1. Suppose you flip a fair coin three times and record the outcomes with Hs and Ts. Describe the following events in words (your description should be as efficient as possible):

   (a) $E = \{HHH, TTT\}$

   (b) $E = \{HHT, HTH, THH\}$

   (c) $E = \{HHH, HHT, HTH, HTT\}$

   (d) $E = \{HHT, HTH, HTT, THH, THT, TTH, TTT\}$

2. A box contains 4 marbles: 2 red, 1 green, and 1 blue.

   (a) Consider an experiment that consists of taking 1 marble from the box, putting it back and drawing a second marble from the box (recording both choices in order). Describe the sample space for this experiment, so that all the outcomes are equally likely.

   (b) Suppose you didn't put the first marble back before you drew the second marble. Describe the sample space in this context, so that all the outcomes are equally likely.

3. Suppose you perform an experiment where there are eight possible outcomes. Assuming that every subset of outcomes constitutes an event, how many distinct events are there?

   Hint: First try this problem in the situation where there are two, three or four outcomes, and look for a pattern.

4. Suppose you roll a fair die repeatedly until a 4 turns up. You record the number of rolls it takes to roll a 4. Describe a probability space for this experiment. Verify that you have constructed a probability space (the point of this problem is for you to figure out what you have to do to "verify" you have a probability space).

5. Verify the following set indentities, called **De Morgan's Laws**, using Venn diagrams:

   (a) $(A \cap B)^C = A^C \cup B^C$;

   (b) $(A \cup B)^C = A^C \cap B^C$.

6. Let $(\Omega, \mathcal{A}, P)$ be a probability space. Prove (using the definition of probability space) that if E and F are events with $E \subseteq F$, then $P(E) \leq P(F)$.

   Hint: Write F as the union of the two sets $E \cap F$ and $E' \cap F$.

7. Prove that there is no such thing as a uniform distribution on $\mathbb{N} = \{1, 2, 3, 4, ...\}$.

   Hint: Prove this by contradiction: suppose there was a uniform distribution on $\mathbb{N}$. This means that $P(m) = P(n)$ for every $m, n \in \mathbb{N}$. There are two possibilities: either $P(1) = 0$ or $P(1) > 0$. Explain why both of these cases are impossible (use the fact that $P(\Omega)$, which in this case is $P(\mathbb{N})$, must be 1).

8. Suppose a point (x, y) is picked at random (with the uniform distribution) from the triangle in the $xy-$plane with vertices at $(0, 0), (4, 0)$, and $(4, 4)$.

   (a) What is the probability that $x \geq 2$?

   (b) What is the probability that $x < y^2$?

9. Suppose two fair dice are rolled and that the 36 possible outcomes are equally likely. Find the probability that the sum of the numbers on the two faces is even.

10. Prove Bonferonni's Inequality, which says that given any two events E and F, $P(E \bigcap F) \geq P(E) + P(F) - 1$.

11. (AE) (The "AE" means this is, or closely resembles, an old actuarial exam problem.) The probability that a small fire in a kitchen destroys a microwave oven is 70%. The probability that a small fire in a kitchen destroys a refrigerator is 50%. If the probability that a small fire destroys both is 45%, find the probability that the fire destroys neither the microwave nor the refrigerator.

12. Suppose a point is picked uniformly from the square whose vertices are $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. Let E be the event that the selected point is in the triangle bounded by the lines $y = 0$, $x = 1$ and $x = y$, and let F be the event that it is in the rectangle with vertices $(0, 0), (1, 0), (1, \frac{1}{2})$, and $(0, \frac{1}{2})$.

    (a) Find $P(E)$.

    (b) Find $P(F)$.

    (c) Find $P(E \cup F)$.

    (d) Find $P(E \cap F)$.

    (e) Find $P(E|F)$.

    (f) Find $P(F|E)$.

    (g) Are E and F independent? Why or why not? (You need a proof.)

13. (a) Suppose events A and B are such that $P(A) = \frac{2}{5}$ and $P(B) = \frac{2}{5}$. If you also know $P(A \cup B) = \frac{1}{2}$, find $P(A \cap B)$.

    (b) If $P(A) = .7$, $P(A \bigcap B^C) = .6$ and A and B are independent, what is $P(B)$?

14. A coin is tossed three times. Consider the following events:
    - $A =$ heads on the first toss
    - $B =$ tails on the second toss
    - $C =$ heads on the third toss
    - $D =$ all three outcomes the same
    - $E =$ exactly one head turns up.

    (a) Which one or ones of the following pairs of these events are independent? A and B, A and D, A and E, D and E (A proof is not required here... use the heuristic idea of independence.)

    (b) Which one or ones of the following triples of these events are independent? A, B and C; A, B and D; C, D and E (A proof is not required here... use the heuristic idea of independence.)

15. Choose one of (a) or (b):

    (a) Suppose E and F are independent. Prove that E^c and F^c are independent.

    (b) If an event E is pairwise independent with itself, what must be true about E? (Prove your statement.)

16. A point is chosen uniformly from the unit square $[0,1] \times [0,1]$. Find a positive number c so that the events $E = \{(x,y) : y + cx \leq 1\}$ and $F = \{(x,y) : y \leq 2x/3\}$ are independent.

    Hint: There are two values of c which solve this problem; you need to find one or the other, not both.

17. Choose one of (a) or (b):

    (a) Three players, Al, Bal, and Cal, take turns flipping a fair coin (Al goes first followed by Bal, then Cal, then Al again, then Bal, etc.). The first player to flip a head wins. What is the probability of each player winning?

    (b) (This is a famous problem in probability called *The Triangle Problem*.) Suppose you take a stick of length 1 and break it into three pieces, choosing the break points uniformly and independently. What is the probability that the three pieces can be used to form a triangle?

    Hint: In a triangle, the sum of the lengths of any two sides must be at least the length of the third side.

18. There are three boxes, labeled I, II and III. Box I contains 2 white balls and 2 black balls; box II contains 2 white balls and 1 black ball; and box III contains 1 white ball and 3 black balls.

(a) One ball is selected from each box (the draws are independent of one another). Find the probability of drawing all white balls.

(b) Suppose you have five slips of paper, two labeled "I", two labeled "II" and one labeled "III". One of these five slips is drawn uniformly and then a ball is drawn from the box indicated by the slip of paper chosen. Calculate the probability that the drawn ball is white.

19. Suppose a student takes a multiple choice exam where each question has 5 possible answers, exactly one of which is correct. If the student knows the answer to the question, she selects the correct answer. Otherwise, she guesses uniformly from the 5 possible answers. Assume that the student knows the answer to 70% of the questions.

    (a) What is the probability that on any single given question, the student gets the correct answer?

    (b) What is the probability that the student knows the answer to a question, given that she got the question correct?

20. (AE) Suppose a factory has two machines A and B which make 60% and 40% of the total production, respectively. Of their output, machine A produces 3% defective items and machine B produces 5% defective items. Find the probability that a given defective part was produced by machine B.

21. (AE) The probability that a randomly chosen male has a blood circulation problem is $.25$. Males who have a circulation problem are twice as likely to be smokers as those who do not have a blood circulation problem. What is the conditional probability that a male has a blood circulation problem, given that he is a smoker?

22. Suppose X is a discrete r.v. with density function f given by

x	-3	-1	0	1	2	3	5	8
$f_X(x)$	$.1$	$.2$	$.15$	$.2$	$.1$	$.15$	$.05$	$.05$

    (a) Find the probability that X is negative.

    (b) Find the probability that X is nonpositive.

    (c) Find the probability that X is even.

    (d) Compute $P(X \in [1, 8])$.

    (e) Compute $P(X = -3 \,|\, X \leq 0)$.

    (f) Compute $P(X \geq 3 \,|\, X > 0)$.

23. Choose two of (a), (b), (c):

(a) Suppose a box has 12 balls numbered 1 to 12. Two balls are selected from the box independently, with replacement. Let X denote the larger of the two numbers on the selected balls. Compute the density of X.

(b) Suppose you choose a zip code (i.e. a 5-digit sequence of numbers) uniformly from all possible zip codes and let X be the number of nonzero digits in the zip code. Find the density function of X.

(c) Suppose you choose three whole numbers from 0 to 9 uniformly and independently. Let X be the first digit of the number you get when you add these whole numbers together. Find the density function of X.

24. (AE) Among a group of 20000 people, 7200 are below age 40, 8200 are childless and 12300 are male. In the same group, there are 5400 males below age 40, 4700 childless persons below age 40 and 6000 childless males. Finally, there are 3100 childless males below age 40. How many people are females above 40 who have children?

25. A committee of 7, consisting of 3 Democrats, 3 Republicans and 1 Independent, is to be chosen from a group of 20 Democrats, 15 Republicans and 10 Independents. How many different committees are possible?

26. A bus stars with 6 people and stops at 10 different stops. Assuming that passengers are equally likely to depart at any stop, find the probability that the 6 people get off at 6 different stops.

27. My niece's iPod has 100 songs on it, of which 10 are performed by Taylor Swift. If she sets her iPod to shuffle mode, which will play all 100 songs in a random order (without repeating any songs until they are all played once), what is the probability that the first Taylor Swift song my niece hears is the eighth song played?

28. A "domino" is a rectangular block divided into two equal subrectangles as below, where each subrectangle has a number on it:

x	y

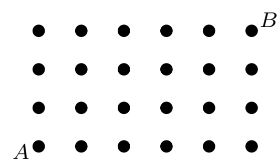
(The numbers x and y might be the same or different.) Since dominos are symmetric, the domino (x, y) is the same as (y, x). How many different domino blocks can be made if the x and y are to be chosen from n different numbers?

Hint: Count the dominos where $x = y$ separately from the dominos where $x \neq y$. Then add these two separate counts.

29. How many distinct arrangements of the letters in the following words are possible?

    (a) COFFEE
    (b) ASSESS
    (c) ARRANGE
    (d) BOOKKEEPER

30. Consider the grid of points shown below. Suppose that starting at the point A you move from point to point, moving only one unit to the right or one unit up at a time, ending at the point B. How many different paths from A to B are possible?



31. How many distinct, non-negative integer-valued vectors $(x_1, x_2, ..., x_5)$ satisfy $x_1 + x_2 + x_3 + x_4 + x_5 = 12$?

    Hint: This has something to do with distinguishable arrangements.

32. Suppose you deal a five-card hand from a standard deck of cards. Find the probability of being dealt each of the following hands:

    (a) A royal flush (the A,K,Q,J and 10 of the same suit)
    (b) A flush (any five cards of the same suit)
    (c) Three-of-a-kind, but not a full house or four-of-a-kind
    (d) A straight (five cards in a sequence, regardless of suit)
    (e) A hand which contains no pair (nor three- nor four-of-a-kind)

33. A fair die is rolled 12 times (independently). Find the probability of rolling exactly 2 sixes, and the probability of rolling at most 2 sixes.

34. Experience shows that 20% of the people reserving tables at a certain restaurant never show up. If the restaurant has 50 tables and takes 52 reservations, what is the probability that it will be able to accomodate everyone who shows up?

35. A circular target of radius 1 is divided into four annular zones (an "annular" shape is like a ring) of outer radii $1/4$, $1/2$, $3/4$ and 1, respectively. Suppose 10 shots are fired independently, and that each shot hits a random point in the target chosen uniformly.

    (a) Find the probability that exactly four shots land in the region of radius $1/4$.

(b) What is the probability that at most three shots land in the zone bounded on the inside by the circle of radius $1/2$ and on the outside by the circle of radius $3/4$?

(c) If exactly 5 shots land inside the circle of radius $1/2$, find the probability that at least one shot lands inside the circle of radius $1/4$.

36. (AE) You own a business that gets bolts from two bolt manufacturers: A and B (you get 70% of your bolts from A and 30% from B). Suppose that 5% of all bolts from manufacturer A are defective, and that 20% of all bolts from manufacturer B are defective. You get a shipment of 10 bolts from one of the two manufacturers. If exactly 3 of the 10 bolts are defective, what is the probability that the shipment came from manufacturer B?

37. There are 40 gumballs in a bag (20 are red, 10 are orange, 8 are green, and 2 are purple).

(a) If you draw 10 gumballs from the bag without replacement, what is the probability that you draw 5 red, 3 orange, and 2 purple gumballs?

(b) If you draw 7 gumballs from the bag without replacement, what is the probability that you draw exactly 4 green gumballs?

(c) If you draw 7 gumballs from the bag with replacement, what is the probability that you draw exactly 4 green gumballs?

(d) If you draw 6 gumballs from the bag without replacement, what is the probability you draw at least 5 orange gumballs?

(e) If you draw 10 gumballs from the bag with replacement, what is the probability that you draw 3 orange gumballs?

(f) If you draw 15 gumballs from the bag without replacement and take a bite out of them, then put them back in the bag, and if you subsequently draw 5 gumballs from the bag with replacement, what is the probability that you drew 3 gumballs that had been bitten?

(g) Suppose you draw gumballs from the bag repeatedly, with replacement. What is the probability that the first time you draw a purple gumball is on the 9th draw?

(h) Suppose you draw gumballs from the bag repeatedly, with replacement. What is the probability that the fifth time you draw a red gumball is on the 14th draw?

(i) Suppose you draw gumballs from the bag 2 at a time, putting each group back after you draw it. What is the probability that the first time you draw 2 red gumballs is the 4th time you draw 2 gumballs from the bag?

38. Suppose a box has 6 red balls and 4 black balls in it. A random sample of size n is selected; let X denote the number of red balls in the sample.
    (a) Find the density function of X if the sampling is without replacement.
    (b) Find the density function of X if the sampling is with replacement.

39. Suppose X has a geometric distribution with $p = .8$. Compute the following:
    (a) $P(X > 3)$
    (b) $P(4 \leq X \leq 7 \text{ or } X > 9)$
    (c) $P(X \leq 2 \,|\, X \leq 3)$
    (d) $P(X \geq 85 \,|\, X \geq 80)$

40. Suppose you choose a real number X from the interval $[2, 10]$ with a density function of the form $f_X(x) = Cx$ where C is some constant.
    (a) Find C.
    (b) Find $P(X > 5)$ and $P(X \leq 7)$.

41. Let X be a random variable whose distribution function is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{3} & \text{if } 0 \leq x < 1 \\ \frac{x}{2} & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

    (a) Find $P(\frac{1}{2} \leq X \leq \frac{3}{2})$.
    (b) Find $P(\frac{1}{2} \leq X \leq 1)$.
    (c) Find $P(\frac{1}{2} \leq X < 1)$.
    (d) Find $P(1 \leq X \leq \frac{3}{2})$.
    (e) Find $P(1 < X < 2)$.
    (f) Find $P(X \text{ is an integer})$.

42. Suppose X is a random variable whose distribution function is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{1}{10} & \text{if } 1 \leq x < 3 \\ \frac{x}{10} & \text{if } 3 \leq x < 4 \\ K - \frac{2}{x} & \text{if } x \geq 4 \end{cases}$$

where K is a constant.
    (a) Find K.
    (b) Find $P(X = 3)$.

(c) Find $P(2 < X < 3)$.
(d) Find $P(X > 1)$.
(e) Find $P(X > 4 \,|\, X \geq 4)$.
(f) Find $P(X < 3.5 \,|\, X \leq 4)$.
(g) Find $P(X > 2 \,|\, X > 3)$.

43. Choose (a) or (b).

    (a) Let p be a point chosen uniformly from the interior of a disc of radius R. Let X denote the distance from p to the center of the disc; compute the distribution and density functions of X.

    (b) Let a point be chosen uniformly from the interior of a triangle having a base of length l and a height h (measured perpendicular to the base). Let X be defined as the distance from the point chosen to the base of the triangle (this means the length of a perpendicular drawn from the point to the base); compute the distribution and density functions of X.

44. Let X be a discrete r.v. with density function f_X defined as follows:

$$f_X(-1) = \frac{1}{5}, f_X(0) = \frac{1}{5}, f_X(1) = \frac{2}{5}, f_X(2) = \frac{1}{5}.$$

    (a) Find a density function for the r.v. $Y = 2X + 1$.

    (b) Find a density function for $Z = X^2$.

45. A point is chosen uniformly from the interval $(-10, 10)$. Let X be the random variable defined so that X denotes the coordinate of the chosen point if the point is in $[-5, 5]$, $X = -5$ if the point is in the interval $(-10, -5)$, and $X = 5$ if the point is in the interval $(5, 10)$. Find the distribution function of X. Does X have a density function? Why or why not?

46. Let X be a continuous, real-valued r.v. with distribution function F_X and density function f_X.

    (a) Compute (in terms of F_X) the distribution function of $Y = e^X$.

    (b) Compute (in terms of f_X) the density function of $Y = e^X$.

47. Let X be a continuous real-valued r.v. with density function f_X. Compute (in terms of f_X) the density function of $Y = |X|$.

48. Suppose a point (X, Y) is chosen uniformly from the rectangle whose vertices are $(1, 1)$, $(7, 1)$, $(1, 3)$ and $(7, 3)$. Find the density function of $Z = XY$.

49. (a) The number of bad checks that a bank receives during a 5-hour business day is a Poisson r.v. with $\lambda = 2$. What is the probability that it will receive no more than 2 bad checks in its business day?

    (b) The mileage (in thousands of miles) that car owners get with a certain kind of radial tire is a random variable whose distribution is exponential with parameter 40. Find the probability that one of these tires will last at least 20,000 miles.

50. (AE) The loss due to a fire in a commercial building is modeled by a continuous r.v. X with density function $f(x) = k(20 - x)$ for $0 < x < 20$ ($f(x) = 0$ otherwise). Given that a fire loss exceeds 8, what is the probability that it exceeds 16?

51. (AE) You are given the following information about N, the annual number of claims for a randomly selected insured person:

$$P(N = 0) = \frac{1}{2}; \quad P(N = 1) = \frac{1}{3}; \quad P(N = 2) = \frac{1}{6}.$$

Let S denote the total annual claim amount for an insured. When $N = 1$, S is exponentially distributed with parameter $\frac{1}{5}$. When $N > 1$, S is exponentially distributed with parameter $\frac{1}{8}$. Find $P(4 < S < 8)$.

Hint: Use the Law of Total Probability.

52. Suppose that events occur according to a Poisson process with hourly rate $\lambda = 3$.

    (a) Let p be the probability that no events occur between 8 AM and 10 AM.

        i. Compute p using the density function of an appropriate discrete r.v.

        ii. Compute p using the density function of an appropriate continuous r.v.

    (b) Suppose you are given that v events occur between times 0 and t. Let $s < t$; find the probability that exactly x of the v events occur between times 0 and s.

    (c) Suppose you are given that v events occur between times 0 and t. Let $s < t$. If X is the number of events occurring between times 0 and s, what kind of random variable is X? Include its parameters.

        Hint: You computed the density function of X in part (b). Simplify this density function and identify it as the density function of a common r.v.

    (d) Suppose eight events occur between 8 AM and noon. What is the probability that (exactly) three of those events occurred after 11 AM?

(e) What is the conditional probability that at least one event takes place between 8 AM and noon, given that no events take place between 8 AM and 10 AM?

(f) What is the probability that exactly one event occurs between 8 and 9 AM and exactly one event occurs between 2 and 4 PM?

(g) What is the probability that exactly one event occurs between 8 and 10 AM and exactly one event occurs between 9 and 11 AM?

53. Suppose X is exponential with parameter λ, where λ is such that $P(X \geq .01) = \frac{1}{2}$. Find the number t such that $P(X \geq t) = \frac{9}{10}$.

54. Suppose the number of claims filed by an insurance policyholder is a Poisson random variable. If the filing of (exactly) one claim is three times as likely as the filing of (exactly) two claims, find the probability the policyholder files exactly four claims.

55. Choose (a) or (b):

(a) Let X have an exponential density with parameter λ. Compute the density of $Y = cX$, where $c > 0$ is a positive constant.

(b) Let X have the Cauchy density. Compute the density of $Y = a + bX$, where a and b are constants such that $b > 0$.

56. (a) Find the exact values of $\Gamma(7)$ and $\Gamma(3.5)$.

(b) Simplify $\frac{\Gamma(3.2)}{\Gamma(5.2)}$.

(c) A useful and amazing fact to know about the gamma function is the following:

$$\Gamma(r)\Gamma(1-r) = \frac{\pi}{\sin(\pi r)}.$$

Use this fact to evaluate $\Gamma(1/3)\Gamma(2/3)$ and $\Gamma(7/6)\Gamma(5/6)$.

57. Suppose Z has the standard normal distribution. Find decimal approximations to the following probabilities (trust me, there are no typos in these inequalities):

(a) $P(Z < 1.33)$

(b) $P(Z \geq .79)$

(c) $P(.55 < Z < 1.22)$

(d) $P(-1.90 \geq Z \geq .44)$

(e) $P(Z > -.2)$

(f) $P(-.63 \leq Z < .3)$

(g) $P(Z < -1.3$ or $Z > .58)$

58. Suppose X is normal with parameters $\mu = 20$ and $\sigma^2 = 100$. Find decimal approximations to the following probabilities:

(a) $P(X \geq 27)$
(b) $P(X < 13)$
(c) $P(X = 22)$
(d) $P(X < 21 \,|\, X \leq 26)$
(e) $P(X \geq 17 \,|\, X < 24)$
(f) $P(X < 15 \text{ or } X \geq 22)$
(g) $P(Y \geq 54)$, assuming $Y = 3X+9$

59. Suppose X is normal $n(\mu, \sigma^2)$ and let $c > 0$. Compute, in terms of Φ, μ and σ, $P(|X - \mu| < c)$.

60. Let f be the density function of the normal r.v. with parameters μ and σ^2. Show that f has its maximum when $x = \mu$, and that the $x-$coordinates of the inflection points of f are $x = \mu \pm \sigma$ (this is one reason why we parameterize normal r.v.s in terms of μ and σ).

61. Suppose that during periods of transcendental meditation the reduction of a person's oxygen consumption is a normal $n(37.6 \text{ cc/min}, 4.6^2 \text{ cc/min})$. Find (a decimal approximation to) the probability that during a period of transcendental meditation a person's oxygen consumption will be reduced by at least 44.5 cc/min, and find (a decimal approximation to) the probability that during a period of transcendental meditation a person's oxygen consumption will be reduced by anywhere from 30 to 40 cc/min.

62. An expert witness in a paternity suit testifies that the length (in days) of a pregnancy, from conception to delivery, is normally distributed with parameters $\mu = 270$, $\sigma^2 = 100$. The defendant in the suit is able to prove that he was out of the country during the period from 253 to 278 days before the birth of the child. Based on this information, what is the probability the defendant was in the country when the child was conceived? (Give an answer should be in terms of Φ, and a decimal approximation.)

63. Use Stirling's formula to show $\binom{2n}{n}$ is approximately equal to $\frac{4^n}{\sqrt{\pi n}}$ for large n (we will use this fact in Math 416).

64. Suppose X and Y are discrete, integer-valued r.v.s with joint density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{9}\frac{2^x}{3^{x+y}} & x \geq 0, y \geq 0 \\ 0 & x < 0 \text{ or } y < 0 \end{cases}$$

(a) Verify that this $f_{X,Y}$ is in fact a density function.
(b) Find the probability that ($X = 3$ and $Y = 4$).
(c) Find the probability that $X = 2$.
(d) Find a density function of the marginal Y. What name would we use to describe the random variable Y?

65. Suppose you have two dice numbered 1 to 6 that you can load however you want (i.e. you can assign whatever probabilities you want to each number on each die). Is it possible to load the dice in such a manner that makes every sum from 2 to 12 equally likely when the dice are rolled independently? If so, explain how. If not, explain why not.

    Hint: It is useful to look at the probabilities of rolling 1 and 6 with each die.

66. Let X and Y be random variables having joint density function given by the following table:

$Y \backslash X$	-1	0	2	6
-2	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{27}$	$\frac{1}{9}$
1	$\frac{2}{9}$	0	$\frac{1}{9}$	$\frac{1}{9}$
3	0	0	$\frac{1}{9}$	$\frac{4}{27}$

    (a) Find the probability that X is even.
    (b) Find the probability that XY is odd.
    (c) Find the probability that $X > 0$ and $Y \geq 0$.
    (d) Compute the density function of the marginals X and Y.

67. Suppose X and Y are discrete random variables, **each taking values on the nonnegative integers**, with joint density function $f_{X,Y}$. For each given probability, write a sum which gives the probability. As an example, if asked to find $P(0 \leq X \leq 5, 2 \leq Y \leq 4)$, one possible correct answer is

$$P(0 \leq X \leq 5, 2 \leq Y \leq 4) = \sum_{x=0}^{5} \sum_{y=2}^{4} f_{X,Y}(x, y).$$

    (a) $P(3 \leq X < 8, 0 < Y < 5)$
    (b) $P(X = 2, 3 \leq Y)$ (here, the comma means "and")
    (c) $P(X = 2 \text{ or } Y \geq 3)$
    (d) $P(X = 2)$
    (e) $P(0 \leq X, 2 \leq Y \leq 5)$
    (f) $P(X + Y = 8)$
    (g) $P(X - Y = 7)$

68. Same directions as the previous question:

    (a) $P(0 \leq X \leq Y \leq 10)$
    (b) $P(0 \leq X \leq Y)$

(c) $P(0 \leq Y \leq -X)$
(d) $P(0 \leq X \leq Y^2)$
(e) $P(X + Y \leq 12)$
(f) $P(X + Y = z)$ where z is a constant
(g) $P(X - Y = z)$ where z is a constant
(h) $P(X \geq 0)$
(i) $P(X \leq 10, Y \leq 10, X + Y \geq 10)$

69. Suppose X and Y are independent random variables, each being uniform on the discrete set $\{1, 2, ..., N\}$. Compute the density of $X + Y$.

70. Let X and Y be independent random variables, where X is Poisson with parameter λ_1 and Y is Poisson with parameter λ_2. Prove that $X+Y$ is Poisson; what is its parameter? (The way you do this for now is to explicitly compute the density function of $X + Y$.) **The fact you are proving in this problem should be memorized** (and will be generalized later).

71. Let X and Y be independent random variables, where X is geometric with parameter p and Y is geometric with parameter q (there is no relationship between p and q in this problem).
    (a) Find $P(X = Y)$.
    (b) Find $P(X \geq Y)$.

72. Let X be the uniform distribution on $\{0, 1\}$ and Y the uniform distribution on $\{0, 1\}$. Characterize all possible joint distributions of X and Y. For each of these joint distributions, find the density of $X + Y$.

73. Suppose X and Y are discrete, integer-valued r.v.s with joint density function

$$f_{X,Y}(x, y) = \begin{cases} \frac{2}{9}\frac{2^x}{3^{x+y}} & x \geq 0, y \geq 0 \\ 0 & x < 0 \text{ or } y < 0 \end{cases}$$

    (a) Find the probability that $X + Y = 8$.
    Hint: I want an answer with no "Σ"s in it; you will need the following formula for a finite geometric sum:

$$\sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r} \quad \text{(this sum is also equal to } \frac{r^{N+1} - 1}{r - 1}\text{)}$$

    (b) Find the probability that $X + Y \geq 12$ (again, no "Σ"s in your answer are allowed).

74. Suppose X and Y have the density function of the previous problem (# 68). Find the density of $Z = X - Y$.

75. There are 40 gumballs in a bag (20 are red, 10 are orange, 8 are green, and 2 are purple).

    (a) Suppose you draw 15 gumballs, one at a time, with replacement. What is the probability you draw 5 red, 5 orange, and 5 green gumballs?

    (b) Suppose you draw 15 gumballs simultaneously. What is the probability you draw 5 red, 5 orange, and 5 green gumballs?

76. Suppose X and Y are continuous random variables such that $X \geq 0$ and $Y \geq 0$, with joint density function $f_{X,Y}$. For each given probability, write an expression involving integrals which gives the probability. As an example, if asked to find $P(0 \leq X \leq 5, 2 \leq Y \leq 4)$, one possible correct answer is

$$P(0 \leq X \leq 5, 2 \leq Y \leq 4) = \int_0^5 \int_2^4 f_{X,Y}(x,y)\, dy\, dx.$$

    (a) $P(3 \leq X < 8, 0 < Y < 5)$
    (b) $P(X \geq 4)$
    (c) $P(X + Y \leq 8)$
    (d) $P(\min(X,Y) \leq 6)$
    (e) $P(X \leq Y)$
    (f) $P(Y/X < 5)$
    (g) $P(X - 2Y > 5)$
    (h) $P(Y \leq 2X^2)$

77. Repeat Problem 76, but under the extra assumptions that X and Y take values only in the square whose vertices are $(0,0)$, $(10,0)$, $(0,10)$ and $(10,10)$.

78. Suppose X and Y are continuous random variables such that $0 < Y < X$, with joint density function $f_{X,Y}$. For each given probability, write an expression involving integrals which gives the probability.

    (a) $P(5 \leq X \leq 8, 3 \leq Y \leq 10)$
    (b) $P(3 \leq X \leq 10, 5 \leq Y \leq 8)$
    (c) $P(X + Y \leq 8)$
    (d) $P(Y \geq \frac{1}{4}X)$
    (e) $P(X \geq 11)$
    (f) $P(Y \leq 2)$
    (g) $P(X - Y > 7)$

79. Suppose X and Y are two real-valued random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} C(x^2 + \frac{xy}{2}) & \text{if } 0 < x < 1, 0 < y < 2 \\ 0 & \text{else} \end{cases}$$

where C is some constant.

(a) Find C.
(b) Find the density functions of each of the marginals.
(c) Find $P(X > Y)$.
(d) Find $P(Y > \frac{1}{2} \mid X < \frac{1}{2})$.

80. Let Δ be the triangle in the $xy-$plane whose vertices are $(0,0)$, $(2,0)$ and $(0,2)$. Suppose X and Y are random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} cx^2y & \text{if } (x,y) \in \Delta \\ 0 & \text{else} \end{cases}$$

where c is some constant.

(a) Find c.
(b) Find the probability that $X \geq 1$.
(c) Find the probability that both X and Y are greater than $\frac{1}{2}$.
(d) Find the marginal density of Y.
(e) Are X and Y independent? Why or why not?

81. (AE) A device runs until either of two components fails, at which point the device stops running. The joint density function of the lifetimes of the two components, both measured in hours, is

$$f(x,y) = \begin{cases} \frac{1}{8}(x+y) & 0 \leq x, y \leq 2 \\ 0 & \text{else} \end{cases}$$

What is the probability that the device fails during its first hour of operation?

82. (AE) Let X and Y be continuous r.v.s with joint density function

$$f(x,y) = \begin{cases} 15y & x^2 \leq y \leq x \\ 0 & \text{else} \end{cases}$$

Let g be the marginal density function of Y. Find a formula for $g(y)$ where $0 \leq y \leq 1$.

83. (AE) An insurance company insures a large number of drivers. Let X be the random variable representing the company's losses under collision insurance, and let Y represent the companys losses under liability insurance. X and Y have joint density function

$$f(x,y) = \begin{cases} \frac{1}{4}(2x+2-y) & x \in (0,1), y \in (0,2) \\ 0 & \text{else} \end{cases}$$

What is the probability that the total loss is at least 1?

84. Suppose X and Y are real-valued r.v.s with joint density

$$f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda y} & 0 \leq x \leq y \\ 0 & \text{else} \end{cases}.$$

   (a) Find the marginal densities of X and Y.
   (b) Find the probability that $Y \leq 4$.

85. Let X and Y denote the coordinates of a point chosen uniformly from the unit square. Let $Z_1 = X^2$ and $Z_2 = Y^2$. Let $Z_3 = X + Y$.

   (a) Are Z_1 and Z_2 independent? Why or why not? (Give a heuristic argument only.)
   (b) Are Z_1 and Z_3 independent? Why or why not? (Give a heuristic argument only.)

86. Suppose X and Y are continuous r.v.s with joint density

$$f_{X,Y}(x,y) = \begin{cases} xe^{-x(y+1)} & \text{if } 0 < x, 0 < y \\ 0 & \text{else.} \end{cases}$$

   Find the conditional density of X given Y.

87. Suppose X and Y are discrete r.v.s, taking values in the integers, whose joint density is

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{x!y!}\lambda^x e^{-\lambda-x-1}(x+1)^y & \text{if } 0 \leq x, 0 \leq y \\ 0 & \text{else.} \end{cases}$$

   Find the conditional density of Y given $X = 3$.

88. (AE) An insurance company supposes that each person has an accident parameter a and that the yearly number of accidents of someone who has accident parameter a is a Poisson random variable X with parameter a. The company also supposes that the parameter of a newly insured person is itself a $\Gamma(r, \lambda)$ random variable. If a newly insured person has n accidents in his first year,

   (a) Find the conditional density of his accident parameter.
   (b) Identify the conditional density in part (a) as the density of a common random variable. What random variable is it? What are its parameters?

89. Let Y be exponential with parameter λ. Suppose λ is itself a r.v. Λ with the gamma density $\Gamma(r, \beta)$.

(a) Find the marginal density of Y.

(b) Find the conditional density of Λ given $Y = y$.

90. (AE) The distribution of Y, given X, is uniform on $[0, X]$. The marginal density of X is $f_X(x) = 2x$ for $0 < x < 1$ ($f_X(x) = 0$ otherwise). Find the conditional density of X given $Y = y$ (where this conditional density is positive).

91. Find the conditional density $f_{Y|X}$, for the joint density given in Problem 84.

92. (AE) An auto insurance policy will pay for damage to both the policyholder's car and the other driver's car in the event that the policyholder is responsible for an accident. Assume that the size X of the payment for damage to the policyholder's car is uniform on $(0, 1)$, and that given $X = x$, the size Y of the payment to the other driver's car is uniform on $(x, x + 1)$. If the policyholder is responsible for an accident, what is the probability that the payment for damage to the other driver's car is greater than $\frac{1}{2}$?

93. (AE) Let X and Y be continuous r.v.s with joint density function

$$f(x, y) = \begin{cases} 24xy & \text{if } 0 < x < 1 \text{ and } 0 < y < 1 - x \\ 0 & \text{else} \end{cases}$$

Calculate $P[Y < X \,|\, X = \frac{1}{3}]$.

94. Suppose X and Y are discrete r.v.s whose joint density is given in the chart in Problem 66.

(a) Find $P(X < 4 \,|\, Y = 1)$.

(b) Find $P(Y < 3 \,|\, X = 6)$.

95. Suppose (X, Y) have joint density

$$f_{X,Y}(x, y) = \begin{cases} 4xy & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$$

Compute the density of $W = X + Y$.

Hint: The computation requires separate cases, depending on whether $W \geq 1$ or $W < 1$.

96. (AE) A company offers earthquake insurance. Annual premiums are modeled by an exponential random variable with parameter 1. Annual claims are modeled by an exponential random variable with parameter 2. Assume that the annual premiums and claims are independent; let X denote the ratio of claims to premiums. What is the density function of X?

97. Let X be $\Gamma(r, \lambda)$. Find the density of $Y = \sqrt{X}$.

98. (AE) The time T that a computer is not working is a r.v. whose cumulative distribution function is $F(t) = 1 - \frac{1}{4}t^{-2}$ for $t > 2$. The resulting cost X to the business as a result of the computer malfunctioning is $X = T^2$. Find the density function of X (when $X > 4$).

99. Let X and Y be independent standard normal r.v.s. What common random variable is Y/X?

    Hint: First, find the joint density of X and Y/X.

100. Let X and Y be continuous r.v.s with some unknown joint density function f. Find (in terms of f) the joint density of X and $Z = X + Y$.

101. Let X and Y be continuous r.v.s with some unknown joint density function f. Find the joint density (in terms of f) of W and Z where $W = Y/X$ and $Z = X + Y$.

102. Let X and Y be independent Poisson r.v.s, with respective parameters λ and μ. Let $Z = X + Y$.

    (a) Find the joint density of X and Z.

    Hint: In terms of X and Z, the joint density of X and Z is $f_{X,Z}(x, z) = P(X = x, Z = z)$. Back-substitute to see what this is in terms of X and Y.

    (b) Find the conditional density of X given Z.

    Hint: You should know what the density of Z is without computing its marginal again (since you studied this situation in a previous homework problem).

103. Suppose X and Y are continuous, real-valued r.v.s having some unknown joint density function $f_{X,Y}$. Let $W = a + bX$ and $Z = c + dY$ where a, b, c and d are constants, and $b > 0$ and $d > 0$.

    (a) Find the joint density function of W and Z.

    (b) Prove or disprove: if $X \perp Y$, then $W \perp Z$.

104. Suppose $X_1, ..., X_d$ are independent random variables.

    (a) Let $MIN = \min(X_1, ..., X_d)$. Find a formula for F_{MIN} in terms of the F_{X_j}.

    (b) Prove that if $X_1, ..., X_d$ are independent exponential r.v.s with respective parameters $\lambda_1, ..., \lambda_d$, then $\min(X_1, ..., X_d)$ is exponential with parameter $\lambda_1 + ... + \lambda_d$.

(c) Let $MAX = \max(X_1, ..., X_d)$ be the maximum of the X_js. Find a formula for F_{MAX} in terms of the F_{X_j}.

(d) (AE) A company decides to accept the highest of five sealed bids on a property. The sealed bids are regarded as five independent random variables, each with common cumulative distribution function

$$F(x) = \frac{(x-3)^2}{4} \text{ for } 3 \leq x \leq 5.$$

Find the density function of the accepted bid.

Note: The results of parts (a)-(c) of this problem are good to memorize for the actuarial exam. The maximum and minimum of the random variables are part of what is called the **order statistics** of the X_j.

105. Find the expected value of X in each of these cases:

(a) X has density function $f(x)$ defined by $f(x) = \frac{3}{4}(1-x^2)$ for $x \in (-1,1)$ and $f(x) = 0$ otherwise;

(b) X has cdf $F_X(x)$ defined by $F_X(x) = 1 - \frac{5}{x}$ if $x \geq 5$ and $F_X(x) = 0$ otherwise;

(c) X is the marginal of the joint distribution obtained when one selects a point (X, Y) uniformly from the triangle with vertices $(0,0)$, $(4,0)$ and $(0,4)$.

NOTE: in all homework problems from this point forward, you may assume without proof that all r.v.s under consideration have finite expectation.

106. Choose three of (a),(b),(c),(d):

(a) Prove that the expected value of the uniform distribution on the interval $[a,b]$ is $\frac{a+b}{2}$.

(b) Prove that the expected value of an exponential random variable with parameter λ is $\frac{1}{\lambda}$.

(c) Verify that the expected value of a hypergeometric r.v. with parameters n, r, k is $\frac{kr}{n}$.
Hint: Vandermonde's Identity may be useful.

(d) Prove that the expected value of a normal random variable with parameters μ and σ^2 is μ.
Hint: A hard (but doable) way to do this is to directly compute

$$\int_{-\infty}^{\infty} x f_X(x)\, dx.$$

There is a much more clever approach, however.

107. (a) Suppose Z is binomial with parameters $n = 4$ and $p = \frac{1}{3}$. Find $E \sin\left(\frac{\pi Z}{2}\right)$ (simplify your answer).

     (b) Suppose X is Poisson with parameter $\lambda = 5$. Find the mean of $(1+X)^{-1}$.

     (c) Let Y be the sine of an angle chosen uniformly from $(-\pi/2, \pi/3)$. Find the expected value of Y.

108. If X has expected value 3 and Y has expected value -1, what is the expected value of $3X - 5Y$?

109. Suppose that the density function f_X of X is:

$$f_X(x) = \begin{cases} a + bx^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{else} \end{cases} .$$

If $E(X) = \frac{3}{5}$, find a and b.

110. Suppose you play a carnival game that works like this: there are two bags, each with discs numbered 1 to 5 in them. You draw one disc uniformly from each bag. Whatever disc is the smaller number you draw, you win that amount of money (for example, if you draw a 2 and a 4, you would win 2).

     (a) How much would you expect to win if you played this game 100 times?

     (b) How much should the person running the game charge you if she expects to make a profit of $.30$ per game?

     (c) Suppose that there were n discs in each bag, numbered 1 to n. How much would you now expect to win if you played the same game 100 times?

     Hint: The summation formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \text{ and } \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

     may be useful.

111. (AE) A new plasma TV costs $\$800$. The lifetime of the TV is exponentially distributed with mean 4 years. Best Buy sells a warranty where they give a full refund to a buyer if the TV fails within the first two years, they give a half refund to a buyer if the TV fails during the third or fourth year, and they give no refund otherwise. How much should Best Buy expect to pay in refunds, if they sell 1000 plasma TVs?

112. (AE) Let T_1 be the time between a car accident and the reporting of a claim to an insurance company; let T_2 be the time between the reporting of this claim and the payment of this claim. Assume that (T_1, T_2) is uniform on the region of points (t_1, t_2) satisfying $0 < t_1 < 10$; $0 < t_2 < 10$; $0 < t_1 + t_2 < 16$. Find the expected amount of time between the accident and the payment of the claim.

113. (a) Let X be exponentially distributed with parameter λ. Find $E(X^2)$ directly (using the change of variables formula together with the Gamma integral formula) and use your answer to verify that the variance of X is $\frac{1}{\lambda^2}$.
   (b) Verify that the variance of the uniform distribution on the interval (a, b) is $\frac{(b-a)^2}{12}$.
   (c) Verify that the mean of a $\Gamma(r, \lambda)$ r.v. is $\frac{r}{\lambda}$ and that the variance of a $\Gamma(r, \lambda)$ r.v. is $\frac{r}{\lambda^2}$.
   (d) Suppose X has density f defined by $f(x) = cx^4$ for $0 < x < 2$ and $f(x) = 0$ otherwise. Find the variance of X.

114. Let X be a r.v. with finite expectation and finite variance. Prove:
   (a) $Var(aX) = a^2 Var(X)$ for any constant a.
   (b) $Var(X + b) = Var(X)$ for any constant b.

115. (a) Suppose X and Y are two independent random variables such that $EX^4 = 2$, $EY^2 = 1$, $EX^2 = 1$ and $EY = 0$. Compute the variance of X^2Y.
   (b) Let S and T be two independent random variables with finite expectation and finite variance. Let $W = 2S + 3T$; compute the mean and varance of W in terms of the means and variances of S and T.

116. Choose two of (a),(b),(c):
   (a) (AE) An actuary has discovered that policyholders are four times as likely to file three claims as they are to file four claims. If the number of claims filed has a Poisson distribution, what is the variance of the number of claims filed?
   (b) (AE) A company has two electric generators. The time until failure for each generator is exponential with mean 10. The company will begin using the second generator immediately after the first one fails. What is the variance of the total time the generators produce electricity?
   (c) (AE) The profit for a new product is given by $Z = 2X - 3Y + 7$, where $X \perp Y$, $Var(X) = 1$ and $Var(Y) = 2$. What is the variance of Z?

117. Let (X, Y) be a point chosen uniformly from the finite set of points

$$\{(0,1),(1,0),(0,-1),(-1,0)\}$$

(each point is chosen with probability $1/4$). Prove that X and Y are uncorrelated but not independent.

118. (a) Suppose a box contains three balls numbered 1 to 3. Two balls are selected without replacement from the box. Let U be the number on the first ball selected, and let V be the number on the second ball selected. Compute $Cov(U, V)$ and $\rho(U, V)$.

   (b) Find the covariance of X and Y if they have joint density

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } x > 0, y > 0, x + y < 1 \\ 0 & \text{else} \end{cases}$$

119. (AE) Let X and Y denote the price of two stocks at the end of a five-year period. Suppose X is uniform on $[0, 8]$ and that given $X = x$, Y is uniform on $[0, x]$. Determine $Cov(X, Y)$.

120. (AE) Let X denote the size of a surgical claim, and let Y denote the size of the associated hospital claim. An actuary is using a model in which $EX = 5$, $EX^2 = 27.4$, $EY = 7$, $EY^2 = 51.4$ and $Var(X + Y) = 8$. Let $C_1 = X + Y$ be the size of the combined claims before the application of a 20% surcharge on the hospital portion of the claim, and let C_2 denote the size of the combined claims after the surcharge. Calculate $Cov(C_1, C_2)$.

121. Prove any two of the following three statements:

   (a) $Cov(X, Y) = Cov(Y, X)$;

   (b) $Cov(X_1 + X_2, Y) = Cov(X_1, Y) + Cov(X_2, Y)$;

   (c) $Cov(aX, Y) = a\,Cov(X, Y)$.

   Note: These three statements, together with induction, prove the following important property of covariance called *bilinearity*:

$$Cov\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j Cov(X_i, Y_j)$$

122. Suppose a and b are positive constants, and that c and d are constants (not necessarily positive). Show that $\rho(aX+c, bY+d) = \rho(X, Y)$, i.e. that the correlation between two quantities does not depend on the units used to measure those quantities.

123. (a) Prove that if $Y = aX + b$ for constants a and b (with $a \neq 0$), then $\rho(X,Y) = \pm 1$. Under what conditions is $\rho(X,Y) = 1$ (as opposed to -1)?

   (b) In this problem, we will prove that if $\rho(X,Y) = \pm 1$, then $Y = aX + b$ where a and b are constants. To start, let $\widehat{X} = \frac{1}{\sqrt{Var(X)}}(X - EX)$ and let $\widehat{Y} = \frac{1}{\sqrt{Var(Y)}}(Y - EY)$.

   i. Compute $E[\widehat{X}]$, $E[\widehat{Y}]$, $E[\widehat{X}^2]$ and $E[\widehat{Y}^2]$.
   ii. Prove that $\rho(X,Y) = Cov(\widehat{X}, \widehat{Y})$.
   iii. Prove that $Cov(\widehat{X}, \widehat{Y}) = E[\widehat{X}\widehat{Y}]$.
   iv. Use parts (i)-(iii) to prove that $E[(\widehat{Y} - \rho(X,Y)\widehat{X})^2] = 1 - \rho(X,Y)^2$.
   v. Use part (iv) to prove that if $\rho(X,Y) = \pm 1$, then $Y = aX + b$ where a and b are constants.

124. Suppose X and Y are discrete r.v.s whose joint density is given in the chart in Problem 66.

   (a) Find $E(Y \,|\, X)$.
   (b) Find $E(X^3 \,|\, Y = 1)$.
   (c) Find $Var(X \,|\, Y = 3)$.

125. Let (X,Y) be chosen uniformly from the triangle whose vertices are $(0,0)$, $(2,0)$ and $(1,2)$. Find the conditional expectation of Y given X.

126. (AE) A fair die is rolled repeatedly. Let X be the number of rolls needed to obtain a 5 and let Y be the number of rolls needed to obtain a 6. Calculate $E[X \,|\, Y = 2]$.

127. Let X and Y be independent, where X is $\Gamma(r, \lambda)$ and Y is $\Gamma(s, \lambda)$. Find $E[X \,|\, X + Y]$.

   Hint: First find the joint density of X and $X + Y$.

128. (AE) Let N_1 and N_2 represent the numbers of claims submitted to a life insurance company in January and February, respectively. The joint density function of N_1 and N_2 is

$$f_{N_1,N_2}(n_1,n_2) = \begin{cases} \frac{2}{3}\left(\frac{1}{3}\right)^{n_1} e^{-n_1-1}\left(1 - e^{-n_1-1}\right)^{n_2} & \text{for } n_1, n_2 \in \{0,1,2,3,...\} \\ 0 & \text{otherwise} \end{cases}$$

   Calculate the expected number of claims that will be submitted to the company in February if exactly 2 claims were submitted in January.

129. (AE) A driver and a passenger are in a car accident. Each of them independently has a probability $.3$ of being hospitalized. If they are hospitalized, the loss is uniform on $[0, 1]$. When two hospitalizations occur, the losses are independent. Calculate the expected number of people who are hospitalized, given that the total loss due to hospitalizations from the accident is less than 1.

130. (a) Prove that the two formulas given in the notes as definitions of conditional variance are the same.
     (b) Prove the Law of Total Variance, which says:
$$E[Var(X|Y)] + Var[E(X|Y)] = Var(X).$$

131. (a) (AE) The number of workplace injuries, N, occuring in a factory on any given day is Poisson with mean λ. The parameter λ is itself a random variable depending on the level of activity in the factory, and is assumed to be uniformly distributed on the interval $[0, 3]$. Find $Var(N)$.
     (b) (AE) The stock prices of two companies at the end of any given year are modeled with r.v.s X and Y whose joint density function is
$$f(x, y) = \begin{cases} 2x & \text{for } 0 < x < 1,\, x < y < x + 1 \\ 0 & \text{otherwise} \end{cases}.$$
     What is the conditional variance of Y given $X = x$?

132. Let X be binomial(n, p). Use the probability generating function of X to compute the expected value and variance of X.

133. (a) Compute the moment generating function of a Poisson r.v. with parameter λ.
     (b) Compute the moment generating function of a gamma $\Gamma(r, \lambda)$ r.v.

134. (AE) Let X, Y and Z be i.i.d. r.v.s, each taking the value 0 with probability p and the value 1 with probability $(1-p)$. Find the moment generating function of $W = XYZ$.

135. Let X be a continuous r.v. having the density $f_X(x) = \frac{1}{2}e^{-|x|}$ for all x. Find the moment generating function of X.

136. (a) Find the first and second moments of X if its moment generating function is $M_X(t) = \frac{1}{\sqrt{1-4t}}$ for $t < \frac{1}{4}$.
     (b) Suppose X is exponential with parameter 1 and Y is exponential with parameter 3. If $X \perp Y$, what is the moment generating function of $4X + Y$?

(c) (AE) Assume that the number of claims related to traffic accidents on a certain road is a r.v. X whose moment generating function is $M_X(t) = (1-2500t)^{-4}$. Find the standard deviation of the claim size for this class of accidents.

137. Explain why each of the following functions *cannot* be the moment generating function of a real-valued random variable X:

(a) $h(t) = \frac{e^{-t}}{2-t}$ for $t < 2$;

(b) $j(t) = \frac{1+t}{1-t}$ for $t < 1$;

(c) $k(t) = \exp(\frac{-t^2}{2})$ for $-\infty < t < \infty$.

138. (AE) Let X represent the number of customers arriving during the morning hours, and let Y be the number of customers arriving during the evening hours to a restaurant. Assuming that X and Y are both Poisson, and that the first moment of X is 8 less than the first moment of Y, and that the second moment of X is 60% of the second moment of Y, what is the variance of Y?

139. (AE) The number N of babies born in a hospital during any one week is a random variable satisfying $P(N = n) = \frac{1}{2^n}$, for $n \in \{0, 1, 2, ...\}$. Suppose that the number of babies born in any one week is independent of the number of babies born in any other week. Determine the probability that exactly seventeen babies are born in a given four-week period.

140. In this problem we will derive the **Beta integral formula** (which we have used before to solve certain expected value and conditional expectation problems):

$$\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

(a) Let X be $\Gamma(\alpha, \lambda)$ and let Y be $\Gamma(\beta, \lambda)$. Suppose that $X \perp Y$. Determine the density function of $Z = X + Y$ using moment generating functions.

(b) Given X and Y as above, compute the joint density function of X and $Z = X + Y$ by the transformation method of Packet 414-5.

(c) Use your answer to part (b) to compute the marginal density of Z (write your answer as an integral with respect to x).

(d) Derive the Beta integral formula by equating the answers to part (a) and (c) of this problem, and solving the resulting equation for the Beta integral above.

Hint: in the integral you obtain from part (c), use the $u-$substitution $u = \frac{x}{z}$.

141. A **Beta random variable** with parameters $r_1 > 0$ and $r_2 > 0$ (denoted $B(r_1, r_2)$) is a continuous r.v. whose density is

$$f(x) = \begin{cases} \frac{\Gamma(r_1+r_2)}{\Gamma(r_1)\Gamma(r_2)} x^{r_1-1}(1-x)^{r_2-1} & \text{if } 0 < x < 1 \\ 0 & \text{else} \end{cases}.$$

(a) Prove that the function above is in fact a density function.

(b) Find the expected value of a Beta $B(r_1, r_2)$ r.v.

142. (AE) Suppose X and Y are independent r.v.s which have the same moment generating function: $M_X(t) = M_Y(t) = e^{t^2}$. Determine the joint moment generating function of $W = X + Y$ and $Z = Y - X$.

143. (AE) Let X and Y be i.i.d. r.v.s such that the moment generating function of $X + Y$ is

$$M_{X+Y}(t) = .09e^{-2t} + .24e^{-t} + .34 + .24e^{t} + .09e^{2t}$$

for all t. Calculate $P(X \leq 0)$.

144. Suppose (X, Y) have the following bivariate normal density:

$$f_{X,Y}(x,y) = C \exp\left[\frac{-1}{54}\left(x^2 + 4y^2 + 2xy + 2x + 8y + 4\right)\right]$$

Find the means and variances of X and Y, and find $Cov(X, Y)$ and $\rho(X, Y)$. Also find C and the covariance matrix Σ.

145. Let X and Y have the density given in the preceding problem (#144). Compute the conditional density of Y given $X = x$, and the conditional density of X given $Y = -1$.

146. Let X be a r.v. whose density is

x	1	2	3
$f_X(x)$	$\frac{1}{18}$	$\frac{16}{18}$	$\frac{1}{18}$

Show that when $\delta = 1$, $P(|X - \mu| \geq \delta) = \frac{Var(X)}{\delta^2}$ (the point of this problem is to show that in general, the $\leq$ sign in Chebyshev's inequality cannot be replaced by a $<$).

147. A bolt manufacturer knows that 5% of his production is defective. He gives a guarantee on his shipment of 10000 parts by promising that no more than a bolts are defective. Use Chebyshev's inequality to find the smallest number a can be so that the manufacturer is assured of not paying a refund more than 1% of the time.

148. Let X be Poisson with mean λ. Use Chebyshev's inequality to verify that $P(X \leq \frac{\lambda}{2}) \leq \frac{4}{\lambda}$.

149. Let X be Poisson with mean λ. Use Chebyshev's inequality to verify that $P(X \geq 2\lambda) \leq \frac{1}{\lambda}$.

150. Suppose X is a random variable with mean and variance both equal to 20. What can be said about $P(0 < X < 40)$? (In particular, what is the maximum or minimum value of this expression?)

151. Suppose X and Y are two real-valued r.v.s with

$$EX = 75,\ EY = 75,\ \text{Var}(X) = 10,\ \text{Var}(Y) = 12,\ \text{Cov}(X, Y) = -3.$$

What can be said about $P(|X - Y| \geq 15)$?

152. For each $\lambda > 0$, let X_λ be Poisson with parameter λ and let $Y_\lambda = \frac{X_\lambda - \lambda}{\sqrt{\lambda}}$.

(a) Show that for all t,

$$\lim_{\lambda \to \infty} M_{Y_\lambda}(t) = \exp(t^2/2).$$

(b) Fix $c > 0$ and use part (a) to estimate, for large λ, the value of $P(X_\lambda \leq c\lambda)$. Your answer should be in terms of Φ, the cumulative distribution function of the standard normal random variable.

(c) Find $\lim_{\lambda \to \infty} P(X_\lambda \leq c\lambda)$.
Hint: There may be different answers depending on the value of c.

153. Fix $\lambda > 0$ and for each $r > 0$ let X_r be $\Gamma(r, \lambda)$ and define $Y_r = \frac{X_\lambda - \left(\frac{r}{\lambda}\right)}{\left(\frac{\sqrt{r}}{\lambda}\right)}$.

(a) Find the expected value and variance of Y_r.

(b) Show that for all t,

$$\lim_{r \to \infty} M_{Y_r}(t) = \exp(t^2/2).$$

(c) Find $\lim_{r \to \infty} P(Y_r \leq x)$ in terms of Φ, the cumulative distribution function of the standard normal random variable.

(d) Find $\lim_{r \to \infty} P(X_r \leq r/\lambda)$.

154. Suppose $X_1, X_2, ...$ are i.i.d. random variables (taking only positive real values), each having finite mean μ. Show that with probability 1, the geometric averages of the X_j converge, where the *geometric average* of $X_1, ..., X_n$ is

$$G_n = \sqrt[n]{\prod_{j=1}^{n} X_j}.$$

Find $\lim_{n\to\infty} G_n$.

Hint: Apply the SLLN to $\log G_n$ (here $\log$ means natural logarithm).

155. The amount of liquid a student puts in their drink at the Rock is a random variable with mean 350 mL and variance 1500 mL. Use the Central Limit Theorem to estimate the probability that a randomly selected group of 12 students put an average of 320 mL or more in their drinks. Give both the exact answer in terms of Φ and a decimal approximation to the answer.

156. 1000 fair dice are rolled independently. Use the Central Limit Theorem to estimate the probability that the sum of these 1000 rolls is at least 3450 and no greater than 3650. Give both the exact answer in terms of Φ and a decimal approximation to the answer.

157. You play a game where you lose $\$1$ with probability .7, you lose $\$2$ with probability .2, and win $\$10$ with probability .1. If you play this game 10000 times, what is the probability that you will be ahead (that is, you have won more money than you have lost)? (You are to approximate this answer using the Central Limit Theorem; give both the exact answer in terms of Φ and a decimal approximation to the answer.)

158. Let X be a Poisson random variable with mean 20. Let $p = P(X \geq 26)$. Approximate p using the Central Limit Theorem, by approximating X as the sum of 20 i.i.d. random variables. Give both the exact answer in terms of Φ and a decimal approximation to the answer.

159. A tobacco company claims that the amount of nicotine in one of its cigarettes is a random variable with mean 2.2 mg and standard deviation .8 mg. Use the Central Limit Theorem to estimate the probability that 100 randomly chosen cigarettes would have an average nicotine content of at most 2.09 mg. Give both the exact answer in terms of Φ and a decimal approximation to the answer.