 176
imaginary numbers, 173
imaginary part (of a complex number), 174
inductor, 96
infectives (SIR model), 211
infinitely differentiable (function), 12
initial value, 8
initial value (n^{th}-order), 224
initial value problem, 8
initial value problem (n^{th}-order), 224
integral equation, 39
integral equation (systems), 116
integrating factor, 66
inverse (of a matrix), 126
inverse of 1×1 matrix, 126
inverse of 2×2 matrix, 127
inverses, properties of, 126
invertible (matrix), 126
IVP, 8
IVP (n^{th}-order), 224

Kirchoff's Laws, 248
Kirchoff's laws, 95

limiting capacity, 50
linear differential operator, 17
linear numerical systems, characterization of, 125
linear ODE, 18
linear operator, 13
linear operators on $\mathbb{R}^d$, characterization of, 125
linear system of numerical equations, 117
linear system of ODEs, 131
linear systems, structure of solution set, 143
linear transformation, 13
linearization, 245
linearly dependent, 137
linearly independent, 137
logistic equation, 50
logistic equation, solution of, 80
logistic population model, 87
Lotka-Volterra equations, 209
lower triangular, 119

Malthusian population model, 87
mass-spring systems, 236
mass-spring systems, coupled, 242
matrix, 118
matrix addition, 120
matrix derivative, 121
matrix exponential, 153
matrix exponential of diagonal matrix, 156
matrix exponentials, *Mathematica* code for, 169, 324
matrix exponentials, computing, 155, 161
matrix multiplication, 121
matrix operations on *Mathematica*, 129, 324
matrix operations, properties of, 123
matrix scalar multiplication, 120
matrix, diagonal, 118
method of successive approximations (equations), 39, 116
mini-tangent line, 24
mixing problems (one tank), 89
mixing problems (two tanks), 207
modulus (of a complex number), 176
multiplication (in $\mathbb{C}$), 179
multiplication in $\mathbb{C}$, 175
multiplication of matrices, 121

nth order (ODE), 8
nth order differential operator, 17
neutral, 199
neutral (equilibrium), 46
Newton's Law of Cooling and Heating, 91
Newton's Second Law (of motion), 90
Newtonian mechanics, 90
node, stable, 202
node, unstable, 202
nonlinear ODE, 18
norm (of a complex number), 176
notation for systems of ODEs, 110
notebook (*Mathematica*), 311
numerical analysis, 30
numerical method, 30

ODE, definition of, 5
ODEs, examples of, 5
Ohm's Law, 96, 248
operator, 13
operator, differential, 17
operator, linear, 13
order (of an ODE), 8
ordinary differential equation, definition of, 5
oscillator, 237
overdamped (oscillator), 239, 240

parameter, 52
parametric equations, 109
parametric equations, graph of, 110
partial derivative, 82

partial differential equation, 9
particular solution (of an ODE), 8
particular solution (of ODE), 8
PDE, 9
pendulum equation, undamped, 245
pendulum, double, 246
pendulum, single, 244
phase line, 43
phase plane, 147
phase planes, *Mathematica* code for, 149, 333
phase shift, 238
Picard's method (equations), 39, 116
Picard's method, *Mathematica* code for, 330
pitchfork bifurcation, 54
plane, trace-determinant, 205
polar coordinates (of a complex number), 177
population model, logistic, 87
population model, Malthusian, 87
powers of a diagonal matrix, 123
predator-prey model, 108
predator-prey systems, 209
product (of matrices), 121
properties of matrix exponentials, 153
properties of matrix inverses, 126
properties of matrix operations, 123
proportionality constant, 21
pure imaginary number, 174

quadratic formula, 173

rate (of exponential growth/decay), 21
rate of reproduction (logistic equation), 50
RC circuit, 96
real axis, 176
real part (of a complex number), 174
real vector space, 11
reciprocals (in $\mathbb{C}$), 177
recovered (SIR model), 211
reduction of order, 221
repeated eigenvalues, 187
repeated roots (of characteristic equation), 229
repelling (equilibrium), 45
resistor, 96
RL circuit, 97
RLC circuit, 248

saddle, 203
saddle-node bifurcation, 53
scalar, 11
scalar multiplication, 11
scalar multiplication (of matrices), 120
semistable (equilibrium), 46, 199
separable (ODE), 76
separation of variables, 76
simple harmonic motion, 246
simple oscillator, 237
simple oscillator, solution of, 238
sine (of a complex number), 178
single pendulum, 244
sink, 45, 199
SIR model, 211
slope field, 24
slope fields (2×2 systems), 145
slope fields, *Mathematica* code for, 26, 28, 149, 327, 333
slope fields, reading pictures of, 27
solution (of a system), 109
solution of first-order linear system, 67
solution of general first-order linear equation, 71
solution of homogeneous, first-order linear equation, 62
solving equations with *Mathematica*, 316
solving exact equations, 86
solving first-order linear ODEs, 67
solving separable ODEs, 77
source, 45
span (of a collection of vectors), 135
span (of a single vector), 63
span (of a vector), 135

spiral, stable, 202
spiral, unstable, 203
square (matrix), 118
stable (equilibrium), 45, 199
stable node, 202
stable spiral, 202
step size, 112
step size (in Euler's method), 32
structure of solution set of a linear system, 143
subspace, 63, 134
successive approximations (equations), 39, 116
susceptibles (SIR model), 211
system of numerical equations, matrix language, 124
system of ODEs, linear, 131
systems, notation for, 110

Taylor polynomials, computing with *Mathematica*, 315
Taylor series of e^t, 153
total derivative, 198
trace, 119
trace-determinant plane, 205
transcritical bifurcation, 55
transformation, linear, 13
triangular, 119

undamped (oscillator), 237
undamped pendulum equation, 245
underdamped (oscillator), 239
undetermined coefficients, 72, 75
undetermined coefficients (n^{th}-order equations), 230
undetermined coefficients (systems), 196
unstable (equilibrium), 45, 199
unstable node, 202
unstable spiral, 203
upper triangular, 119

variation of parameters, 232, 234
vector, 11
vector field, 24
vector field with solution curve through specified initial value, *Mathematica* code, 28, 328
vector field with solution curves, *Mathematica* code, 28, 327
vector field, *Mathematica* code, 26, 327
vector field, interpretation of, 27
vector space, 11
voltage, 96
voltage law (electrical circuits), 95

Wronskian, 138