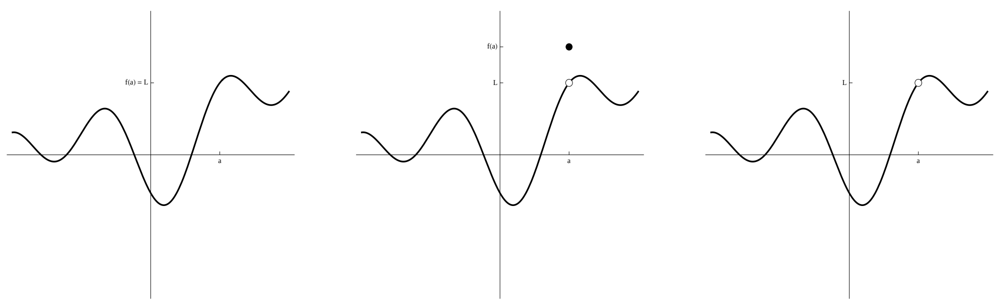
1 Limits

Suppose $f : \mathbb{R} \to \mathbb{R}$. To say

$$\lim_{x \to a} f(x) = L$$

means that as x gets closer and closer to a, then $f(x)$ gets closer and closer to L. This suggests that the graph of f looks like one of the following three pictures:



The graph on the left is "continuous" at a; the other two graphs are not. More precisely,

Definition 1.1 *A function $f : \mathbb{R} \to \mathbb{R}$ is* **continuous at** *a if $\lim_{x \to a} f(x) = f(a)$. f is called* **continuous** *if it is continuous at every point in its domain.*

Most "reasonable" functions are continuous, as seen in the following theorem:

Theorem 1.2 *Any function which is the sum, difference, product, composition, and/or quotient of functions made up of constants, powers of x, sines, cosines, arcsines, arctangents, exponentials and/or logarithms is continuous everywhere except where any denominator is zero.*

This theorem suggests that to evaluate most limits, you should start by plugging in a for x. If you get a number, that is usually the answer.

When you plug in a for x and you get 0 in a denominator, you have to work a bit harder to evaluate a limit.

- **When you get $\frac{\textbf{nonzero}}{0}$:** answer is $\pm\infty$ (this indicates the presence of a **vertical asymptote** at $x = a$); check the signs of the top and bottom carefully to see whether the answer is ∞ or $-\infty$.
- **When you get $\frac{0}{0}$:** answer could be anything

  Techniques for dealing with $\frac{0}{0}$:

  - Factor and cancel
  - Conjugate square roots
  - Clear denominators of "inside" fractions
  - L'Hôpital's Rule (see below)

Theorem 1.3 (L'Hôpital's Rule) *Suppose f and g are differentiable functions. Suppose also that either*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \text{ or } \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty.$$

Then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} \stackrel{L}{=} \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Limits at infinity: Suppose $f : \mathbb{R} \to \mathbb{R}$. To say

$$\lim_{x\to\infty} f(x) = L$$

means that as x gets larger and larger without bound, then $f(x)$ gets closer and closer to L; equivalently this means $y = L$ is a **horizontal asymptote** of f. To evaluate a limit at infinity, "plug in " ∞ for x and use the following arithmetic rules with ∞:

$$\infty \pm c = \infty \qquad \infty \cdot \infty = \infty \qquad \infty^\infty = \infty \qquad c \cdot \infty = \begin{cases} \infty & \text{if } c > 0 \\ -\infty & \text{if } c < 0 \end{cases}$$

$$\sqrt{\infty} = \infty \qquad \ln \infty = \infty \qquad e^\infty = \infty \qquad \infty^c = \begin{cases} \infty & \text{if } c > 0 \\ 0 & \text{if } c < 0 \end{cases}$$

$$\frac{c}{\infty} = 0 \qquad \frac{\infty}{0} = \pm\infty \qquad \frac{c}{0} = \pm\infty \text{ so long as } c \neq 0$$

In the last two situations, you need to analyze the sign carefully to determine whether the answer is ∞ or $-\infty$.

Warning: The following expressions are indeterminate forms, i.e. they can work out to be different things depending on the context (these are evaluated using L'Hôpital's Rule):

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0 \cdot \infty \qquad \infty - \infty \qquad 1^\infty \qquad \infty^0 \qquad 0^0$$

Limits at infinity for rational functions can be determined immediately by way of the following theorem:

Theorem 1.4 (Limits at infinity for rational functions) *Suppose f is a rational function, i.e. has form*

$$f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + ... + a_2 x^2 + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + ... + b_2 x^2 + b_1 x + b_0}.$$

Then:

1. *If $m < n$ (i.e. largest power in numerator < largest power in denominator), then $\lim_{x\to\infty} f(x) = 0$.*
2. *If $m > n$ (i.e. largest power in numerator > largest power in denominator), then $\lim_{x\to\infty} f(x) = \pm\infty$.*
3. *If $m = n$ (i.e. largest powers in numerator and denominator are equal), then $\lim_{x\to\infty} f(x) = \frac{a_m}{b_n}$.*

2 Derivatives

Definition 2.1 (Limit definition of the derivative) *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let x be in the domain of f. If the limit*

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists and is finite, say that f is **differentiable at** x. *In this case, we call the value of this limit* **the derivative of** f *and denote it by $f'(x)$ or $\frac{df}{dx}$ or $\frac{dy}{dx}$.*

Differentiable functions are smooth, i.e. are continuous and do not have sharp corners, vertical tangencies or cusps.

Assuming it exists, the derivative $f'(x)$ computes:

1. the slope of the line tangent to f at x;
2. the slope of the graph of f at x;
3. the instantaneous rate of change of y with respect to x;
4. the instantaneous velocity at time x (if f is the position of the object at time x).

We do not compute derivatives using the limit definition given above. We use **differentiation rules** (given on the next two pages), which consist of a list of functions whose derivative we memorize and a list of rules which tell us how to differentiate more complicated functions made up of pieces whose derivatives we know.

Derivatives have many applications. The most important (for Math 230 purposes) is that given a differentiable function f, you can approximate values of f near a using the tangent line to f at a:

Definition 2.2 *Given a differentiable function f and a number a at which f is differentiable, the* **tangent line** *to f at a is the line whose equation is*

$$y = f(a) + f'(a)(x-a) \text{ (a.k.a. } L(x) = f(a) + f'(a)(x-a)).$$

For values of x near a, $f(x) \approx L(x)$; approximating $f(x)$ via this procedure is called **linear approximation**.

You can also conclude many things about the graph of f from looking at its derivative and its higher-order derivatives, as defined below:

Definition 2.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function.*

- *The* **zeroth derivative** *of f, sometimes denoted $f^{(0)}$, is just the function f itself.*
- *The* **first derivative** *of f, sometimes denoted $f^{(1)}$ or $\frac{dy}{dx}$, is just f'.*
- *The* **second derivative** *of f, denoted f'' or $f^{(2)}$ or $\frac{d^2y}{dx^2}$, is the derivative of f': $f'' = (f')'$.*
- *More generally, the n^{th}* **derivative** *of f, denoted $f^{(n)}$ or $\frac{d^ny}{dx^n}$,is the derivative of $f^{(n-1)}$:*

$$f^{(n)} = ((((f')')\cdots')'$$

The first derivative of a function measures its **tone** (when $f' > 0$, the function is increasing; when $f' < 0$, the function is decreasing). The second derivative of a function measures its **concavity** (when $f'' > 0$, the function is concave up; when $f'' < 0$, the function is concave down). The second derivative of a function which gives the position of an object gives the acceleration of the object.

Derivatives of functions that you should memorize:

Constant Functions	$\frac{d}{dx}(c) = 0$
Power Rule	$\frac{d}{dx}(x^n) = nx^{n-1}$ (so long as $n \neq 0$)
Special cases of the Power Rule:	$\frac{d}{dx}(mx+b) = m$
	$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
	$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$
	$\frac{d}{dx}(x^2) = 2x$
Trigonometric Functions	$\frac{d}{dx}(\sin x) = \cos x$
	$\frac{d}{dx}(\cos x) = -\sin x$
	$\frac{d}{dx}(\tan x) = \sec^2 x$
	$\frac{d}{dx}(\cot x) = -\csc^2 x$
	$\frac{d}{dx}(\sec x) = \sec x \tan x$
	$\frac{d}{dx}(\csc x) = -\csc x \cot x$
Exponential Function	$\frac{d}{dx}(e^x) = e^x$
Natural Log Function	$\frac{d}{dx}(\ln x) = \frac{1}{x}$
Inverse Trig Functions	$\frac{d}{dx}(\arctan x) = \frac{1}{x^2+1}$
	$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

Rules that tell you how to differentiate more complicated functions:

Sum Rule	$(f+g)'(x) = f'(x) + g'(x)$
Difference Rule	$(f-g)'(x) = f'(x) - g'(x)$
Constant Multiple Rule	$(kf)'(x) = k \cdot f'(x)$ for any constant k
Product Rule	$(fg)'(x) = f'(x)\,g(x) + g'(x)\,f(x)$
Quotient Rule	$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\,g(x) - g'(x)\,f(x)}{[g(x)]^2}$
Chain Rule	$(f \circ g)'(x) = f'(g(x))\,g(x)$

3 Integrals

Definition 3.1 *Given function $f : [a,b] \to \mathbb{R}$, the* **definite integral** *of f from a to b is*

$$\int_a^b f(x)\,dx = \lim_{||\mathcal{P}|| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k,$$

where the expression inside the limit is a Riemann sum for f.

Note: In Math 230, the limit above always exists (but it doesn't always exist for crazy functions f... take Math 430 for more on that).

The definite integral of a function is a **number** which gives the signed area of the region between the graph of f and the x-axis. Area above the x-axis is counted as positive area; area below the x-axis is counted as negative area. (A definite integral also gives the displacement of an object from time a to time b, if the object's velocity at time x is the integrand $f(x)$.)

As with derivatives, we do not compute integrals with this definition. We use the Fundamental Theorem of Calculus, which tells us to evaluate integrals using antiderivatives:

Definition 3.2 *Given function f, an* **antiderivative** *of f is a function F such that $F' = f$.*

Every continuous function has an antiderivative (although you may not be able to write its formula down); any two antiderivatives of the same function must differ by a constant (so if you know one antiderivative, you know them all by adding a $+C$ to the one you know).

Definition 3.3 *Given function f, the* **indefinite integral** *of f, denoted*

$$\int f(x)\,dx,$$

is the set of all antiderivatives of f.

Theorem 3.4 (Fundamental Theorem of Calculus Part II) *Let f be continuous on $[a,b]$. Suppose F is* ***any*** *antiderivative of f. Then*

$$\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a).$$

Despite the similar notation, $\int f(x)\,dx$ and $\int_a^b f(x)\,dx$ are very different objects. The first object is a **set** of **functions**; the second object is a **number**.

Theorem 3.5 (Integration Rules to Memorize)

$$\int 0\,dx = C$$
$$\int M\,dx = Mx + C$$
$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \textit{(so long as } n \neq -1\textit{)}$$
$$\int \frac{1}{x}\,dx = \ln|x| + C \quad \textit{(I don't care so much about the } |\ | \textit{)}$$
$$\int \sin x\,dx = -\cos x + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \csc^2 x\,dx = -\cot x + C$$
$$\int \sec x \tan x\,dx = \sec x + C$$
$$\int \csc x \cot x\,dx = -\csc x + C$$
$$\int e^x\,dx = e^x + C$$
$$\int \frac{1}{x^2+1}\,dx = \arctan x + C$$
$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$$
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

Theorem 3.6 (Linearity of Integration) *Suppose f and g are integrable functions. Then:*

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx;$$
$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx;$$
$$\int_a^b [k\cdot f(x)]\,dx = k\int_a^b f(x)\,dx \textit{ for any constant } k;$$

(and the same rules hold for indefinite integrals).

Here are some other basic properties of integrals:

Definition 3.7 *Let $a < b$ and let $f : [a,b] \to \mathbb{R}$ be integrable. Then*

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \quad \textit{and} \quad \int_a^a f(x)\,dx = 0.$$

Theorem 3.8 (Additivity property of integrals) *Suppose f is integrable. Then for any numbers a, b and c,*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Difficult integrals require more advanced techniques. u-substitutions are useful to integrate functions which are the product of two or more related terms:

Theorem 3.9 (Integration by $u-$substitution - Indefinite Integrals)

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$$

by setting $u = g(x)$.

Theorem 3.10 (Integration by $u-$substitution - Definite Integrals)

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

by setting $u = g(x)$.

It is helpful to memorize the following idea, which comes from using a u-substitution $u = mx + b$:

Theorem 3.11 (Linear Replacement Principle) *Suppose you know*

$$\int f(x)\,dx = F(x) + C.$$

Then for any constants m and b,

$$\int f(mx+b)\,dx = \frac{1}{m}F(mx+b) + C.$$

Integrals whose integrands are products of unrelated terms are often computed using integration by parts:

Theorem 3.12 (Integration by Parts (IBP) Formula)

$$\int u\,dv = uv - \int v\,du.$$

A last technique useful to evaluate some integrals is to find the **partial fraction decomposition** of the integrand; to use this technique, the denominator of the integrand should be "factorable" and the degree of the numerator should be less than the degree of the denominator.

4 Infinite Series

An **infinite series** is an attempt to add an infinite list of numbers

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$$

If the infinite list of numbers has a finite sum, we say the series converges; otherwise we say the series diverges. More formally:

Definition 4.1 *Let $\sum a_n$ be an infinite series. For each N, let S_N be the N^{th} partial sum of the series; this is defined to be the sum of all the a_n for which $n \leq N$. Then:*

1. *If L is a real number such that $\lim\limits_{N \to \infty} S_N = L$, then we say the infinite series $a_1 + a_2 + a_3 + ...$* **converges (to L)** *and write $\sum a_n = L$. In this setting L is called the* **sum** *of the series.*
2. *If $\lim\limits_{N \to \infty} S_N = \pm\infty$ or if $\lim\limits_{N \to \infty} S_N$ DNE, then we say the infinite series $\sum a_n$* **diverges**.

Important: There is a big difference between saying "$\sum a_n$ converges" and saying "a_n converges". Without the Σ, you aren't adding the numbers. Therefore, you should never omit the Σ when describing whether or not an infinite series converges.

We divide convergent series into two classes as follows:

Definition 4.2 *Let $\sum a_n$ be an infinite series. We say the series is* **absolutely convergent** *(or that the series* **converges absolutely***) if $\sum |a_n|$ converges.*

If $\sum a_n$ converges but $\sum |a_n|$ diverges, then we say $\sum a_n$ is **conditionally convergent** *(or that the series* **converges conditionally***).*

The reason we care whether a series converges absolutely or conditionally is the following theorem:

Theorem 4.3 (Rearrangement Theorem) *Suppose $\sum a_n$ is an infinite series.*

1. *If $\sum a_n$ converges conditionally, then the terms of that series can be rearranged so that the rearranged series converges to any number you like! (The series can also be rearranged so that the rearranged series diverges.)*
2. *If $\sum a_n$ converges absolutely to L, then no matter how the terms of the series are regrouped or rearranged, the rearranged series still converges absolutely to L.*

The major application of series is to obtain alternate representations of functions which can be used for approximations:

Definition 4.4 *Suppose f is a function which can be differentiated over and over again at $x = 0$. The* **Taylor series (centered at 0) of** *f (a.k.a.* **Maclaurin series of** *f is the power series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

If we truncate this series at the N^{th} power term, we obtain a partial sum of the Taylor series called the N^{th} **Taylor polynomial (centered at 0)** *of f. This polynomial is denoted $P_N(x)$.*

General properties of Taylor polynomials:

1. $P_N(x)$ is a polynomial of degree $\leq N$;
2. $P_0(x)$ is the constant function of height $f(0)$;
3. $P_1(x)$ is the tangent line to f when $x = 0$;
4. $P_N(x)$ is the best N^{th} degree polynomial approximation to f near 0.

The point of Taylor series and Taylor polynomials is that if a function f has N^{th} Taylor polynomial P_N, then $P_N(x) \approx f(x)$ (this approximation improves as N increases, but is pretty good even for small N).

Theorem 4.5 (Uniqueness of power series) *Suppose f is a function which can be differentiated over and over again at $x = 0$. Then if we write f as a power series of the form*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

then the coefficients a_n must satisfy

$$a_n = \frac{f^{(n)}(0)}{n!} \text{ for all } n.$$

*In other words, the **only** power series of the form $\sum a_n x^n$ which can represent f is its Taylor series centered at 0.*

Commonly used Taylor series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots \quad \text{(holds for all } x\text{)}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad \text{(holds for all } x\text{)}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad \text{(holds for all } x\text{)}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \quad \text{(holds when } x \in (-1,1)\text{)}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad \text{(holds when } x \in (-1,1]\text{)}$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad \text{(holds when } x \in [-1,1]\text{)}$$