Algebra and calculus review

1. Solve for y in terms of t:

   (a) $\frac{1}{\sqrt{y+1}} = 3t^2 + 2$

   (b) $\frac{1}{y} = 2t - 3$

   (c) $\frac{1}{y^2} = \frac{1}{t} + 2$

   (d) $\arctan y = 5t - 4$

2. Solve for y in terms of t:

   (a) $y^2 - 2y = t + 4$

   (b) $\sin 3y = 4\cos t$

   (c) $t^2 + 2y^2 = 8$

   (d) $\ln(y+1) = \ln(t+1) - 2$

3. Solve for y in terms of t; simplifying your answer as much as possible.

   (a) $\sqrt{ty} = t$

   (b) $\ln y = 4\ln t + 3$

   (c) $e^{y-2} = 3e^{t-1}$

   (d) $e^{3y} = e^t + 1$

4. Compute the following limits:

   (a) $\lim\limits_{t\to\infty} e^t$

   (b) $\lim\limits_{t\to\infty} e^{-2t}$

   (c) $\lim\limits_{t\to\infty} \frac{3}{2+e^t}$

   (d) $\lim\limits_{t\to\infty} \frac{3}{2+e^{-t}}$

5. Compute the following limits:

   (a) $\lim\limits_{t\to\infty} \left(e^{-3t} + 2e^{-t}\right)$

   (b) $\lim\limits_{t\to\infty} \sin 2t$

   (c) $\lim\limits_{t\to\infty} e^t \sin t$

   (d) $\lim\limits_{t\to\infty} e^{-2t}\cos t$

6. Find the derivative of each function:

   (a) $f(t) = 3t^2 - 7t + 5$

   (b) $f(t) = (t^2+3)^8$

   (c) $f(t) = t^2e^{2t} + 4te^{2t}$

   (d) $f(t) = 4te^{-t}$

7. Find the derivative of each function:

   (a) $f(t) = \sin 4t$

   (b) $f(t) = e^{-t}\sin 2t$

8. Find the derivative of each function:

   (a) $f(t) = \sin^3(4t^2)$

   (b) $f(t) = e^{3t}\cos 4t - 2e^{3t}\sin 4t$

9. Find the first, second and third derivative of each function:
   (a) $f(t) = e^{-2t}$
   (b) $f(t) = \sin 4t + \cos 4t$
   (c) $f(t) = t\cos t$
   (d) $f(t) = -te^{3t} + 2e^{3t}$

10. Compute the following integrals:
    (a) $\int t\,dt$
    (b) $\int (3t^4 - 2t^7)\,dt$
    (c) $\int t^{-1}\,dt$
    (d) $\int t^{-2}\,dt$

11. Compute the following integrals:
    (a) $\int \frac{4}{t}\,dt$
    (b) $\int \frac{1}{t}\,dt$
    (c) $\int \frac{6}{t^2}\,dt$
    (d) $\int \frac{1}{\sqrt{t}}\,dt$

12. Compute the following integrals:
    (a) $\int 12\sqrt{t}\,dt$
    (b) $\int \sqrt[3]{t}\,dt$
    (c) $\int (\sqrt{t} + 1)\,dt$
    (d) $\int \sqrt{t+1}\,dt$

13. Compute the following integrals:
    (a) $\int 2\sin t\,dt$
    (b) $\int (\sin 3t + \cos t)\,dt$
    (c) $\int \frac{1}{t^2+1}\,dt$
    (d) $\int \left(11e^{2t} - 18e^{-3t} + 7e^{-t}\right)\,dt$

14. Compute the following integrals. Your answer will have a t in it.
    (a) $\int_0^t (2s+1)\,ds$
    (b) $\int_1^t 2s^{-1/2}\,ds$
    (c) $\int_0^t e^{-s}\,ds$
    (d) $\int_1^t \frac{4}{s}\,ds$

15. Compute the following integrals (using integration by parts):
    (a) $\int te^{-3t}\,dt$
    (b) $\int t\ln t\,dt$

16. Compute the following integrals:
    (a) $\int \frac{2}{t^2-4}\,dt$
    (b) $\int \sin t\cos t\,dt$

17. Compute the following integrals:
    (a) $\int \frac{e^t}{e^t+1}\,dt$
    (b) $\int \frac{e^t+1}{e^t}\,dt$

18. Compute the following integrals:

    (a) $\int \frac{t^2+2}{(t-1)^2}\,dt$

    (b) $\int \frac{(t-2)^2}{t}\,dt$

19. Some sequences in mathematics are defined by a method called **recursion**: this means that each term in the sequence is given by a formula which depends on previous terms. For example, suppose you are given

    $$x_0 = 3 \text{ and } x_{n+1} = 2x_n.$$

    Then you can find $x_1, x_2, x_3, \ldots$ by repeatedly plugging in to the second equation as follows:

    $$\begin{aligned} x_1 &= 2x_0 = 2(3) = 6 \\ x_2 &= 2x_1 = 2(6) = 12 \\ x_3 &= 2x_2 = 2(12) = 24 \\ &\vdots \qquad \vdots \end{aligned}$$

    In each part of this problem, you are given a recursive formula and a value of x_0. Based on this formula, write down x_1, x_2, x_3, and x_4.

    (a) $x_0 = 1; x_{n+1} = -x_n$

    (b) $x_0 = 18; x_{n+1} = \frac{1}{3}x_n + 3$

    (c) $x_0 = 1; x_{n+1} = 2^{x_n}$

    (d) $x_0 = 3; x_{n+1} = 2 - \frac{1}{2}x_n$

20. Suppose $x_0 = 0$, $x_1 = 1$ and $x_{n+2} = x_{n+1} + 2x_n$. Find x_7.

Problems from Section 1.1

21. In each part, you are given an ODE and a possible solution of that ODE. Determine whether or not the possible solution is actually a solution of the ODE.

    Hint: Plug the possible solution in to both sides of the ODE and see if they are equal.

    (a) *ODE:* $\frac{dy}{dt} = \frac{y^2-1}{t^2+2t}$
    Possible solution: $y = 2t + 1$

    (b) *ODE:* $y'' + y = t^2 + 2$
    Possible solution: $y = \sin t + t^2$

    (c) *ODE:* $\frac{d^2y}{dt^2} - \frac{dy}{dt} = 2y$
    Possible solution: $y = e^{2t} - 3e^{-t}$

22. In each part, you are given an ODE and a possible solution of that ODE. Determine whether or not the possible solution is actually a solution of the ODE

    Hint: To compute derivatives of the possible solutions, differentiate implicitly.

    (a) *ODE:* $\frac{dy}{dt} = \frac{t}{y}$
    Possible solution: $t^2 + y^2 = 9$

    (b) *ODE:* $\frac{dy}{dt} = \frac{2ty}{y-1}$
    Possible solution: $y - \ln y - 1 = t^2$

23. In each part, you are given an initial value problem and a possible solution of that IVP. Determine whether or not the possible solution is actually a solution of the IVP:

(a) *IVP:* $\begin{cases} y' = 6y \\ y(0) = 2 \end{cases}$
Possible solution: $y = 2e^{6t}$

(b) *IVP:* $\begin{cases} y' = 6y \\ y(0) = -1 \end{cases}$
Possible solution: $y = 2e^{6t} + -3$

24. In each part, you are given an initial value problem and a possible solution of that IVP. Determine whether or not the possible solution is actually a solution of the IVP:

(a) *IVP:* $\begin{cases} y' = \frac{-t}{y} \\ y(4) = 3 \end{cases}$
Possible solution: $t^2 + y^2 = 20$

(b) *IVP:* $\begin{cases} y' = \frac{-t}{y} \\ y(4) = 3 \end{cases}$
Possible solution: $t^2 + y^2 = 25$

Problems from Section 1.2

25. Determine whether each of the formulas T given below defines a linear operator on $C^\infty(\mathbb{R}, \mathbb{R})$ (no proof is required, just write "linear" or "not linear"):

(a) $T(y) = y + t$
(b) $T(y) = 4y^{(7)}$
(c) $T(y) = y'' + ty' - (3\sin t)y$
(d) $T(y) = (y')^2 + 2y - 4y''$

26. Suppose T is the linear differential operator defined by

$$T(y) = e^t y''' - e^{-t} y'' + e^{4t} y' - 2y.$$

(a) What is the order of this operator?
(b) Find $T(2e^t)$.
(c) Find $T(e^{2t})$.
(d) Find $T(4)$.

27. Suppose T is the fourth-order linear differential operator defined by setting $p_0(t) = 2t^2$, $p_1(t) = t^3$, $p_2(t) = 8t^4$, $p_3(t) = 0$ and $p_4(t) = 2t^6$ and using the formula given in Definition 1.12 of the lecture notes.

(a) Write the formula for $T(y)$.
(b) Find $T(2t^6)$.

In each part of Problems 28-31 you are given an ODE. For each equation:
(i) give the order of the equation;
(ii) give the number of arbitrary constants you would expect in the general solution;
(iii) classify the equation as linear or nonlinear;
(iv) if the equation is linear, determine whether or not it is homogeneous;
(v) if the equation is linear, determine whether or not it is constant-coefficient.

28. (a) $\frac{dy}{dt} + 5t\frac{dy}{dt} - 3y^2 = y^7 \sin t$
(b) $4y - \cos y^3 + y'''' - t^2 y'' = e^t y'' - y^2$
(c) $e^{t+y}\frac{d^2y}{dt^2} + \cos t \frac{dy}{dt} = 0$

29. (a) $t^2\frac{dy}{dt} + t^4 y = 3t\frac{dy}{dt} - 2t$
    (b) $y^8 + y' = e^{3t}$
    (c) $y^{(8)} + y' = e^{3t}$

30. (a) $2y\frac{d^2y}{dt^2} - y^2\frac{dy}{dt} = 0$
    (b) $e^t\frac{d^2y}{dt^2} + \cos t\frac{dy}{dt} = 0$
    (c) $2\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 6y = 0$

31. (a) $8y'' - 5y' + 3y = 6ty''$
    (b) $y^{(4)} + y''' + y'' + y' + y = e^t$
    (c) $\frac{y'}{y} = 2$

Problems from Section 1.3

32. Suppose that some quantity is changing as time passes so that the rate of increase of the quantity is equal to 4.7 times the size of the quantity.
    (a) Write the ODE represented by this model.
    (b) Write the general solution of this ODE.
    (c) Write the particular solution of this ODE corresponding to the situation where you start with 5 units of the quantity.
    (d) Write the particular solution of this ODE corresponding to the situation where at time 3, you have 12 units of the quantity.

33. Suppose that $y(t)$ is some quantity which grows at a rate proportional to its size.
    (a) Suppose that initially, I have exactly twice as much of the quantity as you do. After 50 years, will I have more than twice as much as you, exactly twice as much as you, or less than twice as much as you? Explain.
    (b) Suppose that initially, I have exactly one more unit of the quantity than you do (we both have a positive amount). After 50 years, will the difference between our holdings be greater than one, equal to one, or less than one? Explain.

34. The amount of money in a retirement account grows proportionally to the amount of money in the account. Suppose initially that there is \$100 in the account, and that five years later, there is \$118 in the account.
    (a) Find a formula for the amount of money $y = y(t)$ in the account t years after the account is opened.
    (b) Find the (exponential) rate of growth of the account, expressed as an annual percentage rate.
    (c) Find the amount of money in the account thirty-five years after it is opened.

35. The **half-life** of a substance which decays exponentially is the amount of time it takes for the substance to decay to half of its original amount. Carbon-14 has a half-life of 5730 years.
    (a) Find the (exponential) rate of decay of carbon-14.

(b) If you have 25 grams of carbon-14 now, how much will you have 20000 years from now? (A decimal approximation is OK.)

(c) If you wanted to have 5 g of carbon-14 10000 years from now, how much should you start with now? (A decimal approximation is OK.)

Problems from Section 1.4

36. Consider the ODE $y' = t - 2y$.

    (a) Find four points (t, y) where $y' = 0$.

    (b) Find y' at the points $(2, 3)$, $(0, 3)$, $(0, -2)$, $(2, 2)$, $(-2, 0)$, $(-1, 0)$, and $(-2, 1)$.

    (c) Sketch the mini-tangent lines corresponding to the points used in parts (a) and (b). (There should be one picture with all the mini-tangents on it.)

37. Consider the ODE $y' = \frac{1}{35}\left(y^4 + y^3 - 20y^2\right)$.

    (a) Use *Mathematica* to draw a picture of the slope field associated to this ODE, in the viewing window $[-8, 8] \times [-8, 8]$. Attach a printout of this picture; please label the picture "37".

    (b) Find the equation of three explicit solutions to this ODE.

    (c) Let $g(t)$ be the solution to this ODE passing through $(-2, -3)$.

       i. Find $\lim\limits_{t\to\infty} g(t)$.

       ii. Find $\lim\limits_{t\to-\infty} g(t)$.

    (d) Sketch (by hand) the graph of the solution to this ODE satisfying $y(2) = 4$ on the picture you printed in part (a).

38. Consider the ODE $y' = \frac{1}{10}\left(y^3 - ty^2 + 2y^2 - 9y + 9t - 18\right)$, and let $h(t)$ be the solution to this ODE satisfying $h(0) = 1$.

    (a) Use *Mathematica* to draw a picture of the slope field associated to this ODE, in the viewing window $[-6, 10] \times [-8, 8]$, with the stream line for h shown. Attach a printout of this picture; please label the picture "38".

    (b) Estimate $h(3)$ and $h(5)$.

    (c) Estimate $h'(2)$.

    (d) Estimate all $t > 0$ (if any) for which $h(t) = 2$.

    (e) Estimate all $t > 0$ (if any) for which $h(t) = 6$.

    (f) Find $\lim\limits_{t\to\infty} h(t)$.

    (g) Find $\lim\limits_{t\to-\infty} h(t)$.

39. Consider the initial value problem

$$\begin{cases} y' = y - \arctan t \\ y(0) = y_0 \end{cases}$$

where y_0 is a constant.

(a) Use *Mathematica* to draw a picture of the slope field associated to this ODE, in the viewing window $[0,10] \times [-5,5]$, with several stream lines shown. Attach a printout of this picture; please label the picture "39".

(b) Suppose you knew that y_0 was between 1.5 and 2.5. Is this information sufficient to describe the qualitative behavior of y for large t (i.e. to find $\lim_{t\to\infty} y(t)$)? Explain.

(c) Suppose you knew that y_0 was between 0 and 1. Is this information sufficient to describe the qualitative behavior of y for large t (i.e. to find $\lim_{t\to\infty} y(t)$)? Explain.

Problems from Section 1.5

40. Consider the initial value problem

$$\begin{cases} y' = 5 + 2t - 3y \\ y(0) = 5 \end{cases}.$$

Perform Euler's method by hand (show your work) to compute (t_1, y_1), (t_2, y_2), (t_3, y_3) and (t_4, y_4) for $\Delta t = 1$.

41. Consider the initial value problem

$$\begin{cases} y' = \frac{1}{2} - t + 2y \\ y(0) = 2 \end{cases}.$$

Perform Euler's method by hand (show your work) to estimate $y(2)$, using four steps.

42. Consider the initial value problem

$$\begin{cases} y' = 5 - 3\sqrt{y} \\ y(0) = 2 \end{cases}.$$

(a) Use *Mathematica* to implement Euler's method to estimate $y(3)$, using $\Delta t = .01$.

(b) Attach a printout of the graph of the points obtained from Euler's method in part (a). Label the picture "42".

43. Consider the initial value problem

$$\begin{cases} y' = \frac{4-ty}{1+y^2} \\ y(0) = 3 \end{cases}.$$

(a) Use *Mathematica* to implement Euler's method to estimate $y(60)$ using 300 steps.

(b) Attach a printout of the graph of the points obtained from Euler's method in part (a). Please label the picture "43 (b)".

(c) Based on the picture you get in part (b), do you trust your answer from part (a)? Why or why not?

(d) Use *Mathematica* to implement Euler's method to estimate $y(60)$ using 3000 steps.

(e) Attach a printout of the graph of the points obtained from Euler's method in part (d). Please label the picture "43 (e)".

(f) Based on the picture you get in part (e), do you trust your answer from part (d)? Why or why not?

(g) **(Optional; extra credit)** Explain thoroughly why you got the picture you did in part (b). (Saying "I didn't use enough steps" is insufficient; you need to describe the particular properties of the vector field of this ODE that lead to the specific picture you get in part (b).)

Problems from Section 1.6

44. Consider the initial value problem

$$\begin{cases} y' = 1 - y^3 \\ y(0) = 0 \end{cases} .$$

(a) Use Picard's method of successive approximations, with $f_0(t) = 0$, to find $f_1(t)$, $f_2(t)$, and $f_3(t)$.

(b) Graph f_3 (for $t \geq 0$ only) as obtained in part (a). If you use a *Mathematica* graph, label it as "44 (b)".

(c) Plot the vector field associated to this ODE (viewing window $[0, 4] \times [0, 4]$) with the stream line passing through the given initial value; label the picture "44 (c)".

(d) Based on your pictures, would you say that the function f_3 obtained in part (a) is, or is not, a good approximation of the solution $y = f(t)$ of the IVP?

(e) Would it be reasonable to estimate $y(4)$ by computing $f_3(4)$? Why or why not?

45. Consider the initial value problem

$$\begin{cases} y' = -y - 1 \\ y(0) = 0 \end{cases} .$$

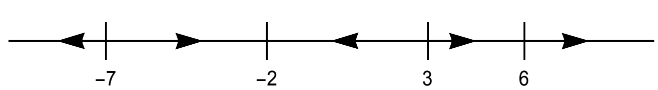
(a) Use Picard's method of successive approximations, with $f_0(t) = 0$, to find $f_1(t)$, $f_2(t)$, $f_3(t)$ and $f_4(t)$.

(b) Based on your answers to part (a), write $f(t) = \lim_{j \to \infty} f_j(t)$ as an infinite series.

(c) Recognize the infinite series from part (b) as being related to a "common" one, and find a formula for $f(t)$.
Hint: the series you get will probably have initial index 1, not 0. Take this into account.

Problems from Section 1.7

In problems 46-50, assume that this is the phase line of some autonomous ODE $y' = \phi(y)$:



46. (a) Write the equation of one explicit solution of this ODE.
    (b) Find all sinks of this ODE (if there aren't any, say so).
    (c) Find all unstable equilibria of this ODE (if there aren't any, say so).
    (d) Find all semistable equilibria of this ODE (if there aren't any, say so).

47. (a) Suppose $y(0) = 0$. Find $\lim\limits_{t\to\infty} y(t)$.
    (b) Suppose $y(0) = -10$. Find $\lim\limits_{t\to\infty} y(t)$.
    (c) Suppose $y(2) = 5$. Find $\lim\limits_{t\to-\infty} y(t)$.
    (d) Suppose $y(-1) = 6$. Find $\lim\limits_{t\to-\infty} y(t)$.

48. (a) Suppose $y(0) = 4$. Is $y(t)$ an increasing function or decreasing function?
    (b) Suppose $y(0) = y_0$. For what values of y_0 is $\lim\limits_{t\to\infty} y(t) = -2$?
    (c) Suppose $y(0) = y_0$. For what values of y_0 is $\lim\limits_{t\to\infty} y(t) > 0$?

49. (a) Sketch a (possible) graph of the function ϕ.
    (b) Sketch a (possible) slope field for this ODE.
    (c) Sketch a graph of the particular solution of this ODE whose initial condition is $y(1) = 0$.
    (d) **(Optional; extra credit)** Write down a formula for ϕ which would produce the phase line shown before Problem 46.

50. For each quantity, determine whether the quantity is positive, negative or zero:

    Hint: Your answer to Problem 49 (a) may be helpful.

    (a) $\phi(-7)$ (c) $\phi(4)$ (e) $\phi'(3)$
    (b) $\phi(0)$ (d) $\phi'(-2)$ (f) $\phi'(6)$

In each part of Problems 51-54, you are given an autonomous equation. For each equation:
(i) find all equilibria of the equation;
(ii) classify each equilibrium as stable, unstable or semistable;
(iii) sketch the phase line of the equation.

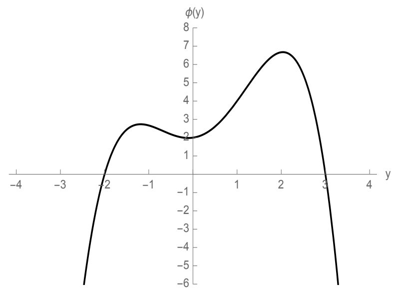
51. (a) $y' = y^2 - 9$ (b) $\frac{dy}{dt} = 2\cos y$

52. (a) $y' = e^y - 1$ (b) $y' = e^{-2y} - 1$

53. (a) $\frac{dy}{dt} = 2y\ln(6/y)$ (b) $y' = y^3(y-2)^2(y+3)(y-5)$

    Hint: In part (b) of Problem 53, graph the function, and estimate the values of its derivative at the equilibria by looking at the graph.

54. $y' = \phi(y)$, where ϕ is a function whose graph is as follows:



55. Suppose that the population y of a certain species of fish in a given area of the ocean (i.e. in a "fishery") would be described by a logistic equation, if there was no fishing. Since fish are delicious, we want to catch some of these fish so we can eat them. But if we catch too many fish, the population may be driven to extinction. That would be bad. This question deals with a model for managing the fishery (which is actually used in the real world, by the way).

Let E be a constant which denotes the level of **effort** being put into fishing (the greater E is, the harder people work at catching fish). It is reasonable to assume that the rate at which they catch fish depends on the fish population y (because the more fish there are, the easier to catch them). So a simple expression which can be used to describe the rate at which fish are caught is E times y. To model the fish population including this effect, you adapt the logistic equation by subtracting the rate at which fish are caught, leading to what is called the **Schaefer model**, used in biology and environmental science. Here is that model:

$$\frac{dy}{dt} = ry(L-y) - Ey.$$

(a) Find the two equilibria of this equation (in terms of the constants r, L and E).

(b) Suppose $E < Lr$. What will the fish population be in the long run? Explain.
Hint: This has something to do with whether or not the equilibria you found in part (a) are stable or unstable.

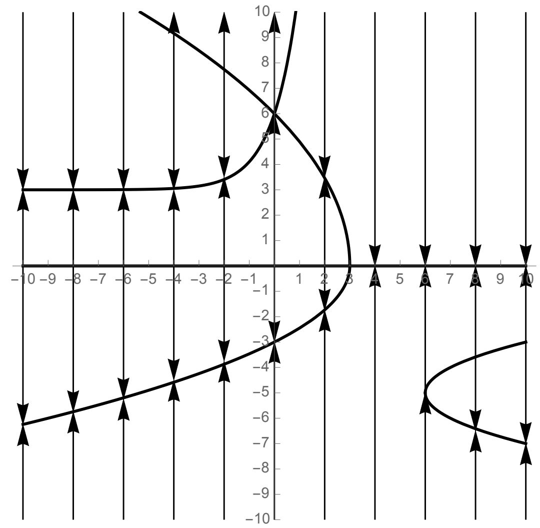
(c) Suppose $E > Lr$. What will the fish population be in the long run? Explain.

(d) A **sustainable yield** Y of the fishery is a quantity of fish that can be caught indefinitely. It is the product of the effort E and the stable equilibrium fish population corresponding to effort E. Find Y as a function of E (this is called the **yield-effort** curve).

(e) Determine the value of E which maximizes Y (and thereby produces the maximum sustainable yield).
Hint: Maximize the function $Y(E)$ using the method you learn in Calculus 1.

Problems from Section 1.8

In Questions 56 and 57, use the following bifurcation diagram for a parameterized family of ODEs $y' = \phi(y; r)$:



56. (a) Find all values of r (if any) at which the family has a saddle-node bifurcation.
    (b) Find all values of r (if any) at which the family has a transcritical bifurcation.
    (c) Find all values of r (if any) at which the family has a pitchfork bifurcation.
    (d) Sketch the phase line corresponding to the equation where $r = 2$.
    (e) Find all equilibria corresponding to the situation where $r = 6$, and classify them as stable, unstable or semistable.
    (f) Suppose $r = -2$ and $y(0) = 1$. Find $\lim_{t\to\infty} y(t)$.

57. (a) Suppose you know that r is somewhere between 5 and 7 and that $y(0) > 0$. Can you accurately predict the long-term behavior of $y(t)$? If so, what is this behavior? If not, why not?
    (b) Suppose you know that r is somewhere between -7 and -6 and that $y(0)$ is somewhere between 1 and 5. Can you accurately predict the long-term behavior of $y(t)$? If so, what is this behavior? If not, why not?
    (c) Suppose you know that r is somewhere between 5 and 7 and that $y(0) = -5$. Can you accurately predict the long-term behavior of $y(t)$? If so, what is this behavior? If not, why not?

In each part of Problems 58-61 you are given a parameterized family of ODEs (r is the parameter). For each parameterized family of ODEs:

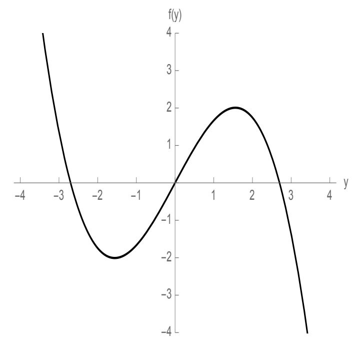
(i) find all value(s) of r at which bifurcations occur;
(ii) classify each bifurcation as saddle-node, pitchfork, transcritical or degenerate;
(iii) sketch a bifurcation diagram for the family.

58. (a) $y' = y^2 + 6y + r$ (b) $y' = y^2 - r^2$

59. (a) $y' = y + \frac{ry}{1+y^2}$ (b) $y' = (y^2 - r)(y^2 - 16)$

60. **(Optional; extra credit)** $y' = ry + y^3 - y^5$

61. $y' = f(y) + r$, where f has the graph given below:



Hint: As r changes, how does the graph of $f(y) + r$ change? What changes in this graph "cause" a bifurcation?

Problems from Section 2.1

62. Find the general solution of each ODE:

(a) $y' - 2y = 0$ (b) $7y' = y$

63. Find the general solution of each ODE:

(a) $y' = y\sqrt{t}$ (b) $\frac{dy}{dt} = y \cos t$

64. Find the particular solution of each IVP:

(a) $\begin{cases} y' \ln t = \frac{y}{t} \\ y(e) = 3 \end{cases}$ (b) $\begin{cases} y'e^t + y' = ye^t \\ y(0) = 4 \end{cases}$

65. Suppose you know that $y = 6e^{4t} \sin 2t$ is a solution to some homogeneous, linear, first-order ODE.

(a) Is $y = 9e^{4t} \sin 2t$ also a solution of this ODE? Why or why not?
(b) Is $y = 6e^{3t} \sin 2t$ also a solution of this ODE? Why or why not?

Problems from Section 2.2

66. Solve each given differential equation or initial value problem, using the method of integrating factors:

    (a) $\begin{cases} y' - 2y = 3e^t \\ y(0) = 4 \end{cases}$

    (b) $\frac{dy}{dt} = te^{-t} + 1 - y$

67. Solve each given differential equation or initial value problem, using the method of integrating factors:

    (a) $ty' - y = t^2 e^{-t}$

    (b) $\begin{cases} t^3 y' + 4t^2 y = e^{-t} \\ y(-1) = 0 \end{cases}$

68. Solve each given differential equation or initial value problem:

    (a) $\begin{cases} y' = \frac{\cos t - 2ty}{t^2} \\ y(\pi) = 0 \end{cases}$

    (b) $t\frac{dy}{dt} + 2y = \sin t$

Problems from Section 2.3

69. Suppose you have a first-order linear ODE, where $y = 3e^{2t}\sin 2t$ is a solution of the equation and $y = 2e^{4t}\sin 2t$ is a solution of the corresponding homogeneous equation. Write the general solution of the ODE.

70. Solve each given differential equation or initial value problem, using the method of undetermined coefficients:

    (a) $y' + 3y = e^{-2t}$

    (b) $\begin{cases} y' - 4y = 2\cos 2t \\ y(0) = -3 \end{cases}$

71. Solve each given differential equation or initial value problem, using the method of undetermined coefficients:

    (a) $\begin{cases} y' + y = te^{2t} \\ y(0) = 24 \end{cases}$

    (b) $y' - 4y = 2t^2 + 8t$

72. Consider the ODE $y' + 7y = 20e^{-7t}$.

    (a) Find a nonzero solution of the corresponding homogeneous equation.
    (b) Try the method of undetermined coefficients with guess $y_p = Ae^{-7t}$. Does this work? Explain.
    (c) Try the method of undetermined coefficients with guess $y_p = Ate^{-7t}$. Does this work? Explain.
    (d) Solve the ODE $y' - 3y = 9e^{3t}$, by first solving the corresponding homogeneous equation and then using the method of undetermined coefficients. To find the appropriate formula to guess for y_p, use the previous example in this question as a guide.

(e) For each given ODE:
 (i) find the solution y_h of the corresponding homogeneous equation; and
 (ii) write down what you would need to guess for y_p to utilize the method of undetermined coefficients.

You do not need to solve the equation.

i. $y' + 6y = e^{4t}$
ii. $y' + 6y = e^{-6t}$
iii. $y' + 6y = e^{2t}$
iv. $y' - 2y = e^{2t}$

(f) Write down a general rule which tells you when your "normal" guess won't work (when attempting the method of undetermined coefficients), and how to adapt your guess so that it will work.

You are responsible for implementing this rule on Exam 1.

Problems from Section 2.4

73. Find the general solution of each ODE:

 (a) $y' = \sqrt{y(t+1)}$
 (b) $\frac{dy}{dt} = \sec y \sin t$

74. Find the general solution of each ODE:

 (a) $y' = \frac{t^2}{y(4+t^3)}$
 (b) $ty' = 1 - y^2$

75. Find the general solution of each ODE; write your answer as a function $y = f(t)$.

 (a) $e^y y' = 4$
 (b) $\frac{dy}{dt} = (y \sec t)^2$

76. Find the particular solution of each initial value problem; write your answer as a function $y = f(t)$:

 (a) $\begin{cases} \frac{dy}{dt} = \frac{1-2t}{y} \\ y(1) = -2 \end{cases}$
 (b) $\begin{cases} yy' = 4t \\ y(1) = -3 \end{cases}$

77. (a) Solve the following initial value problem: $\begin{cases} 2(y-1)y' = e^t \\ y(0) = 2 \end{cases}$

 (b) **(Optional; extra credit)** Write your answer to part (a) as a function $y = f(t)$, simplified as much as possible.

78. Let $y = f(t)$ be the solution of the initial value problem $\begin{cases} y' = ty^2 - 2y^2 \\ y(0) = 1 \end{cases}$. Find $f(4)$.

Problems 79-82 introduce some methods to solve second-order ODEs. A second-order ODE always has the form

$$\phi(t, y, y', y'') = 0 \quad \text{a.k.a. } \phi\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right) = 0.$$

I haven't said anything in class about how to solve second-order ODEs, but in two special cases you can use a trick, together with our techniques for solving first-order ODEs, to solve a second-order equation. **You should be able to solve these types of second-order equations on Exam 1.**

- *Case 1:* The equation has no y in it, i.e. the equation is of the form
$$\phi(t, y', y'') = 0.$$
To solve this, let $v(t) = y'$. Then $y'' = v'(t)$ so by substituting v for y' and v' for y'', the equation can be rewritten as $\phi(t, v, v') = 0$. This rewritten equation is first-order! Solve it for v, then integrate v to get the solution $y(t) = \int v(t)\, dt$ of the original second-order equation.

- *Case 2:* The equation has no t in it, i.e. the equation is of the form
$$\phi(y, y', y'') = 0 \quad \text{a.k.a. } \phi\left(y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right) = 0.$$
To solve this, pretend y is an independent variable and let $v = v(y) = \frac{dy}{dt}$. Then
$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}(v(y)) = \frac{dv}{dy} \cdot \frac{dy}{dt} = \frac{dv}{dy}v.$$
This means you can replace $\frac{dy}{dt}$ with v and $\frac{d^2y}{dt^2}$ with $\frac{dv}{dy}v$ in the original equation to get
$$\phi\left(y, v, \frac{dv}{dy}v\right) = 0.$$
This is a first-order equation in v, which can be solved by usual methods to find $v = v(y)$. Then solve the equation $\frac{dy}{dt} = v(y)$ (this equation is usually separable) to find y in terms of t.

In Problems 79-82, solve the given second-order ODE or IVP. Write your final answer in the form $y = f(t)$.

79. (a) $t\frac{d^2y}{dt^2} = 2\frac{dy}{dt} + 2$ (b) $ty''e^{y'} = e^{y'} - 1$

80. (a) $\begin{cases} \frac{d^2y}{dt^2} + 2\left(\frac{dy}{dt}\right)^2 \tan y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$ (b) $y'' = -2t(y')^2$

81. (a) $y'' = -y$ (b) $(y^2 + 1)y'' = 2y(y')^2$

Hint: In part (a), you will need the following integration rule, which you can use without proof:
$$\int \frac{1}{\sqrt{C - y^2}}\, dy = \arcsin\left(\frac{y}{\sqrt{C}}\right) + D.$$

82. (a) $y'' - 2y' = 12\sin t$ (b) $y'' + 4y' = 10e^{3t}$

Hint: Combine this new technique with the method of undetermined coefficients.

Problems from Section 2.5

83. Determine whether or not each of the given differential equations is exact. If the equation is exact, find its solution.
   (a) $(2t+3) + (2y-2)\frac{dy}{dt} = 0$
   (b) $(2t+4y) + (2t-2y)y' = 0$
   (c) $(e^t \cos y + 2\cos t)\frac{dy}{dt} + e^t \sin y - 2y \sin t = 0$
   (d) $y' = \frac{e^t \sin y + 3y}{3t - e^t \sin y}$

84. Find the solution of each initial value problem:
   (a) $\begin{cases} 2t - y + 2yy' - ty' = 0 \\ y(1) = 3 \end{cases}$
   (b) $\begin{cases} \frac{y}{t} + 6t + (\ln t)y' = 0 \\ y(1) = 7 \end{cases}$
   Note: this IVP could also be written as $\frac{y}{t} + 6t + (\ln t)y' = 0$, $y(1) = 7$.

85. Consider the differential equation $y + (2t - e^y)y' = 0$.
   (a) Show this equation is not exact.
   (b) Show that after you multiply through the entire equation by y, the equation becomes exact.
   (c) Solve the equation.

86. **(Optional; extra credit)** Find the general solution of the differential equation

$$(t+2)\sin y + t\cos y \frac{dy}{dt} = 0.$$

Hint: You have to figure out something to multiply the equation through by to make the equation become exact.

Problems from Section 2.6

In Problems 87-92, use the following setup: A nitric acid solution flows at a constant rate of 6 L/min into a large tank that initially holds 200 L of a 0.5% nitric acid solution. The solution inside the tank is kept well stirred, and flows out of the tank at a rate of 8 L/min. The solution entering the tank is 20% nitric acid. Let $y(t)$ be the amount (i.e. volume) of nitric acid in the tank at time t (where t is in minutes).

87. (a) Determine the volume of solution in the tank as a function of t.
   Hint: The tank starts with 200 L of solution; 6 L of solution flows in per minute and 8 L of solution flows out per minute.

(b) Draw a compartmental diagram (with boxes and arrows) representing this situation (the compartment should be $y(t)$).

WARNING: The "rate out" is tricky. To get this, you have to multiply the rate at which fluid flows out of the tank by the concentration of fluid in the tank; the concentration of fluid in the tank has to take into account the fact that the volume of fluid in the tank is not constant.

(c) Write an initial value problem which models this setup.

(d) Using the initial value problem you wrote in part (c), use Euler's method with $\Delta t = 1$ to estimate the amount of fluid in the tank at time 60.

(e) Using the initial value problem you wrote in part (c), use Euler's method with $\Delta t = .001$ to estimate the amount of fluid in the tank at time 60.

88. (a) Use *Mathematica* to sketch the slope field for this differential equation, with the stream line corresponding to $y(t)$ included. Use the viewing window $[-5, 100] \times [-5, 50]$. Attach a printout of your slope field, labelled as "88 (a)".

(b) Based on your slope field, estimate the amount of nitric acid in the tank at time 20.

(c) Based on your slope field, estimate all times t where there is exactly 10 L of nitric acid in the tank.

(d) Based on your slope field, estimate the maximum amount of nitric acid that is in the tank at any one instant.

(e) Based on your slope field, estimate the time when the amount of nitric acid in the tank is maximized.

89. Solve the IVP of Problem 87 (c), writing your answer in the form $y = f(t)$.

Hint: Use integrating factors.

90. (a) Have *Mathematica* sketch the graph of the solution you obtain in Problem 89 (where $0 \leq t \leq 100$). Attach a printout of this graph, labelled as "90 (a)". (Check that the graph you get is consistent with the picture you obtained in Problem 88 (a); if it isn't, something is wrong.)

(b) Use your answer to Problem 89 to find the exact value of $y(60)$.

(c) Find a decimal value of the value of $y(60)$, and compare it to what you got in parts (d) and (e) of Problem 88. Were those estimates accurate?

(d) **(Optional; extra credit)** Find the exact answers (not decimal approximations) to the questions asked in parts (d) and (e) of Problem 88.

91. (a) Find a formula for the <u>concentration</u> of nitric acid at time t.

Hint: This is tricky, because you have to divide the amount of nitric acid in the tank by the volume, and unlike the example in class, the volume of solution in the tank is not a constant.

(b) Have *Mathematica* sketch the graph of the function you obtain in part (a) (where $0 \leq t \leq 100$). Attach a printout of this graph, labelled as "91 (b)".

(c) From the graph produced in part (b), estimate the first time t when the solution in the tank is 10% nitric acid.

(d) **(Optional; extra credit)** Find the exact value of the first time at which the solution in the tank is 10% nitric acid.

92. Suppose you wanted to know how much nitric acid was in the tank at time 150. I claim that this is not computable by figuring $y(150)$. Why not? What is the amount of nitric acid in the tank at time 150? What is the concentration of nitric acid in the tank at time 150? Explain your answers.

In Problems 93-96, use the following setup: A parachutist has mass 75 kg. She jumps out of a helicopter (assume her initial velocity is 0) which is 2000 m above the ground, and falls toward the ground under the influence of two forces: gravity (which is 9.8 m/sec^2) and air resistance (drag coefficient 30 N sec/m).

93. (a) Draw a free body diagram representing this situation.
    (b) Write an initial value problem which models this situation. Set the problem up so that $v > 0$ corresponds to downward motion.
    (c) Sketch the phase line for the equation you wrote in part (b).
    (d) What is the terminal velocity of the parachutist (assuming she never pulls her rip cord)?
    (e) Use *Mathematica* to sketch the slope field for this differential equation, with the stream line corresponding to $v(t)$ included (use the viewing window $[-5, 50] \times [-5, 50]$). Attach a printout of your slope field, labelled as "93 (d)".

94. (a) Solve the IVP you wrote down in Question 93 (b).
    (b) Use *Mathematica* to sketch the graph of $v(t)$ (use the viewing window $[-5, 50] \times [-5, 50]$). Attach a printout of this graph, labelled as "94 (b)". (Check that this graph is consistent with the slope field obtained in Problem 93 (e); if it isn't, something is wrong.)

95. (a) Find a formula which gives the parachutist's <u>height</u> above the ground at time t. *Hint:* In Problem 94 (a), you found the parachutist's <u>velocity</u> at time t. You know what the parachutist's height at time 0 is (that is given in the setup). How do you get from an object's velocity back to its position (the parachutists' height is like her position)?

    WARNING: Keep in mind that $v > 0$ corresponds to the parachutist falling, given how we have set this problem up.
    (b) Her parachute is set to open automatically when her velocity reaches 20 m/sec. How many seconds after she jumps out of the helicopter will her parachute open? (A decimal answer is OK; it might be useful to have *Mathematica* solve an equation here.)
    (c) How high above the ground will she be when her parachute opens? (Again, a decimal answer is OK.)

96. **(Optional; extra credit)**

    (a) After the parachutist's chute opens, assume that the forces affecting her velocity are gravity (still 9.8 m/sec^2) and air resistance (but now, the drag coefficient is 90 N sec/m rather than 30 because the chute provides greater air resistance).

    How many seconds after her parachute opens will she hit the ground? (A decimal answer is OK.)

    Hint: To solve this question, you need to write an appropriate initial value problem, and then solve it.

    (b) What will her velocity be when she hits the ground? (A decimal answer is OK.)

    (c) Graph the parachutist's velocity against elapsed time, starting at $t = 0$ (when she jumps out of the helicopter) and ending when she hits the ground.

97. A bottle of champagne is originally at room temperature ($70°$). It is chilled in ice ($32°$). Suppose that it takes 15 minutes for the champagne to chill to $60°$. How long (including the original 15 minutes) will it take for the wine to reach $46°$? (A decimal answer is OK, but you need to actually solve the equation... don't rely on pictures or Euler's method.)

    Hint: This is a heating and cooling problem with $U(t) = H(t) = 0$, so this is closely related to Example 1 on page 92 of the Fall 2017 lecture notes.

98. Choose one of problems (a), (b):

    (a) An RC electrical circuit (see page 96 of the lecture notes) with a $1\,\Omega$ resistor and a 10^{-4} F capacitor is driven by a voltage $E_S(t) = \sin 2t$ V. If the initial capacitor voltage is zero, find:

    i. the voltage across the capacitor at time t;
    ii. the voltage across the resistor at time t (to get this, apply Kirchoff's voltage law to the answer to (i); and
    iii. the current at time t (to get this, apply Ohm's law to the answer to part (ii).

    (b) An RL electrical circuit (see page 97 of the lecture notes) with a $1\,\Omega$ resistor and a $.01$ H inductor is driven by a generator whose voltage at time t is $E_S(t) = \sin 2t$ V. If the initial current across the inductor is 0, find:

    i. the current at time t;
    ii. the voltage across the resistor at time t (to get this, apply Ohm's Law to the answer to (i); and
    iii. the voltage across the inductor at time t.

Problems from Section 3.1

99. In each part, you are given a system of first-order ODEs and a possible solution of that system. Determine whether or not the possible solution is actually a solution of the system.

    (a) $\begin{cases} x'(t) = 3x - 2y \\ y'(t) = 2x - 2y \end{cases}$ POSSIBLE SOL'N: $\begin{cases} x(t) = 8e^{2t} + e^{-t} \\ y(t) = 4e^{2t} + 2e^{-t} \end{cases}$

(b) $\begin{cases} x'(t) = x + 3y \\ y'(t) = -4x - y \end{cases}$ POSSIBLE SOL'N: $\begin{cases} x(t) = 2\cos 11t + 2\sin 11t \\ y(t) = -8\sin 11t \end{cases}$

100. In each part, you are given an IVP, together with a possible solution. Determine whether or not the possible solution is actually a solution of the IVP.

(a) IVP: $\begin{cases} \begin{cases} x'(t) = 2y \\ y'(t) = x - y \end{cases} \\ \mathbf{y}(0) = (2, -3) \end{cases}$ POSSIBLE SOL'N: $\begin{cases} x(t) = \frac{2}{3}e^{-2t} - \frac{2}{3}e^{t} \\ y(t) = \frac{-2}{3}e^{-2t} - \frac{1}{3}e^{t} \end{cases}$

(b) IVP: $\begin{cases} \begin{cases} x'(t) = 2y \\ y'(t) = x - y \end{cases} \\ \mathbf{y}(0) = (2, -3) \end{cases}$ POSSIBLE SOL'N: $\begin{cases} x(t) = \frac{8}{3}e^{-2t} - \frac{2}{3}e^{t} \\ y(t) = \frac{-8}{3}e^{-2t} - \frac{1}{3}e^{t} \end{cases}$

101. In each part, you are given a set of parametric equations $\mathbf{y}(t)$. For each set of parametric equations:

(i) Write the set of parametric equations out, coordinate by coordinate (see the answers if you don't understand what this means).
(ii) Find the values of $\mathbf{y}(0)$, $\mathbf{y}(1)$ and $\mathbf{y}(2)$.

(a) $\mathbf{y}(t) = (t^2 - 2, t + 3)$ (b) $\mathbf{y}(t) = (2t, 5t, -7t)$

In each part of Problems 102-104, you are given a set of two parametric equations of the form $\mathbf{y}(t) = (x(t), y(t))$. For each set of parametric equations:

(i) Use the following command in *Mathematica* to sketch the graph of these parametric equations in the xy-plane:

```
ParametricPlot[{formula for x(t), formula for y(t)},
 {t, -100, 100}, PlotRange -> {{xmin, xmax}, {ymin, ymax}}]
```

This command plots the graph of the parametric equations in the viewing window $[xmin, xmax] \times [ymin, ymax]$. You are responsible for choosing an appropriate viewing window for each set of equations (I'd start with $[-10, 10] \times [-10, 10]$ and then zoom in or out as necessary).

(ii) Print the graph you get in *Mathematica* (labelling it with the problem number/letter).
(iii) On the graph you print, draw (by hand) an arrow on the curve indicating the direction of motion.
(iv) Indicate (by hand) the point on the graph corresponding to $t = 0$ by drawing a thick point and labelling that point "$t = 0$".

102. (a) $\mathbf{y}(t) = (\cos t, \sin t)$ (b) $\mathbf{y}(t) = (e^t, 3e^{-t})$

Hint: In *Mathematica*, e is `E`, not `e`.

103. (a) $\mathbf{y}(t) = (e^t \cos 2t, e^t \sin 2t)$ (b) $\mathbf{y}(t) = (-2\sin 3t, \cos 3t)$

104. (a) $\mathbf{y}(t) = (3e^t, e^{2t})$ (b) $\mathbf{y}(t) = \left(\frac{4e^t+e^{-t}}{10}, \frac{2e^t-5e^{-t}}{10}\right)$

Problems from Section 3.2

105. Consider the initial value problem

$$\begin{cases} \begin{cases} x' = x - y \\ y' = 2x + y \end{cases} \\ \mathbf{y}(0) = (2,1) \end{cases}.$$

Perform Euler's method by hand (show your work), to compute the points $(t_1, \mathbf{y}_1)$, $(t_2, \mathbf{y}_2)$ and $(t_3, \mathbf{y}_3)$ when $\Delta t = 1$.

106. Consider the initial value problem

$$\begin{cases} \begin{cases} x' = -y \\ y' = x \end{cases} \\ \mathbf{y}(0) = (3,0) \end{cases}.$$

Perform Euler's method by hand (show your work) to estimate $\mathbf{y}(2)$, using four steps.

107. Consider the initial value problem

$$\begin{cases} \begin{cases} x' = x + 2y \\ y' = -3x + y \end{cases} \\ \mathbf{y}(0) = (-2,1) \end{cases}.$$

Use *Mathematica* to implement Euler's method to estimate $\mathbf{y}(4)$, using 1000 steps.

108. Consider the initial value problem

$$\begin{cases} \begin{cases} x' = -4x - 8y \\ y' = 8x + 4y \end{cases} \\ \mathbf{y}(0) = (1,1) \end{cases}.$$

(a) Use *Mathematica* to implement Euler's method to estimate $\mathbf{y}(30)$, using 10000 steps.

(b) Attach a printout of the graph of the points obtained from Euler's method in part (a) (use the viewing window $[-10, 10] \times [-10, 10]$). Label the graph "108 (b)".

(c) Based on the picture you get in part (b), describe (in your own words) the qualitative behavior of the solutions as t increases.

109. Consider the initial value problem

$$\begin{cases} \begin{cases} x' = -3x - 6y \\ y' = 6x - 3y \end{cases} \\ \mathbf{y}(0) = (-8,7) \end{cases}.$$

(a) Use *Mathematica* to implement Euler's method to estimate $\mathbf{y}(50)$, using 1000 steps.

(b) Attach a printout of the graph of the points obtained from Euler's method in part (a) (use the viewing window $[-10, 10] \times [-10, 10]$). Label the graph "109 (b)".

(c) Based on the picture you get in part (b), describe (in your own words) the qualitative behavior of the solutions as t increases.

110. Consider the initial value problem

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} x' = -y - z \\ y' = x \\ z' = x - z \end{array} \right. \\ \mathbf{y}(0) = (1, 2, -1) \end{array} \right. .$$

(a) Use *Mathematica* to implement Euler's method to estimate $\mathbf{y}(5)$, using 5 steps.

(b) Use *Mathematica* to implement Euler's method to estimate $\mathbf{y}(5)$, using 100 steps.

(c) Based on what you get in part (b), would you say your answer to (a) is accurate? Why or why not?

(d) Use *Mathematica* to implement Euler's method to estimate $\mathbf{y}(5)$, using 1000 steps.

(e) Based on what you get in part (d), would you say your answer to (b) is accurate? Why or why not?

(f) Use *Mathematica* to implement Euler's method to estimate $\mathbf{y}(5)$, using 10000 steps.

(g) Use *Mathematica* to implement Euler's method to estimate $\mathbf{y}(5)$, using 100000 steps. (This took my computer 34 seconds to run.)

(h) Suppose you needed an approximation of $\mathbf{y}(5)$ correct to two decimal places. Based on what you have done in this problem, is it sufficient to use Euler's method with 10000 steps? Explain.

(i) If you needed an approximation of $\mathbf{y}(5)$ correct to three decimal places. Based on what you have done in this problem, is it sufficient to use Euler's method with 10000 steps? Explain.

Problems from Section 3.4

In Problems 111-117, use the following matrices and vectors to compute the indicated quantities. If the quantity does not exist, just write "does not exist".

$$A = \begin{pmatrix} 5 & -2 \\ 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} -3 & -1 \\ 0 & 4 \end{pmatrix} \qquad C = \begin{pmatrix} 4 & 2 \\ -8 & -4 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$G = \begin{pmatrix} 1 & -3 & 0 \\ 2 & 2 & -5 \\ 4 & 1 & -2 \end{pmatrix} \qquad H = \begin{pmatrix} 4 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$K = \begin{pmatrix} \cos 2t & \sin 3t \\ 4 \sin 2t & -\sin 3t \end{pmatrix} \qquad \mathbf{x} = \mathbf{x}(t) = (e^t \cos t, e^t \sin t) \qquad \mathbf{z} = \mathbf{z}(t) = (\sin t, 2\cos 3t, 4e^{7t})$$

111. (a) the trace of C
 (b) the trace of G
 (c) $3A$
 (d) $2A - B + C$
 (e) $A - 2I$ (*Hint:* I is not given above. You should know what I is.)
 (f) $C - \lambda I$ (your answer should be in terms of λ, the Greek letter "lambda".)

112. (a) K' (c) AB (e) $A\mathbf{x}$
 (b) $\mathbf{z}'(t)$ (d) G^2 (f) $BC\mathbf{x}$

113. (a) AK (b) D^8 (c) J^{20} (d) H^3

114. (a) H^n (your answer should be in terms of n).
 (b) $(Dt)^n$ (your answer should be in terms of n and t).

115. (a) A^{-1} (b) C^{-1} (c) $AB^{-1}A$

116. (a) $\det A$ (b) $\det D$ (c) $\det G$

117. (a) $\det(A - \lambda I)$ (b) $\det(B - \lambda I)$

Hint: Your answers should be in terms of λ.

118. (a) Is the matrix A given before Problem 111 a diagonal matrix?
 (b) Is the matrix H given before Problem 111 a diagonal matrix?

Problems from Section 3.7

119. For each given set of functions, compute the Wronskian. Then, determine whether or not the functions are linearly independent.
 (a) $f_1(t) = t^2 + 5t$, $f_2(t) = t^2 - 5t$
 (b) $f_1(t) = e^{3t}$, $f_2(t) = e^{3(t-1)}$
 (c) $f_1(t) = 1$, $f_2(t) = t$, $f_3(t) = t^2$
 (d) $f_1(t) = \sin x$, $f_2(x) = \cos x$, $f_3(x) = \sin 2x$, $f_4(x) = \cos 2x$
 Hint: To compute a 4×4 determinant, use *Mathematica* (there's also a command which computes the Wronskian of a set of functions directly).

120. Show that the Wronskian of the functions $\mathbf{f}_1(t) = (e^t, 2e^t - e^{-t}, 3e^t)$, $\mathbf{f}_2(t) = (4e^t, e^{-t}, 0)$ and $\mathbf{f}_3(t) = (-2e^t, 4e^t - 3e^{-t}, 6e^t)$ is zero. Then show these functions are linearly dependent by finding constants c_1, c_2 and c_3 (not all zero) such that $c_1\mathbf{f}_1 + c_2\mathbf{f}_2 + c_3\mathbf{f}_3 = \mathbf{0}$.

121. (a) Prove that if a and b are different constants, then e^{at} and e^{bt} are linearly independent.
 Hint: Compute the Wronskian of the two functions in terms of a and b.

(b) Prove that if a, b and c are three different constants, then $\{e^{at}, e^{bt}, e^{ct}\}$ is a linearly independent set.

Hint: Compute the Wronskian of the functions, and then use *Mathematica* to factor the Wronskian using the `Factor[ ]` command. This will explain why the Wronskian is not equal to zero.

Note: the facts proved in this problem generalize: if $a_1, ..., a_d$ are different constants, then $\{e^{a_1 t}, e^{a_2 t}, ..., e^{a_d t}\}$ is a linearly independent set. (You should remember this fact, so you don't have to re-verify it over and over.)

122. **(Optional; extra credit)** Prove that for any constants a and b (with $b \neq 0$), the functions $e^{at}\sin bt$ and $e^{at}\cos bt$ are linearly independent.

123. Suppose that $\mathbf{y}_p(t) = (2e^{5t}, -3e^{5t}, e^{5t})$ is a solution of some linear system $A_1\mathbf{y}' + A_0\mathbf{y} = \mathbf{q}$, and suppose that the solution of the corresponding homogeneous system $A_1\mathbf{y}' + A_0\mathbf{y} = \mathbf{0}$ is

$$\mathbf{y}_h(t) = C_1 e^{4t}\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + C_3 e^{t}\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}.$$

(a) How many equations comprise this system?

(b) Write the general solution of this system in vector form.

(c) Write the general solution coordinate-wise.

(d) Write the particular solution corresponding to $C_1 = 3$, $C_2 = -2$ and $C_3 = 2$.

(e) Find the value of $\mathbf{y}(3)$ for the solution found in part (d).

(f) Find the value of $x(2)$ for the solution found in part (d).

(g) Find the particular solution satisfying $\mathbf{y}(0) = (-1, 4, 2)$.

Problems from Section 3.8

124. Consider the system of ODEs

$$\begin{cases} x' = x + 4y \\ y' = x - 2y \end{cases}.$$

For each point (x, y) below, compute the value of $\frac{dy}{dx}$ at that point. Then sketch the mini-tangents corresponding to those points (there should be one picture, with all the vectors on it; each mini-tangent should have an arrowhead on it, drawn in the manner of the example on page 138 of the Fall 2017 lecture notes).

$$(1,0) \qquad (0,2) \qquad (0,-2) \qquad (-3,0)$$

$$(1,3) \qquad (-2,-3) \qquad (3,-1) \qquad (-3,2)$$

$$(-2,2) \qquad (-1,1) \qquad (1,-1) \qquad (2,-2)$$

125. For each given initial value problem, use *Mathematica* to sketch the slope field of the system and the graph of the solution of the IVP (the intent here is for you to use Command 3 from the file `phaseplanes.nb`). Label your pictures with the problem number/letter:

    (a) $\begin{cases} x' = 3y \\ y' = x \end{cases}$ $\begin{cases} x(0) = -2 \\ y(0) = 3 \end{cases}$

    Note: this system of IVPs could also be written $\mathbf{y}' = (3y, x)$; $\mathbf{y}(0) = (-2, 3)$.

    (b) $\mathbf{y}' = (x - xy, x + 2y^2 - x^2y)$; $\mathbf{y}(0) = (1, 0)$

126. Same directions as the previous problem:

    (a) $\mathbf{y}' = (y \sin(x + y), xe^{-x-y})$; $\mathbf{y}(0) = (1, -1)$

    (b) $\mathbf{y}' = (2xy, x - 2y + x^2y^3)$; $\mathbf{y}(2) = \left(-1, \frac{5}{4}\right)$

127. Find all equilibria of each autonomous system of ODEs:

    (a) $\mathbf{y}' = (y^2 - 3y - 4, 2x + 3y)$

    (b) $\mathbf{y}' = (y - x, y^2 + xy - 4)$

128. Find all equilibria of each autonomous system of ODEs:

    (a) $\begin{cases} x' = 4x^2 - y \\ y' = 1 - y^2 \end{cases}$

    (b) $\begin{cases} x' = (x - 2)(y + 3) \\ y' = (y - 1)(x + 4) \end{cases}$

129. Consider the system of two ODEs:

$$\begin{cases} x' = x - 3 \\ y' = 2 - 2y \end{cases}$$

    (a) Use *Mathematica* to draw the phase plane (not the slope field) for this system (use the viewing window $[-8, 8] \times [-8, 8]$ and ask *Mathematica* to draw at least 200 solution curves). Print this picture, labelling it as "129".

    (b) On your picture, sketch the graph of the solution to this system satisfying $\mathbf{y}(0) = (0, 4)$.

    (c) Either by looking at your picture or by doing some algebra, find all constant functions which solve the system.

    (d) Suppose $\mathbf{y}(0) = (0, -5)$. In this situation:

        i. Is $x(t)$ an increasing function, a decreasing function, or a constant function of t?

        ii. Is $y(t)$ an increasing function, a decreasing function, or a constant function of t?

        iii. What is $\lim_{t\to\infty} x(t)$?

        iv. What is $\lim_{t\to\infty} y(t)$?

    (e) Suppose $\mathbf{y}(1) = (5, 5)$. Answer the same questions (i)-(iv) as in part (d).

(f) Suppose $\mathbf{y}(0) = (3, -2)$. Answer the same questions (i)-(iv) as in part (d).

130. Consider the system of two ODEs:

$$\begin{cases} x' = 4y - xy - x + 4 \\ y' = 2x - xy - y + 2 \end{cases}$$

(a) Use *Mathematica* to draw the phase plane (not the slope field) for this system (use the viewing window $[-10, 10] \times [-8, 8]$ and ask *Mathematica* to draw at least 200 solution curves). Print this picture, labelling it as "130".

(b) Either by looking at your picture or by doing some (nontrivial) algebra, find all constant functions which solve the system.

(c) Suppose $\mathbf{y}(0) = (7, -7)$.

i. Which of these statements best describes the behavior of $x(t)$ in this situation?
   A. $x(t)$ increases for all t
   B. $x(t)$ decreases for all t
   C. initially, $x(t)$ is increasing, but then it becomes decreasing
   D. initially, $x(t)$ is decreasing, but then it becomes increasing

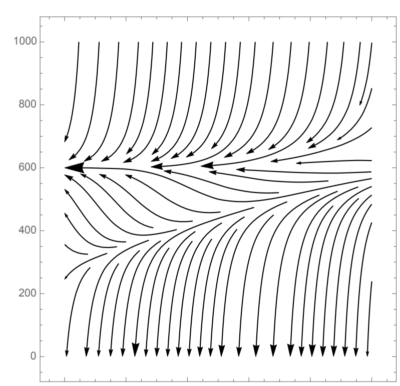
ii. Which of these statements best describes the behavior of $y(t)$ in this situation?
   A. $y(t)$ increases for all t
   B. $y(t)$ decreases for all t
   C. initially, $y(t)$ is increasing, but then it becomes decreasing
   D. initially, $y(t)$ is decreasing, but then it becomes increasing

(d) Suppose $\mathbf{y}(1) = (1, -6)$. Answer the same questions (i) and (ii) as in part (c).

(e) Suppose $\mathbf{y}(0) = (8, 9)$. Answer the same questions (i) and (ii) as in part (c).

(f) Let E be the set of points (x_0, y_0) such that $\lim_{t\to\infty} x(t) = 4$ if $\mathbf{y}(0) \in E$. On the picture you obtained in part (a), shade the set of points which belong to E.

131. A biologist is studying the population of two species, X and Y. He lets $x(t)$ and $y(t)$ represent the population of these species at time t, and based on his biology research, he comes up with a system of ODEs modeling this situation. He asks *Mathematica* to sketch a picture of the phase plane for this system, and *Mathematica* produces this:



(a) Suppose that the current population of species X is somewhere between 700 and 800, and that the current population of species Y is somewhere between 800 and 900. Based on this model, can the biologist say what will happen to the population of these species in the long run? If so, what will happen? If not, why can't he tell?

(b) Suppose that the current population of species X is somewhere between 400 and 500, and that the current population of species Y is somewhere between 200 and 250. Based on this model, can the biologist say what will happen to the population of these species in the long run? If so, what will happen? If not, why can't he tell?

(c) Suppose that the current population of species X is somewhere between 600 and 700, and that the current population of species Y is somewhere between 400 and 600. Based on this model, can the biologist say what will happen to the population of these species in the long run? If so, what will happen? If not, why can't he tell?

132. **(Optional; extra credit)** The observed growth of tumors can be explained by the following mathematical model. Let $N(t)$ be the number of cells in the tumor at time t (so the bigger N is, the worse the tumor is). Some of the cells in the tumor "proliferate" (i.e. they split to make more cancerous cells, growing the tumor); let $P(t)$ be the number of proliferating cells in the tumor at time t. The functions P and N are modeled by the following system of ODEs:

$$\begin{cases} P' = cP - r(N)P \\ N' = cP \end{cases}$$

where c is a positive constant and $r : \mathbb{R} \to \mathbb{R}$ is an increasing function which represents the rate at which proliferating cells become non-proliferating (c and r depend on things like the type of cancer, the patient's age, weight, height, sex, etc.). Suppose that a patient starts with a tumor consisting of one proliferating cancerous cell (so that the initial condition is $P(0) = 1, N(0) = 1$) and suppose that $c = 30$.

For each of the following functions $r(N)$, determine the long-term size of the tumor, i.e. the number of cancerous cells the patient will end up with in the long run.

(a) $r(N) = N^2$

(b) $r(N) = N$

(c) $r(N) = \sqrt{N} \ln N$

(d) $r(N) = \ln N$

Based on what you observe in parts (a)-(d), describe (in English) some property (or properties) of the function r and the long-term size of the tumor.

Hint: in parts (a)-(d), have *Mathematica* sketch the phase line starting at the point $(1, 1)$ for the appropriate system(s), and figure out where each solution curve ends.

Problems from Section 3.9

Note: In Problems 133-136, you can (and probably should) use *Mathematica* to check your answers, but I want to see all the steps worked out by hand.

133. Compute the exponential of the following matrix:

$$\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$$

134. Find the general solution of this system of ODEs:

$$\begin{cases} x' = x + y \\ y' = 4x + y \end{cases}$$

135. Find the particular solution of the system

$$\begin{cases} \mathbf{y}' = (2x - y, 3x - 2y) \\ \mathbf{y}(0) = (-6, 4) \end{cases}$$

136. Find the general solution of this system of ODEs:

$$\begin{cases} x' = 4x - 3y \\ y' = 8x - 6y \end{cases}$$

137. Consider the 2×2 system of ODEs $\mathbf{y}' = A\mathbf{y}$ where

$$A = \begin{pmatrix} -7 & 1 \\ -6 & -2 \end{pmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Find a basis of the solution set, and verify that the functions in this basis are linearly independent by computing their Wronskian.

(c) Find the general solution of the system $\mathbf{y}' = A\mathbf{y}$.

(d) If $\mathbf{y} = (x, y)$ is any solution of the system $\mathbf{y}' = A\mathbf{y}$, find $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.

(e) Find the Cartesian equation(s) of all straight-line solutions of the system.

(f) Find the particular solution of $\mathbf{y}' = A\mathbf{y}$ satisfying $\mathbf{y}(0) = (2, -1)$.

(g) For the solution found in part (f), find (the exact values of) $\mathbf{y}(2)$ and $\mathbf{y}(-2)$.

138. Consider the 3×3 initial value problem $\begin{cases} \mathbf{y}' = A\mathbf{y} \\ \mathbf{y}(0) = (1, 3, -2) \end{cases}$ where

$$A = \begin{pmatrix} 3 & 0 & -1 \\ -2 & 2 & 1 \\ 8 & 0 & -3 \end{pmatrix}.$$

(a) Use *Mathematica* to find the particular solution of this system (recall that the command `MatrixExp[ ]` computes matrix exponentials).

(b) Based on the solution you got in part (a), what are the eigenvalues of the matrix A? Explain how you answered this question by looking at the solution to part (a).

139. Use *Mathematica* to find the general solution of each following system of ODEs:

(a)

$$\begin{cases} x' = 3x + 2y + 4z \\ y' = 2x \qquad\quad + 2z \\ z' = 4x + 2y + 3z \end{cases}$$

(b)

$$\begin{cases} x' = x + y + z \\ y' = 2x + y - z \\ z' = -8x - 5y - 3z \end{cases}$$

(c)

$$\begin{cases} w' = 20w - 101x - 80y - 5z \\ x' = 7w - 7x - 28y - 28z \\ y' = -14w - 49x + 56y - 7z \\ z' = 3w - 75x - 12y + 36z \end{cases}$$

Problems from Sections 3.11 to 3.14

140. Find the general solution of this system of ODEs:

$$\begin{cases} x' = 3x - 2y \\ y' = 4x - y \end{cases}$$

141. Find the particular solution of this initial value problem:

$$\begin{cases} \mathbf{y}' = (x - 5y, x - 3y) \\ \mathbf{y}(0) = (1, 1) \end{cases}$$

142. (a) Find the general solution of this system of ODEs:

$$\begin{cases} x' = 3x - 4y \\ y' = x - y \end{cases}$$

(b) Write the Cartesian equation(s) of any straight-line solutions of this system.

143. Find the particular solution of this initial value problem:

$$\begin{cases} \mathbf{y}' = (x - 4y, 4x - 7y) \\ \mathbf{y}(0) = (3, 2) \end{cases}$$

144. Find the general solution of this system of ODEs:

$$\begin{cases} x' = 2x - y + 8e^{2t} \\ y' = 3x - 2y + 20e^{2t} \end{cases}$$

Write your answer coordinate-wise, i.e. as $\begin{cases} x(t) = \text{ something} \\ y(t) = \text{ something} \end{cases}$.

145. Find the particular solution of this initial value problem:

$$\begin{cases} \mathbf{y}' = (-3x + 2y, x - 2y + 3) \\ \mathbf{y}(0) = (1, 2) \end{cases}$$

146. Find the general solution of this system of ODEs:

$$\begin{cases} x' = 2x - 5y - \cos 3t \\ y' = x - 2y + \sin 3t \end{cases}$$

Write your answer coordinate-wise.

147. Suppose you are given a system of ODEs $\mathbf{y}' = A\mathbf{y}$ where A has the indicated properties. Write the general solution of the system (write each answer coordinate-wise):

(a) A is a 4×4 matrix with the following eigenvalues and eigenvectors:

$$\lambda = -3 \leftrightarrow \begin{pmatrix} 3 \\ 0 \\ 1 \\ 4 \end{pmatrix} \quad \lambda = 2 \leftrightarrow \begin{pmatrix} 2 \\ 1 \\ -3 \\ 1 \end{pmatrix} \quad \lambda = 4 \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \lambda = -6 \leftrightarrow \begin{pmatrix} 0 \\ 3 \\ 5 \\ 1 \end{pmatrix}.$$

(b) A is a 3×3 matrix with eigenvalues $\lambda = 2$ and $\lambda = 4$ ($\lambda = 4$ is repeated twice). An eigenvector corresponding to $\lambda = 2$ is $(1, 5, 0)$ and an eigenvector corresponding to $\lambda = 4$ is $(1, 2, -1)$. A generalized eigenvector for $\lambda = 4$ is $(0, 3, -4)$.

(c) A is a 4×4 matrix with eigenvalues $\lambda = 3 \pm 2i$ and $\lambda = 0$ ($\lambda = 0$ is repeated twice). An eigenvector corresponding to $\lambda = 3 + 2i$ is $(1 - i, 1 + 2i, i, 0)$ and an eigenvector corresponding to $\lambda = 0$ is $(1, 3, -2, 4)$. A generalized eigenvector corresponding to $\lambda = 0$ is $(0, 1, 4, -2)$.

148. Find the solution of each system (using *Mathematica* to compute quantities as necessary). Write the answers coordinate-wise, and simplify them as much as possible. In particular, your answers should not contain i.

(a) $\begin{cases} x' = y + z \\ y' = x + z \\ z' = x + y \end{cases}$

(b) $\begin{cases} \mathbf{y}' = (-45x - 90y - 45z, 16x - 116y - 4z, 76x - 191y - 244z) \\ \mathbf{y}(0) = (1, 3, -2) \end{cases}$

(c) $\begin{cases} \mathbf{y}' = (y_1 - y_2 + y_3, 2y_1 - 2y_2 + 3y_3, \frac{4}{3}y_1 - y_2 + \frac{2}{3}y_4, 3y_1 - y_2 - 2y_3 + y_4) \\ \mathbf{y}(0) = (2, 0, 1, -1) \end{cases}$

Hint: use the `ExpToTrig[ ]` command, then the `Simplify[ ]` command to get rid of the imaginary numbers in part (c).

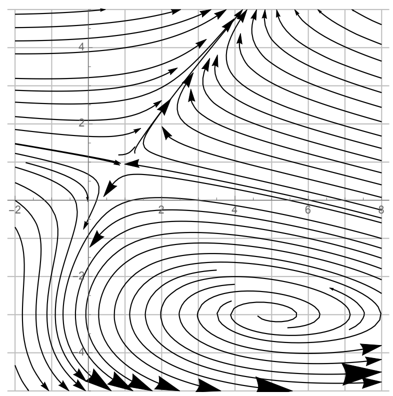
Problems from Section 3.15

In Problems 149-151, find and classify the equilibria of each autonomous system. Your classification should be specific (i.e. your answer should be "saddle", "stable node", "unstable node", "stable spiral", "unstable spiral" or "center").

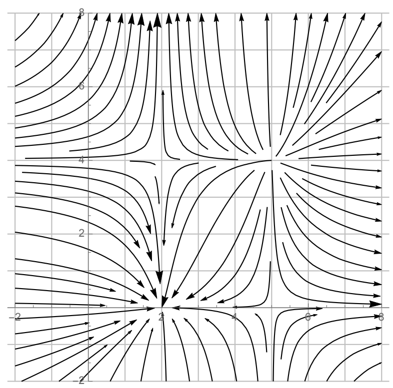
149. (a) $\begin{cases} x' = 2x + y - 3 \\ y' = x + 4y + 1 \end{cases}$ (b) $\begin{cases} x' = -2y - x - 1 \\ y' = x + 3 \end{cases}$

150. (a) $\begin{cases} x' = (x-1)(4+y) \\ y' = -(x+2)(y-3) \end{cases}$ (b) $\begin{cases} x' = e^x - e^{-y} \\ y' = e^{2x} - e^y \end{cases}$

151. (a) The system whose phase plane is



(b) The system whose phase plane is



Problems from Section 3.16

152. Consider a 2×2 autonomous system $\mathbf{y}' = \Phi(\mathbf{y})$; let $\mathbf{y}_0$ be an equilibrium of this system. Classify the equilibrium $\mathbf{y}_0$ based on the given information:

(a) $\text{tr}D\Phi(\mathbf{y}_0) = 8$ and $\det D\Phi(\mathbf{y}_0) = 1$.

(b) $\text{tr}D\Phi(\mathbf{y}_0) = -3$ and $\det D\Phi(\mathbf{y}_0) = -2$.

(c) $\text{tr}D\Phi(\mathbf{y}_0) = -6$ and $\det D\Phi(\mathbf{y}_0) = 3$.

(d) $\text{tr}D\Phi(\mathbf{y}_0) = 0$ and $\det D\Phi(\mathbf{y}_0) = 4$.

153. Consider the 2×2 system $\mathbf{y}' = A\mathbf{y}$ where A is a 2×2 matrix with constant entries. Classify the equilibrium at the origin based on the given information:

(a) $\text{tr}A = -1$ and $\det A = 7$.

(b) $\text{tr}A = 5$ and $\det A = -1$.

(c) $\text{tr}A = 2$ and $\det A = 3$.

(d) $\text{tr}A = 0$ and $\det A = -6$.

Problems from Section 3.17

154. Two large tanks (call them X and Y) each hold 100 L of liquid. They are interconnected by pipes which pump liquid from tank X to tank Y at 3 L/min and from tank Y to tank X at 1 L/min. A brine solution of concentration 0.2 kg/L of salt flows into tank X at a rate of 5 L/min; the solution flows out of the system of tanks via two pipes (one pipe allows flow out of tank X at 2 L/min and another pipe allows flow out of tank Y at 2 L/min). Suppose that initially, tank Y contains pure water but tank X contains 40 kg of salt; assume that at all times the liquids in each tank are kept mixed.

    (a) Draw a compartmental diagram which models this situation.

    (b) Write an initial value problem modeling the situation, where $x(t)$ and $y(t)$ represent the amount of salt in tanks X and Y respectively, at time t, and $\mathbf{y} = (x, y)$.

    (c) Use *Mathematica* to draw a picture of the vector field for your system, with the solution curve to your initial value problem indicated. Use the viewing window $[0, 60] \times [0, 60]$; attach a printout of this picture, labelled as "154 (c)".

    (d) Based on your picture from part (c), estimate the amount of salt in tank X at the instant when tank Y has 10 kg of salt in it.

    (e) Estimate the amount of salt in each tank 8 minutes after the initial situation by having *Mathematica* implement Euler's method with 10000 steps.

    (f) Solve the initial value problem you wrote down in part (b). Write your answer coordinate-wise.

    (g) Have *Mathematica* graph the functions $x(t)$ and $y(t)$ on the same axes (in the viewing window $[0, 100] \times [0, 60]$). The appropriate *Mathematica* commands are

    `x[t_] =` *whatever formula you get for* $x(t)$
    `y[t_] =` *whatever formula you get for* $y(t)$
    `Plot[{x[t],y[t]}, {t, 0, 100}, PlotRange -> {0,60}]`

    Attach a printout of these graphs (labelled as "154 (g)"). Make sure you indicate which graph is $x(t)$ and which is $y(t)$.

    (h) Find the amount of salt in each tank at time 8 (I want both the exact amount and a decimal approximation). Compare your answer to part (e); how accurate was your estimate in part (e)?

    (i) Find the concentration of salt in tank X at time 8.

155. In this problem, we explore a model (called the **Gause competitive exclusion model**) which describes how the populations of two species behave, when the species are in competition for the same resource. Let X and Y be two species whose populations at time t are $x(t)$ and $y(t)$, respectively, where x and y satisfy the system of differential equations

$$\mathbf{y}' = \Phi(\mathbf{y}) \text{ a.k.a. } \begin{cases} x' = r_X x(L_X - x - \alpha y) \\ y' = r_Y y(L_Y - y - \beta x) \end{cases} .$$

In this setting, α is a nonnegative number called the **competition coefficient of Y on X**; it measures the degree to which increased numbers of species Y negatively impact the ability of species X to survive. Similarly, $\beta \geq 0$ is the competition coefficient of X on Y.

(a) If $\alpha = \beta = 0$ (i.e. the species are not in competition), the populations of X and Y change according to what model? In light of this, what do the constants r_X, L_x, r_Y and L_Y represent?

(b) This system of ODEs has four equilibria: three of them are $(0,0)$ (i.e. both species are extinct), $(0, L_Y)$ (species X is extinct), and $(L_X, 0)$ (species Y is extinct).

   i. Find the fourth equilibrium.

   ii. Explain why this fourth equilibrium is called the **coexistence equilibrium**.

(c) Find $D\Phi(0,0)$. Compute the eigenvalues of $D\Phi(0,0)$ and show that $(0,0)$ is an unstable equilibrium.

   Hint: a matrix is called **triangular** if all the entries below its diagonal are zero, or if all the entries above its diagonal are zero. The eigenvalues of a triangular matrix are always its diagonal entries.

(d) Find $D\Phi(0, L_Y)$. Compute the eigenvalues of $D\Phi(0, L_Y)$ (the hint of part (c) is useful).

(e) Based on your computations in part (d), show that $(0, L_Y)$ is a stable equilibrium if and only if $\alpha > \frac{L_X}{L_Y}$.

(f) Find $D\Phi(L_X, 0)$. Compute the eigenvalues of $D\Phi(L_X, 0)$.

(g) Based on your computations in part (f), show that $(L_X, 0)$ is a stable equilibrium if and only if $\beta > \frac{L_Y}{L_X}$.

(h) Suppose that neither $(0, L_Y)$ nor $(L_X, 0)$ are stable.

   i. Show that the following three inequalities hold:

$$L_X - \alpha L_Y > 0 \qquad \beta L_X - L_Y < 0 \qquad \alpha\beta - 1 < 0$$

   ii. Compute $D\Phi$ at the coexistence equilibrium.

   iii. Compute and simplify $\det D\Phi$ and tr $D\Phi$ at the coexistence equilibrium using *Mathematica*.

   iv. Use the inequalities described in part (i) to show that the coexistence equilibrium must be stable.

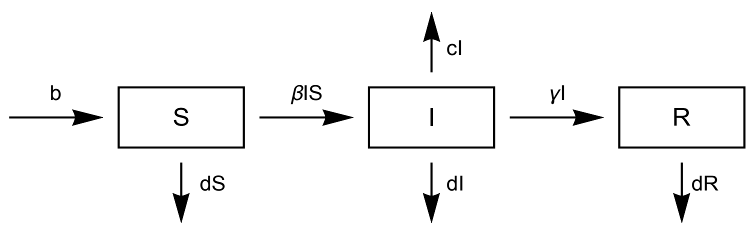
(i) If the two species occupy the same ecological niche (i.e. they eat the same food, live in the same places, etc.), then it is reasonable to assume that $\beta = \frac{1}{\alpha}$. For example, if X and Y are species of beetles where each Y beetle eats twice as much grain as each X beetle, then α would be 2 and β would be $\frac{1}{2}$. What happens to the coexistence equilibrium in this case? What does that mean about the two species?

156. In the SIR model we studied in class, we assumed that the disease acted quickly (before any births or deaths could occur). Now let's assume that the disease acts

more slowly, so that we can account for births and deaths in our model. Let:

$$\begin{aligned}
S &= \text{ the population of the susceptible class (as before)} \\
I &= \text{ the population of the infective class (as before)} \\
R &= \text{ the population of the recovered class (as before)} \\
b &= \text{ the birth rate (assume this is constant)} \\
d &= \text{ the death rate due to factors other than the disease} \\
c &= \text{ the death rate due to the disease} \\
\beta &= \text{ the rate at which infections take place} \\
\gamma &= \text{ the rate at which infected patients recover}
\end{aligned}$$

This leads to the following compartmental diagram:



Note that we can no longer assume that $S + I + R = 1$, because the total population may change as time passes.

(a) Write the system of differential equations for S, I and R. (In the rest of this problem, we will call this system $\mathbf{y}' = \Phi(\mathbf{y})$ where $\mathbf{y} = (S, I, R)$.)

(b) There is one equilibrium of the system which has the form $(S^{\#}, 0, 0)$, where $S^{\#} \neq 0$ (this is called the **disease-free equilibrium**; it represents the situation where no one has the disease). Find $S^{\#}$ in terms of the variables listed above.

(c) Find $D\Phi(S^{\#}, 0, 0)$, and have *Mathematica* compute the eigenvalues of this matrix.

(d) Show that the disease-free equilibrium is stable if and only if $\beta b < d(c + d + \gamma)$.

(e) There is a second equilibrium called the **endemic equilibrium** (because in this situation the disease persists), which has the form (S^*, I^*, R^*). Find S^*, I^* and R^* in terms of the variables listed above.

(f) Find $D\Phi(S^*, I^*, R^*)$ and have *Mathematica* compute the eigenvalues of this matrix.

(g) Show that the endemic equilibrium is stable if and only if $\beta b > d(\gamma + c + d)$.

(h) Explain why your answers to parts (d) and (g) make sense, given what the variables in the problem mean.

Problems from Section 4.1

157. Rewrite this third-order ODE as a first-order system of the form $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$. Be sure to carefully identify what $\mathbf{y}$, A and $\mathbf{q}$ are.

$$y''' - 4y'' + 7y' - 8y = \cos t$$

158. Rewrite this second-order system of equations as a first-order system of the form $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$. Be sure to carefully identify what $\mathbf{y}$, A and $\mathbf{q}$ are.

$$\begin{array}{cccccccc} x'' & & +2x' & -3y' & +4x & & = 0 \\ & y'' & -x' & +4y' & -7x & +3y & = 0 \end{array}$$

Problems from Section 4.2

159. Find the general solution of each differential equation:

(a) $y'' - 8y' + 12y = 0$
(b) $y'' = -7y' + 18y$
(c) $y''' + 11y'' + 30y' = 0$

160. Find the general solution of each differential equation:

(a) $y'' + 4y' + 11y = 0$
(b) $y''' + 9y' = 0$
(c) $y'' - 12y' + 36y = 0$

161. Find the general solution of each differential equation:

(a) $y'' - 4y' - 21y = e^{4t}$
(b) $y'' + y = \sin 2t$
(c) $y'' - 25y = t$

162. Find the particular solution of each initial value problem:

(a) $\begin{cases} y'' - 8y' + 15y = 0 \\ y(0) = 3 \\ y'(0) = 2 \end{cases}$ (b) $\begin{cases} y'' + 4y' + 4y = 0 \\ y(0) = -6 \\ y'(0) = -1 \end{cases}$ (c) $\begin{cases} y'' + 2y' + 5y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

163. Find the particular solution of this initial value problem:

$$\begin{cases} y'' - 16y' + 63y = e^t \\ y(0) = 11 \\ y'(0) = -5 \end{cases}$$

164. Find the general solution of this differential equation:

$$y^{(5)} + y^{(4)} - 4y''' - 8y'' - 32y' - 48y = 0.$$

Hint: use *Mathematica* to factor the characteristic equation.

Problems from Section 4.4

165. (a) Explain why the α on page 223 of the lecture notes must be negative.
(b) Explain why the λ_1 and λ_2 on page 223 of the lecture notes must both be negative.

166. Consider a 12 kg mass attached to a fixed point by a spring (like the picture on the top of page 220 of the notes). If the spring constant is $\frac{1}{4}$ N/m and the damping coefficient (i.e. coefficient of friction) is $\frac{7}{2}$ N sec/m, and the entire system (at time t) is subject to an external force of $257 \cos 2t$ N, find the position of the mass at time t. Assume that at time 0, the mass has no initial velocity but is 8 m to the right of where it would be at rest.

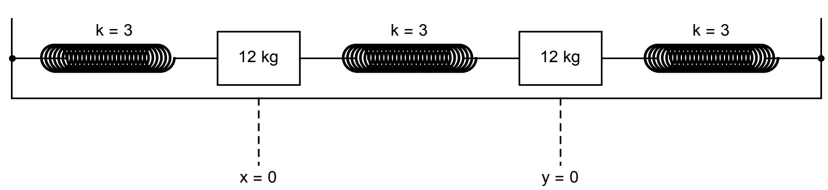
167. Consider a coupled mass-spring system like the one pictured at the top of page 226 in the notes. Assume that $m_1 = 6$ kg and $m_2 = 2$ kg, and the spring constants are $k_1 = 3$ N/m and $k_2 = 2$ N/m. If the first mass is displaced 4 m left of its equilibrium position and the second mass is displaced 2 m right of its equilibrium position, and then the masses are released with no initial velocity:

   (a) Find the position of each mass at time t.

   (b) Assuming that the spring between the two masses is 10 m long at rest, have *Mathematica* sketch a graph showing the positions of the masses at time t in the viewing window $[0, 50] \times [-5, 20]$ (attach a printout of this graph, labelled as "168"). The vertical axis should be scaled so that height 0 corresponds to the first mass being at equilibrium, as in the second picture on page 228 in the lecture notes.

   To graph two functions at once; see Problem 154 (g) for the appropriate commands.

168. Suppose that three identical springs, each with spring constant $k = 3$ and two identical masses, each of mass 12, are attached in a straight line, with the ends of the outside springs fixed. Here is a picture of the system:



Suppose that the masses move along a frictionless surface; let $x(t)$ and $y(t)$ be the displacement of the masses at time t, where $x = 0$ and $y = 0$ correspond to the system being at rest, and positive values of x and y indicate displacement to the right.

   (a) Write down a second-order system of differential equations describing the system (coming from Newton's laws).

   (b) Convert the system from part (a) to a first-order system of equations.

   (c) Find the general solution of the system you wrote down in part (a).

   Hint: Use *Mathematica* to find eigenvalues and eigenvectors for the system you obtained in part (b).

(d) Suppose that initially, the left-hand mass starts at $x = 2$ and the right-hand mass starts at $y = -1$, and that both masses have an initial rightward velocity of 1.

i. Find $x(t)$ and $y(t)$ in this case.

ii. Graph x and y on the same axes (see Problem 154 (g) for the appropriate *Mathematica* commands); attach a printout of this graph, labelled as "168 (d)". You are responsible for choosing a reasonable viewing window.

iii. Find the positions of the masses (i.e. x and y) when $t = 3$.

169. Consider an undamped pendulum of length 2 m and mass 45 kg.

(a) Write the undamped pendulum equation in this case.

(b) Write the linearization of the equation you wrote in part (a).

(c) Find the general solution of the linearized equation you wrote in part (b).

(d) Suppose that at time 0, the pendulum is at angle $\theta = .05$ radians and has angular velocity $-.3$ radians/second. What is the largest angle obtained by the pendulum as it swings?

Hint: Find the particular solution corresponding to these initial conditions; one aspect of this particular solution gives the answer to the question.

170. **(Optional; extra credit)** Consider the linearized double pendulum equation (coming from page 231 of the lecture notes; repeated here for convenience):

$$\begin{cases} (m_1 + m_2)l_1\theta_1'' + m_2 l_2 \theta_2'' + (m_1 + m_2) g\theta_1 = 0 \\ m_2 l_2 \theta_2'' + m_2 l_1 \theta_1'' + m_2 g \theta_2 = 0 \end{cases}$$

(a) Rewrite this system of two second-order equations as a first-order system of four equations $\mathbf{y}' = A\mathbf{y}$.

Hint: To get started, pretend this is a system of two equations in the two variables θ_1'' and θ_2'' and solve for those two quantities in terms of the other variables.

(b) Use *Mathematica* to find the eigenvalues and eigenvectors of the matrix A, and write the general solution of the equation $\mathbf{y}' = A\mathbf{y}$ you wrote in part (a).

(c) Write the formulas for θ_1 and θ_2.

171. A series RLC circuit (as in page 233 of the Fall 2017 lecture notes) has a voltage source given by $E_S(t) = 10\cos 20t$ volts, a resistor of $2\,\Omega$, an inductor of $\frac{1}{4}$ H, and a capacitor of $\frac{1}{13}$ F. If the initial current is zero and the initial charge on the capacitor is 3.5 coulombs, find the current in the circuit as a function of time t.

Selected answers

1. (b) $y = \frac{1}{2t-3}$

2. (a) $y = \sqrt{t+3} + 1$
   (c) $y = \pm\frac{1}{2}\sqrt{8-t^2}$

3. (c) $y = 2 + \ln 3 + t - 1$

4. (a) ∞
   (b) 0

6. (a) $f'(t) = 6t - 7$

9. (a) $f'(t) = -2e^{-2t}$
   $f''(t) = 4e^{-2t}$
   $f'''(t) = -8e^{-2t}$

10. (a) $\frac{1}{2}t^2 + C$

12. (b) $\frac{3}{4}t^{4/3} + C$
    (c) $\frac{2}{3}t^{3/2} + t + C$

14. (a) $t^2 + t$
    (b) $4\sqrt{t} - 4$

15. (b) $\frac{-1}{4}t^2 + \frac{1}{2}t^2 \ln t + C$

16. (a) $\frac{1}{2}\ln(2-x) - \frac{1}{2}\ln(2+x) + C$
    (b) $\frac{1}{2}\sin^2 t + C$

18. (a) $\frac{-3}{t-1} + (t-1) + 2\ln(t-1) + C$

19. (a) $x_1 = -1$; $x_2 = 1$; $x_3 = -1$; $x_4 = 1$

20. 43

21. (a) not a solution

22. (a) solution

24. (a) not a solution

25. (a) not linear
    (b) linear

28. (b) order 4;
    4 constants in gen. sol'n;
    nonlinear

29. (a) order 1;
    1 constant in gen. sol'n;
    linear;
    not homogeneous;
    not constant-coefficient

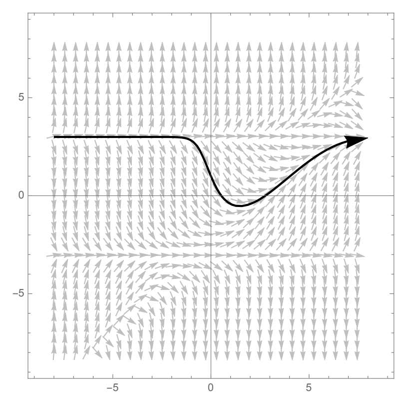
32. (d) $y = \frac{12}{e^{14.1}}e^{4.7t}$

34. (c) $318.55.

35. (c) 16.7621 grams.

37. (b) $y = 4$, $y = 0$, $y = -5$

38. (a)



40. $(t_4, y_4) = (4, 61)$.

41. $y(2) \approx 33$.

46. (a) There are four possible answers: $y = -7$, $y = -2$, $y = 3$ or $y = 6$.
    (b) $y = -2$ is the only sink.

47. (a) -2

50. (a) $\phi(-7)$ is negative
    (c) $\phi(4)$ is positive
    (d) $\phi'(-2)$ is negative

51. (a) $y = -3$ is stable; $y = 3$ is unstable

52. (a) $y = 0$ is unstable

53. (a) $y = 0$ is unstable; $y = 6$ is stable

55. (a) $y = 0$ and $y = \frac{rL-e}{r}$.
    (e) $E = \frac{1}{2}rL$

57. (a) Yes; if $5 \le r \le 7$ and $y(0) > 0$ we know that $\lim_{t\to\infty} y(t) = 0$, so the long-term value of y is zero.

58. (a) There is only one bifurcation at $r = 9$; it is a saddle-node bifurcation.

62. (a) $y = Ce^{2t}$

63. (b) $y = Ce^{\sin t}$

64. (a) $y = 3\ln t$

66. (a) $y = 7e^{2t} - 3e^t$

67. (a) $y = -te^{-t} + Ct$

70. (a) $y = Ce^{-3t} + e^{-2t}$

71. (b) $y = Ce^{4t} - \frac{1}{2}t^2 - \frac{9}{4}t - \frac{9}{16}$

72. (a) $y_h = e^{-7t}$
    (b) No; on the left-side you get 0 which cannot equal $20e^{-7t}$.
    (d) $y = Ce^{3t} + 9te^{3t}$

74. (a) $\frac{1}{2}y^2 = \frac{1}{3}\ln(4+t^3) + C$
    (b) $y = \frac{Ct^2-1}{Ct^2+1}$

75. (a) $y = \ln(4t + C)$

79. (a) $y = Ct^3 - t + D$

80. (a) $y = \arctan(Ct + D)$

81. (a) $y = C\cos t + D\sin t$

82. (a) $y = Ce^{2t} - 2t^3 - 3t^2 - t + D$

83. (a) Exact; solution is $y^2-2y+t^2+3t = C$
    (b) Not exact

84. (a) $y^2 - ty + t^2 = 7$

85. (c) $y^2t - ye^y + e^y = C$

87. (a) $200 - 2t$
    (c) $\begin{cases} y' = \frac{6}{5} - \frac{4y}{100-t} \\ y(0) = 1 \end{cases}$
    (e) 15.0017

88. (d) $y_{max} \approx 20$
    (e) $t \approx 35$

89. $y = \frac{-2}{5}(t-100) - \frac{39}{10^8}(t-100)^4$.

90. (b) $f(60) = \frac{9376}{25}$.

91. (a) $\frac{\frac{-2}{5}(100-t) - \frac{39}{10^8}(100-t)^4}{200-2t}$.
    (c) $t \approx 20$.

93. (b) $\begin{cases} \frac{dv}{dt} = 9.8 - \frac{30}{75}v \\ v(0) = 0 \end{cases}$
    (d) 24.5

94. $v = \frac{49}{2} - \frac{49}{2}e^{-(2/5)t}$.

95. (a) $h(t) = \frac{8245}{4} - \frac{49}{2}t - \frac{245}{4}e^{-(2/5)t}$.
    (b) $t = \frac{2}{5}\ln\frac{49}{9} \approx 4.2365$.

96. $t = \frac{15\ln\frac{7}{19}}{\ln\frac{14}{19}} \approx 49.0466$

99. (a) solution
    (b) not a solution

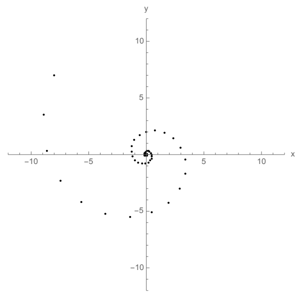
101. (a) (i) $\begin{cases} x(t) = t^2 - 2 \\ y(t) = t + 3 \end{cases}$
    (ii) $\mathbf{y}(0) = (-2,3)$; $\mathbf{y}(1) = (-1,4)$; $\mathbf{y}(2) = (2,5)$.

102. (a) Use this code (all executed in one cell):
```
ParametricPlot[{Cos[t], Sin[t]}, {t, -100,100},
 PlotRange -> {{-2,2},{-2,2}}]
```
You will find that the graph is a circle.

105. $(t_1, \mathbf{y}_1) = (1,(3,6))$; $(t_2, \mathbf{y}_2) = (2,(0,18))$; $(t_3, \mathbf{y}_3) = (3,(-18,36))$.

107. The x-coordinate when $t = 4$ is 92.1414.

109. (a) $\mathbf{y}(50) \approx (-6.531 \cdot 10^{-45}, 5.708 \cdot 10^{-45}) \approx (0, 0)$.

(b) 

110. (d) $\mathbf{y}(5) \approx (.515676, .362851, .326229)$.

(e) No, when more steps are used, the coordinates of $\mathbf{y}(5)$ change significantly.

111. (a) 0

(c) $\begin{pmatrix} 15 & -6 \\ 3 & 9 \end{pmatrix}$

(e) $\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 4-\lambda & 2 \\ -8 & -4-\lambda \end{pmatrix}$

112. (a) $\begin{pmatrix} -2\sin 2t & 3\cos 3t \\ 8\cos 2t & -3\cos 3t \end{pmatrix}$

(b) $(\cos t, -6\sin 3t, 28e^{7t})$

(c) $\begin{pmatrix} -15 & -13 \\ -3 & 11 \end{pmatrix}$

113. (b) $\begin{pmatrix} 2^8 & 0 \\ 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

115. (a) $\begin{pmatrix} \frac{3}{17} & \frac{2}{17} \\ \frac{-1}{17} & \frac{5}{17} \end{pmatrix}$

116. (a) 17

(b) -2

117. (a) $\lambda^2 - 8\lambda + 17$

118. (a) No

(b) Yes

119. (a) $W(t) = 20t^2$; the functions are linearly independent (since $W(1) \neq 0$).

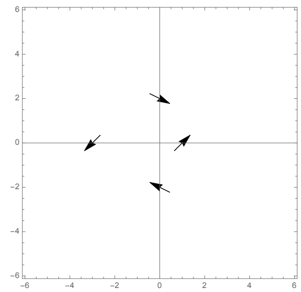
(c) $W(t) = 2$; the functions are linearly independent (since $W(0) \neq 0$).

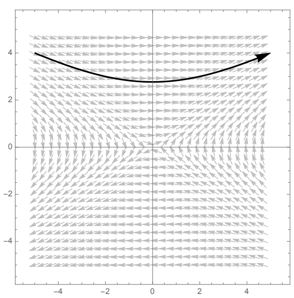
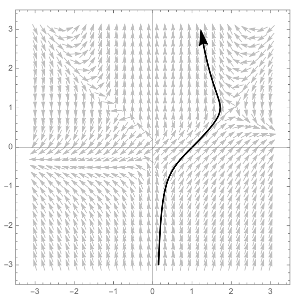
121. (a) First, compute the Wronskian:

$$W(t) = \det\begin{pmatrix} e^{at} & e^{bt} \\ (e^{at})' & (e^{bt})' \end{pmatrix} = \det\begin{pmatrix} e^{at} & e^{bt} \\ ae^{at} & be^{bt} \end{pmatrix} = be^{(a+b)t} - ae^{(a+b)t} = (b-a)e^{(a+b)t}.$$

Since $a \neq b$, $W(t) \neq 0$, so the functions e^{at} and e^{bt} are linearly independent, as desired.

124. $\left.\frac{dy}{dx}\right|_{(1,0)} = 1$; $\left.\frac{dy}{dx}\right|_{(0,2)} = \frac{-1}{2}$; $\left.\frac{dy}{dx}\right|_{(0,-2)} = \frac{-1}{2}$; $\left.\frac{dy}{dx}\right|_{(-3,0)} = 1$. The slope field with these four mini-tangents is:



125. (a)  (b) 

127. (a) $(-6, 4)$ and $\left(\frac{3}{2}, -1\right)$

128. (a) $\left(\frac{1}{2}, 1\right)$ and $\left(\frac{-1}{2}, 1\right)$

129. (c) $\mathbf{y}(t) = (3, 1)$.
   (d) i. $x(t)$ is decreasing
      ii. $y(t)$ is increasing
      iii. $\lim_{t\to\infty} x(t) = -\infty$
      iv. $\lim_{t\to\infty} y(t) = 1$

135. $\mathbf{y} = C_1 e^t \binom{1}{1} + C_2 e^{-t} \binom{1}{3}$

137. (c) $\mathbf{y} = C_1 e^{-4t} \binom{1}{3} + C_2 e^{-5t} \binom{1}{2}$.

141. $\mathbf{y} = \binom{e^{-t}\cos t - 3e^{-t}\sin t}{e^{-t}\cos t - e^{-t}\sin t}$

143. $\mathbf{y} = \binom{3e^{-3t} + 4te^{-3t}}{2e^{-3t} + 4te^{-3t}}$

144. $\begin{cases} x(t) = 4e^{2t} + C_1 e^t + C_2 e^{-t} \\ y(t) = 8e^{2t} + C_1 e^t + 3C_2 e^{-t} \end{cases}$

147. (a) $\begin{cases} y_1(t) = 3C_1 e^{-3t} + 2C_2 e^{2t} \\ y_2(t) = 2C_2 e^{2t} + 3C_4 e^{-6t} \\ y_3(t) = C_1 e^{-3t} - 3C_2 e^{2t} + 5C_4 e^{-6t} \\ y_4(t) = 4C_1 e^{-3t} + C_2 e^{2t} + C_3 e^{4t} + C_4 e^{-6t} \end{cases}$

148. (a) $\begin{cases} x(t) = C_1 e^{2t} - C_2 e^{-t} - C_3 e^{-t} \\ y(t) = C_1 e^{2t} + C_3 e^{-t} \\ z(t) = C_1 e^{2t} + C_2 e^{-t} \end{cases}$

149. (a) The only equilibrium is $\left(\frac{13}{7}, \frac{-5}{7}\right)$, which is an unstable node.

151. (b) There are four equilibria: $(2,0)$ (stable node), $(5,0)$ (unstable saddle), $(5,4)$ (unstable node) and $(2,4)$ (unstable saddle).

152. (a) Unstable node
   (b) Unstable saddle
   (c) Stable node

154. (b) $\begin{cases} \begin{cases} x' = \frac{-1}{20}x + \frac{1}{100}y + 1 \\ y' = \frac{3}{100}x - \frac{3}{100}y \end{cases} \\ \mathbf{y}(0) = (40,0) \end{cases}$

155. (a) When $\alpha = \beta = 0$, the populations of X and Y behave according to logistic models.
   (b) i. The fourth equilibrium is $\left(\frac{\alpha L_Y - L_X}{\alpha\beta - 1}, \frac{\beta L_X - L_Y}{\alpha\beta - 1}\right)$.
   (c) $D\Phi(0,0) = \begin{pmatrix} r_X L_X & 0 \\ 0 & r_Y L_Y \end{pmatrix}$. The eigenvalues of this matrix are $r_X L_X$ and $r_Y L_Y$, both of which are positive, so $(0,0)$ is unstable.
   (d) $D\Phi(0, L_Y) = \begin{pmatrix} r_X L_X - \alpha r_X L_Y & 0 \\ -\beta r_Y L_Y & -r_Y L_Y \end{pmatrix}$. One eigenvalue of this matrix is $-r_Y L_Y < 0$; the other is $r_X(L_X - \alpha L_Y)$.
   (e) If $\alpha > \frac{L_X}{L_Y}$, then $L_X - \alpha L_Y < 0$ so both eigenvalues of $D\Phi(0, L_Y)$ are negative, in which case $(0, L_Y)$ is stable.
   (i) In this case, there is no coexistence equilibrium (since the denominators in the coexistence equilibrium would have to be zero).

156. (b) $S^\# = \frac{b}{d}$
   (c) $D\Phi(\frac{b}{d}, 0, 0) = \begin{pmatrix} -d & -\beta\frac{b}{d} & 0 \\ 0 & & \\ 0 & \beta\frac{b}{d} - c - d - \gamma & 0 \\ 0 & \gamma & -d \end{pmatrix}$;
   the eigenvalues are $-d$ (repeated twice) and $\frac{1}{d}(b\beta - d(c+d+\gamma))$.
   (e) $S^* = \frac{c+d+\gamma}{\beta}$; $I^* = \frac{b\beta - d(c+d+\gamma)}{\beta(c+d+\gamma)}$; $R^* = \frac{\gamma(b\beta - d(c+d+\gamma))}{\beta d(c+d+\gamma)}$.

157. $\mathbf{y} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$; $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -7 & 4 \end{pmatrix}$; $\mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ \cos t \end{pmatrix}$.

159. (a) $y = C_1 e^{2t} + C_2 e^{6t}$.

160. (a) $y = C_1 e^{2t} \cos\left(\sqrt{7}t\right) + C_2 e^{2t} \sin\left(\sqrt{7}t\right)$.

161. (a) $y = \frac{-1}{21}e^{4t} + C_1 e^{7t} + C_2 e^{-3t}$.

162. (a) $y = \frac{13}{2}e^{3t} - \frac{7}{2}e^{5t}$.

165. (a) If you convert the second-order equation $mx''(t) + bx'(t) + kx(t) = 0$ to a first-order system via reduction of order, the system becomes

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-b}{m} \end{pmatrix} \mathbf{x} = A\mathbf{x}.$$

We have $tr(A) = \frac{-b}{m} < 0$ and $\det A = \frac{k}{m} > 0$. Since the trace is negative and the determinant is positive, the equilibrium **0** is either a stable spiral or stable node, so both of its eigenvalues must have negative real part.

166. $x(t) = \frac{-336}{37265}\cos 2t - \frac{2292}{37265}\sin 2t - \frac{1302}{145}e^{-t/6} + \frac{2824}{257}e^{-t/8}$.

167. (a) $\begin{cases} x(t) = \frac{-16}{7}\cos\left(t\sqrt{\frac{3}{2}}\right) - \frac{12}{7}\cos\left(\frac{t}{\sqrt{3}}\right) \\ y(t) = \frac{32}{7}\cos\left(t\sqrt{\frac{3}{2}}\right) - \frac{18}{7}\cos\left(\frac{t}{\sqrt{3}}\right) \end{cases}$

168. (a) $\begin{cases} 12x'' = -3x + 3(y-x) \\ 12y'' = -3(y-x) - 3y \end{cases}$ (or something equivalent to this)

(d) i. $\begin{cases} x(t) = \frac{1}{2}\cos\frac{t}{2} + \frac{3}{2}\cos\frac{t\sqrt{3}}{2} + 2\sin\frac{t}{2} \\ y(t) = \frac{1}{2}\cos\frac{t}{2} - \frac{3}{2}\cos\frac{t\sqrt{3}}{2} + 2\sin\frac{t}{2} \end{cases}$

169. (a) $\theta'' + \frac{9.8}{2}\sin\theta = 0$.

(b) $\theta'' + \frac{9.8}{2}\theta = 0$.

(d) $\frac{\sqrt{409}}{140}$ radians (which is about 0.1444).

171. $I(t) = \frac{1}{2}\cos 2t - \frac{3}{2}\sin 2t - \frac{1}{2}e^{-4t}\cos 6t - \frac{47}{2}e^{-4t}\sin 6t$.

1. **The Snowplow Problem:** Suppose that one morning it starts snowing hard, but at a constant rate. A snowplow sets out at 9 AM to clear a road. At 11 AM, it has cleared 15 miles, and at 1 PM, it has cleared an additional 10 miles. When did it start snowing?

   To solve this problem, carry out the following steps:

   (a) Let t be the time, measured in hours, with $t = 0$ corresponding to 9 AM. Let $h = h(t)$ be the height of the snow at time t. We will assume the rate of snowfall is constant, and equal to r. Write down a differential equation which could be used to solve for $h(t)$, if there was no plowing.

   (b) Solve the equation from part (a) for h in terms of t (there will be an arbitrary constant; let's call this constant B so everyone is using the same notation).

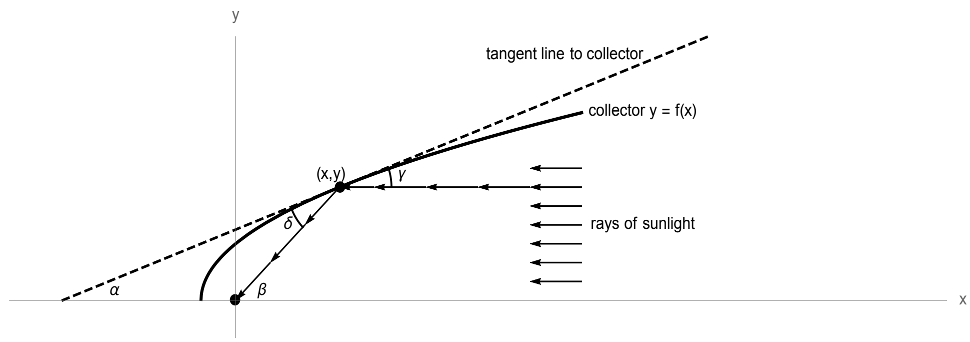
   (c) Let $x(t)$ be the distance the snowplow has travelled at time t. It is reasonable to assume that the deeper the snow is, the slower the snowplow has to travel. A simple mathematical model for this is to assume that the velocity of the snowplow at time t is inversely proportional to the height $h(t)$. Use this to write a differential equation satisfied the function $x(t)$ (let's agree to use the letter k for the proportionality constant).

   (d) Solve the differential equation of part (c) to obtain a formula for $x(t)$ (let's agree to call the arbitrary constant that appears here D).

   (e) Use the initial conditions of this problem (given before part (a)) to solve for B, D and k.

   (f) Answer the original question: when did it start snowing? Round your answer to the nearest minute.

2. **The Solar Collector Problem:** Suppose you want to design a solar collector which will concentrate the sun's rays at a point. The collector will be in the shape of a curve $y = f(x)$, and the point we want to concentrate the rays at will be $(0, 0)$. Suppose that the sun is located at the extreme positive x-axis, and that the light coming from the sun hits the collector, travelling in a horizontal path (see the picture below).



(a) From physics, the law of reflection for rays of light says that angles γ and δ are equal. Use this, together with some facts from high-school geometry, to explain why $\beta = 2\alpha$.

(b) From calculus, the slope of the tangent line at (x, y) is $\frac{dy}{dx}$. Use this to explain why $\frac{dy}{dx} = \tan\alpha$. Using similar logic, find $\tan\beta$ in terms of x and y.

(c) Use the double-angle identity for tangent (look this up via Google if you don't know it) to show that

$$\frac{y}{x} = \frac{2\frac{dy}{dx}}{1 - \left(\frac{dy}{dx}\right)^2}.$$

(d) Solve for $\frac{dy}{dx}$ in the above equation to show that the curve must satisfy

$$\frac{dy}{dx} = \frac{-x + \sqrt{x^2 + y^2}}{y}.$$

(e) Solve the equation obtained in part (e) of this problem.
Hint: To solve this equation, you need a trick. Start by letting $u = x^2 + y^2$ and differentiate implicitly to obtain a formula for $\frac{du}{dx}$ in terms of x, y and $\frac{dy}{dx}$. Use this formula, together with the equation from part (d), to write a separable differential equation of the form $\frac{du}{dx} = \textit{something}$. Then solve this equation for $u = u(x)$, and use that to recover y in terms of x.

(f) What shape are the solutions?