Old Math 330 Exams

David M. McClendon

Department of Mathematics
Ferris State University

Last updated to include exams from Fall 2017

Contents

Chapter 1

General information about these exams

These are the exams I have given in differential equations courses. Each exam is given here, followed by what I believe are the solutions (there may be some number of computational errors or typos in these answers).

Typically speaking, tests labelled “Exam 1” cover Chapter 1 in my differential equations lecture notes; tests labelled “Exam 2” cover Chapter 2 (except for the last section), and tests labelled “Exam 3” cover the last section of Chapter 2 and all of Chapter 3.

Chapter 2

Exams from Fall 2016

2.1 Fall 2016 Exam 1

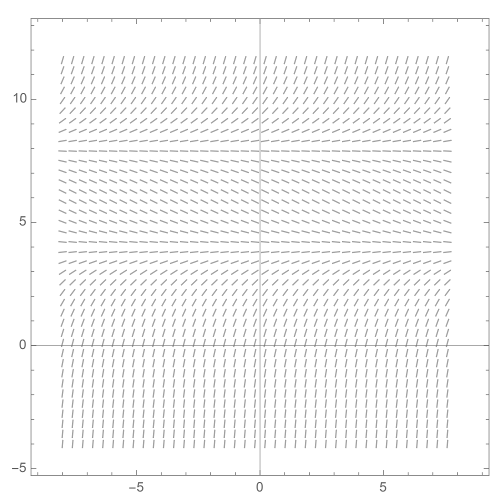
1. Briefly explain what is meant by "existence / uniqueness" in the context of ordinary differential equations.

2. Let $y = y(t)$ be the solution of the initial value problem

$$\begin{cases} y' = y + 2t \\ y(1) = -2 \end{cases}.$$

   Suppose you wanted to estimate $y(31)$ using Euler's method with 10 steps. Find the points (t_1, y_1) and (t_2, y_2) obtained by this method.

3. Here is the picture of the slope field associated to an autonomous ODE $y' = \phi(y)$:



a) Suppose $y(0) = 7$. Estimate $y(2)$.

b) Suppose $y(-1) = 2$. Find $\lim_{t\to\infty} y(t)$.

c) On the picture above, sketch the graph of the solution satisfying the initial condition $y(0) = 6$. Label the graph "(d)".

d) Find all equilibria of this equation, and classify them as stable, semistable or unstable.

4. Find the general solution of the following ODE:

$$\frac{dy}{dt} - 2y = 14e^{4t}$$

5. Find the particular solution of the following initial value problem:

$$\begin{cases} t^2 y' = y^2 \\ y(1) = 2 \end{cases}$$

Write your answer as a function $y = f(t)$.

6. Find the particular solution of the following initial value problem:

$$\begin{cases} 2ty\frac{dy}{dt} = t^2 - y^2 \\ y(3) = 1 \end{cases}$$

7. Find the general solution of the following ODE:

$$ty'' = y'$$

Write your answer as a function $y = f(t)$.

Solutions

1. The Existence/Uniqueness Theorem for first-order ODEs says if the function ϕ is "nice" (i.e. ϕ and $\frac{\partial \phi}{\partial y}$ are continuous), then the initial value problem

$$\begin{cases} y' = \phi(t, y) \\ y(t_0) = y_0 \end{cases}$$

has one and only one solution, which is of the form $y = f(t)$.

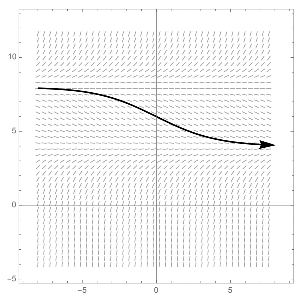
2. First, $\Delta t = \frac{1}{n}(t_n - t_0) = \frac{1}{10}(31 - 1) = 3$. Next, we are given $(t_0, y_0) = (1, -2)$. Now $\phi(t_0, y_0) = -2 + 2(1) = 0$ so

$$\begin{aligned} t_1 &= t_0 + \Delta t = 1 + 3 = 4 \\ y_1 &= y_0 + \phi(t_0, y_0)\Delta t = -2 + 0(3) = -2. \end{aligned}$$

Therefore $(t_1, y_1) = (4, -2)$. Now $\phi(t_1, y_1) = -2 + 2(4) = 6$ so

$$\begin{aligned} t_2 &= t_1 + \Delta t = 4 + 3 = 7 \\ y_2 &= y_1 + \phi(t_1, y_1)\Delta t = -2 + 6(3) = 16. \end{aligned}$$

Therefore $(t_2, y_2) = (7, 16)$.

3. a) Starting at the point $(0, 7)$ and following the vector field, we come to the point $(2, 6)$ so $y(2) \approx 6$.
   b) Starting at the point $(-1, 2)$ and moving to the extreme right, we see that $\lim\limits_{t\to\infty} y(t) = 4$.
   c) 

   d) $y = 4$ is stable; $y = 8$ is unstable.

4. *Method 1 (integrating factors):* The integrating factor is $\mu(t) = \exp\left[\int_0^t (-2)\, ds\right] = e^{-2t}$. After multiplying through by $\mu(t)$, the equation becomes

$$\begin{aligned} \frac{dy}{dt}e^{-2t} - 2e^{-2t}y &= 14e^{4t}(e^{-2t}) \\ \frac{d}{dt}\left(ye^{-2t}\right) &= 14e^{2t} \\ ye^{-2t} &= \int 14e^{2t}\, dt \\ ye^{-2t} &= 7e^{2t} + C \\ y &= e^{2t}\left(7e^{2t} + C\right) \\ y &= 7e^{4t} + Ce^{2t}. \end{aligned}$$

Method 2 (undetermined coefficients): The corresponding homogeneous equation is $\frac{dy}{dt} - 2y = 0$ which has solution $y_h = e^{2t}$ (exponential growth model).

Now, guess $y_p = Ae^{4t}$ and plug into the left-hand side of the equation to get $4Ae^{4t} - 2Ae^{4t} = 14e^{4t}$. That means $4A - 2A = 14$, i.e. $A = 7$. Therefore $y_p = 7e^{4t}$ so $y = y_p + Cy_h$, i.e.

$$y = 7e^{4t} + Ce^{2t}.$$

5. Start with the ODE, which is separable:

$$\begin{aligned} t^2 \frac{dy}{dt} &= y^2 \\ y^{-2}\,dy &= t^{-2}\,dt \\ \int y^{-2}\,dy &= \int t^{-2}\,dt \\ -\frac{1}{y} &= -\frac{1}{t} + C \end{aligned}$$

   Next, I will solve for C using the initial condition (you could have solved for y first):

$$-\frac{1}{2} = -\frac{1}{1} + C \quad \Rightarrow \quad C = \frac{1}{2}$$

   Therefore the particular solution is

$$-\frac{1}{y} = -\frac{1}{t} + \frac{1}{2}.$$

   Solve for y by first multiplying through by -1 and then taking reciprocals to get

$$y = \frac{1}{\frac{1}{t} - \frac{1}{2}} \quad \text{(this answer is fine)}$$

   which, if you multiply through the numerator and denominator by $2t$ simplifies to

$$y = \frac{2t}{2-t}.$$

6. Rewrite this equation as $(y^2 - t^2) + 2ty\frac{dy}{dt}$. Letting $M = y^2 - t^2$ and $N = 2ty$, we see that

$$M_y = 2y = N_t$$

   so the equation is exact. Now

$$\begin{aligned} \psi(t,y) &= \int M\,dt = \int (y^2 - t^2)\,dt = y^2 t - \frac{1}{3}t^3 + A(y) \\ &= \int N\,dy = \int 2ty\,dy = ty^2 + B(t). \end{aligned}$$

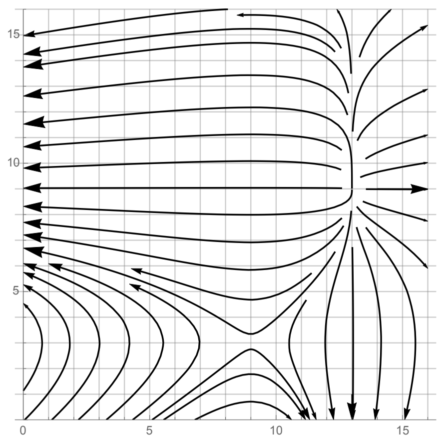
   By setting $B(t) = -\frac{1}{3}t^3$ and $A(y) = 0$, we reconcile these integrals to obtain $\psi(t,y) = y^2 t - \frac{1}{3}t^3$. Thus the general solution is $\psi(t,y) = C$, i.e. $y^2 t - \frac{1}{3}t^3 = C$. Plugging in the initial condition and solving for C, we see $1^2(3) - \frac{1}{3}(3^3) = C$, i.e. $C = 3 - 9 = -6$. So the particular solution is

$$y^2 t - \frac{1}{3}t^3 = -6.$$

7. This equation is second-order with no y. To solve it, let $v = \frac{dy}{dt}$ so that the equation becomes
$$tv' = v \quad \text{i.e.} \quad t\frac{dv}{dt} = v.$$
This is separable: rewrite it as $\frac{1}{v}\,dv = \frac{1}{t}\,dt$ and integrate both sides to obtain $\ln v = \ln t + C$. Solving for v, we get
$$v = e^{\ln t + C} = e^{\ln t}e^C = te^C = Ct.$$
Last, since $v = \frac{dy}{dt}$, integrate to obtain y:
$$y = \int v(t)\,dt = \int Ct\,dt = \frac{1}{2}Ct^2 + D.$$
Renaming the first constant, this can be written as $y = Ct^2 + D$.

2.2 Fall 2016 Exam 2

1. Here is a picture of the phase plane of an autonomous, first-order 2×2 system of ODEs $\mathbf{y}' = \Phi(\mathbf{y})$, where as usual, $\mathbf{y} = (x, y)$:



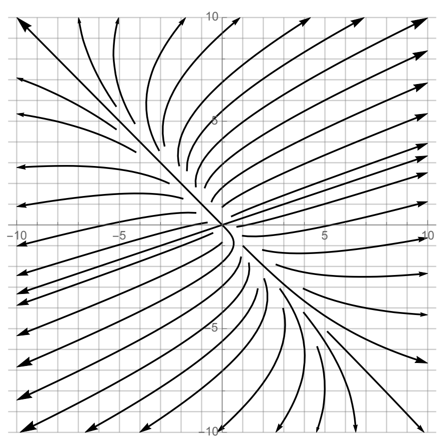
   a) Find the two equilibria of this system and classify each of them as a center, node, spiral or saddle.

   b) Suppose $x(0) = 10$ and $y(0) = 13$. Find the following four limits:

$$\lim_{t\to\infty} x(t) = \qquad \lim_{t\to\infty} y(t) = \qquad \lim_{t\to-\infty} x(t) = \qquad \lim_{t\to-\infty} y(t) =$$

   c) Suppose $\mathbf{y}(0) = (3, 0)$. Estimate the maximum value obtained by $x(t)$.

2. Here is the phase plane of a first-order, constant-coefficient, homogeneous linear 2×2 system of ODEs $\mathbf{y}' = A\mathbf{y}$:



   a) Find two eigenvectors of A (corresponding to distinct eigenvalues).

b) How many positive, real eigenvalues does A have?

c) How many negative, real eigenvalues does A have?

d) How many non-real eigenvalues does A have?

3. Find the general solution of the following system of ODEs:

$$\begin{cases} x' = & 2x & + & 4y & + & 2e^{2t} \\ y' = & x & - & y & + & e^{2t} \end{cases}$$

4. Find the general solution of the following system of ODEs:

$$\begin{cases} x' = -3x + y \\ y' = -x - 5y \end{cases}$$

Write your final answer coordinate-wise.

5. Find the particular solution of the following initial value problem:

$$\begin{cases} \mathbf{y}' = (x - 4y, 2x + 5y) \\ \mathbf{y}(0) = (3, -1) \end{cases}$$

Solutions

1. a) From looking at the picture, the two equilibria are $(9, 3)$ (which is a **saddle**) and $(13, 9)$ (which is a (unstable) **node**).

   b) By following the curve passing through $(10, 13)$ forwards and backwards, we see that

$$\lim_{t\to\infty} x(t) = -\infty \qquad \lim_{t\to\infty} y(t) = 9 \qquad \lim_{t\to-\infty} x(t) = 13 \qquad \lim_{t\to-\infty} y(t) = 9.$$

   c) The maximum value obtained by $x(t)$ is the right-most point on the curve passing through $(3, 0)$, which is approximately 5.

2. a) The eigenvectors of A go in the direction of the straight-line solutions; from the picture these are $(-1, 1)$ (or any multiple of $(-1, 1)$) and $(3, 1)$ (or any multiple of $(3, 1)$).

   b) Since **0** is an unstable node (from the picture), both eigenvalues of A are real and positive, so the answer is **two**.

   c) **None** (since the eigenvalues of A are both positive).

   d) **None** (since the eigenvalues of A are both positive and real).

3. Let $A = \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix}$. We start by finding the solution of the homogeneous equation $\mathbf{y}' = A\mathbf{y}$. First, the eigenvalues of A:

$$\begin{aligned} \det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 4 \\ 1 & -1-\lambda \end{pmatrix} &= (2-\lambda)(-1-\lambda) - 4 \\ &= \lambda^2 - \lambda - 6 \\ &= (\lambda - 3)(\lambda + 2) \end{aligned}$$

so the eigenvalues are $\lambda = 3$ and $\lambda = -2$. Now for the eigenvectors (let $\mathbf{v} = (x, y)$):

$$\lambda = 3 : A\mathbf{v} = \lambda\mathbf{v} \Rightarrow \begin{cases} 2x + 4y = 3x \\ x - y = 3y \end{cases} \Rightarrow x = 4y \Rightarrow (4, 1)$$

$$\lambda = -2 : A\mathbf{v} = \lambda\mathbf{v} \Rightarrow \begin{cases} 2x + 4y = -2x \\ x - y = -2y \end{cases} \Rightarrow x = -y \Rightarrow (1, -1)$$

Therefore the general solution of the homogeneous is

$$\mathbf{y}_h = C_1 e^{3t} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Now, we find a particular solution $\mathbf{y}_p$ using undetermined coefficients. Since $\mathbf{q} = (2e^{2t}, e^{2t}$, we guess $\mathbf{y}_p = (Ae^{2t}, Be^{2t}$. Plugging this into the system, we get

$$\begin{cases} 2Ae^{2t} & = 2(Ae^{2t}) + 4(Be^{2t}) + 2e^{2t} \\ 2Be^{2t} & = Ae^{2t} - Be^{2t} + e^{2t} \end{cases}$$

Dividing through by e^{2t}, we get

$$\begin{cases} 2A & = 2A + 4B + 2 \\ 2B & = A - B + 1 \end{cases}$$

In the first equation, the As cancel, so we can solve for B to get $B = \frac{-1}{2}$. From the second equation, we have $A = 3B - 1 = \frac{-5}{2}$ so

$$\mathbf{y}_p = \begin{pmatrix} Ae^{2t} \\ Be^{2t} \end{pmatrix} = \begin{pmatrix} \frac{-5}{2}e^{2t} \\ \frac{-1}{2}e^{2t} \end{pmatrix}.$$

Last, the solution is

$$\begin{aligned} \mathbf{y} &= \mathbf{y}_p + \mathbf{y}_h \\ &= \begin{pmatrix} \frac{-5}{2}e^{2t} \\ \frac{-1}{2}e^{2t} \end{pmatrix} + C_1 e^{3t} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{-5}{2}e^{2t} + 4C_1 e^{3t} + C_2 e^{-2t} \\ \frac{-1}{2}e^{2t} + C_1 e^{3t} - C_2 e^{-2t} \end{pmatrix}. \end{aligned}$$

4. Let $A = \begin{pmatrix} -3 & 1 \\ -1 & -5 \end{pmatrix}$ so that the system is $\mathbf{y}' = A\mathbf{y}$. First, find the eigenvalues of A:

$$\begin{aligned} \det(A - \lambda I) = \det \begin{pmatrix} -3-\lambda & 1 \\ -1 & -5-\lambda \end{pmatrix} &= (-3-\lambda)(-5-\lambda) + 1 \\ &= \lambda^2 + 8\lambda + 16 = (\lambda+4)^2 \end{aligned}$$

so the only eigenvalue is $\lambda = -4$ (repeated twice). Now for the eigenvector(s); let $\mathbf{v} = (x, y)$:

$$A\mathbf{v} = \lambda\mathbf{v} \Rightarrow \begin{cases} -3x + y = -4x \\ -3x + y = -4y \end{cases} \Rightarrow y = -x \Rightarrow \mathbf{v} = (1, -1)$$

We will also need a generalized eigenvector $\mathbf{w}$, which satisfies $(A-\lambda I)\mathbf{w} = \mathbf{v}$:

$$\begin{aligned}
&(A - \lambda I)\mathbf{w} = \mathbf{v} \\
\Rightarrow &\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
\Rightarrow &\begin{cases} x + y = & 1 \\ -x - y = & -1 \end{cases}
\end{aligned}$$

Any $\mathbf{w} = (x, y)$ satisfying $x + y = 1$ works in both these equations; let's use $x = 1$, $y = 0$ so that $\mathbf{w} = (1, 0)$.

Now, applying the formula from Theorem 2.68 from the lecture notes (which should be on your index card), we see that the solution has the form

$$\begin{aligned}
\mathbf{y} &= C_1 e^{\lambda t}\mathbf{v} + C_2 \left[e^{\lambda t}\mathbf{w} + t e^{\lambda t}\mathbf{v} \right] \\
&= C_1 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 \left[e^{4t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \\
&= \begin{pmatrix} C_1 e^{-4t} + C_2 e^{-4t} + C_2 t e^{-4t} \\ -C_1 e^{-4t} - C_2 t e^{-4t} \end{pmatrix} \\
&= \begin{pmatrix} (C_1 + C_2)\, e^{-4t} + C_2 t e^{-4t} \\ -C_1 e^{-4t} - C_2 t e^{-4t} \end{pmatrix}.
\end{aligned}$$

Writing this coordinate-wise as requested, we have the solution

$$\begin{cases} x(t) = (C_1 + C_2)\, e^{-4t} + C_2 t e^{-4t} \\ y(t) = -C_1 e^{-4t} - C_2 t e^{-4t} \end{cases}$$

5. Let $A = \begin{pmatrix} 1 & -4 \\ 2 & 5 \end{pmatrix}$. We start by finding the solution of the homogeneous equation $\mathbf{y}' = A\mathbf{y}$. First, the eigenvalues of A:

$$\begin{aligned}
\det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & -4 \\ 2 & 5 - \lambda \end{pmatrix} &= (1 - \lambda)(5 - \lambda) + 8 \\
&= \lambda^2 - 6\lambda + 13.
\end{aligned}$$

Setting this equal to zero and solving with the quadratic formula, we get

$$\lambda = \frac{6 \pm \sqrt{36 - 4(1)(13)}}{2 \cdot 1} = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i$$

so the eigenvalues are $\lambda = 3 + 2i$ and $\overline{\lambda} = 3 - 2i$. Now for the eigenvector corresponding to one of the eigenvalues (let $\mathbf{v} = (x, y)$):

$$\begin{aligned}\lambda = 3 + 2i:\ A\mathbf{v} = \lambda\mathbf{v} &\Rightarrow \begin{cases} x - 4y = (3 + 2i)x \\ 2x + 5y = (3 + 2i)y \end{cases} \\ &\Rightarrow -4y = (2 + 2i)x \\ &\Rightarrow -2y = (1 + i)x \\ &\Rightarrow \mathbf{v} = (-2, 1 + i) = (-2, 1) + i(0, 1)\end{aligned}$$

We have $\alpha = 3$, $\beta = 2$, $\mathbf{a} = (-2, 1)$ and $\mathbf{b} = (0, 1)$. So applying the formula from Theorem 2.67 of the lecture notes (which should be on your index card), we obtain the solution

$$\begin{aligned}\mathbf{y} &= C_1\left[e^{\alpha t}\cos(\beta t)\mathbf{a} - e^{\alpha t}\sin(\beta t)\mathbf{b}\right] + C_2\left[e^{\alpha t}\cos(\beta t)\mathbf{b} + e^{\alpha t}\sin(\beta t)\mathbf{a}\right] \\ &= C_1\left[e^{3t}\cos 2t\begin{pmatrix} -2 \\ 1 \end{pmatrix} - e^{3t}\sin 2t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] + C_2\left[e^{3t}\cos 2t\begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{3t}\sin 2t\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right] \\ &= \begin{pmatrix} -2C_1e^{3t}\cos 2t - 2C_2e^{3t}\sin 2t \\ (C_1 + C_2)e^{3t}\cos 2t + (C_2 - C_1)e^{3t}\sin 2t \end{pmatrix}.\end{aligned}$$

Now we plug in the initial condition $\mathbf{y}(0) = (3, -1)$ to find C_1 and C_2: plugging in, we see

$$\begin{aligned}&\begin{cases} 3 = -2C_1e^0\cos 0 - 2C_2e^0\sin 0 \\ -1 = (C_1 + C_2)e^0\cos 0 + (C_2 - C_1)e^0\sin 0 \end{cases} \\ &\Rightarrow \begin{cases} 3 = -2C_1 \\ -1 = C_1 + C_2 \end{cases} \\ &\Rightarrow C_1 = \frac{-3}{2}, C_2 = \frac{1}{2}.\end{aligned}$$

Therefore the particular solution is

$$\begin{aligned}\mathbf{y} &= \begin{pmatrix} -2\left(\frac{-3}{2}\right)e^{3t}\cos 2t - 2\left(\frac{1}{2}\right)e^{3t}\sin 2t \\ (\frac{-3}{2} + \frac{1}{2})e^{3t}\cos 2t + (\frac{1}{2} - \frac{-3}{2})e^{3t}\sin 2t \end{pmatrix} \\ &= \begin{pmatrix} 3e^{3t}\cos 2t - e^{3t}\sin 2t \\ -e^{3t}\cos 2t + 2e^{3t}\sin 2t \end{pmatrix}.\end{aligned}$$

2.3 Fall 2016 Exam 3

1. Suppose you are given a fourth-order, linear ODE. What is meant by an "initial value" of this ODE?

2. Convert this third-order ODE to a first-order system $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$.

$$e^{2t}y''' - 6ty'' + 4y' - y = 7\sin 3t$$

3. Find the particular solution of this initial value problem:

$$\begin{cases} y'' + 9y' + 18y = 0. \\ y(0) = -1 \\ y'(0) = 7 \end{cases}$$

4. Find the general solution of this ODE:

$$y^{(4)} - 14y^{(3)} + 49y'' = 0$$

5. Find the general solution of this ODE:

$$y'' - 4y' - 12y = 48e^{6t}$$

6. Find the general solution of this ODE:

$$y'' - 10y' + 34y = 0$$

7. A 4 kg mass is attached to a fixed point by a spring whose spring constant is 40 N/m. The mass moves back and forth along a line, subject to friction where the damping coefficient is 24 N sec/m. Suppose also that initiallly, the mass is 3 m to the right of its equilibrium position, and moving to the right at 2 m/sec.

   a) Suppose the mass is not subject to any external force. Find the position of the mass at time t.

   b) Suppose that the mass is subject to an external force of $20\sin 2t$ newtons. Find the position of the mass at time t.

Solutions

1. An "initial value" of a fourth-order ODE consists of values of y, y', y'' and y''' at the same value of t. In other words,

$$\mathbf{y}_0 = \begin{pmatrix} y(t_0) \\ y'(t_0) \\ y''(t_0) \\ y'''(t_0) \end{pmatrix}.$$

2. First, solve for y''' to get $y''' = e^{-2t}y - 4e^{-2t}y' + 6te^{-2t}y'' + 7e^{-2t}\sin 3t$. Then let $\mathbf{y} = (y, y', y'')$; then

$$\mathbf{y}' = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} 0y + 1y' + 0y'' \\ 0y + 0y' + 1y'' \\ e^{-2t}y - 4e^{-2t}y' + 6te^{-2t}y'' + 7\sin 3t \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ e^{-2t} & -4e^{-2t} & 6te^{-2t} \end{pmatrix}\mathbf{y} + \begin{pmatrix} 0 \\ 0 \\ 7e^{-2t}\sin 3t \end{pmatrix}.$$

   So by setting

$$\mathbf{y} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}, A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ e^{-2t} & -4e^{-2t} & 6te^{-2t} \end{pmatrix} \text{ and } \mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ 7e^{-2t}\sin 3t \end{pmatrix},$$

   the system becomes $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$, as desired.

3. The characteristic equation is $\lambda^2 + 9\lambda + 18 = (\lambda + 6)(\lambda + 3)$, which has roots -6 and -3, so the general solution is $y = C_1e^{-6t} + C_2e^{-3t}$. Differentiating, we get $y' = -6C_1e^{-6t} - 3C_2e^{-3t}$, so by plugging in the given initial conditions we get

$$\begin{cases} y(0) = -1 \\ y'(0) = 7 \end{cases} \Rightarrow \begin{cases} -1 = C_1 + C_2 \\ 7 = -6C_1 - 3C_2 \end{cases} \Rightarrow C_1 = \frac{-4}{3}, C_2 = \frac{1}{3}$$

   Therefore the particular solution is $y = \frac{-4}{3}e^{-6t} + \frac{1}{3}e^{-3t}$.

4. The characteristic equation is $l^4 - 14\lambda^3 + 49\lambda^2 = \lambda^2(\lambda - 7)^2$, which has roots 0 and 7 (both repeated twice) so the general solution is $y = C_1 + C_2t + C_3e^{7t} + C_4te^{7t}$.

5. The characteristic equation is $\lambda^2 - 4\lambda - 12 = (\lambda - 6)(\lambda + 2)$, which has roots 6 and -2, so the solution of the homogeneous is $y_h = C_1e^{6t} + C_2e^{-2t}$.

   Now for a particular solution y_p of the non-homogeneous equation. Since $q = 48e^{6t}$, we'd ordinarily guess $y_p = Ae^{6t}$, but since e^{6t} is part of the solution of the homogeneous, we need to multiply by t and guess $y_p = Ate^{6t}$. Thus by the Product Rule, $y_p' = Ae^{6t} + 6Ate^{6t}$ and $y_p'' = 12Ae^{6t} + 36Ate^{6t}$. Plugging in the original equation, we get

$$\begin{aligned} y_p'' - 4y_p' - 12y_p &= 48e^{6t} \\ 12Ae^{6t} + 36Ate^{6t} - 4(Ae^{6t} + 6Ate^{6t}) - 12Ate^{6t} &= 48e^{6t} \\ 8A &= 48 \\ A &= 6 \end{aligned}$$

Therefore $y_p = 6te^{6t}$, so the general solution is $y = y_p + y_h$, i.e.

$$y = 6te^{6t} + C_1e^{6t} + C_2e^{-2t}.$$

6. The characteristic equation is $\lambda^2 - 10\lambda + 34 = 0$ which has solutions

$$\lambda = \frac{10 \pm \sqrt{100 - 4(34)}}{2} = \frac{10 \pm \sqrt{-36}}{2} = \frac{10 \pm 6i}{2} = 5 \pm 3i.$$

Therefore the general solution is $y = C_1e^{5t}\cos 3t + C_2e^{5t}\sin 3t$.

7. Throughout this problem, let $x = x(t)$ be the position of the mass at time t. From the oscillator equation, we obtain the second-order ODE

$$\begin{aligned} mx'' + bx' + kx &= F_{ext}(t) \\ 4x'' + 24x' + 40x &= F_{ext}(t) \end{aligned}$$

Also, throughout the problem, we have the initial value $x(0) = 3, x'(0) = 2$.

a) In this part, assume $F_{ext}(t) = 0$. Then the characteristic equation is $4\lambda^2 + 24\lambda + 40\lambda = 4(\lambda^2 + 6\lambda + 10)$ which has solutions

$$\lambda = \frac{-6 \pm \sqrt{6^2 - 4(10)}}{2} = \frac{-6 \pm \sqrt{-4}}{2} = \frac{-6 \pm 2i}{2} = -3 \pm i.$$

Therefore the general solution of this ODE is $x = C_1e^{-3t}\cos t + C_2e^{-3t}\sin t$. Since $x(0) = 3$, we know $C_1 = 3$. Differentiating, we get

$$x' = -3C_1e^{3t}\cos t - C_1e^{3t}\sin t - 3C_2e^{-3t}\sin t + C_2e^{-3t}\cos t$$

and since $x'(0) = 2$, we get $-3C_1 + C_2 = 2$, i.e. $C_2 = 11$. Therefore the particular solution of this ODE is

$$x(t) = 3e^{-3t}\cos t + 11e^{-3t}\sin t.$$

b) In this part, assume $F_{ext}(t) = 20\sin 2t$. From part (a), the solution of the homogeneous is

$$x_h = C_1 e^{-3t}\cos t + C_2 e^{-3t}\sin t.$$

Now we need to find the x_p. Since $q = 20\sin 2t$, guess $x_p = A\sin 2t + B\cos 2t$. Differentiating, we get $x_p' = 2A\cos 2t - 2B\sin 2t$ and $x_p'' = -4A\sin 2t - 4B\cos 2t$, and by plugging in to the original equation we get

$$\begin{aligned} 4x_p'' + 24x_p' + 40x_p &= 20\sin 2t \\ 4(-4A\sin 2t - 4B\cos 2t) + 24(2A\cos 2t - 2B\sin 2t) + 40(A\sin 2t + B\cos 2t) &= 20\sin 2t \\ (24A - 48B)\sin 2t + (24B + 48A)\cos 2t &= 20\sin 2t \end{aligned}$$

Therefore we get the system of equations

$$\begin{cases} 24A - 48B = 20 \\ 24B + 48A = 0 \end{cases} \Rightarrow B = -2A \Rightarrow A = \frac{1}{6}, B = \frac{-1}{3}$$

so the particular solution is $x_p = \frac{1}{6}\sin 2t - \frac{1}{3}\cos 2t$. That makes the solution of the ODE $x = x_p + x_h$, i.e.

$$x(t) = \frac{1}{6}\sin 2t - \frac{1}{3}\cos 2t + C_1 e^{-3t}\cos t + C_2 e^{-3t}\sin t.$$

Since $x(0) = 3$, we have $C_2 - \frac{1}{3} = 3$, i.e. $C_2 = \frac{10}{3}$. Differentiating, we obtain

$$x'(t) = \frac{1}{3}\cos 2t + \frac{2}{3}\sin 2t - 3C_1 e^{3t}\cos t - C_1 e^{3t}\sin t - 3C_2 e^{-3t}\sin t + C_2 e^{-3t}\cos t$$

and since $x'(0) = 2$, we get $\frac{1}{3} - 3C_1 + C_2 = 2$, i.e. $C_2 = \frac{35}{3}$. Therefore the particular solution is

$$x(t) = \frac{1}{6}\sin 2t - \frac{1}{3}\cos 2t + \frac{10}{3}e^{-3t}\cos t + \frac{35}{3}e^{-3t}\sin t.$$

2.4 Fall 2016 Final Exam

1. a) Explain the difference between the terms "general solution" and "particular solution", in the context of ODEs.

   b) What is Euler's formula? Why is this formula important, in the context of ODEs?

   c) An evil professor tells a student to solve a 3×3 system of second-order, linear, homogeneous ODEs by hand. After hours of work, the student produces the following answer:

$$y = C_1 e^{t\sqrt{3}} + C_2 e^{-t\sqrt{3}} + C_3 e^{7t} + C_4 e^{-t}\cos(t\sqrt{2}) + C_5 e^{-t}\sin(t\sqrt{2}).$$

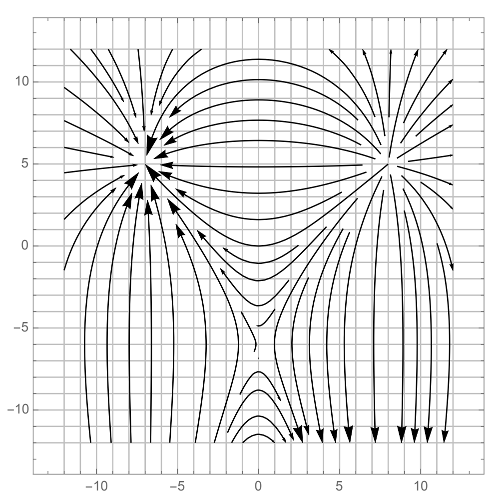
   Despite not having done the problem himself, the professor knows this answer is wrong, just by looking at it. Why?

2. Consider the initial value problem

$$\begin{cases} \mathbf{y}' = (y + 2t, x - y) \\ \mathbf{y}(0) = (1, 2) \end{cases}.$$

   Suppose you wanted to estimate $\mathbf{y}(100)$ using Euler's method with 20 steps. Compute the first two points (other than the given initial condition) obtained by this method.

3. Here is a picture of the phase plane of a 2×2 first-order system $\mathbf{y}' = \Phi(\mathbf{y})$:



   a) How many stable equilibria does this system have?

   b) How many unstable equilibria does this system have?

   c) Give the equation of any constant solution of the system.

d) Let $\mathbf{y}(t) = (x(t), y(t))$ be the solution to this system satisfying $\mathbf{y}(0) = (4, -1)$.

   i. Which statement best describes the behavior of the function $x(t)$?
      A. $x(t)$ is increasing for all t.
      B. $x(t)$ is decreasing for all t.
      C. $x(t)$ is increasing for small t, but decreasing for large t.
      D. $x(t)$ is decreasing for small t, but increasing for large t.

   ii. Which statement best describes the behavior of the function $y(t)$?
      A. $y(t)$ is increasing for all t.
      B. $y(t)$ is decreasing for all t.
      C. $y(t)$ is increasing for small t, but decreasing for large t.
      D. $y(t)$ is decreasing for small t, but increasing for large t.

   iii. Find $\lim\limits_{t\to\infty} x(t)$.

   iv. Find $\lim\limits_{t\to-\infty} y(t)$.

4. Find and classify all equilibria of the system

$$\begin{cases} x' = y - 2x \\ y' = y^2 - 2y - 80 \end{cases}.$$

5. Consider the parameterized family of ODEs $y' = \phi(y; r)$, where

$$\phi(y; r) = y^2 - r^2.$$

   Find the location(s) of any bifurcation(s) occurring in this family, classify the bifurcation(s), and sketch the bifurcation diagram for the family.

6. a) Find the particular solution of this initial value problem:

$$\begin{cases} y' = \frac{y+1}{t+1} \\ y(2) = 3 \end{cases}$$

   Write your answer as a function $y = f(t)$.

   b) Find the general solution of this ODE:

$$\frac{dy}{dt} = \sqrt{ty}$$

7. a) Find the general solution of this ODE:

$$ty' + 2y = 4t^2$$

b) Find the particular solution of this initial value problem:

$$\begin{cases} y'' - 9y' - 22y = 0 \\ y(0) = 9 \\ y'(0) = 8 \end{cases}$$

8. a) Find the general solution of this ODE:

$$y''' + y'' - 20y' = 56e^{2t}$$

b) Find the general solution of this system:

$$\begin{cases} x' = 3y \\ y' = 2x - y \end{cases}$$

9. Find the particular solution of this initial value problem:

$$\begin{cases} x' = 6x - y \\ y' = x + 4y \\ \mathbf{y}(0) = (3, -1) \end{cases}$$

10. Find the general solution of this ODE:

$$y'' = e^{-y}y'$$

Write your answer as a function $y = f(t)$.

11. Find the particular solution of this initial value problem:

$$\begin{cases} x' = 5x + 2y \\ y' = -5x - y \\ \mathbf{y}(0) = (1, 1) \end{cases}$$

12. A 40 L tank contains fresh water initially. A saline solution containing $.02$ kg of salt per liter is pumped into the tank at a rate of 4 L/min. At the same time, the tank drains through a pipe which removes solution from the tank at a rate of 4 L/min. Assuming the tank is kept well-stirred, how much salt is in the tank 3 minutes after this procedure starts?

13. An RLC series circuit consists of a $12\ \Omega$ resistor, a 4 H inductor, and a $\frac{1}{25}$ F capacitor. Assume that at time 0, the charge across the resistor is 13 coulombs, and the current running through the system is 3 amperes. If an external power supply of $102 \cos \frac{t}{2}$ V is applied to the circuit, find the charge in the circuit at time t.

Solutions

1. a) The **general solution** of an ODE (or system of ODEs) is a description of all its solutions (this description has arbitrary constants in it). Given an initial value problem, you can plug in the initial conditions to the general solution and solve for the constants, obtaining a **particular solution** of the IVP (which has no constants in it).

   b) **Euler's formula** says that for any complex number t, $e^{it} = \cos t + i \sin t$. This formula is important in solving systems of ODEs (and higher-order ODEs) because it tells you how to rewrite solutions obtained from complex eigenvalues and eigenvectors in terms of cosines and sines.

   c) If you reduce the order of a 3×3 second-order system, you will get a 6×6 first-order linear, homogeneous system (because $6 = 3 \cdot 2$) The solution of any 6×6 first-order linear, homogeneous system is a 6-dimensional subspace, so it has to have six arbitrary constants in it. The student's answer only has five arbitrary constants, so it has to be wrong.

2. First, let $\Delta t = \frac{t_n - t_0}{n} = \frac{100-0}{20} = 5$. Next, to establish notation, let the system be $\mathbf{y}' = \Phi(t, \mathbf{y}) = (\phi_1(t,(x,y)), \phi_2(t,(x,y)))$. We are given $(t_0, \mathbf{y}_0) = (0,(1,2))$. Therefore $t_1 = t_0 + \Delta t = 0 + 5 = 5$ and

$$\left\{\begin{array}{ll} x_1 & = x_0 + \phi_1(0,(1,2))\Delta t \\ & = 1 + (2 + 2(0))5 \\ & = 1 + 10 = 11 \\ y_1 & = y_0 + \phi_2(0,(1,2))\Delta t \\ & = 2 + (1-2)5 \\ & = 2 - 5 = -3 \end{array}\right. \Rightarrow (t_1, \mathbf{y}_1) = (5,(11,-3))$$

Next, $t_2 = t_1 + \Delta t = 5 + 5 = 10$ and

$$\left\{\begin{array}{ll} x_2 & = x_1 + \phi_1(0,(1,2))\Delta t \\ & = 11 + (-3 + 2(5))5 \\ & = 11 + 35 = 46 \\ y_2 & = y_1 + \phi_2(0,(1,2))\Delta t \\ & = -3 + (11 - (-3))5 \\ & = -3 + 70 = 67 \end{array}\right. \Rightarrow (t_2, \mathbf{y}_2) = (10,(46,67)).$$

3. a) The system has **one** stable equilibrium (at $(-7,5)$).

   b) The system has **two** unstable equilibria (at $(8,5)$ and $(0,-6)$).

   c) The equilibria are constant solutions, so any of these three answers are valid:

$$\left\{\begin{array}{l} x = -7 \\ y = 5 \end{array}\right. \qquad \left\{\begin{array}{l} x = 8 \\ y = 5 \end{array}\right. \qquad \left\{\begin{array}{l} x = 0 \\ y = -6 \end{array}\right.$$

   (There are other ways to write this; for example, $\mathbf{y} = (-7,5)$, etc.)

d) Let $\mathbf{y}(t) = (x(t), y(t))$ be the solution to this system satisfying $\mathbf{y}(0) = (4, -1)$.

i. From the phase plane, $x(t)$ decreases then increases. The answer is **D**.

ii. From the phase plane, $y(t)$ is always decreasing. The answer is **B**.

iii. From the phase plane, $\lim\limits_{t\to\infty} x(t) = 8$.

iv. From the phase plane, $\lim\limits_{t\to-\infty} y(t) = 5$.

4. Thinking of the system as $\mathbf{y}' = \Phi(\mathbf{y})$, we set $\Phi(\mathbf{y}) = \mathbf{0}$ and solve for x and y:

$$\begin{cases} 0 = y - 2x \\ 0 = y^2 - 2y - 80 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}y \\ 0 = (y-10)(y+8) \end{cases}$$

From the second equation, $y = 10$ or $y = -8$. From the first equation, the corresponding x-values are 5 and -4, so the two equilibria are $(5, 10)$ and $(-4, -8)$.

To classify the equilibria, compute the total derivative:

$$D\Phi = \begin{pmatrix} \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_1}{\partial y} \\ \frac{\partial \phi_2}{\partial x} & \frac{\partial \phi_2}{\partial y} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & 2y - 2 \end{pmatrix}$$

Since this matrix is upper triangular, its eigenvalues are -2 and $2y - 2$. That means that for the equilibrium $(5, 10)$, the eigenvalues of $D\Phi(5, 10)$ are -2 and 18. Since there is one positive and one negative eigenvalue, $(5, 10)$ is an **unstable saddle**.

For the equilibrium $(-4, -8)$, the eigenvalues of $D\Phi(-4, -8)$ are -2 and -18. Since both eigenvalues are negative and real, $(-4, -8)$ is a **stable node**.

5. We start by finding the equilibria of the system in terms of r: set $\phi(y; r) = 0$ and solve for y to get

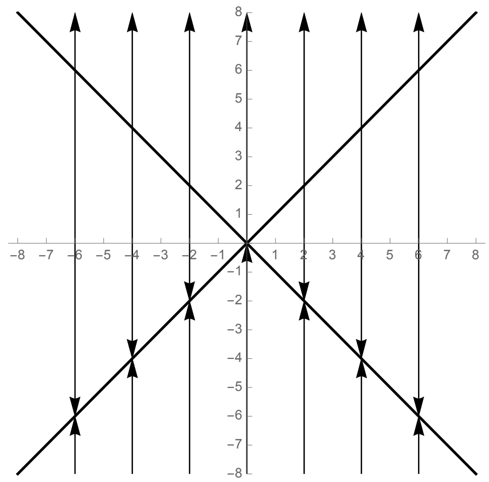
$$0 = y^2 - r^2 \Rightarrow 0 = (y - r)(y + r) \Rightarrow y = r, y = -r.$$

Next, classify these equilibria using the sign of ϕ'. $\phi'(y) = 2y$, so we conclude

$$\phi'(r) = 2r \Rightarrow y = r \text{ is } \begin{cases} \text{stable if } r < 0 \\ \text{semistable if } r = 0 \\ \text{unstable if } r > 0 \end{cases}$$

$$\phi'(-r) = -2r \Rightarrow y = -r \text{ is } \begin{cases} \text{unstable if } r < 0 \\ \text{semistable if } r = 0 \\ \text{stable if } r > 0 \end{cases}$$

We can now sketch the bifurcation diagram:



That means there is a **transcritical bifurcation** at $r = 0$ because the two equilibria cross and change behavior.

6. a) Separate the variables and integrate both sides:

$$\begin{aligned}
\frac{dy}{dt} &= \frac{y+1}{t+1} \\
\frac{1}{y+1}\,dy &= \frac{1}{t+1}\,dt \\
\int \frac{1}{y+1}\,dy &= \int \frac{1}{t+1}\,dt \\
\ln(y+1) &= \ln(t+1) + C \\
y + 1 &= e^{\ln(t+1)+C} = C(t+1) \\
y &= C(t+1) - 1
\end{aligned}$$

Now plug in the initial condition $y(2) = 3$ to get $3 = C(2+1) - 1$ and solve for C to get $C = \frac{4}{3}$. Thus the particular solution is $y = \frac{4}{3}(t+1) - 1$, i.e.

$$y = \frac{4}{3}t + \frac{1}{3}.$$

b) Separate the variables and integrate both sides:

$$\begin{aligned}\frac{dy}{dt} &= \sqrt{ty}\\ \frac{1}{\sqrt{y}}\,dy &= \sqrt{t}\,dt\\ \int \frac{1}{\sqrt{y}}\,dy &= \int \sqrt{t}\,dt\\ 2\sqrt{y} &= \frac{2}{3}t^{3/2} + C.\end{aligned}$$

(If you solved for y, you'd get $y = (\frac{1}{3}t^{3/2} + C)^2 = \frac{1}{9}t^3 + \frac{2}{3}Ct\sqrt{t} + C^2$.)

7. a) First, divide through by t to write the equation as $y' + \frac{2}{t}y = 4t$. Then, compute the integrating factor:

$$\mu(t) = e^{\int_0^t p_0(s)\,ds} = e^{\int_0^t \frac{2}{s}\,ds} = e^{2\ln t} = t^2.$$

Multiply through by μ to obtain the equation

$$\begin{aligned}\frac{d}{dt}(y\mu) &= 4t(t^2)\\ \frac{d}{dt}\left(yt^2\right) &= 4t^3\\ yt^2 &= t^4 + C\\ y &= t^2 + Ct^{-2}.\end{aligned}$$

b) The characteristic equation is $0 = \lambda^2 - 9\lambda - 22 = (\lambda - 11)(\lambda + 2)$ which has roots $\lambda = 11$ and $\lambda = -2$. Therefore the general solution is $y = C_1e^{11t} + C_2e^{-2t}$. To find the particular solution, differentiate to get $y' = 11C_1e^{11t} - 2C_2e^{-2t}$ and plug in the initial values to get

$$\begin{cases} 9 = C_1 + C_2 \\ 8 = 11C_1 - 2C_2 \end{cases} \Rightarrow C_1 = 2, C_2 = 7$$

Therefore the particular solution is $y = 2e^{11t} + 7e^{-2t}$.

8. a) The characteristic equation is $\lambda^3 + \lambda^2 - 20\lambda = \lambda(\lambda + 5)(\lambda - 4)$ which has roots $\lambda = 0$, $\lambda = 4$ and $\lambda = -5$. Therefore the general solution of the homogeneous is $y = C_1 + C_2e^{4t} + C_3e^{-5t}$. Since $q = 56e^{2t}$, guess $y_p = Ae^{2t}$ and plug in the original equation to get

$$8Ae^{2t} + 4Ae^{2t} - 40Ae^{2t} = 56e^{2t} \Rightarrow -28A = 56 \Rightarrow A = -2.$$

Therefore $y_p = -2e^{2t}$ so the general solution is $y = y_p + y_h$, i.e.

$$y = -2e^{2t} + C_1 + C_2e^{4t} + C_3e^{-5t}.$$

b) Thinking of the system as $\mathbf{y}' = A\mathbf{y}$, start with the eigenvalues of A:

$\det(A-\lambda I) = (0-\lambda)(-1-\lambda)-2(-3) = \lambda^2+\lambda+6 = (\lambda+3)(\lambda-2) \Rightarrow \lambda = -3, \lambda = 2$

Next, eigenvectors. Let $\mathbf{v} = (x, y)$ and solve $A\mathbf{v} = \lambda\mathbf{v}$ for each eigenvalue:

$$\lambda = -3 : \begin{cases} 3y = -3x \\ 2x - y = -3y \end{cases} \Rightarrow y = -x \Rightarrow \mathbf{v} = (1, -1)$$

$$\lambda = 2 : \begin{cases} 3y = 2x \\ 2x - y = 2y \end{cases} \Rightarrow 3y = 2x \Rightarrow \mathbf{v} = (3, 2)$$

Thus the general solution is

$$\mathbf{y} = C_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Written coordinate-wise (not required), this is

$$\begin{cases} x(t) = C_1 e^{-3t} + 3C_2 e^{2t} \\ y(t) = -C_1 e^{-3t} + 2C_2 e^{2t} \end{cases}$$

9. Thinking of the system as $\mathbf{y}' = A\mathbf{y}$, start by finding eigenvalues of A:

$$\det(A - \lambda I) = (6 - \lambda)(4 - \lambda) + 1 = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2$$

so the only eigenvalue of A is $\lambda = 5$. Next, find eigenvector(s): let $\mathbf{v} = (x, y)$ and set $A\mathbf{v} = \lambda\mathbf{v}$ to get

$$\begin{cases} 6x - y = 5x \\ x + 4y = 5y \end{cases} \Rightarrow x = y \Rightarrow \mathbf{v} = (1, 1)$$

Now, find a generalized eigenvector $\mathbf{w} = (x, y)$ by solving $(A - \lambda I)\mathbf{w} = \mathbf{v}$:

$$\begin{cases} x - y = 1 \\ x - y = 1 \end{cases} \Rightarrow \mathbf{w} = (1, 0)$$

Now, our theorem on repeated eigenvalues tells us that the general solution is

$$\mathbf{y} = C_1 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \left[e^{5t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right].$$

Coordinate-wise, this is

$$\begin{cases} x = (C_1 + C_2)e^{5t} + C_2 te^{5t} \\ y = C_1 e^{5t} + C_2 te^{5t} \end{cases}$$

To find the particular solution, plug in the initial condition to get $3 = C_1 + C_2$ and $-1 = C_1$. Therefore $C_2 = 4$ so the particular solution is

$$\begin{cases} x = 3e^{5t} + 4te^{5t} \\ y = -e^{5t} + 4te^{5t} \end{cases}.$$

10. This is a second-order, non-linear equation with no t in it. Think of y as the independent variable and let $v = y' = \frac{dy}{dt}$. Then $y'' = v\frac{dv}{dy}$ so the equation becomes
$$v\frac{dv}{dy} = e^{-y}v.$$
This equation can be solved by separating variables and integrating both sides:
$$dv = e^{-y}\,dy \Rightarrow v = -e^{-y} + C.$$
Now back-substitute for v to get the equation
$$\frac{dy}{dt} = -e^{-y} + C = \frac{-1}{e^y} + C = \frac{Ce^y - 1}{e^y}.$$
This equation is separable and can be rewritten as
$$\frac{e^y}{Ce^y - 1}\,dy = dt;$$
integrate both sides (you need the u-substitution $u = Ce^y - 1$ on the left-hand side) to get
$$\frac{1}{C}\ln(Ce^y - 1) = t + D.$$
Solve for y to get
$$y = \ln\left[\frac{1}{C}(e^{C(t+D)} + 1)\right], \text{ i.e. } y = \ln(e^{Ct+D} + 1) - \ln C.$$

11. Think of the system $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$ and start by finding the eigenvalues of A:
$$\det(A-\lambda I) = (5-\lambda)(-1-\lambda)+10 = \lambda^2-4\lambda+5 \Rightarrow \lambda = \frac{4 \pm \sqrt{4^2 - 4(5)}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2\pm i$$
Solve for the eigenvector $\mathbf{v} = (x, y)$ corresponding to $2 + i$:
$$\begin{cases} 5x + 2y = (2+i)x \\ -5x - y = (2+i)y \end{cases} \Rightarrow 2y = (-3+i)x \Rightarrow \mathbf{v} = \begin{pmatrix} 2 \\ -3+i \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$
We have $\alpha = 2$, $\beta = 1$, $\mathbf{a} = (2, -3)$ and $\mathbf{b} = (0, 1)$, so by the theorem governing solutions with complex eigenvalues we have
$$\mathbf{y} = C_1\left[e^{2t}\cos t\begin{pmatrix} 2 \\ -3 \end{pmatrix} - e^{2t}\sin t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] + C_2\left[e^{2t}\cos t\begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{2t}\sin t\begin{pmatrix} 2 \\ -3 \end{pmatrix}\right].$$
To find the particular solution, plug in the initial condition to get
$$\begin{cases} 1 = 2C_1 \\ 1 = -3C_1 + C_2 \end{cases} \Rightarrow C_1 = \frac{1}{2}, C_2 = \frac{5}{2}$$

Therefore the particular solution, written coordinate-wise, is

$$\begin{cases} x(t) = e^{2t}\cos t + 5e^{2t}\sin t \\ y(t) = e^{2t}\cos t - 8e^{2t}\sin t \end{cases}.$$

12. Let $y(t)$ be the amount of salt in the tank at time t. The rate at which salt enters the tank is $.02$ kg/L $\times 4$ L/min $= .08 = \frac{2}{25}$ kg/min, and the rate at which salt leaves the tank is $y/40$ kg/L $\times 4$ L/min = $\frac{1}{10}y$ kg/min. This leads to the initial value problem

$$\begin{cases} y' = \frac{2}{25} - \frac{1}{10}y \\ y(0) = 0 \end{cases}$$

($y(0) = 0$ because the water is initially fresh.) To solve the ODE, one can use integrating factors or undetermined coefficients. Using undetermined coefficients, the corresponding homogeneous equation is $y' = \frac{-1}{10}y$ which has solution $y_h = Ce^{-t/10}$. Since $q = \frac{2}{25}$, guess $y_p = A$ and plug in to obtain $0 = \frac{2}{25} - \frac{A}{10}$. Solve for A to get $A = \frac{4}{5}$, so $y_p = \frac{4}{5}$ and the general solution is therefore

$$y = y_p + y_h = Ce^{-t/10} + \frac{4}{5}.$$

Solve for C using the initial condition $y(0) = 0$ to get the particular solution $y = -\frac{4}{5}e^{-t/10} + \frac{4}{5}$. Therefore at time 3, the amount of salt in the tank is

$$y(3) = -\frac{4}{5}e^{-3/10} + \frac{4}{5}.$$

13. The RLC series circuit equation is

$$Lq''(t) + Rq'(t) + \frac{1}{C}q(t) = E_S(t)$$

where q is the charge. In this problem, the above equation leads to the initial value problem

$$\begin{cases} 4q'' + 12q' + 25q = 102\cos\frac{t}{2} \\ q(0) = 13 \\ q'(0) = 3 \end{cases}$$

To solve the IVP, start with the characteristic equation $0 = 4\lambda^2 + 12\lambda + 25 = 4(\lambda^2 + 3\lambda + \frac{25}{4})$ which has roots

$$\lambda = \frac{-3 \pm \sqrt{9-25}}{2} = \frac{-3}{2} \pm 2i$$

Therefore the general solution of the homogeneous is

$$q_h = C_1 e^{-3t/2}\cos 2t + C_2 e^{-3t/2}\sin 2t.$$

To find q_p, guess $q_p = A\cos\frac{t}{2} + B\sin\frac{t}{2}$. Then $q_p' = -\frac{A}{2}\sin\frac{t}{2} + \frac{B}{2}\cos\frac{t}{2}$ and $q_p'' = \frac{-A}{4}\cos\frac{t}{2} - \frac{B}{4}\sin\frac{t}{2}$. Plugging in the original equation, we get

$$4\left(\frac{-A}{4}\cos\frac{t}{2} - \frac{B}{4}\sin\frac{t}{2}\right) + 12\left(\frac{-A}{2}\sin\frac{t}{2} + \frac{B}{2}\cos\frac{t}{2}\right) + 25\left(A\cos\frac{t}{2} + B\sin\frac{t}{2}\right) = 102\cos\frac{t}{2}$$
$$(24A + 6B)\cos\frac{t}{2} + (24B - 6A)\sin\frac{t}{2} = 102\cos\frac{t}{2}$$

Therefore $24A + 6B = 102$ and $24B - 6A = 0$; solving for A and B we get $B = 1$ and $A = 4$. Therefore $y_p = 4\cos\frac{t}{2} + \sin\frac{t}{2}$ so the general solution is

$$q = q_p + q_h = 4\cos\frac{t}{2} + \sin\frac{t}{2} + C_1e^{-3t/2}\cos 2t + C_2e^{-3t/2}\sin 2t.$$

To find C_1 and C_2, differentiate to get

$$q' = -2\sin\frac{t}{2} + \frac{1}{2}\cos\frac{t}{2} - \frac{3}{2}C_1e^{-3t/2}\cos 2t - 2C_1e^{-3t/2}\sin 2t - \frac{3}{2}C_2e^{-3t/2}\sin 2t + 2C_2e^{-3t/2}\cos 2t;$$

then plug in the initial conditions to get

$$\begin{cases} 13 = 4 + C_1 \\ 3 = \frac{1}{2} - \frac{3}{2}C_1 + 2C_2 \end{cases} \Rightarrow C_1 = 9, C_2 = 8$$

Therefore the particular solution is $q(t) = q_p + q_h$, i.e.

$$q(t) = 4\cos\frac{t}{2} + \sin\frac{t}{2} + 9e^{-3t/2}\cos 2t + 8e^{-3t/2}\sin 2t.$$

Chapter 3

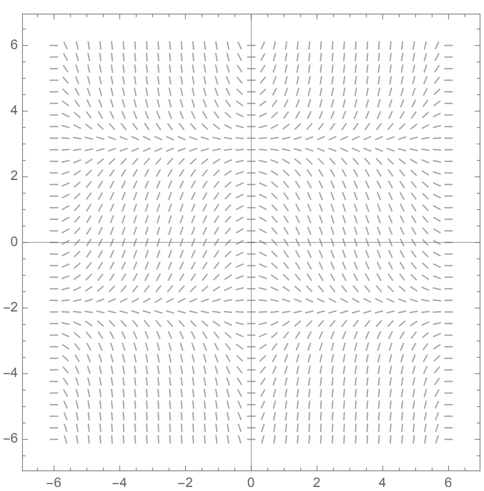
Exams from Fall 2017

3.1 Fall 2017 Exam 1

1. Briefly explain the difference between the terms "general solution" and "particular solution" in the context of ordinary differential equations.

2. Sketch the phase line for the autonomous ODE $y' = 5y^2 - y^3$.

3. Here is the picture of the slope field associated to some first-order ODE $y' = \phi(t, y)$:



   a) Write the equation of any one solution of this ODE.
   b) Suppose $y(0) = -5$. Find $\lim_{t\to\infty} y(t)$.
   c) Suppose $y(-1) = 1$. Estimate $y(2)$.
   d) On the picture above, sketch the graph of the solution satisfying the initial condition $y(3) = -4$.

e) Let $y = h(t)$ be the solution of the initial value problem

$$\begin{cases} y' = \phi(t, y) \\ y(0) = 2 \end{cases}$$

Suppose you used Euler's method to estimate $h(12)$ using 3 steps. What are the coordinates of the point you would you obtain as (t_1, y_1)?

4. Find the general solution of the following ODE:

$$\frac{dy}{dt} = 5y + 20e^{5t}$$

Write your answer as a function $y = f(t)$.

5. Find the particular solution of the following initial value problem:

$$\begin{cases} \frac{dy}{dt} = \frac{ty}{t^2+1} \\ y(0) = 4 \end{cases}$$

Write your answer as a function $y = f(t)$.

6. Find the particular solution of the following initial value problem:

$$\begin{cases} y' = \frac{y}{t} - te^{-t} \\ y(-1) = 0 \end{cases}$$

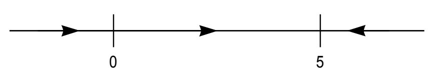
7. Find the general solution of the following ODE:

$$y' = \frac{\cos t - y - \cos y}{t - t\sin y}$$

Solutions

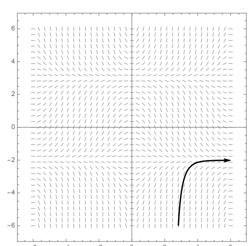
1. Given an ODE, the set of all solutions of that ODE is called the **general solution** of the ODE. This solution will have one or more arbitrary constants in it. If you are given an initial value, you can plug that initial value into the general solution, solving for the constant in the general solution. This produces a solution of the ODE with no arbitrary constants, which is called a **particular solution** of the ODE.

2. $\phi(y) = 5y^2 - y^3$; set $\phi(y) = 0$ and factor to solve for y: this gives $0 = y^2(5-y)$ so the two equilibria are $y = 0$ and $y = 5$. To classify them, compute derivatives of ϕ and plug in the equilibria. When $y = 5$, $\phi'(5) = 50 - 75 < 0$ so 5 is stable; when $y = 0$; $\phi'(0) = 0$ but $\phi''(0) = 10 \neq 0$ so 0 is semistable. Therefore the phase line looks like this:



3. Here is the picture of the slope field associated to some first-order ODE $y' = \phi(t, y)$:

   a) $y = 3$ and $y = -2$ are solutions.
   b) If $y(0) = -5$, then $\lim\limits_{t\to\infty} y(t) = -2$.
   c) If $y(-1) = 1$, then $y(2) \approx -1$.
   d) Here is the graph:



   e) Notice that $\phi(0, 2)$ is the slope of the vector field at the point $(0, 2)$, which is 0. So if you used Euler's method with 3 steps, $\Delta t = \frac{t_n - t_0}{n} = \frac{12-0}{3} = 4$ so
$$\begin{cases} t_1 = t_0 + \Delta t = 0 + 4 = 3 \\ y_1 = y_0 + \phi(t_0, y_0)\Delta t = 2 + \phi(0, 2) \cdot 4 = 2 + 0 \cdot 4 = 2 \end{cases}$$
   and therefore $(t_1, y_1) = (4, 2)$.

4. *Solution # 1:* Rewrite in standard form as $y' - 5y = 20e^{5t}$; then the integrating factor is
$$\mu = \exp\left(\int -5\, dt\right) = e^{-5t}.$$

After multiplying through by the integrating factor, the equation becomes

$$\frac{d}{dt}\left(ye^{-5t}\right) = 20e^{5t}e^{-5t} = 20.$$

Integrate both sides to get

$$ye^{-5t} = 20t + C;$$

solve for y to get $y = 20te^{5t} + Ce^{5t}$.

Solution # 2: Rewrite in standard form as $y' - 5y = 20e^{5t}$ and use undetermined coefficients. The corresponding homogeneous equation is $y' - 5y = 0$ which has solution $y_h = e^{5t}$. Since the right-hand side of the ODE is $20e^{5t}$, a normal guess for the particular solution would be $y_p = Ae^{5t}$ but since this is the same as y_h up to a constant, you need to multiply the guess by t, i.e. $y_p = Ate^{5t}$. Plugging in the equation, we get

$$\begin{aligned}
y' - 5y &= 20e^{5t} \\
(Ate^{5t})' - 5(Ate^{5t}) &= 20e^{5t} \\
Ae^{5t} + 5Ate^{5t} - 5Ate^{5t} &= 20e^{5t} \\
Ae^{5t} &= 20e^{5t} \\
A &= 20
\end{aligned}$$

Therefore $y_p = 20te^{5t}$, so the solution is $y = y_p + Cy_h = 20te^{5t} + Ce^{5t}$.

5. This equation is separable; divide both sides by y and multiply both sides by dt to obtain

$$\frac{1}{y}\,dy = \frac{t}{t^2+1}\,dt.$$

Then integrate both sides (you need the u-sub $u = t^2 + 1$ on the right) to get

$$\ln y = \frac{1}{2}\ln(t^2+1) + C;$$

solving for y by exponentiating both sides gives

$$y = e^{\frac{1}{2}\ln(t^2+1)+C} = C\sqrt{t^2+1}.$$

Now, plug in the initial condition $t = 0, y = 4$ to get $4 = C\sqrt{0^2+1}$, i.e. $C = 4$. Thus the particular solution is $y = 4\sqrt{t^2+1}$.

6. This equation is first-order linear; rewrite it as

$$y' - \frac{1}{t}y = -te^{-t}.$$

From this point there are two methods of solution:

Solution # 1: The integrating factor is

$$\mu = \exp\left(\int \frac{-1}{t}\,dt\right) = \exp(-\ln t) = t^{-1} = \frac{1}{t};$$

after multiplying through the equation by the integrating factor we get

$$\frac{d}{dt}\left(\frac{y}{t}\right) = -e^{-t}.$$

Integrate both sides to get

$$\frac{y}{t} = e^{-t} + C.$$

Now plug in the initial condition $t = -1, y = 0$ to get $0 = e + C$, i.e. $C = -e$. Thus the particular solution is

$$\frac{y}{t} = e^{-t} - e.$$

If you solved for y (not required), this can be rewritten as $y = te^{-t} - et$.

Solution # 2: The corresponding homogeneous equation is $y' - \frac{1}{t}y = 0$ which has solution

$$y_h = \exp\left(\int \frac{1}{t}\,dt\right) = e^{\ln t} = t.$$

For the particular solution, guess $y_p = Ate^{-t} + Be^{-t}$; plugging in the equation we get

$$\begin{aligned} y' - \frac{1}{t}y &= -e^{-t} \\ (Ate^{-t} + Be^{-t})' - \frac{1}{t}(Ate^{-t} + Be^{-t}) &= -e^{-t} \\ Ae^{-t} - Ate^{-t} - Be^{-t} - Ae^{-t} + B\frac{1}{t}e^{-t} &= -e^{-t} \\ -Ate^{-t} - Be^{-t} + \frac{B}{t}e^{-t} &= -e^{-t} \end{aligned}$$

Therefore $-A = -1$ so $A = 1$ and $B = 0$, so $y_p = te^{-t}$. That makes the general solution $y = y_p + Cy_h = te^{-t} + Ct$. Plugging in the initial condition as in Solution # 1 gives the same particular solution: $C = -e$ and therefore $y = te^{-t} - et$.

7. Rewrite the equation as

$$y + \cos y - \cos t + (t - t\sin y)y' = 0.$$

Now let $M = y + \cos y - \cos t$ and let $N = t - t\sin y$. We see that $M_y = N_t = 1 - \sin y$ so the equation is exact. Now find ψ by integrating:

$$\psi = \int M\,dt = \int (y + \cos y - \cos t)\,dt = yt + t\cos y - \sin t + A(y)$$
$$\psi = \int N\,dy = \int (t - t\sin y)\,dy = ty + t\cos y + B(t)$$

To reconcile these answers, set $B(t) = -\sin t$ and $A(y) = 0$ so that $\psi(t, y) = ty + t\cos y - \sin t$. The general solution is therefore $\psi(t, y) = C$, i.e. $ty + t\cos y - \sin t = C$.

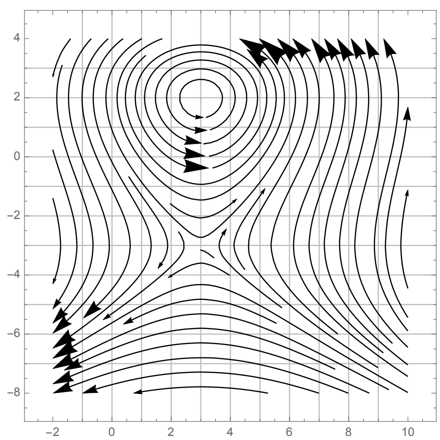
3.2 Fall 2017 Exam 2

1. Let $\mathbf{y}(t) = (x(t), y(t))$ be the solution of the initial value problem

$$\begin{cases} \mathbf{y}' = (x - 2y + t, y - 3x) \\ \mathbf{y}(0) = (2, 1) \end{cases}.$$

Suppose you wanted to estimate $\mathbf{y}(80)$ using Euler's method with 40 steps. Compute the points $\mathbf{y}_1$ and $\mathbf{y}_2$ that would be obtained by this method.

2. Here is the phase plane of a first-order, autonomous 2×2 system of ODEs $\mathbf{y}' = \Phi(\mathbf{y})$:



Use this picture to answer the following questions:

a) Find all saddles of this system (if there are no saddles, say so).

b) Find all nodes of this system (if there are no nodes, say so).

c) Find all centers of this system (if there are no centers, say so).

d) Is the trace of $D\Phi(3, 2)$ positive, negative or zero?

e) Is the determinant of $D\Phi(3, 2)$ positive, negative or zero?

f) Suppose $\mathbf{y}(0) = (5, -6)$. In this situation, which statement best describes the behavior of $x(t)$?

A. $x(t)$ increases for all t
B. $x(t)$ decreases for all t
C. initially, $x(t)$ is increasing, but then it becomes decreasing
D. initially, $x(t)$ is decreasing, but then it becomes increasing

g) Suppose $\mathbf{y}(0) = (5, -6)$. In this situation, which statement best describes the behavior of $y(t)$?

A. $y(t)$ increases for all t

B. $y(t)$ decreases for all t
C. initially, $y(t)$ is increasing, but then it becomes decreasing
D. initially, $y(t)$ is decreasing, but then it becomes increasing

3. Find the particular solution of the following initial value problem:

$$\begin{cases} x' = -7x + 5y \\ y' = x - 3y \end{cases} \qquad \begin{cases} x(0) = 2 \\ y(0) = 4 \end{cases}$$

4. Find the general solution of the following system of ODEs:

$$\begin{cases} x' &= 5x &- &4y &+ &3e^{-t} \\ y' &= 4x &- &5y &+ &4e^{-t} \end{cases}$$

5. Find the particular solution of the following initial value problem:

$$\begin{cases} \mathbf{y}' = (-7x + 2y, -25x + 3y) \\ \mathbf{y}(0) = (1, -5) \end{cases}$$

Write your final answer coordinate-wise.

Solutions

1. First, $\Delta t = \frac{t_n - t_0}{n} = \frac{80-0}{40} = 2$. Now, $(t_0, \mathbf{y}_0) = (0,(2,1))$ and $\Phi(\mathbf{y}_0) = (2-2(1)+0,\ 1-3(2)) = (0,-5)$ so by the Euler's method formula,

$$\begin{cases} t_1 = t_0 + \Delta t = 0 + 2 = 2 \\ \mathbf{y}_1 = \mathbf{y}_0 + \Phi(t_0, \mathbf{y}_0)\Delta t = (2,1) + (0,-5)2 = (2,-9)\,. \end{cases}$$

Now, $\Phi(t_1, \mathbf{y}_1) = (2-2(-9)+2, -9-3(2)) = (22,-15)$ so

$$\begin{cases} t_2 = t_1 + \Delta t = 2 + 2 = 4 \\ \mathbf{y}_2 = \mathbf{y}_1 + \Phi(t_1, \mathbf{y}_1)\Delta t = (2,-9) + (22,-15)2 = (46,-39)\,. \end{cases}$$

2. a) The only saddle is at $(3,-3)$.
   b) This system has no nodes.
   c) The only center is at $(3,2)$.
   d) Since $(3,2)$ is a center, $tr(D\Phi(3,2)) = 0$.
   e) Since $(3,2)$ is a center, $\det(D\Phi(3,2))$ is positive.
   f) **B.** Since the graph of the solution always moves to the left (in the direction of increasing t), $x(t)$ decreases for all t.
   g) **C.** In the direction of increasing t, the graph of the solution initially goes upwards, then downwards.

3. Let $A = \begin{pmatrix} -7 & 5 \\ 1 & -3 \end{pmatrix}$ so that the system is $\mathbf{y}' = A\mathbf{y}$. First, find the eigenvalues of A:

$$\begin{aligned} \det(A - \lambda I) = \det \begin{pmatrix} -7-\lambda & 5 \\ 1 & -3-\lambda \end{pmatrix} &= (-7-\lambda)(-3-\lambda) - 5 \\ &= \lambda^2 + 10\lambda + 16 = (\lambda+8)(\lambda+2) \end{aligned}$$

so the eigenvalues are $\lambda = -8$ and $\lambda = -2$. Now for the eigenvectors; let $\mathbf{v} = (x,y)$:

$$\lambda = -8 : A\mathbf{v} = \lambda\mathbf{v} \Rightarrow \begin{cases} -7x + 5y = -8x \\ x - 3y = -8y \end{cases} \Rightarrow -x = 5y \Rightarrow (5,-1)$$

$$\lambda = -2 : A\mathbf{v} = \lambda\mathbf{v} \Rightarrow \begin{cases} -7x + 5y = -2x \\ x - 3y = -2y \end{cases} \Rightarrow x = y \Rightarrow (1,1)$$

Therefore the general solution is

$$\mathbf{y} = C_1 e^{-8t} \begin{pmatrix} 5 \\ -1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

To find the particular solution, plug in the initial value $\mathbf{y}(0) = (2, 4)$ to get

$$\begin{cases} 2 = 5C_1 + C_2 \\ 4 = -C_1 + C_2 \end{cases} \Rightarrow 6C_1 = -2 \Rightarrow C_1 = \frac{-1}{3}, C_2 = \frac{11}{3}.$$

Therefore the particular solution is

$$\begin{aligned} \mathbf{y} &= \frac{-1}{3} e^{-8t} \begin{pmatrix} 5 \\ -1 \end{pmatrix} + \frac{11}{3} e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{-5}{3} e^{-8t} + \frac{11}{3} e^{-2t} \\ \frac{1}{3} e^{-8t} + \frac{11}{3} e^{-2t} \end{pmatrix}. \end{aligned}$$

4. Let $A = \begin{pmatrix} 5 & -4 \\ 4 & -5 \end{pmatrix}$ so that the corresponding homogeneous system is $\mathbf{y}' = A\mathbf{y}$. First, find the eigenvalues of A:

$$\begin{aligned} \det(A - \lambda I) = \det \begin{pmatrix} 5 - \lambda & -4 \\ 4 & -5 - \lambda \end{pmatrix} &= (5 - \lambda)(-5 - \lambda) + 16 \\ &= \lambda^2 - 9 = (\lambda + 3)(\lambda - 3) \end{aligned}$$

so the eigenvalues are $\lambda = -3$ and $\lambda = 3$. Now for the eigenvectors; let $\mathbf{v} = (x, y)$:

$$\begin{aligned} \lambda = -3 : A\mathbf{v} = \lambda \mathbf{v} &\Rightarrow \begin{cases} 5x - 4y = -3x \\ 4x - 5y = -3y \end{cases} \Rightarrow y = 2x \Rightarrow (1, 2) \\ \lambda = 3 : A\mathbf{v} = \lambda \mathbf{v} &\Rightarrow \begin{cases} 5x - 4y = 3x \\ 4x - 5y = 3y \end{cases} \Rightarrow x = 2y \Rightarrow (2, 1) \end{aligned}$$

Therefore the general solution of the homogeneous is

$$\mathbf{y}_h = C_1 e^{-3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Now, we find a particular solution $\mathbf{y}_p$ using undetermined coefficients. Since $\mathbf{q} = (3e^{-t}, 4e^{-t})$, we guess $\mathbf{y}_p = (Ae^{-t}, Be^{-t})$. Plugging this into the system, we get

$$\begin{cases} -Ae^{-t} &= 5Ae^{-t} - 4Be^{-t} + 3e^{-t} \\ -Be^{-t} &= 4Ae^{-t} - 5Be^{-t} + 4e^{-t} \end{cases} \Rightarrow \begin{cases} -A &= 5A - 4B + 3 \\ -B &= 4A - 5B + 4 \end{cases}$$

From the second equation $4B = 4A + 4$, i.e. $B = A + 1$. Plugging this into the first equation gives $-A = 5A - 4(A + 1) + 3$, i.e. $-A = A - 1$ so $A = \frac{1}{2}$ and $B = \frac{3}{2}$. Therefore

$$\mathbf{y}_p = \begin{pmatrix} Ae^{-t} \\ Be^{-t} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} e^{-t} \\ \frac{3}{2} e^{-t} \end{pmatrix}.$$

Last, the solution is

$$\begin{aligned}
\mathbf{y} &= \mathbf{y}_p + \mathbf{y}_h \\
&= \begin{pmatrix} \frac{-7}{10}e^{-t} \\ \frac{3}{10}e^{-t} \end{pmatrix} + C_1 e^{-3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}e^{-t} + C_1 e^{-3t} + 2C_2 e^{3t} \\ \frac{3}{2}e^{-t} + 2C_1 e^{-3t} + C_2 e^{3t} \end{pmatrix}.
\end{aligned}$$

5. Let $A = \begin{pmatrix} -7 & 2 \\ -25 & 3 \end{pmatrix}$ so that the system is $\mathbf{y}' = A\mathbf{y}$. First, find the eigenvalues of A:

$$\begin{aligned}
\det(A - \lambda I) = \det \begin{pmatrix} -7-\lambda & 2 \\ -25 & 3-\lambda \end{pmatrix} &= (-7-\lambda)(3-\lambda) + 50 \\
&= \lambda^2 + 4\lambda + 29.
\end{aligned}$$

By the quadratic formula, the eigenvalues are

$$\lambda = \frac{-4 \pm \sqrt{16 - 4(29)}}{2} = \frac{-4 \pm \sqrt{-100}}{2} = \frac{-4 \pm 10i}{2} = -2 \pm 5i$$

so $\alpha = -2$ and $\beta = 5$. Next, find the eigenvectors; let $\mathbf{v} = (x, y)$:

$$\begin{aligned}
\lambda = -2 + 5i: \ A\mathbf{v} = \lambda\mathbf{v} &\Rightarrow \begin{cases} -7x + 2y = (-2+5i)x \\ 25x + 3y = (-2+5i)y \end{cases} \\
&\Rightarrow 2y = (5+5i)x \\
&\Rightarrow (2, 5+5i) = (2,5) + i(0,5).
\end{aligned}$$

Therefore $\mathbf{a} = (2,5)$ and $\mathbf{b} = (0,5)$. By the theorem from the lecture notes, the general solution is therefore

$$\begin{aligned}
\mathbf{y} &= C_1 \left[e^{\alpha t}\cos(\beta t)\mathbf{a} - e^{\alpha t}\sin(\beta t)\mathbf{b} \right] + C_2 \left[e^{\alpha t}\cos(\beta t)\mathbf{b} + e^{\alpha t}\sin(\beta t)\mathbf{a} \right] \\
&= C_1 \left[e^{-2t}\cos 5t \begin{pmatrix} 2 \\ 5 \end{pmatrix} - e^{-2t}\sin 5t \begin{pmatrix} 0 \\ 5 \end{pmatrix} \right] + C_2 \left[e^{-2t}\cos 5t \begin{pmatrix} 0 \\ 5 \end{pmatrix} + e^{-2t}\sin 5t \begin{pmatrix} 2 \\ 5 \end{pmatrix} \right] \\
&= \begin{pmatrix} 2C_1 e^{-2t}\cos 5t + 2C_2 e^{-2t}\sin 5t \\ (5C_1 + 5C_2)e^{-2t}\cos 5t + (5C_2 - 5C_1)e^{-2t}\sin 5t \end{pmatrix}.
\end{aligned}$$

Now for the particular solution. Plugging in $t = 0$, $\mathbf{y} = (1, -5)$, we get

$$\begin{cases} 1 = 2C_1 \\ -5 = 5C_1 + 5C_2 \end{cases} \Rightarrow C_1 = \frac{1}{2}, C_2 = \frac{-3}{2}.$$

Therefore the particular solution is

$$\mathbf{y} = \begin{pmatrix} e^{-2t}\cos 5t - 3e^{-2t}\sin 5t \\ -5e^{-2t}\cos 5t - 10e^{-2t}\sin 5t \end{pmatrix}.$$

Written coordinate-wise, this is

$$\begin{cases} x(t) &= e^{-2t}\cos 5t - 3e^{-2t}\sin 5t \\ y(t) &= -5e^{-2t}\cos 5t - 10e^{-2t}\sin 5t \end{cases}.$$

3.3 Fall 2017 Final Exam

1. a) Write down an example of a 2×2 matrix A such that the constant-coefficient system $\mathbf{y}' = A\mathbf{y}$ has a stable node at the origin.

   b) Consider the third-order differential equation $y''' + y' - 5y = e^t$. Convert this equation to a first-order system of the form $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$, clearly defining what $\mathbf{y}$, A and $\mathbf{q}$ are.

   c) Let $y' = \phi(t, y)$ be a first-order, linear ODE and suppose that $y_1(t) = 2t + t^2$ and $y_2(t) = t + t^3$ are particular solutions of $y' = \phi(t, y)$. Find the general solution of the ODE $y' = \phi(t, y)$.

   d) Find $\lim\limits_{t\to\infty} h(t)$, where $y = h(t)$ is the solution of the IVP

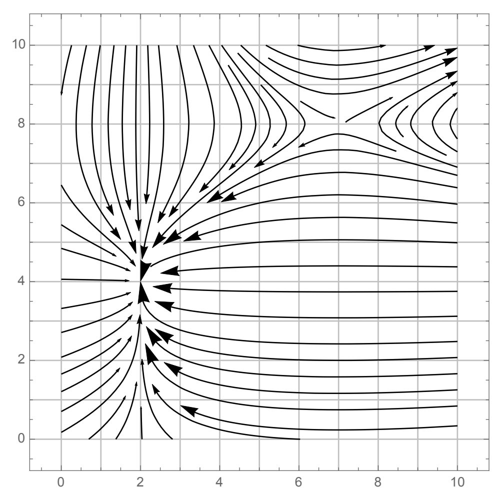
$$\begin{cases} y' = (y-2)(y-9) \\ y(0) = 5 \end{cases}.$$

2. Consider the initial value problem

$$\begin{cases} \frac{dy}{dt} = \frac{t-1}{y+t} \\ y(1) = 2 \end{cases}.$$

   Estimate $y(10)$ by using Euler's method with 3 steps.

3. Here is a picture of the phase plane of some 2×2 first-order system $\mathbf{y}' = \Phi(\mathbf{y})$:



   Use this picture to answer the following questions:

   a) Give the location of all equilibria of the system. Classify each equilibrium.

   b) Let $\mathbf{y} = (x(t), y(t))$ be the solution of this system satisfying $\mathbf{y}(0) = (6, 2)$.

i. Find $\lim\limits_{t\to\infty} x(t)$.

ii. Find $\lim\limits_{t\to\infty} y(t)$.

iii. Find $\lim\limits_{t\to-\infty} x(t)$.

c) Let $\mathbf{y} = (f(t), g(t))$ be the solution of this system satisfying $f(0) = 1$ and $g(0) = 6$.

i. Is $f'(0)$ positive, negative or zero?

ii. Is $g'(0)$ positive, negative or zero?

4. Find and classify all equilibria of this system:

$$\begin{cases} x' = x^2 - y - 1 \\ y' = y^2 - 3y \end{cases}$$

5. Find the particular solution of this initial value problem:

$$\begin{cases} y' = \frac{10}{6y^2+1} \\ y(4) = 3 \end{cases}$$

6. Find the general solution of this ODE writing your answer as a function $y = f(t)$.

$$\frac{dy}{dt} + 6ty = 12t$$

7. Find the particular solution of this initial value problem:

$$\begin{cases} y'' = -16y \\ y(0) = 3 \\ y'(0) = 5 \end{cases}$$

8. Find the general solution of this ODE:

$$y'' - 7y' - 18y = -54t + 15$$

9. Find the particular solution of this initial value problem:

$$\begin{cases} \mathbf{y}' = (-3x + 4y, -x - 7y) \\ \mathbf{y}(0) = (7, -2) \end{cases}$$

10. Find the general solution of this system, and write your answer coordinate-wise:

$$\begin{cases} x' = -7x + 25y \\ y' = -5x + 13y \end{cases}$$

11. A 4 kg mass is attached to the end of a spring with spring constant 2 N/cm. Assume that the damping coefficient is 6 N sec/cm and the entire system is subject at time t to an external force of $2\cos\frac{t}{2} + 6\sin\frac{t}{2}$. If at time $t = 0$, the mass is moving with initial velocity -4 cm/sec and has position 2, find the position of the mass at time $t = \pi$.

12. Two large tanks each hold some volume of liquid. Tank X holds 100 L of sulfuric acid solution which is initially 4% hydrochloric acid; tank Y holds 50 L of solution which is initially 12% hydrochloric acid. Pure water flows into tank Y at a rate of 3 L/min and pure water flows into tank X at a rate of 1 L/min. Tank Y drains into tank X at a rate of 2 L/min, and drains out of the system at a rate of 1 L/min. Tank X drains out of the system at a rate of 3 L/min, and does not drain into tank Y.

    Assuming that at all times the solution in each tank is kept mixed, find the concentration of hydrochloric acid in tank X at time 25.

Solutions

1. a) Any 2×2 matrix with two negative, real eigenvalues works: the easiest thing is to write down any triangular (or diagonal) matrix with negative numbers along the diagonal, like (for example) $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

   b) Rewrite the equation as $y''' = -y' + 5y + e^t$. Then, let $\mathbf{y} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$, let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -1 & 0 \end{pmatrix}$ and let $\mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ e^t \end{pmatrix}$. The equation becomes $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$ as desired.

   c) From the theory of first-order linear equations, we know that the difference of any two solutions of a linear ODE solves the corresponding homogeneous equation. So $y_1(t) - y_2(t) = t + t^2 - t^3$ is a solution of the corresponding homogeneous equation. Furthermore, we know that for a homogeneous first-order linear ODE, the solution set is the span (i.e. set of multiples) of any one nonzero solution, so the homogeneous equation has as its solution set $y_h = C(t + t^2 - t^3)$. The solution set of the original equation is therefore any one particular solution of the equation (like either y_1 or y_2) plus the solution set of the homogeneous, i.e. the general solution is

$$y = 2t + t^2 + C(t + t^2 - t^3).$$

   d) Let $\phi(y) = (y-2)(y-9)$. Clearly $\phi(y) = 0$ when $y = 2$ or $y = 9$. Then $\phi'(y) = 2y - 11$, so $\phi'(2) = -7 < 0$ and $\phi'(9) = 7 > 0$. Thus 2 is a stable equilibrium and 9 is an unstable equilibrium of this equation. As $t \to \infty$, $h(t)$ will approach the stable equilibrium, so $\lim\limits_{t\to\infty} h(t) = 2$.

2. Let $\phi(t,y) = \frac{t-1}{y+t}$ so that the equation becomes $y' = \phi(t,y)$. Next, $\Delta t = \frac{t_n - t_0}{n} = \frac{10-1}{3} = 3$. We are given $(t_0, y_0) = (1, 2)$. Therefore $\phi(3, 10) = \frac{1-1}{2+1} = 0$ so

$$\begin{cases} t_1 = t_0 + \Delta t = 1 + 3 = 4 \\ y_1 = y_0 + \phi(t_0, y_0)\Delta t = 2 + 0(3) = 2 \end{cases} \Rightarrow (t_1, y_1) = (4, 2).$$

   Next, $\phi(t_1, y_1) = \frac{4-1}{4+2} = \frac{1}{2}$ so

$$\begin{cases} t_2 = t_1 + \Delta t = 4 + 3 = 7 \\ y_2 = y_1 + \phi(t_1, y_1)\Delta t = 2 + \frac{1}{2}(3) = \frac{7}{2} \end{cases} \Rightarrow (t_2, y_2) = (7, \frac{7}{2}).$$

Last, $\phi(t_2, y_2) = \frac{7-1}{\frac{7}{2}+7} = \frac{6}{\frac{21}{2}} = \frac{4}{7}$ so

$$\begin{cases} t_3 = t_2 + \Delta t = 7 + 3 = 10 \\ y_3 = y_2 + \phi(t_2, y_2)\Delta t = \frac{7}{2} + \frac{4}{7}(3) = \frac{7}{2} + \frac{12}{7} = \frac{73}{14} \end{cases} \Rightarrow (t_3, y_3) = (10, \frac{73}{14}).$$

So $y(10) \approx \frac{73}{14}$.

3. a) $(2, 4)$ is a stable node; $(7, 8)$ is a saddle.
   b) i. $\lim_{t\to\infty} x(t) = 2$.
      ii. $\lim_{t\to\infty} y(t) = 4$.
      iii. $\lim_{t\to-\infty} x(t) = \infty$.
   c) i. $f'(0) = \frac{dx}{dt}\Big|_{x=1,y=6} > 0$ since the graph of $\mathbf{y}$ is moving to the right at $t = 0$.
      ii. $g'(0) = \frac{dy}{dt}\Big|_{x=1,y=6} < 0$ since the graph of $\mathbf{y}$ is moving downward at $t = 0$.

4. Let $\Phi(x, y) = (x^2 - y - 1, y^2 - 3y)$. Setting $\Phi(x, y) = \mathbf{0}$, we obtain

$$\begin{cases} 0 = x^2 - y - 1 \\ 0 = y^2 - 3y \end{cases}$$

From the second equation (which factors as $0 = y(y-3)$), either $y = 0$ or $y = 3$. When $y = 0$, from the first equation $x^2 = 1$, so $x = \pm 1$. When $y = 3$, from the first equation $x^2 = 4$, so $x = \pm 2$. We therefore have four equilibria, which we classify by looking at the eigenvalues of the total derivative $D\Phi(x, y) = \begin{pmatrix} 2x & -1 \\ 0 & 2y-3 \end{pmatrix}$:

- $(1, 0)$: $D\Phi(1, 0) = \begin{pmatrix} 2 & -1 \\ 0 & -3 \end{pmatrix}$ has eigenvalues 2 and -3, making $(1, 0)$ a **saddle**.
- $(-1, 0)$: $D\Phi(-1, 0) = \begin{pmatrix} -2 & -1 \\ 0 & -3 \end{pmatrix}$ has eigenvalues -2 and -3, making $(-1, 0)$ a **stable node**.
- $(2, 3)$: $D\Phi(2, 3) = \begin{pmatrix} 4 & -1 \\ 0 & 3 \end{pmatrix}$ has eigenvalues 4 and 3, making $(2, 3)$ an **unstable node**.
- $(-2, 3)$: $D\Phi(2, 3) = \begin{pmatrix} -4 & -1 \\ 0 & 3 \end{pmatrix}$ has eigenvalues -4 and 3, making $(-2, 3)$ a **saddle**.

5. This equation is separable: rewrite it as $(6y^2 + 1)\,dy = 10\,dt$. Then integrate both sides to get $2y^3 + y = 10t + C$. Plug in the initial condition $(4, 3)$ to get $2(27) + 3 = 10(4) + C$, i.e. $C = 17$. Thus the particular solution is $2y^3 + y = 10t + 17$.

6. Multiply through by the integrating factor $\mu = \exp\left(\int 6t\,dt\right) = \exp(3t^2)$ to obtain the equation $\frac{d}{dt}\left(ye^{3t^2}\right) = 12te^{3t^2}$. Now integrate both sides (you need the u-substitution $u = 3t^2, du = 6t\,dt$ on the right-hand side) to get $ye^{3t^2} = 2e^{3t^2} + C$. Solving for y, we obtain $y = 2 + Ce^{-3t^2}$.

7. Rewrite the equation as $y'' + 16y = 0$ and solve the characteristic equation $\lambda^2 + 16 = 0$ to obtain $\lambda = \pm 4i$. Thus the general solution is $y = C_1 \cos 4t + C_2 \sin 4t$. To find the particular solution, differentiate to obtain $y' = -4C_1 \sin 4t + 4C_2 \cos 4t$. Plugging in $y(0) = 3$ and $y'(0) = 5$ yields

$$\begin{cases} 3 = C_1 \\ 5 = 4C_2 \end{cases} \Rightarrow C_1 = 3, C_2 = \frac{5}{4}.$$

Thus the particular solution is $y = 3\cos 4t + \frac{5}{4}\sin 4t$.

8. First, solve the homogeneous equation $y'' - 7y' - 18y = 0$ by considering the characteristic equation $0 = \lambda^2 - 7\lambda - 18 = (\lambda - 9)(\lambda + 2)$. This equation has solutions $\lambda = 9, \lambda = -2$ so the solution of the homogeneous is $y_h = C_1 e^{9t} + C_2 e^{-2t}$.

   Now find a particular solution using undetermined coefficients; guess $y_p = At + B$ and plug into the equation to obtain

$$0 - 7(A) - 18(At + B) = -54t + 15,$$

   i.e. $-18At + (-7A - 18B) = -54t + 15$. Thus $-18A = -54$, so $A = 3$. Then $-7(3) - 18B = 15$ so $-18B = 36$ so $B = -2$. Thus $y_p = 3t - 2$ so the general solution of the equation is

$$y = y_p + y_h = 3t - 2 + C_1 e^{9t} + C_2 e^{-2t}.$$

9. First, find the eigenvalue(s):

$$\det \begin{pmatrix} -3 - \lambda & 4 \\ -1 & -7 - \lambda \end{pmatrix} = (-3 - \lambda)(-7 - \lambda) + 4 = \lambda^2 + 10\lambda + 25 = (\lambda + 5)^2$$

   so the only eigenvalue is $\lambda = -5$ (repeated twice). Next, the eigenvector(s):

$$\begin{cases} -3x + 4y = -5x \\ -x - 7y = -5y \end{cases} \Rightarrow x = -2y \Rightarrow \mathbf{v} = (2, -1).$$

Since there is only one linearly independent eigenvector, we find a generalized eigenvector $\mathbf{w} = (x, y)$:

$$\begin{pmatrix} -3-(-5) & 4 \\ -1 & -7-(-5) \end{pmatrix} \mathbf{w} = \mathbf{v} \Rightarrow \begin{cases} 2x + 4y = 2 \\ -x - 2y = -1 \end{cases} \Rightarrow \mathbf{w} = (1, 0).$$

Now by the formula from the lecture notes, the general solution is

$$\begin{aligned} \mathbf{y} &= C_1 e^{-5t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + C_2 \left[e^{-5t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t e^{-5t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right] \\ &= \begin{pmatrix} (2C_1 + C_2)e^{-5t} + 2C_2 t e^{-5t} \\ -C_1 e^{-5t} - C_2 t e^{-5t} \end{pmatrix}. \end{aligned}$$

Now for the particular solution. Plugging in the initial condition $\mathbf{y}(0) = (7, -2)$, we get

$$\begin{cases} 7 = 2C_1 + C_2 \\ -2 = -C_1 \end{cases}$$

which leads to $C_1 = 2$ and $C_2 = 3$. Thus the particular solution is

$$\mathbf{y} = \begin{pmatrix} 7e^{-5t} + 6te^{-5t} \\ -2e^{-5t} - 3te^{-5t} \end{pmatrix}.$$

10. First, find the eigenvalue(s):

$$\det \begin{pmatrix} -7-\lambda & 25 \\ -5 & 13-\lambda \end{pmatrix} = (-7-\lambda)(13-\lambda) + 125 = \lambda^2 - 6\lambda + 34$$

so by the quadratic formula, the eigenvalues are

$$\lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(34)}}{2} = 3 \pm \frac{1}{2}\sqrt{-100} = 3 \pm 5i.$$

Next, the eigenvector for $3 + 5i$:

$$\begin{cases} -7x + 25y = (3+5i)x \\ -5x + 13y = (3+5i)y \end{cases} \Rightarrow 25y = (10+5i)x \Rightarrow 5y = (2+i)x \Rightarrow \mathbf{v} = (5, 2+i) = (5, 2) + i(0, 1).$$

Now by the formula in the lecture notes, the general solution is

$$\begin{aligned} \mathbf{y} &= C_1 \left[e^{3t} \cos 5t \begin{pmatrix} 5 \\ 2 \end{pmatrix} - e^{3t} \sin 5t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] + C_2 \left[e^{3t} \cos 5t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{3t} \sin 5t \begin{pmatrix} 5 \\ 2 \end{pmatrix} \right] \\ &= \begin{pmatrix} 5C_1 e^{3t} \cos 5t + 5C_2 e^{3t} \sin 5t \\ (2C_1 + C_2) e^{3t} \cos 5t + (-C_1 + 2C_2) e^{3t} \sin 5t \end{pmatrix}. \end{aligned}$$

Written coordinate-wise, this is

$$\begin{cases} x(t) = 5C_1 e^{3t} \cos 5t + 5C_2 e^{3t} \sin 5t \\ y(t) = (2C_1 + C_2) e^{3t} \cos 5t + (-C_1 + 2C_2) e^{3t} \sin 5t \end{cases}.$$

11. From the spring equation $mx''(t) + bx'(t) + kx(t) = F_{ext}(t)$, we obtain, by plugging in the constants given in the problem, the following IVP:

$$\begin{cases} 4x'' + 6x' + 2x = 2\cos\frac{t}{2} + 6\sin\frac{t}{2} \\ x(0) = 2 \\ x'(0) = -4 \end{cases}$$

To solve this, start with the corresponding homogeneous equation $4x''+6x'+2x = 0$ which has characteristic equation $4\lambda^2 + 6\lambda + 2 = 0$. This equation factors as $2(2\lambda+1)(\lambda+1) = 0$ so it has solutions $\lambda = -1$ and $\lambda = \frac{-1}{2}$. Thus the general solution of the homogeneous is $y_h = C_1e^{-t} + C_2e^{-t/2}$.

Next, find a particular solution with undetermined coefficients. Guess $x_p = A\cos\frac{t}{2} + B\sin\frac{t}{2}$; then $x_p' = -\frac{A}{2}\sin\frac{t}{2} + \frac{B}{2}\cos\frac{t}{2}$ and $x_p'' = -\frac{A}{4}\cos\frac{t}{2} - \frac{B}{4}\sin\frac{t}{2}$. Plugging in the original equation, we get

$$\begin{aligned} 4(-\frac{A}{4}\cos\frac{t}{2} - \frac{B}{4}\sin\frac{t}{2}) + 6(-\frac{A}{2}\sin\frac{t}{2} + \frac{B}{2}\cos\frac{t}{2}) + 2(A\cos\frac{t}{2} + B\sin\frac{t}{2}) &= 2\cos\frac{t}{2} + 6\sin\frac{t}{2} \\ \Rightarrow (A+3B)\cos\frac{t}{2} + (-3A+B)\sin\frac{t}{2} &= 2\cos\frac{t}{2} + 6\sin\frac{t}{2} \\ \Rightarrow \begin{cases} A+3B = 2 \\ -3A+B = 6 \end{cases} \Rightarrow A = -\frac{8}{5}, B &= \frac{6}{5}. \end{aligned}$$

Therefore $x_p = -\frac{8}{5}\cos\frac{t}{2} + \frac{6}{5}\sin\frac{t}{2}$ so

$$x = x_p + x_h = -\frac{8}{5}\cos\frac{t}{2} + \frac{6}{5}\sin\frac{t}{2} + C_1e^{-t} + C_2e^{-t/2}.$$

Now find C_1 and C_2 by plugging in the initial conditions: plugging in $x(0) = 2$, we get $2 = -\frac{8}{5} + C_1 + C_2$ so $C_1 + C_2 = \frac{18}{5}$. Plugging in $x'(0) = -4$, we get $-4 = \frac{3}{5} - C_1 - \frac{1}{2}C_2$, i.e. $-C_1 - \frac{1}{2}C_2 = \frac{-23}{5}$; solving the two equations together gives $C_2 = -2, C_1 = \frac{28}{5}$ so the particular solution of the spring equation is

$$x(t) = \frac{-8}{5}\cos\frac{t}{2} + \frac{6}{5}\sin\frac{t}{2} + \frac{28}{5}e^{-t} - 2e^{-t/2}.$$

Finally, answer the question by plugging in $x = \pi$ to obtain

$$\begin{aligned} x(\pi) &= \frac{-8}{5}\cos\frac{\pi}{2} + \frac{6}{5}\sin\frac{\pi}{2} + \frac{28}{5}e^{-\pi} - 2e^{-\pi/2} \\ &= \frac{6}{5} + \frac{28}{5}e^{-\pi} - 2e^{-\pi/2}. \end{aligned}$$

12. Let $x(t)$ and $y(t)$ be the amount of hydrochloric acid in tanks X and Y, respectively, at time t. Let $\mathbf{y} = (x(t), y(t)$; from the given information, we obtain the following IVP:

$$\begin{cases} \mathbf{y}' = (2y/50 - 3x/100, -3y/50) \\ \mathbf{y}(0) = (4, 6) \end{cases}$$

Written in matrix form, this equation is $\mathbf{y}' = A\mathbf{y}$ where

$$A = \begin{pmatrix} \frac{-3}{100} & \frac{1}{25} \\ 0 & \frac{-3}{50} \end{pmatrix}.$$

Since this matrix is triangular, its eigenvalues are its diagonal entries: $\lambda = \frac{-3}{100}, \lambda = \frac{-3}{50}$. Now find eigenvectors:

- $\lambda = \frac{-3}{100}$: $\begin{cases} \frac{-3}{100}x + \frac{1}{25}y = \frac{-3}{100}x \\ \frac{-3}{50}y = \frac{-3}{100}y \end{cases} \Rightarrow y = 0 \Rightarrow (1, 0)$
- $\lambda = \frac{-3}{50}$: $\begin{cases} \frac{-3}{100}x + \frac{1}{25}y = \frac{-3}{50}x \\ \frac{-3}{50}y = \frac{-3}{50}y \end{cases} \Rightarrow \frac{1}{25}y = \frac{-3}{100}x \Rightarrow 4y = -3x \Rightarrow (4, -3)$

Therefore the general solution of this system is

$$\mathbf{y} = C_1 e^{-3t/100} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-3t/50} \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

Plugging in the initial conditions, we get

$$\begin{cases} 4 = C_1 + 4C_2 \\ 6 = -3C_2 \end{cases} \Rightarrow C_1 = 12, C_2 = -2.$$

Thus the particular solution is

$$\mathbf{y} = 12e^{-3t/100} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2e^{-3t/50} \begin{pmatrix} 4 \\ -3 \end{pmatrix},$$

which coordinate-wise is

$$\begin{cases} x(t) = 12e^{-3t/100} - 8e^{-3t/50} \\ y(t) = 6e^{-3t/50} \end{cases}.$$

So the amount of hydrochloric acid in tank X at time 25 is $x(25) = 12e^{-3/4} - 8e^{-3/2}$; the concentration is given by this amount divided by the volume of tank X, which gives

$$\frac{12e^{-3/4} - 8e^{-3/2}}{100} = \frac{7e^{-3/4} - 2e^{-3/2}}{25}.$$