Old Math 414 Exams

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Last updated to include Exams from 2014

Contents

Chapter 1

General information about these exams

These are the exams I have given between 2007 and 2014 in probability courses. Each exam is given here, followed by what I believe are the solutions (there may be some number of computational errors or typos in these answers).

I have edited these exams to remove questions that do not match the current syllabus of Math 414; that's why some of them may contain a less than expected number questions.

Note that I have revised my probability course several times over the years, and what was on "Exam 1" in past years may not match what is on "Exam 1" now. To help give you some guidance on what questions are appropriate, each question on each exam is followed by a section number in parenthesis (like "(3.2)"). That means that question can be solved using material from that section (or from earlier sections) in my *Lectures in Probability*.

Last, my exam-writing style has evolved over the years; generally speaking, the more recent the exam, the more likely you are to see something similar on one of your tests.

Chapter 2

Exams from 2007 to 2010

2.1 Fall 2007 Exam 1

1. Suppose X is geometrically distributed with parameter $3/4$.

   a) (2.4) Find the probability that $X = 3$.

   b) (2.4) Find the probability that $X \geq 7$.

2. Suppose X is a real-valued random variable with the following distribution function:

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{16}x^3 + \frac{1}{2} & \text{if } 0 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

   a) (3.2) Find $P(0 \leq X \leq 1)$.

   b) (3.2) Find a density function of X, or explain why X does not have a density function.

   c) (3.3) Suppose $Y = \sqrt{X}$. Find the distribution function of Y.

3. Suppose you deal a 7 card hand from a standard 52-card deck.

   a) (2.3) What is the probability your hand contains 3 distinct pairs (i.e. your 7 cards have values x, x, y, y, z, z, w where x, y, z and w are different)?

   b) (2.3) What is the probability your hand contains 4 of a kind and three of a kind (i.e. your 7 cards have values x, x, x, x, y, y, y with x and y different)?

4. Your sock drawer contains 24 white socks, 12 grey socks, 8 brown socks, and 6 blue socks (there are a total of 50 socks in the drawer).

a) (2.3) If you grab 4 socks (at once) from the drawer randomly without looking, what is the probability you grabbed exactly 2 white socks?

b) (2.3) If you randomly take 8 socks, one at a time with replacement, what is the probability that at most 2 of the socks you draw are grey?

5. Suppose a point (X, Y) is chosen uniformly from the rectangle with vertices $(0,0)$, $(0,3)$, $(4,0)$ and $(4,3)$.

a) (3.3) Find the probability that $3X - 4Y < 6$.

b) (3.3) Let $W = 3X - 4Y$; find the distribution function F_W.

Solutions

1. The density function is $f(x) = p(1-p)^x = (3/4)(1/4)^x$.

a) $f(3) = (3/4)(1/4)^3 = 3/256$.

b)

$$P(X \geq 7) = \sum_{x=7}^{\infty} f(x) = \frac{3}{4}\left(\frac{1}{4}\right)^7 \sum_{x=0}^{\infty}\left(\frac{1}{4}\right)^x = \frac{3}{4}\left(\frac{1}{4}\right)^7 \frac{1}{1-\frac{1}{4}} = \left(\frac{1}{4}\right)^7.$$

2. a) Here is the calculation:

$$\begin{aligned} P(X \in [0,1]) &= P(X=0) + P(X \in (0,1]) \\ &= (F_X(0) - \lim_{x \to 0^-} F_X(x)) + (F_X(1) - F_X(0)) \\ &= (1/2 - 0) + (1/16 + 1/2 - 1/2) = 9/16. \end{aligned}$$

b) X is not continuous so it does not have a density.

c) First notice that $0 \leq Y \leq \sqrt{2}$ since the range of X is $[0,2]$. Now let $y \in (0, \sqrt{2})$:

$$F_Y(y) = P(Y \leq y) = P(\sqrt{X} \leq y) = P(X \leq y^2) = F_X(y^2) = \frac{1}{16}y^6 + \frac{1}{2}.$$

So the distribution function of Y is

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{1}{16}y^6 + \frac{1}{2} & \text{if } 0 \leq y < \sqrt{2} \\ 1 & \text{if } y \geq \sqrt{2} \end{cases}$$

3. Throughout this problem, $C(a,b)$ means "a choose b".

a) The total number of 7 card hands is $C(52,7)$. To specify a hand with three distinct pairs, you need to give:

- the value of x, y, z and w ($C(13,3)$ choices for the x, y, z and 10 choices for the z)
- the suits of x, y, z and w ($C(4,2)$ choices for x, y and z and $C(4,1) = 4$ choices for w)

The total number of hands with three distinct pairs is therefore $C(13,3) \cdot C(4,2) \cdot C(4,2) \cdot C(4,2) \cdot 10 \cdot C(4,1)$ and so the probability is

$$\frac{\begin{pmatrix} 13 \\ 3 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} 10 \begin{pmatrix} 4 \\ 1 \end{pmatrix}}{C(52,7)}.$$

b) There are 13 choices for x, $C(4,4)$ choices for the suits of x, 12 choices for y, and $C(4,3)$ choices for the suits of y. The answer is

$$\frac{13\begin{pmatrix} 4 \\ 4 \end{pmatrix} 12 \begin{pmatrix} 4 \\ 3 \end{pmatrix}}{C(52,7)} = \frac{13 \cdot 12 \cdot 4}{C(52,7)}.$$

4. a) This is a hypergeometric distribution. There are 24 white socks and therefore 26 non-white socks; the answer is $\dfrac{\begin{pmatrix} 24 \\ 2 \end{pmatrix}\begin{pmatrix} 26 \\ 6 \end{pmatrix}}{\begin{pmatrix} 50 \\ 8 \end{pmatrix}}$.

b) This is a Bernoulli experiment with $n = 8$ and $p = 12/50 = .24$. The probability is therefore $b(8, .24, 0) + b(8, .24, 1) + b(8, .24, 2)$ which is

$$\begin{pmatrix} 8 \\ 0 \end{pmatrix}(.24)^0(.76)^8 + \begin{pmatrix} 8 \\ 1 \end{pmatrix}(.24)^1(.76)^7 + \begin{pmatrix} 8 \\ 2 \end{pmatrix}(.24)^2(.76)^6.$$

5. a) The line $3X - 4Y = 6$ intersects the box $[0,4] \times [0,3]$ at the points $(2,0)$ and $(4,1.5)$. The desired probability corresponds to 1 minus the area of a triangle with vertices at the points $(2,0)$, $(4,0)$ and $(4,1.5)$ divided by the area of the whole rectangle (which is 12). This probability is

$$1 - \frac{\frac{1}{2}(2)(1.5)}{12} = \frac{7}{8}.$$

b) First notice $-12 \leq W \leq 12$ so $F_W(w) = 0$ when $w < -12$ and $F_W(w) = 1$ when $w \geq 12$. Now suppose $-12 \leq w < 12$. In this case,

$$F_W(w) = P(W \leq w) = P(3X - 4Y \leq w) = P\left(Y \geq \frac{3}{4}X - \frac{w}{4}\right).$$

There are two cases:

- $w < 0$. In this case, the line $Y = (3/4)X - (w/4)$ intersects the left side and top of the rectangle; the area of the triangle above this line is $1/2$ times the height of the triangle times its width, which is $(1/2)(3 + w/4)(4 + w/3)$. So

$$F_W(w) = \frac{\frac{1}{2}(3 + w/4)(4 + w/3)}{12} \text{ if } -12 \le w < 0.$$

- $w > 0$. In this case, the line $Y = (3/4)X - (w/4)$ intersects the bottom and right side of the rectangle; the area of the triangle below this line is $1/2$ times its width times its height which is $(1/2)(4 - w/3)(3 - w/4)$. So

$$F_W(w) = 1 - \frac{(1/2)(4 - w/3)(3 - w/4)}{12} \text{ if } 0 \le w < 12.$$

To summarize,

$$F_W(w) = \begin{cases} 0 & \text{if } w < -12 \\ \frac{1}{24}(3 + w/4)(4 + w/3) & \text{if } -12 \le w < 0 \\ 1 - \frac{1}{24}(4 - w/3)(3 - w/4) & \text{if } 0 \le w < 12 \\ 1 & \text{if } w \ge 12 \end{cases}.$$

2.2 Fall 2009 Exam 1

1. (1.4) Let A, B and C be events, each having probability $1/4$. Suppose A and B are independent, A and C are disjoint, and $P(C \,|\, B) = 1/3$. Find $P(A \cup B \cup C)$.

2. You have a fair die and a coin that flips heads $2/3$ of the time. Suppose you roll the die and then flip the coin the number of times that the die shows (i.e. if you roll a 3, you flip the coin three times).

   a) (2.4) What is the probability that you flip exactly four heads?

   b) (2.4) Show that the probability that you rolled a five, given that you flipped exactly four heads, is $5/13$.

3. Let X be exponential with parameter λ.

   a) (3.4) What is the probability X is less than 2 or greater than 4?

   b) (3.4) Let $Y = 1/(X+1)$. Find the distribution function of Y.

   c) (3.4) Let $Z = \sqrt{X}$. Find a density function of Z.

4. Suppose X is geometric with parameter p.

   a) (2.4) Let $0 \le a \le b$. Find $P(X \ge b \,|\, X \ge a)$.

   b) (3.3) Let $Y = \begin{cases} X & \text{if } X \le 3 \\ 4 & \text{if } X \ge 4 \end{cases}$. Find the density function of Y.

5. Suppose you deal a six-card hand from a standard 52-card deck.

   a) (2.3) What is the probability that your six-card hand is a flush (i.e. your cards all belong to the same suit)?

   b) (2.3) What is the probability your hand contains 3 of a kind and a pair (i.e. your six cards have values x, x, x, y, y, and z, where x, y, z are different)?

   c) (2.3) What is the probability your hand contains at least two aces if it contains at least one ace?

Solutions

1. First, we see that since A and C are disjoint, $P(A \cap C) = 0$ and $P(A \cap B \cap C) = 0$. Next, observe $P(B \cap C) = P(B)P(C \mid B) = (1/4)(1/3) = 1/12$ and since $A \perp B$, $P(A \cap B) = P(A)P(B) = (1/4)(1/4) = 1/16$. Now by the generalized inclusion-exclusion rule,

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &\quad -P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &\quad +P(A \cap B \cap C) \\ &= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - \frac{1}{16} - 0 - \frac{1}{12} + 0 = \frac{29}{48}. \end{aligned}$$

2. You have a fair die and a coin that flips heads $2/3$ of the time. Suppose you roll the die and then flip the coin the number of times that the die shows (i.e. if you roll a 3, you flip the coin three times).

   a) Let $E =$ flipping four heads and let A_j be the event of rolling a j. Since the A_j, for $1 \le j \le 6$ form a partition, by the Law of Total Probability we have
$$P(E) = \sum_{j=0}^{6} P(A_j)P(E \mid A_j)$$
   Clearly $P(A_j) = 1/6$ for all j. If $j < 4$, $P(E \mid A_j) = 0$ since there are not enough coin flips to flip four heads. When $j \ge 4$, $P(E \mid A_j)$ is the following binomial expression:
$$b(j, 2/3, 4) = \binom{j}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^{j-4}.$$
   So the answer is
$$\begin{aligned} P(E) &= \sum_{j=4}^{6}\left(\frac{1}{6}\right)\cdot\binom{j}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^{j-4} \\ &= \frac{1}{6}\left[\left(\frac{2}{3}\right)^4 + \binom{5}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right) + \binom{6}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2\right]. \end{aligned}$$

   b) Keeping the same notation as in the previous problem, we are asked to find $P(A_5 \mid E)$. By the definition of conditional probability (or by Bayes'

Law), this is

$$\begin{aligned} P(A_5 \mid E) &= \frac{P(E \cap A_5)}{P(E)} \\ &= \frac{P(A_5)P(E \mid A_5)}{P(E)} \\ &= \frac{\left(\frac{1}{6}\right) \cdot \begin{pmatrix} 5 \\ 4 \end{pmatrix} \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)}{\frac{1}{6}\left[\left(\frac{2}{3}\right)^4 + \begin{pmatrix} 5 \\ 4 \end{pmatrix} \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right) + \begin{pmatrix} 6 \\ 4 \end{pmatrix} \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2\right]}, \end{aligned}$$

using the answers already calculated in part (a). Now, observe that $C(5,4) = 5$ and $C(6,4) = 15$; additionally, you can cancel $(1/6)(2/3)^4$ from both the numerator and denominator to obtain a cleaner answer of

$$P(A_5 \mid E) = \frac{\frac{5}{3}}{1 + \frac{5}{3} + 15 \cdot \frac{1}{9}} = \frac{5}{13}.$$

3. a) The cdf of X is $F_X(x) = 1 - e^{-\lambda x}$. Using the fact that X is continuous so we can switch between $<$ and $\geq$ interchangeably):

$$\begin{aligned} P(X < 2 \text{ or } X > 4) &= P(X < 2) + P(X > 4) \\ &= F_X(2) + [1 - F_X(4)] \\ &= 1 - e^{-2\lambda} + e^{-4\lambda}. \end{aligned}$$

b) First, the largest possible value of Y corresponds to the smallest value of X, this is when $X = 0$ (in this case, $Y = 1$). The smallest possible value of Y corresponds to the largest value of X; this when $X \to \infty$ (and consequently $Y \to 0$). So the range of Y is $(0, 1]$ and therefore

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ ? & \text{if } y \in [0,1) \\ 1 & \text{if } y \geq 1 \end{cases}.$$

Let To find what the "?" is, let $y \in [0, 1)$. By the definition of distribution function,

$$\begin{aligned} F_Y(y) = P(Y \leq y) &= P(\frac{1}{X+1} \leq y) \\ &= P(X+1 \geq \frac{1}{y}) \\ &= P(X \geq \frac{1}{y} - 1) \\ &= 1 - P(X \leq \frac{1}{y} - 1) \\ &= 1 - F_X(\frac{1}{y} - 1) \\ &= 1 - [1 - e^{-\lambda(\frac{1}{y}-1)}] \\ &= e^{-\lambda(\frac{1}{y}-1)}. \end{aligned}$$

c) Z has range $[0, \infty)$; let $z \geq 0$. We see that $F_Z(z) = P(Z \leq z) = P(\sqrt{X} \leq z) = P(X \leq z^2) = F_X(z^2) = 1 - e^{-\lambda(z^2)}$. This holds when $z \geq 0$; when $z < 0$, $F_Z(z) = 0$. Differentiate F_Z to obtain f_Z; the answer is

$$f_Z(z) = \begin{cases} 2\lambda z e^{-\lambda(z^2)} & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases}.$$

Alternate solution: Let $\phi(x) = \sqrt{x}$, on $I = [0, \infty)$, ϕ is monotone increasing and has inverse $\phi^{-1}(z) = z^2$. So by the change of variable formula for density functions,

$$\begin{aligned} f_Z(z) = f_X(\phi^{-1}(z)) \left| \frac{d}{dz}\phi^{-1}(z) \right| &= \lambda e^{-\lambda(z^2)} \, |2z| \\ &= 2\lambda z e^{-\lambda(z^2)}. \end{aligned}$$

This holds for $z \in \phi(I) = [0, \infty)$; when $z < 0$, $f_Z(z) = 0$.

4. a) By the memoryless property of the geometric r.v.,

$$P(X \geq b \,|\, X \geq a) = P(X \geq b - a) = (1-p)^{b-a}.$$

b) The range of Y is $\{0, 1, 2, 3, 4\}$. The density function is defined by $f_Y(y) = P(Y = y)$; for $y = 0, 1, 2, 3$, $f_Y(y) = P(Y = y) = P(X = y) = p(1-p)^y$. For $y = 4$, $f_Y(4) = P(Y = 4) = P(X \geq 4) = (1-p)^4$.

5. There are a total of $C(52, 6)$ six card hands.

a) To specify a flush, we need to specify a suit for all the cards to belong to (4 choices) and a collection of six cards from that suit which comprise the hand; this is tantamount to picking an unordered group of 6 from a group of 13 so this can be done in $C(13,6)$ ways. So this probability is

$$\frac{4 \cdot C(13,6)}{C(52,6)}.$$

b) To specify such a hand, we need to choose (i) the value of x (13 choices); (ii) the particular suits of x which are in the hand ($C(4,3) = 4$ choices); (iii); the value of y (12 choices); (iv) the particular suits of y which are in the hand ($C(4,2)$ choices); (v) the value of z (11 choices) and (vi) the suit of z ($C(4,1) = 4$ choices). Thus the probability is

$$\frac{13 \cdot 4 \cdot 12 \cdot C(4,2) \cdot 11 \cdot 4}{C(52,6)}$$

c) Once we are given that the hand contains at least one ace, the number of possible hands is $C(51,5)$ (the number of groups of five cards other than the specified ace). To specify a hand that contains no other aces, we must specify a group of five non-aces, to be taken from the group of 48 non-aces in the deck (there are $C(48,5)$ such groups. So the probability asked for in the problem is therefore 1 minus the probability that the hand has no other aces; this is

$$1 - \frac{C(48,5)}{C(51,5)}.$$

Alternate solution: First of all,

$$P(\text{hand has} \geq 2 \text{ aces} \mid \text{hand has} \geq 1 \text{ ace})$$

is equal to

$$1 - P(\text{hand has exactly 1 ace} \mid \text{hand has} \geq 1 \text{ ace})$$

by the complement rule. By the definition of conditional probability, this is

$$\mathcal{P} = 1 - \frac{P(\text{hand has exactly 1 ace})}{P(\text{hand has} \geq 1 \text{ ace})}.$$

To specify a hand with exactly one ace, we need to specify (i) which ace is in the hand (4 choices); (ii) which other 5 cards are in the hand (these must be chosen from the 48 non-aces, so there are $C(48,5)$ ways to do this). So

$$P(\text{hand has exactly 1 ace}) = \frac{4 \cdot C(48,5)}{C(52,6)}.$$

The probability that a hand has at least one ace is (by the complement rule) 1 minus the probability it has no aces; to specify a hand with no aces one must specify which six cards are chosen from the 48 non-aces; there are $C(48,6)$ ways to do this. So

$$P(\text{hand has} \geq 1 \text{ ace}) = 1 - \frac{C(48,6)}{C(52,6)}.$$

Finally, the answer is

$$\begin{aligned} \mathcal{P} &= 1 - \frac{\frac{4\cdot C(48,5)}{C(52,6)}}{1 - \frac{C(48,6)}{C(52,6)}} \\ &= 1 - \frac{4 \cdot C(48,5)}{C(52,6) - C(48,6)}. \end{aligned}$$

2.3 Fall 2007 Exam 2

1. Urn I contains 5 blue, 6 green and 7 yellow balls; urn II contains 3 blue, 8 green and 2 yellow balls; urn III contains 4 blue, 2 green and 5 yellow balls. One of the urns is chosen at random and then a ball is randomly selected from the chosen urn.

   a) (2.3) What is the probability that the chosen ball is yellow?

   b) (2.3) Suppose that a yellow ball has been chosen. What is the probability that urn I was picked?

   c) (2.3) Now suppose that the balls from all three urns are put into one urn and 8 balls are chosen simultaneously. What is the probability that 4 are blue, 3 are green and one is yellow?

2. Let X be uniform on the interval $[a, b]$.

   a) (6.2) Show (via calculation) that $EX = \frac{a+b}{2}$ and $Var(X) = \frac{1}{12}(b-a)^2$.

   b) (6.2) Find the mean and variance of X^2.

3. Let X be the noon temperature in degrees Fahrenheit and Y the same temperature in degrees Celsius. Note that $Y = \frac{5}{9}(X-32)$. X is normal with mean 68 and variance 81.

   a) (6.2) Find the mean and variance of Y.

   b) (3.6) Find the density function for Y.

   c) (3.6) What type of random variable is Y? (Your answer should include the values of any necessary parameters).

4. A fair die is rolled repeatedly.

   a) (2.4) Find the probablities that the 3rd six comes on the 5th roll; that the 7th six comes on the 11th roll; and that the 4th six comes on the 6th roll.

   b) (2.4) Are the events that the 3rd six comes on the 5th roll and the 7th six comes on the 11th roll independent?

5. The joint density of two random variable X and Y is given by $f_{X,Y}(x, y) = ce^{-2x-3y}$ if $x > 0$ and $y > 0$ and $f_{X,Y}(x, y) = 0$ otherwise.

   a) (5.1) Show that $c = 6$.

   b) (5.1) Find the joint distribution function $F_{X,Y}(x, y)$.

   c) (5.1) Find the densities $f_X(x)$ and $f_Y(y)$.

   d) (5.1) Are X and Y independent?

Solutions

1. a) Let E_i be the event that urn i is chosen; let Y be the event that the chosen ball is yellow. By the Law of Total Probability,

$$P(Y) = \sum_{i=1}^{3} P(Y|E_i)P(E_i) = \frac{1}{3}\left(\frac{7}{18} + \frac{2}{13} + \frac{5}{11}\right).$$

b) By Bayes' Formula,

$$\begin{aligned} P(E_1|Y) &= \frac{P(Y|E_1)P(E_1)}{P(Y|E_1)P(E_1) + P(Y|E_2)P(E_2) + P(Y|E_3)P(E_3)} \\ &= \frac{7/18}{7/18 + 2/13 + 5/11}. \end{aligned}$$

Notice that all the $(1/3)$ terms were cancelled from this expression.

c) This is $d-$dimensional hypergeometric; there are a total of 42 balls of which 14 are yellow, 12 are blue and 16 are green. So the answer is

$$\frac{\binom{14}{1}\binom{12}{4}\binom{16}{3}}{\binom{42}{8}}.$$

2. Note that the density function of X is $\frac{1}{b-a}$ if $X \in [a, b]$ and is zero otherwise. Hence:

$$EX = \int_a^b x f(x)\,dx = \int_a^b \frac{x}{b-a}\,dx = \frac{1}{2}\frac{b^2-a^2}{b-a} = \frac{a+b}{2}.$$

$$EX^2 = E(X^2) = \int_a^b x^2 f(x)\,dx = \int_a^b \frac{x^2}{b-a}\,dx = \frac{1}{3}\frac{b^3-a^3}{b-a} = \frac{a^2+ab+b^2}{3}.$$

$$\text{Var}(X) = EX^2 - (EX)^2 = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}.$$

$$EX^4 = E((X^2)^2) = \int_a^b \frac{x^4}{b-a}\,dx = \frac{1}{5}\frac{b^5-a^5}{b-a} = \frac{1}{5}(b^4+ab^3+a^2b^2+a^3b+a^4).$$

$$\begin{aligned} \text{Var}(X^2) &= E((X^2)^2) - (E(X^2))^2 \\ &= \frac{1}{5}(b^4+ab^3+a^2b^2+a^3b+a^4) - \left(\frac{a^2+ab+b^2}{3}\right)^2. \end{aligned}$$

3. Let N be the standard normal random variable. Then $X = 68 + 9N$. Solve for Y in terms of N:

$$Y = \frac{5}{9}(X - 32) = \frac{5}{9}(68 + 9N - 32) = 20 + 5N.$$

So Y is normal with parameters $\mu = 20, \sigma = 5$ (this means Y has mean 20, variance $5^2 = 25$, and has density function $f_Y(y) = \frac{1}{5\sqrt{2\pi}} \exp\left(\frac{-(y-20)^2}{2(25)}\right)$).

4. Let A be the event of rolling the third 6 on the 5th roll; let B be the event of rolling the seventh 6 on the 11th roll; let C be the event of rolling the fourth 6 on the 6th roll.

   a) These are all done with the negative binomial distribution, with $p = 1/2$ and varying values of α and x:

$$P(A) = \binom{4}{2}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 \quad P(B) = \binom{10}{4}\left(\frac{1}{6}\right)^7\left(\frac{5}{6}\right)^4$$

$$P(C) = \binom{5}{2}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^2$$

   b) We need to find $P(A \bigcap B)$. For a sequence of rolls to be in this intersection, you need to first roll the third 6 on the 5th roll and then, if you imagine starting the Bernoulli experiment over, roll the 4th 6 (after you start over) on the 6th roll. This would give you a total of $3 + 4 = 7$ sixes in a total of $5 + 6 = 11$ rolls. So

$$P(A \bigcap B) = P(A)P(C).$$

   So if $A \perp B$, this would mean that $P(B)$ and $P(C)$ would have to be equal, which they aren't. So A and B are not independent.

5. a) The joint density function must integrate to 1:

$$\begin{aligned} 1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dA &= c\int_0^{\infty}\int_0^{\infty} e^{-2x}e^{-3y}\,dy\,dx \\ &= c\int_0^{\infty} e^{-2x}\left(\frac{1}{3}\right)\,dx \\ &= c\left(\frac{1}{3}\right)\left(\frac{1}{2}\right). \end{aligned}$$

   So $c/6 = 1$ so $c = 6$.

b) By definition,

$$\begin{aligned} F_{X,Y}(x,y) &= \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\,dA \\ &= \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0 \\ 6\int_0^x\int_0^y e^{-2x}e^{-3y}\,dv\,du & \text{if } x \geq 0, y \geq 0 \end{cases} \end{aligned}$$

It remains to calculate the integral in the second case. In fact,

$$6\int_0^x\int_0^y e^{-2x}e^{-3y}\,dv\,du = (1-e^{-2x})(1-e^{-3y}).$$

so to summarize,

$$F_{X,Y}(x,y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0 \\ (1-e^{-2x})(1-e^{-3y}) & \text{if } x \geq 0, y \geq 0 \end{cases}$$

c) Find the densities by integration:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \\ &= \begin{cases} 0 & \text{if } x < 0 \\ \int_0^\infty 6e^{-2x}e^{-3y}\,dy = 2e^{-2x} & \text{if } x \geq 0 \end{cases} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \\ &= \begin{cases} 0 & \text{if } y < 0 \\ \int_0^\infty 6e^{-2x}e^{-3y}\,dx = 3e^{-3y} & \text{if } y \geq 0 \end{cases} \end{aligned}$$

d) Yes, $X \perp Y$. It is clear that $f_X(x) \cdot f_Y(y) = f_{X,Y}(x,y)$.

2.4 Fall 2009 Exam 2

1. Suppose that a point (X, Y) is chosen uniformly from the interior of the triangle with vertices $(0,0)$, $(0,2)$ and $(2,0)$.

   a) (5.1) Suppose (x, y) is in the interior of the triangle. Find $F_{X,Y}(x, y)$.
   b) (5.1) Find a density function of Y.
   c) (5.4) Let $Z = X + Y$. Find a density function of Z.
   d) (5.1) Are X and Y independent? Why or why not?

2. Let $X_1, X_2, ..., X_{80}$ be i.i.d. exponential random variables, each with the same parameter $\lambda = 4$. Let $S = X_1 + \cdots + X_{80}$.

   a) (8.1) Use the Markov inequality to find an upper bound on $P(S \geq 50)$.
   b) (8.1) Use the Chebyshev inequality to find an upper bound on $P(S \geq 50)$.
   c) (8.3) Use the Central Limit Theorem to approximate $P(S \geq 50)$; leave your answer in terms of Φ, the cumulative distribution function of the standard normal random variable.

3. Let X be a random variable with a density function $f_X(x) = \frac{3}{2}x^2$ for $-1 \leq x \leq 1$ and $f_X(x) = 0$ otherwise.

   a) (6.2) Find the mean and variance of X.
   b) (6.2) Find the mean and variance of X^2.

4. Let X be a r.v. taking values in the non-negative integers $\{0, 1, 2, ...\}$ such that

$$P(X > k) = \frac{1}{k+1} - \frac{1}{k+2} \qquad \text{for } k = 0, 1, 2, ...$$

   a) (2.2) Find $f_X(0)$.
   b) (6.1) Find EX.

5. Let X and Y be two independent random variables such that $EX = EY = 7$ and $Var(X) = Var(Y) = 4$.

   a) (6.2) Find $Var(3X - Y + 1)$.
   b) (6.2) Find $Cov(X - Y, X + Y)$.

Solutions

1. First, the area of the triangle from which X and Y are chosen is $\frac{1}{2}\cdot 2\cdot 2 = 2$ so the joint density function is

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{if } (x,y) \text{ belongs to the triangle} \\ 0 & \text{else} \end{cases}.$$

a) First, $F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$. Notice that for the given x and y, $X \leq x$ and $Y \leq y$ if and only if (X,Y) lies in the rectangle E with vertices $(0,0),(x,0),(0,y),(x,y)$. So

$$F_{X,Y}(x,y) = P((X,Y)\in E) = \int_0^y \int_0^x \frac{1}{2}\,du\,dv = \frac{xy}{2}.$$

This problem could also be done with area calculations.

b) This is a simple calculation:

$$\begin{aligned} f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx &= \begin{cases} \int_0^{2-y} \frac{1}{2}\,dx & \text{if } 0<y<2 \\ \int_{-\infty}^{\infty} 0\,dx & \text{else} \end{cases} \\ &= \begin{cases} \frac{1}{2}(2-y) & \text{if } 0<y<2 \\ 0 & \text{else} \end{cases}. \end{aligned}$$

c) First, find the distribution function of Z; observe that $0 \leq Z \leq 2$ so $F_Z(z) = 0$ when $z < 0$ and $F_Z(z) = 1$ when $z \geq 2$. Now let $z \in [0,2)$:

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P(X+Y \leq z) = \int_0^z \int_0^{z-u} \frac{1}{2}\,dv\,du \\ &= \frac{1}{2}\int_0^z (z-u)\,du = \frac{1}{2}\left[zu - \frac{1}{2}u^2\right]_0^z = \frac{1}{2}\left[z^2 - \frac{1}{2}z^2\right] = \frac{z^2}{4}. \end{aligned}$$

To summarize,

$$F_Z(z) = \begin{cases} 0 & \text{if } z<0 \\ \frac{z^2}{4} & \text{if } z\in[0,2) \\ 1 & \text{if } z \geq 2 \end{cases}.$$

Finally,

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \begin{cases} \frac{z}{2} & \text{if } z \in [0,2) \\ 1 & \text{else} \end{cases}.$$

d) A density function of the $X-$marginal is

$$\begin{aligned} f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy &= \begin{cases} \int_0^{2-x} \frac{1}{2}\,dy & \text{if } 0<x<2 \\ \int_{-\infty}^{\infty} 0\,dy & \text{else} \end{cases} \\ &= \begin{cases} \frac{1}{2}(2-x) & \text{if } 0<x<2 \\ 0 & \text{else.} \end{cases} \end{aligned}$$

We see $f_X(x) \cdot f_Y(y) = \frac{1}{2}(2-x)\frac{1}{2}(2-y) \neq \frac{1}{2} = f_{X,Y}(x,y)$ when (x,y) is in the triangle, so X and Y are not independent.

2. First, notice $E(X_j) = \frac{1}{4}$ for all j, so by linearity $ES = \frac{80}{4} = 20$. Also, since the X_j are independent, $Var(S) = \sum_{j=1}^{100} Var(X_j) = \frac{80}{4^2} = 5$.

   a) The Markov inequality says that $P(X \geq a) \leq \frac{ES}{a}$; in this case ($X = S, a = 50$) that means

$$P(S \geq 50) \leq \frac{20}{50} = \frac{2}{5}.$$

   b) The Chebyshev inequality says that $P(|X - EX| \geq t) \leq \frac{Var(X)}{t^2}$; in this case ($X = S, t = 50$) that means

$$P(S \geq 50) = P(S - 20 \geq 30) \leq P(|S - 20| \geq 30) \leq \frac{5}{30^2} = \frac{5}{900} = \frac{1}{180}.$$

   c) Let $S = S_n$ where $n = 80$; this is an application of the Central Limit Theorem where $\mu = E(X_j) = \frac{1}{4}$, $\sigma = \sqrt{Var(X_j)} = \frac{1}{4}$ and $n = 80$:

$$\begin{aligned}
P(S \geq 50) &= P(S - \frac{1}{4}(80) \geq 30) \\
&= P\left(\frac{S - \frac{1}{4}(80)}{\frac{1}{4}\sqrt{80}} \geq \frac{30}{\frac{1}{4}\sqrt{80}}\right) \\
&= P\left(\frac{S - \frac{1}{4}(80)}{\frac{1}{4}\sqrt{80}} \geq 6\sqrt{5}\right) \\
&= 1 - P\left(\frac{S - \frac{1}{4}(80)}{\frac{1}{4}\sqrt{80}} \leq 6\sqrt{5}\right) \\
&= 1 - P\left(A_n^* \leq 6\sqrt{5}\right) \quad \text{(where } A_n^* = \frac{S_n - n\mu}{\sigma\sqrt{n}}\text{)} \\
&\approx 1 - \Phi(6\sqrt{5}) \quad \text{(by Central Limit Theorem).}
\end{aligned}$$

3. a) We have

$$E(X) = \int_{-1}^{1} x \frac{3}{2} x^2 \, dx = 0$$

because the integrand is an odd function. Hence

$$Var(X) = E(X^2) = \int_{-1}^{1} x^2 \frac{3}{2} x^2 \, dx = \frac{3}{10} x^5 \Big|_{-1}^{1} = \frac{3}{5}.$$

b) We already know $E(X^2)$. Now we need

$$E(X^4) = \int_{-1}^{1} x^4 \frac{3}{2} x^2 \, dx = \frac{3}{14} x^7 \Big|_{-1}^{1} = \frac{3}{7}.$$

Hence

$$Var(X^4) = E(X^4) - E(X^2)^2 = \frac{3}{7} - \left(\frac{3}{5}\right)^2 = \frac{75-63}{175} = \frac{12}{175}.$$

4. a) $P(X=0) = 1 - P(X>0) = 1 - (1-\frac{1}{2}) = \frac{1}{2}$.

b) We have

$$\begin{aligned} EX &= \sum_{k=0}^{\infty} P(X>k) \\ &= (1-\frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \ldots = 1. \end{aligned}$$

5. a) The first question uses properties of variance: $Var(3X - Y + 1) = Var(3X - Y) = Var(3X) + Var(-Y) = 3^2 Var(X) + (-1)^2 Var(Y) = 9 \cdot 4 + 1 \cdot 4 = 40$.

b) Use bilinearity of covariance:

$$\begin{aligned} Cov(X-Y, X+Y) &= Cov(X,X) - Cov(Y,X) + Cov(X,Y) - Cov(Y,Y) \\ &= Var(X) - Cov(X,Y) + Cov(X,Y) - Var(Y) \\ &= 4 - 4 = 0. \end{aligned}$$

2.5 Winter 2008 Exam 1

1. (7.2) Let X be a normal random variable with mean -2 and variance 12. Let Y be a normal random variable with mean 3 and variance 5. Assume that $X \perp Y$ and let $Z = X - 2Y$. Describe Z (giving parameters if necessary).

2. (5.4) Let X be an exponential random variable with parameter λ and let Y be exponential with parameter μ. Find the density of Y/X if X and Y are independent.

Solutions

1. Z is normal with mean $-2 - 2(3) = -8$ and variance $12 + (-2)^2(5) = 32$.

2. First, find the distribution function:

$$\begin{aligned}
F_{Y/X}(z) = P(Y/X \leq z) &= P(Y \leq zX) \\
&= \int_0^\infty \int_0^{zx} f_{X,Y}(x,y)\,dy\,dx \\
&= \int_0^\infty \int_0^{zx} f_X(x) f_Y(y)\,dy\,dx \\
&\quad \text{(since } X \perp Y\text{)} \\
&= \int_0^\infty \int_0^{zx} \lambda e^{-\lambda x} \mu e^{-\mu y}\,dy\,dx \\
&\quad \text{(since } X, Y \text{ exponential)} \\
&= \int_0^\infty -\lambda e^{-\lambda x} e^{-\mu y}\Big|_0^{zx}\,dx \\
&= \int_0^\infty \left(\lambda e^{-\lambda x} - \lambda e^{-\lambda x} e^{-\mu z x}\right)\,dx \\
&= -e^{-\lambda x}\Big|_0^\infty + \frac{\lambda}{\lambda + \mu z} e^{-(\lambda + \mu z)x}\Big|_0^\infty \\
&= 1 - \frac{\lambda}{\lambda + \mu z}.
\end{aligned}$$

Now for the density function:

$$f_{Y/X}(z) = \frac{d}{dz} F_{Y/X}(z) = \frac{d}{dz}\left[1 - \frac{\lambda}{\lambda + \mu z}\right] = \frac{\lambda \mu}{(\lambda + \mu z)^2}.$$

2.6 Winter 2009 Exam 1

1. a) (7.2) Let X be a continuous real-valued random variable with density

$$f_X(x) = \begin{cases} Ce^{2x}, & 0 \le x \le 4 \\ 0, & \text{else,} \end{cases}$$

   where C is some constant chosen so that the density function has total integral 1.

   Find (in terms of C) the moment generating function of X.

   b) (7.2) Suppose Y is a standard normal random variable and let $Z = e^Y$. Find the mean and variance of Z.

2. For each $\lambda > 0$, let X_λ be a Poisson random variable with parameter λ, and let

$$Y_\lambda = \frac{X_\lambda - \lambda}{\sqrt{\lambda}}.$$

   a) (8.3) Show that the characteristic function of Y_λ is given by

$$\phi_{Y_\lambda}(t) = \exp\left(\lambda e^{it/\sqrt{\lambda}} - it\sqrt{\lambda} - \lambda\right).$$

   b) (8.3) Find $\lim_{\lambda\to\infty} \phi_{Y_\lambda}(t)$. *Hint:* Show that

$$\lim_{\lambda\to\infty} \left(\lambda e^{it/\sqrt{\lambda}} - it\sqrt{\lambda} - \lambda\right) = -\frac{t^2}{2}.$$

   c) (8.3) Use your answer to part (b) to find, for fixed y, $\lim_{\lambda\to\infty} P(Y_\lambda \le y)$.

Solutions

1. a) By the definition of moment generating function, we have

$$\begin{aligned} M_X(t) = E[e^{tX}] &= \int_{-\infty}^{\infty} f_X(t) e^{tx}\, dx \\ &= \int_0^4 Ce^{2x} e^{tx}\, dx \\ &= \int_0^4 Ce^{(t+2)x}\, dx \\ &= \begin{cases} \int_0^4 C\, dx & \text{if } t = -2 \\ \frac{C}{t+2} e^{(t+2)x} \Big|_0^4 & \text{else} \end{cases} \\ &= \begin{cases} 4C & \text{if } t = -2 \\ \frac{C}{t+2}\left(e^{4(t+2)} - 1\right) & \text{if } t \neq -2 \end{cases}. \end{aligned}$$

b) Y has moment generating function $M_Y(t) = \exp(t^2/2)$. Now the mean of Z is given by

$$EZ = E[e^Y] = E[e^{1Y}] = M_Y(1) = e^{1/2}.$$

The second moment of Z is

$$EZ^2 = E[(e^Y)^2] = E[e^{2Y}] = M_Y(2) = e^2.$$

Finally the variance of Z is given by

$$\text{Var } Z = EZ^2 - (EZ)^2 = e^2 - (e^{1/2})^2 = e^2 - e.$$

2. a) We know $M_{X_\lambda}(t) = \exp(\lambda(e^t - 1))$. Now by a theorem about MGFs which states $M_{aX+b} = e^{bt} M_X(at)$, we can conclude

$$M_{X_\lambda - \lambda}(t) = \exp(-t\lambda)\, M_{X_\lambda}(t) = \exp(-t\lambda) \exp\left(\lambda(e^t - 1)\right).$$

Then by a second application of the same theorem we have

$$\begin{aligned} M_{Y_\lambda}(t) = M_{\frac{X_\lambda - \lambda}{\sqrt{\lambda}}}(t) &= \phi_{X_\lambda - \lambda}\left(\frac{t}{\sqrt{\lambda}}\right) \\ &= \exp\left(-\left(\frac{t}{\sqrt{\lambda}}\right)\lambda\right) \exp\left(\lambda(e^{\left(\frac{t}{\sqrt{\lambda}}\right)} - 1)\right) \\ &= \exp\left(-t\sqrt{\lambda}\right) \exp\left(\lambda(e^{t/\sqrt{\lambda}} - 1)\right) \\ &= \exp\left(\lambda e^{t/\sqrt{\lambda}} - t\sqrt{\lambda} - \lambda\right). \end{aligned}$$

b) Consider the expression $A = \lambda e^{t/\sqrt{\lambda}} - t\sqrt{\lambda} - \lambda$; we have $M_{Y_\lambda}(t) = e^A$.

$$\lim_{\lambda\to\infty} A = \lim_{\lambda\to\infty} \left(\lambda e^{t/\sqrt{\lambda}} - t\sqrt{\lambda} - \lambda\right).$$

Now, let $s = \frac{t}{\sqrt{\lambda}}$ so that $\lambda = \frac{t^2}{s^2}$; as $\lambda \to \infty$, $s \to 0$ so $\lim\limits_{\lambda\to\infty} A$ can be rewritten as

$$\lim_{\lambda\to\infty} A = \lim_{s\to 0} \frac{t^2}{s^2}e^s - \frac{t^2}{s} - \frac{t^2}{s^2} = \lim_{s\to 0} \frac{t^2e^s - t^2s - t^2}{s^2}.$$

Two applications of L'Hopital's rule (or an analysis using power series) gives this limit as $\frac{t^2}{2}$. So

$$\lim_{\lambda\to\infty} M_{Y_\lambda}(t) = \lim_{\lambda\to\infty} e^A = \exp\left(\frac{t^2}{2}\right).$$

c) Let $n(0,1)$ denote the standard normal distribution. From part (b), we see that $\lim\limits_{\lambda\to\infty} M_{Y_\lambda}(t) = \exp\left(\frac{t^2}{2}\right) = M_{n(0,1)}(t)$. So by the uniqueness of MGFs,

$$\lim_{\lambda\to\infty} P(Y_\lambda \le y) = \lim_{\lambda\to\infty} F_{Y_\lambda}(y) = F_{n(0,1)}(y) = \Phi(y).$$

2.7 Winter 2010 Exam 1

1. (6.3) Suppose X and Y are positive, real-valued r.v.s with joint density function
$$f_{X,Y}(x,y) = xe^{-x(y+1)} \quad \text{whenever } x, y > 0.$$
Show that $E(X \mid Y)(y) = \frac{2}{y+1}$.

2. Let X and Y be independent exponential random variables, each with parameter λ.

   a) (5.4) Verify that $X + Y$ is $\Gamma(2, \lambda)$.

   b) (5.4) Find the density of Y/X.

3. Find the 100^{th} moment of each of these random variables: .

   a) (7.2) $\Gamma(\alpha, \lambda)$.

   b) (7.2) The standard normal $n(0, 1)$.

4. (8.3) The Central Limit Theorem says (heuristically) that averages of n i.i.d. r.v.s are approximately normally distributed for large n, but part (b) of this problem says that averages of i.i.d. Cauchy r.v.s are Cauchy (i.e., they aren't normal). Why doesn't this observation contradict the Central Limit Theorem?

Solutions

1. First, find the density of Y:

$$\begin{aligned} f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx &= \int_0^{\infty} xe^{-x(y+1)}\,dx \\ &= \int_0^{\infty} x^{2-1}e^{-x(y+1)}\,dx \\ &= \frac{\Gamma(2)}{(y+1)^2} \\ &= \frac{1}{(y+1)^2}. \end{aligned}$$

Next, find the conditional density of X given Y:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = (y+1)^2 xe^{-x(y+1)}.$$

This holds for $x, y > 0$. Finally, find the conditional expectation:

$$\begin{aligned} E(X|Y)(y) = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dx &= \int_0^{\infty} (y+1)^2 x^2 e^{-x(y+1)}\,dx \\ &= (y+1)^2 \int_0^{\infty} x^{3-1}e^{-x(y+1)}\,dx \\ &= (y+1)^2 \frac{\Gamma(3)}{(y+1)^3} \\ &= \frac{2}{y+1}. \end{aligned}$$

2. a) Obviously $f_{X+Y}(z) = 0$ for $z \le 0$. For $z > 0$, first find the distribution function:

$$\begin{aligned} F_{X+Y}(z) = P(X+Y \le z) &= \int_0^z \int_0^{z-x} f_X(x)f_Y(y)\,dy\,dx \\ &= \int_0^z \int_0^{z-x} f_X(x)f_Y(y)\,dy\,dx \\ &= \int_0^z \int_0^{z-x} \lambda e^{-\lambda x}\lambda e^{-\lambda y}\,dy\,dx \\ &= \int_0^z -\lambda e^{-\lambda x}e^{-\lambda y}\Big|_0^{z-x}\,dx \\ &= \int_0^z \left[\lambda e^{-\lambda x} - \lambda e^{-\lambda z}\right]\,dx \\ &= \left[-e^{-\lambda x} - \lambda e^{-\lambda z}x\right]_0^z \\ &= 1 - e^{\lambda z} - \lambda z e^{-\lambda z}. \end{aligned}$$

Last, the density function (for $z > 0$) is

$$f_{X+Y}(z) = \frac{d}{dz}F_{X+Y}(z) = -\lambda e^{-\lambda z} + \lambda e^{-\lambda z} + \lambda^2 z e^{-\lambda z} = \lambda^2 z e^{-\lambda z}.$$

This can be rewritten as $f_{X+Y}(z) = \frac{\lambda^2}{\Gamma(2)} z^{2-1} e^{-\lambda z}$, so $X+Y$ is $\Gamma(2,\lambda)$ as desired.

b) Start with the distribution function:

$$\begin{aligned}
F_{Y/X}(z) = P(Y/X \le z) &= P(Y \le zX) \\
&= \int_0^\infty \int_0^{zx} f_{X,Y}(x,y)\,dy\,dx \\
&= \int_0^\infty \int_0^{zx} \lambda e^{-\lambda x} \lambda e^{-\lambda y}\,dy\,dx \\
&= \int_0^\infty -\lambda e^{-\lambda x} e^{-\lambda y}\Big|_0^{zx}\,dx \\
&= \int_0^\infty \left(\lambda e^{-\lambda x} - \lambda e^{-\lambda x} e^{-\lambda z x}\right)\,dx \\
&= -e^{-\lambda x}\Big|_0^\infty + \frac{\lambda}{\lambda + \lambda z} e^{-(\lambda + \lambda z)x}\Big|_0^\infty \\
&= 1 - \frac{1}{1+z}.
\end{aligned}$$

Now for the density function:

$$f_{Y/X}(z) = \frac{d}{dz}F_{Y/X}(z) = \frac{d}{dz}\left[1 - \frac{1}{1+z}\right] = \frac{1}{(1+z)^2}.$$

This holds for $z > 0$ ($f_{Y/X}(z) = 0$ otherwise).

3. a) Let $X = \Gamma(\alpha, \lambda)$. Then

$$\begin{aligned}
EX^{100} = \int_{-\infty}^\infty x^{100} f_X(x)\,dx &= \int_0^\infty x^{100} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\,dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{(100+\alpha)-1} e^{-\lambda x}\,dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+100)}{\lambda^{\alpha+100}} \\
&= \frac{\Gamma(\alpha+100)}{\lambda^{100}\Gamma(\alpha)}.
\end{aligned}$$

b) Let $X = n(0,1)$; the moment generating function of X is

$$\begin{aligned} M_X(t) &= \exp(t^2/2) \\ &= 1 + (t^2/2) + \frac{(t^2/2)^2}{2!} + \frac{(t^2/2)^3}{3!} + ... + \frac{(t^2/2)^{50}}{50!} + ... \\ &= 1 + \frac{t^2}{2} + \frac{t^4}{2! \cdot 2^2} + \frac{t^6}{3! \cdot 2^3} + ... + \frac{t^{100}}{50! \cdot 2^{50}} + ... \\ &= 1 + \frac{1}{2}t^2 + \frac{1}{2! \cdot 2^2}t^4 + \frac{1}{3! \cdot 2^3}t^6 + ... + \frac{1}{50! \cdot 2^{50}}t^{100} + ... \end{aligned}$$

But if μ_r is the r^{th} moment of X, then since $\mu_r = M_X^{(r)}(0)$, then by Taylor's theorem

$$M_X(t) = 1 + \mu_1 t + \frac{\mu_2}{2!}t^2 + ... \frac{\mu_{100}}{100!}t^{100} + ...$$

and by equating coefficients on the t^{100} terms from these power series, we see

$$\frac{1}{50! \cdot 2^{50}} = \frac{\mu_{100}}{100!} \quad \Rightarrow \quad EX^{100} = \mu_{100} = \frac{100!}{50! \cdot 2^{50}}.$$

4. The Central Limit Theorem only applies to random variables with finite mean and variance. The Cauchy r.v. does not have finite mean, so the CLT doesn't apply.

2.8 Winter 2008 Exam 2

1. Let X, Y and Z be i.i.d. random variables each having density f.

   a) (5.4) Find the values of X, Y and Z if $X - Y = a$, $X + Y = b$ and $X + Y + Z = c$.

   b) (5.4) Find $f_{X-Y,X+Y,X+Y+Z}(a, b, c)$.

2. Let X and Y be jointly distributed r.v.s such that X is exponential with parameter λ and Y, when conditioned by $X = x$, is Poisson with mean x.

   a) (5.3) Find the (marginal) density of Y.

   b) (6.3) Find $E(X|Y)$.

Solutions

1. a) We have

$$\begin{cases} a &= x - y \\ b &= x + y \\ c &= x + y + z \end{cases};$$

by adding the first two equations and dividing the result by two we see that $x = \frac{1}{2}(a + b)$; by subtracting the second equation from the third we see that $z = c - b$; by subtracting the first equation from the second and dividing the result by two we see $y = \frac{1}{2}(b - a)$.

   b) Let $g(X, Y, Z) = (A, B, C)$; then g is described by the equations above so the Jacobian of g is

$$J(g) = \det \begin{pmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} & \frac{\partial a}{\partial z} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} & \frac{\partial b}{\partial z} \\ \frac{\partial c}{\partial x} & \frac{\partial c}{\partial y} & \frac{\partial c}{\partial z} \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = -2.$$

Next, by the change of variables theorem for density functions we have

$$\begin{aligned} f_{A,B,C}(a, b, c) &= \frac{1}{|J(g)|} f_{X,Y,Z}(x, y, z) \\ &= \frac{1}{2} f(x)f(y)f(z) \text{ (since } X, Y, Z \text{ i.i.d.)} \\ &= \frac{1}{2} f\left(\frac{a + b}{2}\right) f\left(\frac{b - a}{2}\right) f(c - b). \end{aligned}$$

2. a) First, find the joint density of X and Y:

$$\begin{aligned} f_{X,Y}(x,y) &= f_{Y|X}(y|x)f_X(x) \\ &= \begin{cases} \left(\frac{e^{-x}x^y}{y!}\right)\left(\lambda e^{-\lambda x}\right) & \text{if } x \geq 0, y = 0,1,2,... \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \frac{\lambda e^{-(\lambda+1)x}x^y}{y!} & \text{if } x \geq 0, y = 0,1,2,... \\ 0 & \text{else} \end{cases} . \end{aligned}$$

Next, find the marginal density of Y by integrating:

$$\begin{aligned} f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx &= \begin{cases} \int_0^\infty \frac{\lambda e^{-(\lambda+1)x}x^y}{y!}\,dx & \text{if } y = 0,1,2,... \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \frac{\lambda}{y!}\int_0^\infty e^{-(\lambda+1)x}x^y\,dx & \text{if } y = 0,1,2,... \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \frac{\lambda}{y!}\frac{\Gamma(y+1)}{(\lambda+1)^{y+1}} & \text{if } y = 0,1,2,... \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \frac{\lambda}{(\lambda+1)^{y+1}} & \text{if } y = 0,1,2,... \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \frac{\lambda}{\lambda+1}\left(\frac{1}{\lambda+1}\right)^y & \text{if } y = 0,1,2,... \\ 0 & \text{else} \end{cases} , \end{aligned}$$

in other words Y is geometric with parameter $p = \frac{\lambda}{\lambda+1}$.

b) First, find $f_{X|Y}(x|y)$ using the result of part (a):

$$\begin{aligned} f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{\lambda e^{-(\lambda+1)x}x^y}{y!}}{\frac{\lambda}{(\lambda+1)^{y+1}}} &= \frac{e^{-(\lambda+1)x}x^y(\lambda+1)^{y+1}}{y!} \\ &= \frac{(\lambda+1)^{y+1}}{y!}x^y e^{-(\lambda+1)x}. \end{aligned}$$

for $x \geq 0$ and $y = 0,1,2,...$ ($f_{X|Y}(x|y) = 0$ otherwise). In particular $X|Y$ is $\Gamma(y+1, \lambda+1)$. This means $E(X|Y) = \frac{y+1}{\lambda+1}$ for $y = 0,1,2,....$

2.9 Fall 2007 Final Exam

1. (1.4) There are three events A, B and C. The events B and C are mutually exclusive; the events A and B are independent; and $P(A \mid C) = 1/2$. You know that the events A, B and C are equally likely and the probability of $A \cup B \cup C$ is $13/18$. What is $P(A)$?

2. An urn contains b black balls and r red balls. One of the balls is drawn at random. It is then returned to the urn together with c additional balls of the same colour. A second ball is then drawn at random.

   a) (1.4) What is the probability that the second ball chosen is red?

   b) (1.5) What is the probability that the first ball drawn was black if the second ball drawn is red?

3. Let X be a Poisson random variable with mean 9, and let Y be a Poisson random variable with variance 4. Suppose X and Y are chosen independently.

   a) (7.1) Find $E[X(X-1)(X-2)]$.

   b) (4.1) Find the probability that $X + Y = 12$.

   c) (6.2) Find $Cov(X, Y)$.

4. A continuous random variable X has a density function $f_X(x)$ of the following form:

$$f_X(x) = \begin{cases} cx^3 & \text{if } 0 \leq x \leq 4 \\ 0 & \text{else.} \end{cases}$$

   a) (3.1) Find c.

   b) (3.1) Find $P(0 \leq X \leq 1)$.

   c) (6.2) Find the variance of X.

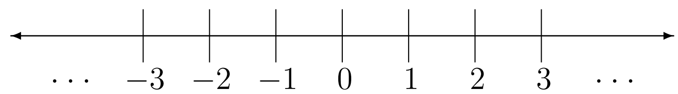
5. There are 15 children in a class, of which 9 are boys and 6 are girls.

   a) (2.3) Suppose the teacher randomly divides the students into two groups, one with 8 students and one with 7 students. What is the probability that the group with 8 students has more boys in it than the group with 7 students?

   b) (2.3) If the teacher randomly selects a group of 5 students, what is the probability that the group is made up of 4 boys and a girl, given that the group contains at least one boy and and at least one girl?

   c) **(Bonus)** (2.3) Suppose the 15 children arrange themselves in a circle randomly. What is the probability that exactly 2 boys have a girl on their immediate left?

6. At 8:00 AM, a frog is at position 0 on the following number line:



Once every minute beginning at 8:01 AM, the frog hops 1 unit either to the left or the right. Each time, the frog hops to the right with probability 2/3 and left with probability 1/3, independent of any previous hops. The frog does not move other than when it hops.

a) (2.4) What is the probability that the frog hops to the right seven times in its first 10 hops?

b) (2.4) What is the probability that the third time the frog hops to the left is at 8:08 AM (i.e. after the frog has hopped 8 times)?

c) (2.4) What is the probability that the frog is at its original position at 8:06 AM?

d) (2.4) What is the probability that the frog is at its original position at 8:07 AM?

e) (6.1) Let X be the position of the frog on the number line at 8:30 AM. What is the expected value of X?

7. Suppose (X, Y) is chosen uniformly from the square in $\mathbb{R}^2$ whose vertices are $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$.

a) (5.4) Let $Z = e^X$. Find the cumulative distribution function of Z.

b) (5.1) Are X and Y independent? (You must justify your answer.)

c) (5.4) Let $S = X + Y$ and $D = X - Y$. Are S and D independent? (You must justify your answer.)

8. The heights of African giraffes are normally distributed with mean 17 ft and standard deviation 1.5 ft. Suppose a biologist measures the height of n giraffes chosen independently. Let A_n be the average of these n heights.

a) (8.3) Find the mean and variance of A_n.

b) (8.3) Use Chebyshev's inequality to find the smallest n which guarantees that the average height of the measured giraffes is 95% likely to be between 16 and 18 feet.

Solutions

1. Let $x = P(A)$; then we have $x = P(B)$ and $x = P(C)$ as well. Furthermore, we have $P(A \cap B) = P(A)\,P(B) = x^2$ since $A \perp B$. Also, $P(B \cap C) = 0$ since B and C are mutually exclusive. Also, $P(A \cap C) = P(A \,|\, C)\,P(C) = \frac{1}{2}x$. Finally by the generalized inclusion-exclusion principle,

$$\begin{aligned}
P(A \bigcup B \bigcup C) &= P(A) + P(B) + P(C) \\
&\quad -P(A \bigcap B) - P(A \bigcap C) - P(B \bigcap C) \\
&\quad +P(A \bigcap B \bigcap C) \\
\frac{13}{18} &= x + x + x - x^2 - \frac{1}{2}x - 0 + 0 \\
\frac{13}{18} &= \frac{5}{2}x - x^2 \\
x^2 - \frac{5}{2}x + \frac{13}{18} &= 0 \\
\left(x - \frac{1}{3}\right)\left(x - \frac{13}{6}\right) &= 0
\end{aligned}$$

so $P(A) = x = \frac{1}{3}$ (throw out $x = \frac{13}{6}$ because it is larger than 1).

2. a) Let R_2 be the event that the second ball chosen is red; let B_1 and R_1 be the events that the first ball chosen is black or red, respectively. Then by the Law of Total Probability,

$$\begin{aligned}
P(R_2) &= P(R_2 \,|\, B_1)\,P(B_1) + P(R_2 \,|\, R_1)\,P(R_1) \\
&= \left(\frac{r}{r+b+c}\right)\left(\frac{b}{r+b}\right) + \left(\frac{r+c}{r+b+c}\right)\left(\frac{r}{r+b}\right) \\
&= \left(\frac{r}{(r+b)(r+b+c)}\right)(b+r+c) = \frac{r}{r+b}.
\end{aligned}$$

   b) Using the notation from part (a), we have

$$\begin{aligned}
P(B_1 \,|\, R_2) &= \frac{P(B_1 \cap R_2)}{P(R_2)} \\
&= \frac{P(R_2 \,|\, B_1)\,P(B_1)}{P(R_2)} \\
&= \frac{\left(\frac{r}{r+b+c}\right)\left(\frac{b}{r+b}\right)}{\frac{r}{r+b}} \qquad \text{(from part (a))} \\
&= \frac{b}{r+b+c}.
\end{aligned}$$

3. The given information implies that X is Poisson with parameter 9, Y is Poisson with parameter 4 and $X \perp Y$.

   a) Let $\Phi_X(t)$ be the probability generating function for X; we know from theory of probability generating functions that
   $$\Phi_X(t) = \sum_{x=0}^{\infty} f_X(x) t^x.$$
   Take three derivatives of both sides of this equation, we get
   $$\Phi_X'''(t) = \sum_{x=0}^{\infty} x(x-1)(x-2) f_X(x) t^x.$$
   When $t = 1$, the right-hand side of this equation is $E(X(X-1)(X-2))$ by the change of variables formula for expected value. So we need only find $\Phi_X'''(1)$. Since X is Poisson with $\lambda = 9$, $\Phi_X'''(t) = e^{9(t-1)}$. So
   $$E(X(X-1)(X-2)) = \Phi_X'''(1) = 9^3 e^{9(t-1)}|_{t=1} = 9^3 = 729.$$
   (This problem can also be done directly with the change of variable formula for expected value but requires some series manipulations to get the answer.)

   b) $X + Y$ is Poisson with parameter $\lambda = 9 + 4 = 13$, so the probability that $X + Y = 12$ is
   $$f_{X+Y}(12) = \frac{e^{-13} 13^{12}}{12!}.$$

   c) Since $X \perp Y$, their covariance is 0.

4. a) The density function must integrate to 1:
   $$1 = \int_0^4 cx^3\,dx = \left[\frac{c}{4}x^4\right]_0^4 = 64c \;\Rightarrow\; c = \frac{1}{64}.$$

   b) Integrate the density function:
   $$P(0 \le X \le 1) = \int_0^1 \frac{1}{64} x^3\,dx = \left[\left(\frac{1}{256}\right) x^4\right]_0^1 = \frac{1}{256}.$$

   c) First, find the expected value and EX^2:
   $$EX = \int_0^4 x c x^3\,dx = \left[\frac{1}{(64)(5)} x^5\right]_0^4 = \frac{16}{5}.$$
   $$EX^2 = \int_0^4 x^2 c x^3\,dx = \left[\frac{1}{(64)(6)} x^6\right]_0^4 = \frac{32}{3}.$$
   So
   $$\mathrm{Var}(X) = EX^2 - (EX)^2 = \frac{32}{3} - \left(\frac{16}{5}\right)^2 = \frac{32}{75}.$$

5. a) Since there are 9 total boys, if the group of 8 has more boys than the group of 7, then the group of 8 must have at least 5 boys in it. We have

$$P(\text{group of 8 has} \geq 5 \text{ boys}) = \sum_{i=5}^{8} P(\text{group of 8 has exactly } n \text{ boys}).$$

Each of the terms on the right can be found with the hypergeometric distribution (we write $C(a,b)$ instead of $\left(\begin{array}{c} a \\ b \end{array} \right)$ throughout this problem); in particular the probability that the group of 8 has exactly n boys is

$$\frac{C(9,n)\,C(6,8-n)}{C(15,8)}.$$

So we have $P(\text{group of 8 has} \geq 5 \text{ boys}) =$

$$\frac{C(9,5)\,C(6,3)}{C(15,8)} + \frac{C(9,6)\,C(6,2)}{C(15,8)} + \frac{C(9,7)\,C(6,1)}{C(15,8)} + \frac{C(9,8)\,C(6,0)}{C(15,8)}.$$

b) First apply the definition of conditional probability to obtain

$$\begin{aligned} P(4B,1G \mid \geq 1B, \geq 1G) &= \frac{P(4B,1G)}{P(\geq 1B, \geq 1G)} \\ &= \frac{P(4B,1G)}{1 - P(0\,B) - P(0\,G)} \\ &= \frac{\frac{C(9,4)\,C(6,1)}{C(15,5)}}{1 - \frac{C(6,5)}{C(15,5)} - \frac{C(9,5)}{C(15,5)}} \\ &= \frac{C(9,4)\,6}{C(15,5) - C(6,5) - C(9,5)}. \end{aligned}$$

c) This question is asking the following: suppose you put 9 Bs and 6 Gs in a circle randomly; what is the probability that exactly 2 Bs are followed by Gs as you go around the circle clockwise?
The answer to this question is

$$P = \frac{\#\text{ ways to put 9 } B\text{s and 6 } G\text{s in a circle with exactly 2 } B\text{s followed by } G\text{s}}{\#\text{ arrangements of the children in a circle}}.$$

The denominator of P is $14!$. This is because there are $15!$ ways to put the kids into the circle (there are 15 places in the circle to put the first child, 14 places to put the second child, etc.). BUT, this counts every circular arrangement of the children 15 times because you don't change an arrangement when you rotate it; so to account for this overcounting you divide $15!$ by 15 to get $14!$. (To see this, write ABCD clockwise in a circle, with A at the bottom. If you start on the left and read the letters, you will read BCDA. Therefore ABCD and BCDA (and also CDAB and DABC) are all the same arrangement and shouldn't be counted separately.)

Now I will count the numerator of P. To do this, I have to pick a place to start my indexing from. I know that a favorable arrangement must have two (and only two) Bs followed by Gs. So I will start my indexing from one of the Bs which is followed by a G. So I know a favorable arrangement must look like this as I go around the circle clockwise:

$$B, G, (\text{some } G\text{s}), (\text{some } B\text{s}), B, G, (\text{some } G\text{s}), (\text{some } B\text{s})$$

Let g_1 be the number of Gs that appear immediately after the first G; notice that since 2 Gs are already accounted for (they follow the Bs); we have $0 \leq g_1 \leq 4$. Let b_1 be the number of Bs that appear immediately after the g_1+1 Gs; we have $0 \leq b_1 \leq 7$. At this point we know a favorable sequence must look like this as I go around the circle clockwise:

$$B, G, (g_1 \text{ more } G\text{s}), (b_1\ B\text{s}), B, G, (\text{ the rest of the } G\text{s}), (\text{ the rest of the } B\text{s})$$

These sequences can be counted by specifying

- which boy is the first B (9 choices);
- which boy is the other boy followed by a G (8 choices);
- which girl follows the first B (6 choices);
- which girl follows the second B (5 choices);
- the value of g_1 (5 choices);
- the value of b_1 (8 choices);
- an ordering of the remaining 4 girls ($4!$ choices); and
- an ordering of the remaining 7 boys ($7!$ choices).

So there are a total of

$$9 \cdot 8 \cdot 6 \cdot 5 \cdot 5 \cdot 8 \cdot 4! \cdot 7! = 9! \cdot 6! \cdot 5 \cdot 8$$

favorable sequences. But wait! These sequences have been overcounted. I said I would list the sequences starting at some point where we see a B followed by a G. There are two such places for each sequence, so I have counted the sequences twice. So there are actually

$$\frac{1}{2}(9! \cdot 6! \cdot 5 \cdot 8) = 9! \cdot 6! \cdot 5 \cdot 4$$

different favorable arrangements. Finally,

$$P = \frac{9! \cdot 6! \cdot 5 \cdot 4}{14!} = \frac{6! \cdot 20}{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10} = \frac{60}{1001}.$$

6. a) The number of times the frog hops to the right in its first 10 hops is a binomial random variable with $n = 10$, $p = 2/3$. So the answer is

$$P(X = 7) = \binom{10}{7} \left(\frac{2}{3}\right)^7 \left(\frac{1}{3}\right)^3.$$

b) The number of hops to the left before the third hop to the right is a negative binomial random variable with $\alpha = 3$, $p = 2/3$. If the frog has hopped 8 total times, then there have been $8 - 3 = 5$ hops to the left. So the answer is

$$P(X = 5) = \binom{5+3-1}{5} \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^5 = \binom{7}{5} \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^5.$$

c) The frog has hopped 6 times. If it is back where it started, it has to have hopped to the right 3 times (and to the left 3 times) so this is binomial (like part (a)) with $n = 6$ and $p = 2/3$. The answer is

$$P(X = 3) = \binom{6}{3} \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^3.$$

d) The frog has hopped 7 times. There is no way it can be back where it started because the number of right hops cannot be equal to the number of hops to the left (if they were equal, there would be an even number of total hops). So this probability is 0.

e) Let X_i be a random variable representing the amount the frog moves in its i^{th} hop. We have

$$E(X_i) = \frac{2}{3}(1) + \frac{1}{3}(-1) = \frac{1}{3}.$$

X, the position of the frog on the number line at 8:30 AM, can be written as

$$X = \sum_{i=1}^{30} X_i$$

so since expected value is linear,

$$EX = \sum_{i=1}^{30} E(X_i) = 30\left(\frac{1}{3}\right) = 10.$$

7. Let Ω be the indicated square; Ω has area 2, so the joint density function of X and Y is

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{2} & \text{if } (x, y) \in \Omega \\ 0 & \text{else} \end{cases}.$$

a) First find the density function of the marginal X. We have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \\ &= \begin{cases} 0 & \text{if } x < -1 \\ \int_{-x-1}^{x+1} \frac{1}{2}\,dy = x+1 & \text{if } -1 \le x < 0 \\ \int_{x-1}^{1-x} \frac{1}{2}\,dy = 1-x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \end{cases} \end{aligned}$$

Now using this we find the cdf of $Z = e^X$. First, $F_Z(z) = P(Z \le z) = P(e^X \le z) = P(X \le \ln z)$. The range of Z is $\left[\frac{1}{e}, e\right]$ so we have $F_Z(z) = 0$ for $z < 1/e$ and $F_Z(z) = 1$ for $z \ge e$. There are two cases when $1/e \le z < e$:

i. $\frac{1}{e} \le z < 1$: In this case

$$P(X \le \ln z) = \int_{-1}^{\ln z} (x+1)\,dx = \left[\frac{x^2}{2} + x\right]_{-1}^{\ln z} = \frac{\ln^2 z}{2} + \ln z + \frac{1}{2}.$$

ii. $1 \le z < e$: In this case

$$P(X \le \ln z) = 1 - \int_{\ln z}^{1} (1-x)\,dx = 1 - \left[x - \frac{x^2}{2}\right]_{\ln z}^{1} = \frac{1}{2} - \frac{\ln^2 z}{2} + \ln z.$$

To summarize,

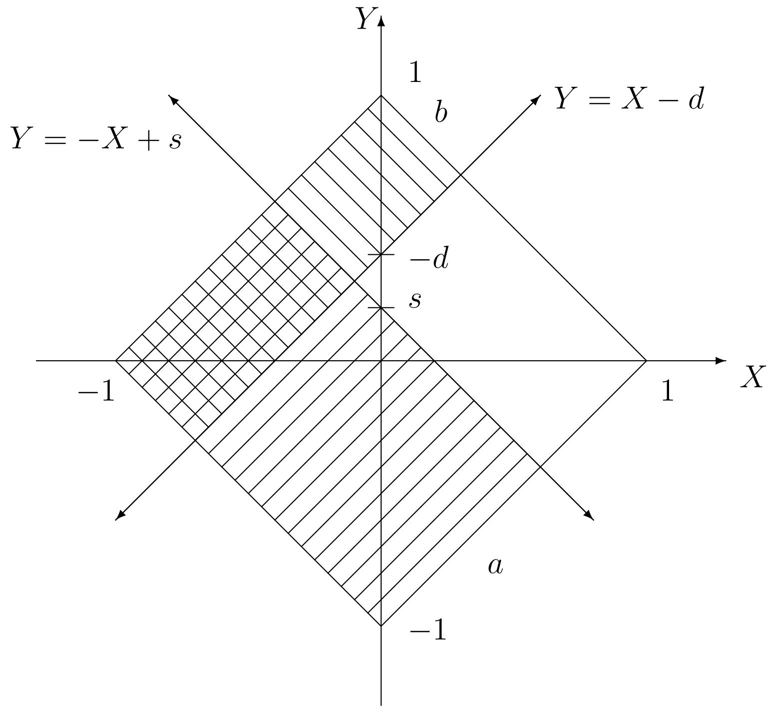
$$F_Z(z) = \begin{cases} 0 & \text{if } z < \frac{1}{e} \\ \frac{\ln^2 z}{2} + \ln z + \frac{1}{2} & \text{if } \frac{1}{e} \le z < 1 \\ \frac{1}{2} - \frac{\ln^2 z}{2} + \ln z & \text{if } 1 \le z < e \\ 1 & \text{if } z \ge e \end{cases}.$$

b) Notice $P(X \le \frac{1}{2}) = \frac{1}{8}$ (by considering areas of regions) but $P(X \le \frac{1}{2} \,|\, Y \ge \frac{1}{2}) = 0 \ne \frac{1}{8}$ so X and Y are not independent.

c) S and D are independent if and only if

$$P(S \le s)\,P(D \le d) = P(S \le s \bigcap D \le d) \tag{2.1}$$

for every s and d. It is sufficient to consider s and d between 0 and 1 for this equation clearly holds otherwise (this is because if the s or d is less than -1, both sides of (2) are 0 and if both s and d are greater than 1, then both sides of (1) are 1). Consider the following picture:



The area of the rectangle shaded by /// is $a\sqrt{2}$, so

$$P(S \leq s) = \frac{a\sqrt{2}}{2} = \frac{a}{\sqrt{2}}.$$

The area of the rectangle shaded by $\backslash\backslash\backslash$ is $b\sqrt{2}$, so

$$P(D \leq d) = \frac{b\sqrt{2}}{2} = \frac{b}{\sqrt{2}}.$$

Finally, the area of the region shaded twice is ab, so

$$P(S \leq s \bigcap D \leq d) = \frac{ab}{2} = P(S \leq s)\, P(D \leq d).$$

Therefore $S \perp D$.

8. Let X_i be the height of the i^{th} giraffe measured. $E(X_i) = 17$ and $\mathrm{Var}(X_i) = (1.5)^2 = 2.25$ for each i.

   a) By using properties of expected value and variance,

$$E(A_n) = E\left(\frac{1}{n}\sum_{i=1}^{n} E(X_i)\right) = \frac{1}{n}(17n) = 17.$$

$$\mathrm{Var}(A_n) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i) = \frac{1}{n^2}(2.25n) = \frac{2.25}{n}.$$

b) Apply Chebyshev's inequality to the random variable $X = A_n$, which has mean 17 and variance $\frac{2.25}{n}$. We want the probability that $|X - \mu| \geq 1$ to be at most $1 - .95 = .05$. To satisfy this, we need to choose n so that $\frac{\sigma^2}{t^2} = \frac{2.25}{n \cdot 1^2} \leq .05$. So $n \geq 2.25(20) = 45$.

2.10 Fall 2009 Final Exam

1. Let A, B, C be events in a probability space with

$$P(A) = a, \quad P(B) = b \text{ and } P(C) = c,$$

where a, b, c are positive. Suppose that A and C are independent, B and C are independent and that A and B are mutually exclusive (i.e disjoint). Find the following (in terms of a, b and c):

   a) (1.4) $P(A \cup B \cup C)$
   b) (1.4) $P(A \mid B \cup C)$
   c) (1.4) $P(A \cap B \cap C)$

2. Let X be a continuous, real-valued random variable with density function

$$f_X(x) = \begin{cases} 2x & \text{if } 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

   a) (3.2) Find the (cumulative) distribution function of X.
   b) (3.1) Find $P(X > 1/2)$.
   c) (3.3) Let $Y = 1/X$. Find a density function of Y.
   d) (6.1) Let $Z = X^2 + X + 1$. Find the mean of Z.

3. A fair die is rolled indefinitely, with the individual rolls being independent of one another.

   a) (2.4) What is the probability that the first time a five is rolled is on an even toss?
   b) (2.4) What is the probability that the eighth five is rolled on the thirteenth toss?

4. A continuous, real-valued random variable X has density function

$$f_X(x) = c\, e^{-9x^2} \qquad -\infty < x < \infty$$

   a) (3.6) Find the value of the constant c.
   b) (6.1) Find the expected value and variance of X.

5. A group of 6 people are asked which day of the week (Monday, Tuesday, etc.) they were born. Assuming each person is equally likely to have been born on each day:

a) (2.3) What is the probability that the six people were born on six different days of the week?

b) (2.3) What is the probability that exactly two of the six people were born on the same day of the week, but all others were born on different days of the week?

c) (2.3) What is the probability that from the six people, two were born on the same day of the week, two others were born on the same day of the week (but not the same day as the first two), and the other two were born on dates different from each other and different from the pairs born on the same date?

6. Suppose X and Y are chosen with joint cumulative distribution function defined by

$$F_{X,Y}(x,y) = 1 - e^{-y} - \frac{1}{x+1}\left(1 - e^{-y(x+1)}\right) \qquad \text{if } x > 0, y > 0$$

and $F_{X,Y}(x,y) = 0$ otherwise.

a) (5.1) Find a joint density function of X and Y.

b) (5.1) Find a density function for each of the marginals.

c) (5.1) Are X and Y independent? Why or why not?

7. (1.5) Suppose that the population of Oshkosh is forty percent male and sixty percent female. In addition, suppose that fifty percent of the males and thirty percent of the females are smokers. Find the probability that a randomly chosen smoker from Oshkosh is male.

8. Flybynight Airlines has a policy of overbooking, i.e. selling more tickets than the 180 seats it has on its aircraft. Assume that each ticket holder has probability .1 of being a "no-show" and probability .9 of being at the departure gate on time. Use the Central Limit Theorem to approximate the answers to these questions:

a) (8.3) If 200 tickets are sold for a certain flight, what is the probability that all ticket holders present can board the flight?

b) (8.3) What is the largest number of tickets the airline can sell, if they want at most a .05 probability that a ticket holder will not be able to board the flight?

Solutions

1. Since $A \perp C$, $P(A \cap C) = P(A)P(C) = ac$. Since $B \perp C$, $P(B \cap C) = P(B)P(C) = bc$. Since A and B are disjoint, $P(A \cap B) = P(\emptyset) = 0$ and also $P(A \cap B \cap C) = P(\emptyset) = 0$.

   a) By the Inclusion-Exclusion Principle,

   $$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) - B(A \cap C) - P(B \cap C) \\ &\quad +P(A \cap B \cap C) \\ &= a + b + c - 0 - ac - bc - 0 \\ &= a + b + c - ac - bc. \end{aligned}$$

   b) By the definition of conditional probability,

   $$\begin{aligned} P(A \,|\, B \cup C) &= \frac{P(A \cap (B \cup C))}{P(B \cup C)} \\ &= \frac{P[(A \cap B) \cup (A \cap C)]}{P(B \cup C)} \quad \text{by DeMorgan's Law} \\ &= \frac{P(A \cap C)}{P(B \cup C)} \quad \text{since } A \cap B = \emptyset \\ &= \frac{P(A \cap C)}{P(B) + B(C) - P(B \cap C)} \quad \text{by Inclusion-Exclusion} \\ &= \frac{ac}{b + c - bc}. \end{aligned}$$

   c) This is zero since $A \cap B \cap C \subseteq A \cap B = \emptyset$.

2. a) If $0 \le x < 1$, then $F_X(x) = P(X \le x) = \int_0^x f_X(t)\,dt = \int_0^x 2t\,dt = x^2$. Together with the obvious cases, we have

   $$\begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } x \in [0, 1) \\ 1 & \text{if } x \ge 1 \end{cases}.$$

   b) $P(X > 1/2) = \int_{1/2}^{\infty} f_X(x)\,dx = \int_{1/2}^{1} 2x\,dx = x^2|_{1/2}^{1} = 1 - (1/2)^2 = 3/4.$

   c) First, find the distribution function of Y. The initial step here is to range Y; the minimum value of Y occurs when X is maximized, this is when $X = 1$ and $Y = 1$. Therefore when $y < 1$, $F_Y(y) = P(Y \le y) = 0$. There is no maximum value of Y because when X is small, Y becomes arbitrarily large. So the range of Y is $[1, \infty)$. Now let $y \in [1, \infty)$:

$$\begin{aligned} F_Y(y) = P(Y \le y) &= P(\frac{1}{X} \le y) \\ &= P(X \ge \frac{1}{y}) \\ &= \int_{1/y}^{1} 2x \, dx \\ &= 1 - \frac{1}{y^2}. \end{aligned}$$

Differentiate the distribution function of Y to obtain a density function:

$$f_Y(y) = \begin{cases} 0 & \text{if } y \le 1 \\ \frac{2}{y^3} & \text{if } y > 1 \end{cases}.$$

d) Let $\varphi(x) = x^2+x+1$ so that $Z = \varphi(X)$. By the change of variable formula for expected value,

$$EZ = \int_{-\infty}^{\infty} \phi(x) f_X(x) \, dx = \int_0^1 (x^2 + x + 1) 2x \, dx = \frac{13}{6}.$$

3. a) Let X record the first time a five is rolled. $X - 1$ records the number of failures before the first five, so $X - 1$ is geometric with parameter $p = 1/6$ and therefore has density $f_{X-1}(x) = (1/6)(5/6)^x$. Consequently, the density of X is given by

$$f_X(x) = P(X = x) = P(X - 1 = x - 1) = (1/6)(5/6)^{x-1}.$$

We want the probability that X is even:

$$\begin{aligned} P(X \text{ is even}) &= \sum_{k=1}^{\infty} P(X = 2k) \\ &= \sum_{k=1}^{\infty} \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{2k-1} \\ &= \frac{1}{6} \cdot \frac{6}{5} \sum_{k=1}^{\infty} \left(\frac{25}{36}\right)^k \\ &= \frac{1}{5} \cdot \left(\frac{1}{1 - \frac{25}{36}} - 1\right) \\ &= \frac{1}{5} \cdot \frac{25}{11} = \frac{5}{11}. \end{aligned}$$

Alternate solution: Let x be the probability that the first five comes on an even roll; then $1 - x$ is the probability that the first five comes on an odd roll. Observe that the probability that the first five comes on an even roll is exactly $5/6$ times the probability the first five comes on an odd roll, because the probability that the first five comes on roll $r+1$ given a non-five on the first roll is the same as the probability that the first five comes on roll r. So $x = \frac{5}{6}(1-x)$ and $x = 5/11$.

b) Define "success" to be rolling a five. If the eighth five comes on the thirteenth roll, that means that there were $13 - 8 = 5$ failures before the eighth success. Therefore this probability can be described by a negative binomial r.v. with parameters $p = 1/6$, $\alpha = 8$. We want the probability that this r.v. takes the value $x = 5$, which is

$$\begin{pmatrix} x+\alpha-1 \\ x \end{pmatrix} p^\alpha (1-p)^x = \begin{pmatrix} 12 \\ 5 \end{pmatrix} (1/6)^8 (5/6)^5.$$

4. a) First, recall that the normal distribution $n(\mu, \sigma^2)$ has density function

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \quad -\infty < x < \infty.$$

Observe that the given density function can be rewritten as

$$c \exp\left(\frac{-x^2}{2 \cdot (1/18)}\right);$$

therefore the given density must be the density of the normal random variable $n(0, \frac{1}{18})$. Hence $c = \frac{1}{\sigma\sqrt{2\pi}} = \frac{1}{(1/\sqrt{18})\sqrt{2\pi}} = \frac{3}{\sqrt{\pi}}$.

b) Observing as in part (a) that X is $n(0, \frac{1}{18})$, it is immediate that $EX = 0$ and $Var(X) = \frac{1}{18}$.

5. a) The number of outcomes where each person is born on a different day is $7 \cdot 6 \cdot 5 \cdots 2$ (7 choices for first person, 6 choices for second person, etc.). So the probability this occurs is

$$\frac{7 \cdots 2}{7^6} = \frac{7!}{7^6}.$$

b) To specify a favorable outcome in this situation, we need to specify:

- The pair of people who have the same birthday ($C(6,2)$ choices for the pair);
- A birthday for the pair (7 choices); and
- Birthdays for everyone else ($6 \cdot 5 \cdot 4 \cdot 3$ choices).

So the probability of this type of outcome is

$$\frac{C(6,2) \cdot 7 \cdot (6 \cdots 3)}{7^6}.$$

c) to specify a favorable outcome here, we need to specify:
   - The first pair of people who have the same birthday ($C(6,2)$ choices for the pair);
   - A birthday for this pair (7 choices);
   - The second pair of people who have the same birthday ($C(4,2)$ choices for this pair);
   - A birthday for this pair (6 choices); and
   - Birthdays for everyone else ($5 \cdot 4$ choices).

   But wait! We can't distinguish between the first and second pair of people here, so this counting overcounts by a factor of two. So the probability of this type of outcome is

$$\frac{\frac{1}{2}(C(6,2) \cdot 7 \cdot C(4,2) \cdot 6 \cdot 5 \cdot 4)}{7^6}.$$

6. a) The joint density is given by the mixed second-order partial:

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) \\ &= \frac{\partial}{\partial x}\left(e^{-y} - e^{-y(x+1)}\right) \\ &= ye^{-y(x+1)}. \end{aligned}$$

   This holds when x and y are positive; otherwise $f_{X,Y}(x,y) = 0$.

   b) First, find the distribution functions of each of the marginals:

$$F_X(x) = \lim_{y \to \infty} F_{X,Y}(x,y) = 1 - \frac{1}{x+1}.$$

$$F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x,y) = 1 - e^{-y}.$$

   These hold when $x > 0$ and $y > 0$, respectively; otherwise the distribution functions are zero. Now differentiate these to obtain densities for X and Y:

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} \frac{1}{(x+1)^2} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} e^{-y} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

   c) We see that $F_X(x) \cdot F_Y(y) \neq F_{X,Y}(x,y)$, so X and Y are not independent.

7. a) This is an application of Bayes' Formula. Let M represent males and F represent females; let S represent smokers and N represent non-smokers. We are given that $P(M) = .4$ and $P(F) = .6$; we are also given $P(S \mid M) = .5$ and $P(S \mid F) = .3$. We are asked to find $P(M \mid S)$ which by Bayes' Formula is

$$P(M \mid S) = \frac{P(S|M)P(M)}{P(S|M)P(M) + P(S|F)P(F)} = \frac{(.5)(.4)}{(.5)(.4) + (.3)(.6)} = \frac{.2}{.38} = \frac{10}{19}.$$

8. Let X_j be equal to 1 if passenger j shows up at the departure gate and let $X_j = 0$ otherwise. The X_j are i.i.d.; each X_j has mean $\mu = .9$ and variance $.9(1 - .9) = .09$. Let $S_n = X_1 + ... + X_n$.

a) We are asked to approximate $P(S_{200} \leq 180)$. This is:

$$\begin{aligned} P(S_{200} \leq 180) &= P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq \frac{180 - (.9)(200)}{\sqrt{.09}\sqrt{200}}\right) \\ &= P\left(A_n^* \leq \frac{180 - 180}{3\sqrt{2}}\right) \\ &\approx P(n(0,1) \leq 0) \quad \text{by CLT} \\ &= \Phi(0) = \frac{1}{2}. \end{aligned}$$

b) We are asked to find n so that $P(S_n \leq 180) = .95$; by the Central Limit Theorem,

$$P(S_n \leq 180) = P\left(\frac{S_n - .9n}{.09\sqrt{n}} \leq \frac{180 - .9n}{.09\sqrt{n}}\right) \approx \Phi\left(\frac{180 - .9n}{.09\sqrt{n}}\right).$$

Now we need to solve

$$\Phi\left(\frac{180 - .9n}{.09\sqrt{n}}\right) = .95$$

for n; apply Φ^{-1} to both sides, then multiply by $.09\sqrt{n}$ to get

$$180 - .9n = .09\sqrt{n}\Phi^{-1}(.95).$$

Then square both sides to get a quadratic equation in n; from the quadratic formula the solution is

$$n = \frac{2(180)(.9) + (.09)^2[\Phi^{-1}(.95)]^2 - \sqrt{\left(2(180)(.9) + (.09)^2[\Phi^{-1}(.95)]^2\right)^2 - 4(180)^2(.9)^2}}{2(.9)^2}$$

which simplifies a bit to

$$n = \frac{324 + .0081[\Phi^{-1}(.95)]^2 - \sqrt{\left(324 + .0081[\Phi^{-1}(.95)]^2\right)^2 - 4(180)^2(.81)}}{.0162}.$$

Note: I didn't realize how terrible the computations would be in this problem. This is kind of a bad question.

2.11 Winter 2008 Final Exam

1. A multiple choice test has 600 questions. A student guesses the answers with a 60% chance of being right on each question (the events of being right on different questions are assumed to be independent).

   a) (6.2) Show that the number of correct answers has mean 360 and standard deviation 12.

   b) (8.1) Use Chebyshev's inequality to find an upper bound on the probability that the student gets less than 300 answers correct.

   c) (8.3) Use the Central Limit Theorem to estimate the probability that the student gets more than 300 answers correct. Leave your answer in terms of Φ, the cumulative distribution function for the standard normal random variable.

2. Suppose X and Y are real-valued random variables whose joint density is given by

$$f_{X,Y}(x,y) = \begin{cases} 10xy^2 & \text{if } 0 < x < y < 1 \\ 0 & \text{else} \end{cases} .$$

   a) (5.4) Let $U = X + Y$ and $V = X/Y$. Find $f_{U,V}(u,v)$.

   b) (5.3) Show that the probability that $X < 1/4$, given that $Y = 1/2$, is $1/4$.

   c) (6.3) Find $E(X|Y)$.

   d) (5.4) Let $Z = XY$; find the density of Z.

3. Let X be a random variable with moment generating function

$$M_X(t) = \exp\left(1 - \sqrt{1-2t}\right) \text{ for } t \le 1/2$$

   (this is in fact the moment generating function of a random variable called an *inverse Gaussian* random variable).

   a) (7.2) Find the expected value of X.

   b) (7.2) Find the mean of $W = e^{-X}$.

Solutions

1. a) The number of correct answers is a binomial r.v. with $n = 600$, $p = .6$. Therefore

$$\mu = np = .60(600) = 360 \text{ and } \sigma = \sqrt{np(1-p)} = \sqrt{360(.4)} = 12.$$

b) We have

$$P(S_n < 300) = P(S_n - 360 < -60) \le P(|S_n - 360| \ge 60).$$

Then by Chebyshev's inequality, we see that

$$P(|S_n - 360| \ge 60) \le \frac{\sigma^2}{t^2} = \frac{144}{60^2} = \frac{144}{3600} = \frac{1}{25}.$$

c) First of all, we define

$$A_n^* = \frac{S_n - \mu n}{\sigma\sqrt{n}}$$

where $\mu = p = .6$, $n = 600$, and $\sigma = \sqrt{p(1-p)} = \sqrt{.6(.4)} = \sqrt{.24}$. (This is not the same μ and σ found in part (a).) Here is the solution:

$$\begin{aligned}
P(S_n > 300) &= P\left(S_n - \mu n > 300 - (.6)(600)\right) \\
&= P\left(\frac{S_n - \mu n}{\sigma\sqrt{n}} > \frac{300 - 360}{\sqrt{.24}\sqrt{600}}\right) \\
&= P\left(A_n^* > \frac{-60}{\sqrt{.24}\sqrt{1200}}\right) \\
&= P\left(A_n^* > \frac{-60}{\sqrt{144}}\right) \\
&= P\left(A_n^* > -5\right) \\
&\approx 1 - \Phi(-5).
\end{aligned}$$

2. a) Let $g(x, y) = (x + y, x/y)$. Then $g(X, Y) = (U, V)$. Since $0 < X < Y < 1$, $0 < U < 2$ and $0 < V < 1$. First, find the Jacobian of g:

$$J(g) = \det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \det\begin{pmatrix} 1 & 1 \\ \frac{1}{y} & \frac{-x}{y^2} \end{pmatrix} = \frac{-(x+y)}{y^2}.$$

Next, we "invert" g by solving for x and y in terms of u and v. Since $v = x/y$, we see that $yv = x$ so by substituting this into the equation for u we see $u = yv + y = y(v + 1)$. Therefore $y = \frac{u}{v+1}$ and $x = yv = \frac{uv}{v+1}$.

Finally, by the change of variable formula for density functions we see

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{|J(g)|} f_{X,Y}(x,y) \\ &= \frac{y^2}{x+y} 10xy^2 \\ &= \frac{u}{(v+1)^2} 10 \left(\frac{uv}{v+1}\right)\left(\frac{u}{v+1}\right)^2 \\ &= \frac{10u^4 v}{(v+1)^5}. \end{aligned}$$

This formula is valid when $v \in (0,1)$ and $u \in (0, v+1)$; otherwise $f_{U,V}(u,v) = 0$.

b) The question is asking for

$$\int_0^{1/4} f_{X|Y}\left(x|\frac{1}{2}\right)\, dx.$$

The first thing we need to do is find the marginal density of Y:

$$f_Y(y) = \int_0^y 10xy^2\, dx = 5x^2y^2|_0^y = 5y^4.$$

Then the conditional density of X given Y is

$$f_{X|Y}(y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{10xy^2}{5y^4} = \frac{2x}{y^2}.$$

Last, we solve the problem:

$$\int_0^{1/4} f_{X|Y}\left(x|\frac{1}{2}\right)\, dx = \int_0^{1/4} \frac{2x}{(1/2)^2}\, dx = \int_0^{1/4} 8x\, dx = \frac{1}{4}.$$

c) Notice that given y, the values of s for which $f_{X,Y}(x,y) > 0$ are $0 < x < y$. Using the work from part (b), we have

$$E(X|Y) = \int_0^y x f_{X|Y}(x|y)\, dx = \int_0^y \frac{2x^2}{y^2}\, dx = \left.\frac{2x^3}{3y^2}\right|_0^y = \frac{2y}{3}.$$

d) Let $Z = XY$; find the density of Z.

The range of Z is $[0,1]$; therefore for $z<0$, $F_Z(z)=0$ and for $z\geq 1$, $F_Z(z)=1$. Now let $z\in[0,1]$ and compute the distribution function:

$$\begin{aligned}
F_Z(z)=P(Z\leq z)=P(XY\leq z) &= P(Y\leq \frac{z}{X}) \\
&= 1-P(Y>\frac{z}{X}) \\
&= 1-\int_{\sqrt{z}}^{1}\int_{z/y}^{y} f_{X,Y}(x,y)\,dx\,dy \\
&= 1-\int_{\sqrt{z}}^{1}\int_{z/y}^{y} 10xy^2\,dx\,dy \\
&= 1-\int_{\sqrt{z}}^{1} 5x^2y^2\Big|_{z/y}^{y}\,dy \\
&= 1-\int_{\sqrt{z}}^{1}\left[5y^4-5z^2\right]\,dy \\
&= 1-\left[y^5-5z^2y\right]_{\sqrt{z}}^{1}\,dy \\
&= 1-\left[(1-5z^2)-(z^{5/2}-5z^{5/2})\right] \\
&= 1-\left[1-5z^2+4z^{5/2}\right] \\
&= 5z^2-4z^{5/2}.
\end{aligned}$$

Last, the density function:

$$f_Z(z)=\frac{d}{dz}F_Z(z)=\frac{d}{dz}\left[5z^2-4z^{5/2}\right]=10z-10z^{3/2}.$$

This holds when $z\in[0,1]$; $f_Z(z)=0$ otherwise.

3. a) $EX=M_X'(0)=[\exp(1-\sqrt{1-2t})(\frac{-1}{2}(1-2t)^{-1/2}(-2))]_{t=0}=1.$

   b) Recall that $M_X(t)=E(e^{tX})$. Then

$$EW=E(e^{-X})=M_X(-1)=\exp(1-\sqrt{3}).$$

2.12 Winter 2009 Final Exam

1. (5.4) Suppose Z_1 and Z_2 are continuous, positive random variables with joint density function $f(z_1, z_2) = f_{Z_1,Z_2}(z_1, z_2)$. Find the joint density (in terms of f) of the random variables W_1 and W_2 defined by

$$W_1 = Z_1 + Z_2 \quad \text{and} \quad W_2 = \frac{Z_1}{Z_1 + Z_2}.$$

Solutions

1. First, write $g(z_1, z_2) = (z_1 + z_2, \frac{z_1}{z_1+z_2})$ so that $g(Z_1, Z_2) = (W_1, W_2)$. Find the Jacobian of g:

$$J(g) = \begin{vmatrix} \frac{\partial w_1}{\partial z_1} & \frac{\partial w_1}{\partial z_2} \\ \frac{\partial w_2}{\partial z_1} & \frac{\partial w_2}{\partial z_2} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{z_2}{(z_1+z_2)^2} & \frac{-z_1}{(z_1+z_2)^2} \end{vmatrix} = \frac{-(z_1 + z_2)}{(z_1 + z_2)^2} = \frac{-1}{z_1 + z_2} = \frac{-1}{w_1}.$$

Since $w_1 = z_1 + z_2 > 0$ (as Z_1 and Z_2 are positive), $J(g) < 0$ so the change of variable formula applies to give

$$f_{W_1,W_2}(w_1, w_2) = \frac{1}{|J(g)|} f_{Z_1,Z_2}(z_1, z_2) = w_1 f(z_1, z_2).$$

Last, we need to find z_1 and z_2 in terms of w_1 and w_2. The equations that define g give

$$w_1 = z_1 + z_2, \quad w_2 = \frac{z_1}{z_1 + z_2}.$$

By substitution, the second equation can be rewritten as $w_2 = z_1/w_1$, so $z_1 = w_1 w_2$. Last, substituting into the first equation we see $z_2 = w_1 - z_1 = w_1 - w_1 w_2 = w_1(1 - w_2)$ and therefore

$$f_{W_1,W_2}(w_1, w_2) = w_1 f(w_1 w_2, w_1(1 - w_2)).$$

2.13 Winter 2010 Final Exam

1. Suppose X is exponential with parameter $\lambda > 0$ and, given $X = x$, Y is Poisson with parameter x.

   a) (5.3) Show that the density of Y is given by $f_Y(y) = \dfrac{\lambda}{(\lambda+1)^{y+1}}$ for $y > 0$ and $f_Y(y) = 0$ for $y \leq 0$.

   b) (6.3) Find the conditional expectation of X given Y.

2. Suppose A and B are i.i.d. normal random variables, each with mean μ and variance σ^2.

   a) (7.2) Prove that $A - B$ is normal with mean 0 and variance $2\sigma^2$.

   b) (7.2) Find $E(A^{100} - B^{100})$ and $E(A^{100} + B^{100})$ if $\mu = 0$.

Solutions

1. a) First, calculate the joint density:

$$f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|x) = \frac{\lambda e^{-\lambda x} e^{-x} x^y}{y!}$$

This holds for $x > 0$ and $y \in \{0, 1, 2, ...\}$, otherwise the joint density is zero. Next, we calculate the density of the $Y-$marginal; for $y < 0$ we see $f_Y(y) = 0$ and for $y \in \{0, 1, 2, ...\}$ we have

$$\begin{aligned} f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx &= \int_0^\infty \frac{\lambda e^{-\lambda x} e^{-x} x^y}{y!}\, dx \\ &= \frac{\lambda}{y!} \int_0^\infty x^y e^{-(\lambda+1)x}\, dx \\ &= \frac{\lambda \Gamma(y+1)}{y!(\lambda+1)^{y+1}} = \frac{\lambda}{(\lambda+1)^{y+1}}. \end{aligned}$$

   b) Next we calculate the conditional density of X given Y; for $x > 0$ and $y \in \{0, 1, 2, ...\}$ this is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{\lambda e^{-\lambda x} e^{-x} x^y}{y!}}{\frac{\lambda}{(\lambda+1)^{y+1}}} = \frac{e^{-(\lambda+1)x} x^y (\lambda+1)^{y+1}}{y!}.$$

Finally, the conditional expectation calculation:

$$\begin{aligned} E(X|Y)(y) &= \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx \\ &= \frac{(\lambda+1)^{y+1}}{y!} \int_0^{\infty} e^{-(\lambda+1)x} x^{y+1}\, dx \\ &= \frac{(\lambda+1)^{y+1}\Gamma(y+2)}{y!(\lambda+1)^{y+2}} = \frac{y+1}{\lambda+1}. \end{aligned}$$

2. a) We see that since $X \perp Y$ and $X \perp (-Y)$, $M_{X-Y}(t) = M_{X+(-Y)}(t) = M_X(t)M_{-Y}(t) = M_X(t)M_Y(-t)$. Therefore

$$M_{X-Y}(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\exp\left(-\mu t + \frac{\sigma^2(-t)^2}{2}\right) = \exp(\sigma^2 t^2)$$

which is the MGF of a normal r.v. with mean 0 and variance $2\sigma^2$. By uniqueness of MGFs, $X - Y$ is normal $n(0, 2\sigma^2)$.

b) $E(A^{100} - B^{100}) = E(A^{100}) - E(B^{100}) = 0$, since A and B have the same density. For the same reason $E(A^{100} + B^{100}) = 2E(A^{100})$. Since $A = n(0, \sigma)$; the moment generating function of X is

$$\begin{aligned} M_A(t) &= \exp(\sigma^2 t^2/2) \\ &= 1 + (\sigma^2 t^2/2) + \frac{(\sigma^2 t^2/2)^2}{2!} + \frac{(\sigma^2 t^2/2)^3}{3!} + \ldots + \frac{(\sigma^2 t^2/2)^{50}}{50!} + \ldots \\ &= 1 + \frac{\sigma^2}{2}t^2 + \frac{\sigma^4}{2!\cdot 2^2}t^4 + \frac{\sigma^6}{3!\cdot 2^3}t^6 + \ldots + \frac{\sigma^{100}}{50!\cdot 2^{50}}t^{100} + \ldots \end{aligned}$$

But if μ_r is the r^{th} moment of A, then since $\mu_r = M_A^{(r)}(0)$, then by Taylor's theorem

$$M_A(t) = 1 + \mu_1 t + \frac{\mu_2}{2!}t^2 + \ldots \frac{\mu_{100}}{100!}t^{100} + \ldots$$

and by equating coefficients on the t^{100} terms from these power series, we see

$$\frac{\sigma^{100}}{50!\cdot 2^{50}} = \frac{\mu_{100}}{100!} \quad \Rightarrow \quad E(A^{100}) = \mu_{100} = \frac{100!\cdot\sigma^{100}}{50!\cdot 2^{50}\cdot}.$$

Hence

$$E(A^{100} + B^{100}) = 2\frac{100!\cdot\sigma^{100}}{50!\cdot 2^{50}} = \frac{100!\cdot\sigma^{100}}{50!\cdot 2^{49}}.$$

Chapter 3

Exams from 2012 to 2014

3.1 Fall 2012 Exam 1

1. a) (1.4) The probability that a drunk driver gets stopped by a police officer is $\frac{1}{10}$. Given that the drunk driver is stopped by an officer, the probability that the driver is subsequently arrested is $\frac{3}{4}$. What is the probability that a drunk driver is stopped by a police officer and subsequently arrested? Please write your answer as a fraction in lowest terms.

   b) (1.4) Suppose A and B are events such that $P(A) = .4$, $P(B) = .5$ and $P(A \cap B) = .2$. Are A and B independent? Why or why not?

   c) (1.4) Suppose E and F are events such that $P(E) = \frac{4}{7}$, $P(F) = \frac{5}{7}$ and $P(E \cup F) = \frac{6}{7}$. Compute $P(E \mid F)$ (please write your answer as a fraction in lowest terms).

2. A bag contains 60 jelly beans in three different flavors: 15 are pineapple, 20 are grape, and 25 are strawberry.

   a) (2.3) If eight jelly beans are drawn from the jar simultaneously, what is the probability that 2 pineapple, 2 grape and 4 strawberry jelly beans are drawn?

   b) (2.3) If eight jelly beans are selected without replacement, what is the probability that exactly four grape jelly beans are drawn?

   c) (2.3) Suppose all the jelly beans are dumped out of the bag and then arranged in a straight line. In how many distinguishable ways can the jelly beans be arranged (assuming jelly beans of the same flavor are indistinguishable)?

   d) (2.3) If eight jelly beans are drawn from the bag all at once, what is the probability that all eight jelly beans are the same color?

e) (2.4) If ten jelly beans are drawn from the bag one at a time with replacement, what is the probability that exactly seven of the ten jelly beans are strawberry?

f) (2.4) If jelly beans are drawn from the bag one at a time with replacement, what is the probability that the first time a grape jelly bean is drawn is on the 15th draw?

3. a) (1.5) Suppose a bin contains flashlights of three types: I, II and III. 70% of all flashlights of type I last at least one year; 40% of all flashlights of type II last at least one year; 30% of all flashlights of type III last at least one year. Suppose 20% of the flashlights in the bin are of type I, 40% of the flashlights in the bin are of type II, and 40% of the flashlights in the bin are of type III. If a flashlight is chosen randomly from the bin, what is the probability that it will last at least one year?

   b) (1.5) Three cooks are working in a kitchen. Cook A and Cook B each burn 10% of their meals, but Cook C burns 30% of the meals he cooks. Suppose Cook A cooks half of all meals served, and suppose also that Cooks B and C each cook the same percentage of meals. If you are served a burnt meal, what is the probability that it was cooked by Cook C?

4. Let X be a Poisson random variable with parameter $\lambda = 16$.

   a) (3.4) Find $P(X = 2)$.

   b) (3.4) Find $P(X = 3 \mid X \leq 3)$.

   c) (3.4) Sketch a rough picture of the density function of X. Indicate on your picture which value x has the greatest probability of occurence.

5. Suppose X is a continuous random variable whose density is given by

$$f_X(x) = \begin{cases} \frac{1}{8}x & x \in [0, b] \\ 0 & \text{else} \end{cases}$$

   where b is some constant.

   a) (3.1) Show that b must equal 4.

   b) (3.1) Find $P(X < 2)$.

   c) (3.1) Find $P(X = 3)$.

   d) (3.3) Let $Y = \sqrt{X}$. Find a density function of Y.

6. (Bonus) (1.4) Suppose I have three cards, identical except for the way in which each side of the card is colored. One card is red on both sides; one

card is red on one side and black on the other; one card is black on both sides. I show you one side of one card, which is red. What is the probability that the other side of the card is red? Explain your answer (to get the bonus, you will need a valid explanation as well as the correct answer).

Solutions

1. a) Let S be the event that the driver is stopped and let A be the event that the driver is arrested. We are given $P(S) = \frac{1}{10}$ and $P(A \mid S) = \frac{3}{4}$. The question asks for $P(A \cap S)$ which is, by a formula from conditional probability, $P(A \cap S) = P(S)P(A \mid S) = \frac{1}{10} \cdot \frac{3}{4} = \frac{3}{40}$.

   b) We need to check whether or nor $P(A \cap B) = P(A)P(B)$. In this case, this holds since $.2 = (.4)(.5)$ so $A \perp B$.

   c) First, by Inclusion-Exclusion, $P(E \cup F) = P(E) + P(F) - P(E \cap F)$. Substituting the given information gives $\frac{6}{7} = \frac{4}{7} + \frac{5}{7} - P(E \cap F)$ so $P(E \cap F) = \frac{3}{7}$. Finally, $P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{3/7}{5/7} = \frac{3}{5}$.

2. a) This is hypergeometric: $\frac{C(15,2)C(20,2)C(25,4)}{C(60,8)}$.

   b) This is also hypergeometric: $\frac{C(20,4)C(40,4)}{C(60,8)}$.

   c) The number of distinguishable arrangements is $\frac{60!}{15!20!25!}$.

   d) We can figure the probability of drawing all jelly beans of each flavor separately and then add to obtain $P(8 \text{ pineapple}) + P(8 \text{ grape}) + P(8 \text{ strawberry})$. Each of these probabilities are hypergeometric, so we obtain $\frac{C(15,8)C(45,0)}{C(60,8)} + \frac{C(20,8)C(40,0)}{C(60,8)} + \frac{C(25,8)C(35,0)}{C(60,8)} = \frac{C(15,8)+C(20,8)+C(25,8)}{C(60,8)}$.

   e) This is binomial with $n = 10$ trials and $p = \frac{25}{60} = \frac{5}{12}$; we want the probability of seven successes which is $b(10, \frac{5}{12}, 7) = C(10,7)(5/12)^7(7/12)^3$.

   f) If we define "success" as drawing a grape jelly bean, then the success probability on each trial is $p = \frac{20}{60} = \frac{1}{3}$. If we want the first success to be on the 15th trial, then we want the number of failures before the first success to be 14. So the probability we are looking for here is the probability that a geometric r.v. with parameter $p = \frac{1}{3}$ takes the value 14; this is $p(1-p)^{14} = \frac{1}{3}\left(\frac{2}{3}\right)^{14}$.

3. a) Let A be the event that a randomly chosen flashlight lasts at least one year. We are given $P(A \mid I) = .7$, $P(A \mid II) = .4$ and $P(A \mid III) = .3$ and are also given $P(I) = .2$, $P(II) = .4$ and $P(III) = .4$. By the Law of

Total Probability, we obtain

$$\begin{aligned} P(A) &= P(A \mid I)P(I) + P(A \mid II)P(II) + P(A \mid III)P(III) \\ &= (.7)(.2) + (.4)(.4) + (.3)(.4) \\ &= .42. \end{aligned}$$

b) Let X be the event that the meal is burnt, and let A, B, C be the events that the meal was cooked by cook A, B or C respectively. We are given $P(X \mid A) = P(X \mid B) = .1$ and $P(X \mid C) = .3$, and we are also given $P(A) = .5$, $P(B) = P(C) = .25$. Now by Bayes' Theorem:

$$\begin{aligned} P(C \mid X) &= \frac{P(X \mid C)P(C)}{P(X \mid A)P(A) + P(X \mid B)P(B) + P(X \mid C)P(C)} \\ &= \frac{(.3)(.25)}{(.1)(.5) + (.1)(.25) + (.3)(.25)} \\ &= \frac{1}{2}. \end{aligned}$$

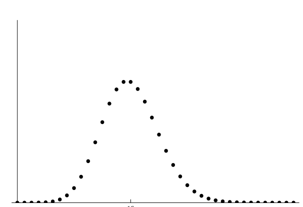
4. The density function of X is $f_X(x) = \frac{16^x e^{-16}}{x!}$ for $x = 0, 1, 2, ...$

a) $P(X = 2) = f_X(2) = \frac{16^2 e^{-16}}{2!}$.

b) By definition of conditional probability:

$$\begin{aligned} P(X = 3 \mid X \leq 3) &= \frac{P(X = 3 \cap X \leq 3)}{P(X \leq 3)} \\ &= \frac{P(X = 3)}{P(X \leq 3)} \\ &= \frac{f_X(3)}{\sum_{x=0}^{3} f_X(3)} \\ &= \frac{\frac{16^3 e^{-16}}{3!}}{\frac{16^0 e^{-16}}{0!} + \frac{16^1 e^{-16}}{1!} + \frac{16^2 e^{-16}}{2!} + \frac{16^3 e^{-16}}{3!}} \\ &= \frac{\frac{16^3}{6}}{1 + 16 + \frac{16^2}{2} + \frac{16^3}{6}} \\ &= \frac{2048}{2083}. \end{aligned}$$

c) Here is a graph of f_X:



5. a) Since f_X is a density, we have $\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^b \frac{1}{8}x\,dx = 1$. Thus $\left[\frac{1}{16}x^2\right]_0^b = \frac{1}{16}b^2 = 1$ so $b^2 = 16$; since $b \geq 0$ we have $b = 4$ as desired.

b) $P(X < 2) = \int_{-\infty}^{2} f_X(x)\,dx = \int_0^2 \frac{1}{8}x\,dx = \frac{1}{16}(2^2) - \frac{1}{16}(0^2) = \frac{1}{4}$.

c) $P(X = 3) = 0$ since X is continuous.

d) First, since X is continuous with range $[0, 4]$, Y is continuous with range $[0, \sqrt{4}] = [0, 2]$. Therefore $F_Y(y) = 0$ for $y < 0$ and $F_Y(y) = 1$ for $y \geq 1$. If $y \in [0, 1)$, then $F_Y(y) = P(Y \leq y) = P(\sqrt{X} \leq y) = P(X \leq y^2) = \int_{-\infty}^{y^2} f_X(x)\,dx = \int_0^{y^2} \frac{1}{8}x\,dx = \left[\frac{1}{16}x^2\right]_0^{y^2} = \frac{1}{16}y^4$. Summarizing, we have

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{16}y^4 & y \in [0, 2) \\ 1 & y \geq 2 \end{cases}$$

Differentiate to get a density function of Y:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} \frac{1}{4}y^3 & y \in [0, 2] \\ 0 & \text{else} \end{cases}$$

6. The trick here is to make sure that the elements of your sample space are the sides of the cards, not the cards themselves (since you could be shown either side of any card). In other words, $\Omega = \{1, 2, 3, 4, 5, 6\}$ where:

- 1 is the red side of the card which is red on one side and black on the other side;
- 2 is the black side of the card which is red on one side and black on the other side;
- 3 and 4 are the two black sides of the card which is black on both sides;
- 5 and 6 are the two red sides of the card which is red on both sides.

Now let E be the event that you are shown a red side; $E = \{1, 5, 6\}$. Also let F be the event that the other side of the card you are shown is red; $F = \{2, 5, 6\}$. You are asked for $P(F \mid E)$ which is $\frac{P(F \cap E)}{P(E)} = \frac{P(\{5,6\})}{P(\{1,5,6\})} = \frac{2}{3}$.

3.2 Fall 2013 Exam 1

1. a) (1.4) Suppose E and F are events in a probability space such that $P(E) = \frac{1}{7}$, and $P(F^C) = \frac{4}{5}$. If E and F are independent, what is $P(E \cup F)$?

   b) (2.2) Suppose that a client of an insurance company will file N claims in a given year, where N is a discrete random variable having the following density:

n	0	1	2	3
$f_N(n)$	.60	.25	.10	.05

   Suppose also that the probability that the insurance company turns a profit on the policy they sell the client is $1 - \frac{N}{3}$, where N is the number of claims filed. If the insurance company turns a profit on this policy, what is the probability that the customer files zero claims?

2. A bag contains a total of 60 bolts of various types:
   - 8 one-inch steel bolts;
   - 12 two-inch steel bolts;
   - 10 one-inch aluminum bolts;
   - 15 two-inch aluminum bolts;
   - 10 one-inch brass bolts;
   - 5 two-inch brass bolts.

   a) (2.3) Suppose ten bolts are drawn from the bag uniformly, without replacement. Compute the probability that of the ten bolts, five are steel and three are brass.

   b) (2.4) Suppose eighteen bolts are drawn from the bag uniformly, with replacement. Compute the probability that eleven of the bolts drawn are aluminum.

   c) (2.3) Suppose three bolts are drawn from the bag uniformly, without replacement. Compute the probability that all three bolts are two inches long, given that all three bolts drawn are steel.

   d) (2.4) Suppose bolts are drawn from the bag one at a time, with replacement. Compute the probability that the third time a one-inch brass bolt is drawn is on the sixteenth draw.

   e) (1.2) Suppose you draw a bolt from the bag, but you feel around before drawing in such a way that you are three times as likely to draw any individual two-inch bolt as you are to draw any individual one-inch bolt. What is the probability that you draw a steel bolt? Simplify your answer.

3. Suppose that X is a continuous random variable whose density function is

$$f_X(x) = \begin{cases} cx^2 & \text{if } x \in [-1,2] \\ 0 & \text{else} \end{cases} .$$

   a) (3.1) Find the value of c.
   b) (3.1) Find the probability that $X = 1$.
   c) (3.1) Find the probability that $X < 1$.
   d) (3.1) Find the probability that $X \geq 0$, given that $X < 1$.

4. a) (3.4) Suppose that the damage caused by tornados in Mecosta County in 2013, in thousands of dollars, is an exponential random variable with parameter $.02$. Find the probability that more than $\$70,000$ worth of damage will be done by tornados in Mecosta County in 2013.
   b) (2.4) Suppose X is a geometric with parameter p. Find the probability that X is at least 30, given that X is at least 12.
   c) (3.4) Suppose that the times at which customers enter a hardware store are distributed according to a Poisson process with rate $\lambda = \frac{1}{6}$. Find the probability that at least two customers enter the store within the first three hours the store is open.
   d) (3.6) Suppose that the length of time, in years, that a circuit will function properly is a normal random variable with mean 25 and variance 13. Find the probability that the circuit will function properly for at least 27 years, but no more than 30 years. Leave your answer in terms of Φ, the cumulative distribution function of the standard normal r.v.

5. (3.3) Suppose a point (X, Y) is chosen uniformly from a triangle whose vertices are $(0,0)$, $(0,4)$ and $(4,4)$. Let $W = \frac{Y}{X}$; compute the density function of W.

Solutions

1. a) First, $P(F) = 1 - P(F^C) = \frac{1}{5}$. Now, since $E \perp F$, $P(E \cap F) = P(E)P(F) = \frac{1}{7} \cdot \frac{1}{5} = \frac{1}{35}$. Last, by Inclusion-Exclusion, $P(E \cup F) = P(E) + P(F) - P(E \cap F) = \frac{1}{7} + \frac{1}{5} - \frac{1}{35} = \frac{11}{35}$.

   b) Let E_0, E_1, E_2 and E_3 be the event that the client files zero, one, two, and three or more claims respectively. Let A be the event that the company turns a profit; we are given $P(A \,|\, E_N) = 1 - \frac{N}{3}$ for all N. Thus, by Bayes' Law,

$$\begin{aligned}
P(E_0 \mid A) &= \frac{P(A \mid E_0)P(E_0)}{\sum_{N=0}^{3} P(A \mid E_N)P(E_N)} \\
&= \frac{1(.6)}{1(.6) + (2/3)(.25) + (1/3)(.1) + 0(.05)} \\
&= \frac{3/5}{3/5 + 1/6 + 1/30} \\
&= \frac{3}{4}.
\end{aligned}$$

2. a) The remaining two bolts must be aluminum, so by the formula for partition problems, the probability is

$$\frac{\binom{20}{5}\binom{15}{3}\binom{25}{2}}{\binom{60}{10}}.$$

   b) This is binomial with success probability $p = \frac{25}{60} = \frac{5}{12}$:

$$b(18, 5/12, 11) = \binom{18}{11}\left(\frac{5}{12}\right)^{11}\left(\frac{7}{12}\right)^{7}.$$

   c) This is a conditional probability problem: let S be the event that all three bolts are steel. Then $P(S) = \frac{\binom{20}{3}}{\binom{60}{3}}$. Let T be the event that all three bolts are two inches long. We have $P(S \cap T) = \frac{\binom{12}{3}}{\binom{60}{3}}$. Therefore, by the definition of conditional probability,

$$P(T \mid S) = \frac{P(S \cap T)}{P(S)} = \frac{\binom{12}{3}}{\binom{20}{3}}.$$

Alternate Solution: If all three bolts are steel, we assume that we are drawing from the 20 steel bolts. Therefore, by the hypergeometric density formula, the probability is

$$P(Hyp(20, 12, 3) = 3) = \frac{\binom{12}{3}\binom{8}{0}}{\binom{20}{3}} = \frac{\binom{12}{3}}{\binom{20}{3}}.$$

d) The number of non-one-inch-brass bolts drawn before the third one-inch brass bolt is $NB(3, \frac{1}{6})$. If the third success is to come on the sixteenth draw, then we need 13 failures before the third success; thus the probability is

$$P(NB(3, 1/12) = 13) = \binom{15}{2}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{13}.$$

e) Notice there are 28 one-inch bolts and 32 two-inch bolts. If you set p to be the probability that you draw any single one-inch bolt, then $3p$ is the probability that you draw any of the two-inch bolts, so we have $28p + 32(3p) = 28p + 96p = 124p = 1$ so $p = \frac{1}{124}$. The probability you draw a steel bolt is therefore

$$8 \cdot \left(\frac{1}{124}\right) + 12 \cdot \left(\frac{3}{124}\right) = \frac{8}{124} + \frac{36}{124} = \frac{44}{124} = \frac{11}{31}.$$

3. a) We have

$$1 = \int_{-1}^{2} cx^2\,dx = \left[\frac{c}{3}x^3\right]_{-1}^{2} = \frac{c}{3}\left(2^3 - (-1)^3\right) = \frac{9c}{3} = 3c.$$

Thus $c = \frac{1}{3}$.

b) Since X is continuous, $P(X = 1) = 0$.

c) Integrate the density function:

$$P(X < 1) = \int_{-1}^{1} f_X(x)\,dx = \int_{-1}^{1} \frac{1}{3}x^2\,dx = \left[\frac{1}{9}x^3\right]_{-1}^{1} = \frac{1}{9} - -\frac{1}{9} = \frac{2}{9}.$$

d) By the definition of conditional probability (and using the answer to part (c) as the denominator),

$$P(X \geq 0 \,|\, X < 1) = \frac{P(0 \leq X < 1)}{P(X < 1)} = \frac{\int_0^1 f_X(x)\,dx}{\frac{2}{9}} = \frac{\frac{1}{9}}{\frac{2}{9}} = \frac{1}{2}.$$

4. a) Let X be the r.v.; $F_X(x) = 1 - e^{-.02x}$ so $P(X > 70) = 1 - F_X(70) = 1 - (1 - e^{-70(.02)}) = e^{-1.4}$.

   b) Since X is geometric, it is memoryless, so

$$\begin{aligned} P(X \geq 30 \,|\, X \geq 12) &= P(X \geq 30 - 12) \\ &= P(X \geq 18) \\ &= (1-p)^{18}. \end{aligned}$$

   The last line comes from a theorem about geometric r.v.s proved in class.

   c) Since the Poisson process has rate $\frac{1}{6}$, the number X of customers arriving in three hours is $Pois(3 \cdot \frac{1}{6}) = Pois(\frac{1}{2})$. Thus

$$\begin{aligned} P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - \frac{e^{-1/2}(1/2)^0}{0!} - \frac{e^{-1/2}(1/2)^1}{1!} \\ &= 1 - \frac{3}{2}e^{-1/2}. \end{aligned}$$

   d) Let X be the time that the circuit functions properly. Since X is $n(25, 13)$, $X = 25 + \sqrt{13}\,Z$ where Z is standard normal. Then $P(27 < X < 30) = P(27 < 25 + \sqrt{13}\,Z < 30) = P(\frac{2}{\sqrt{13}} < Z < \frac{5}{\sqrt{13}}) = \Phi(\frac{5}{\sqrt{13}}) - \Phi(\frac{2}{\sqrt{13}})$.

5. First, the area of the triangle is $\frac{1}{2} \cdot 4 \cdot 4 = 8$ so the probability of any subset of the triangle is $P(E) = \frac{1}{8} \cdot area(E)$. Now, for any point in the triangle, $Y \geq X \geq 0$, so $W = \frac{Y}{X}$ is a continuous random variable with range $[1, \infty)$. Thus $F_W(w) = 0$ when $w < 1$. Now, let $w \in [1, \infty)$. Observe that the region E of points in the triangle which satisfy $Y \leq wX$ has as its complement a triangle with vertices $(0,0)$, $(0,4)$ and $(\frac{4}{w}, 4)$. Thus

$$\begin{aligned} F_W(w) = P(W \leq w) = P(Y/X \leq w) &= P(Y \leq wX) \\ &= P(E) \\ &= 1 - P(E^C) \\ &= 1 - \frac{1}{8} \cdot area(E^C) \\ &= 1 - \frac{1}{8}\left[\frac{1}{2} \cdot \frac{4}{w} \cdot 4\right] \\ &= 1 - \frac{1}{w}. \end{aligned}$$

To summarize,

$$F_W(w) = \begin{cases} 1 - \frac{1}{w} & \text{if } w \geq 1 \\ 0 & \text{else} \end{cases}$$

and therefore

$$f_W(w) = \frac{d}{dw} F_W(w) = \begin{cases} \frac{1}{w^2} & \text{if } w \geq 1 \\ 0 & \text{else} \end{cases} \quad .$$

3.3 Fall 2014 Exam 1

1. A point (X, Y) is chosen uniformly from the rectangle whose vertices are $(0, 0)$, $(1, 0)$, $(0, 2)$ and $(1, 2)$.

   a) (3.3) Find $P(Y = X)$.

   b) (3.3) Find $P(X + Y \geq 2)$.

   c) (3.3) Find $P(X \geq \frac{1}{2} \mid X + Y \geq 2)$.

   d) (3.3) Find $P(Y > X^2)$.

2. a) (1.3) Let A and B be events in a probability space such that $P(A \cup B) = .8$, $P(A \cap B) = .2$ and $P(A) = .6$. Find $P(B)$.

   b) (1.3) Let E, F and G be events in a probability space which are mutually independent. If $P(E) = \frac{2}{3}$, $P(F) = \frac{4}{7}$ and $P(G) = \frac{5}{12}$, find $P(E \cup F \cup G)$. Write your answer as a fraction in lowest terms.

   c) (1.4) Let H and J be events in a probability space such that $P(H \mid J) = P(J \mid H) = \frac{3}{4}$. If $P(H^C \cap J^C) = \frac{1}{8}$. Find $P(H)$. Write your answer as a fraction in lowest terms.

3. Suppose X is a random variable taking values in $\{1, 2, 3, 4\}$ such that $f_X(x) = cx$ for some constant c.

   a) (2.2) Find c.

   b) (2.2) Find $P(X > 2)$.

4. a) (1.5) Suppose that a football team passes the football on 35% of their plays, runs the football on 50% of their plays (and kicks/punts the football otherwise). If the team gains at least five yards on 55% of its pass plays, and gains at least five yards on 25% of its run plays, what is the probability that a play that gained five yards was a pass play?

   b) (1.5) An insurance company insures drivers of all ages. Given the following data on the company's insured drivers:

Age of driver	Probability of accident	Portion of company's insured drivers
16 to 20	.08	.1
21 to 30	.04	.1
31 to 65	.01	.6
66 to 99	.03	.2

   Find the probability that a randomly selected driver this company insures has an accident.

5. a) (2.3) Suppose seven cards are dealt from a standard deck, one at a time without replacement. Compute the probability that of the seven cards, four are diamonds, two are clubs and one is a heart.

   b) (2.3) Suppose six cards are dealt from a standard deck, one at a time without replacement. Compute the probability that the six cards are three pairs (in this setting, four-of-a-kind does not count as two pairs, etc.)

   c) (2.4) Suppose seven cards are dealt from a standard deck, one at a time with replacement. Compute the probability that of those seven cards, two are aces.

   d) (2.3) Suppose eight cards are dealt from a standard deck without replacement. What is the probability that of those eight cards, three are hearts and two are clubs?

Solutions

1. First, since the rectangle has area 2, the probability of any region is its area divided by 2.

   a) $P(Y = X) = 0$ since the area of a line is zero.

   b) Let E be the set of points in Ω satisfying $X + Y \geq 2$. E is a triangle with vertices $(0, 2)$, $(1, 1)$ and $(1, 2)$ so it has area $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$ so its probability is $\frac{1/2}{2} = \frac{1}{4}$.

   c) $P(X \geq \frac{1}{2} \,|\, X + Y \geq 2) = \frac{P(X \geq \frac{1}{2} \cap X+Y \geq 2)}{P(X+Y \geq 2)} = \frac{P(X \geq \frac{1}{2} \cap X+Y \geq 2)}{1/4}$ from part (b). Now, the set of points described in the numerator is the trapezoid with vertices $(\frac{1}{2}, 2)$, $(\frac{1}{2}, \frac{3}{2})$, $(1, 1)$ and $(1, 2)$; this trapezoid has area $\frac{1}{2}(\frac{1}{2})(\frac{1}{2}+1) = \frac{3}{8}$ so it has probability $\frac{3/8}{2} = \frac{3}{16}$. Therefore the whole fraction which gives the conditional probability is $\frac{3/16}{1/4} = \frac{3}{4}$.

   d) Let E be the region of points satisfying $Y > X^2$, then E^C is the set of points under the parabola $Y = X^2$ so the area of E^C is $\int_0^1 x^2 \, dx = \frac{1}{3}$ so the probability of E^C is $\frac{1/3}{2} = \frac{1}{6}$ so $P(E) = 1 - P(E^C) = \frac{5}{6}$.

2. a) By Inclusion-Exclusion, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ so $.8 = .6 + P(B) - .2$ so $P(B) = .4$.

   b) Since the events are mutually independent, their complements are mutually independent. So by applying De Morgan and the definition of independence,

$$\begin{aligned}
P(E \cup F \cup G) = 1 - P(E^C \cap F^C \cap G^C) &= 1 - P(E^C)P(F^C)P(G^C) \\
&= 1 - (1 - \frac{2}{3})(1 - \frac{4}{7})(1 - \frac{5}{12}) \\
&= 1 - \frac{1}{3} \cdot \frac{3}{7} \cdot \frac{7}{12} \\
&= 1 - \frac{1}{12} = \frac{11}{12}.
\end{aligned}$$

c) By definition of conditional probability,

$$\frac{P(H \cap J)}{P(J)} = P(H \mid J) = \frac{3}{4} = P(J \mid H) = \frac{P(H \cap J)}{P(H)}$$

Taking the reciprocal of both sides and then multiplying through by $P(H \cap J)$, we see $P(H) = P(J)$. From the same line, we have $P(H \cap J) = \frac{3}{4}P(H) = \frac{3}{4}P(J)$. Also, since $P(H^C \cap J^C) = \frac{1}{8}$, by taking complements and using De Morgan we have $P(H \cup J) = \frac{7}{8}$. Now by Inclusion-Exclusion, we have

$$\begin{aligned}
P(H \cup J) &= P(H) + P(J) - P(H \cap J) \\
\frac{7}{8} &= P(H) + P(H) - \frac{3}{4}P(H) \\
\frac{7}{8} &= \frac{5}{4}P(H) \\
\frac{7}{10} &= P(H).
\end{aligned}$$

3. a) Since f_X is a density function, we know its values must sum to 1, so $f_X(1) + f_X(2) + f_X(3) + f_X(4) = 1$, i.e. $1c + 2c + 3c + 4c = 10c = 1$, so $c = \frac{1}{10}$.

   b) $P(X > 2) = f_X(3) + f_X(4) = \frac{1}{10}(3) + \frac{1}{10}(4) = \frac{7}{10}$.

4. a) Let Q_1 be passing plays, Q_2 be running plays and let Q_3 be punts/kicks. Next, let F represent gaining five yards on a play. By Bayes' Law,

$$\begin{aligned}
P(Q \mid F) &= \frac{P(F \mid Q_1)P(Q_1)}{P(F \mid Q_1)P(Q_1) + P(F \mid Q_2)P(Q_2) + P(F \mid Q_3)P(Q_3)} \\
&= \frac{(.55)(.35)}{(.55)(.35) + (.25)(.5) + (0)(.15)} = \frac{(.55)(.35)}{(.55)(.35) + (.25)(.5)}.
\end{aligned}$$

   b) Let A be the event that a driver has an accident; by the Law of Total Probability this is

$$P(A) = (.08)(.1) + (.04)(.1) + (.01)(.6) + (.03)(.2)$$

5. a) This is a partition problem:

$$\frac{C(13,4)C(13,2)C(13,1)}{C(52,7)}$$

b) The hand must be of the form $xxyyzz$; to describe such a hand one needs to choose x, y and z ($C(13,3)$ choices); then the suits of x, y and z respectively ($C(4,2)$ choices of suits for each rank). Thus the probability is

$$\frac{C(13,3)[C(4,2)]^3}{C(52,6)}.$$

c) This is the probability of two successes in a Bernoulli experiment with 7 trials, i.e. $b(7, \frac{1}{13}, 2) = \left(\begin{array}{c} 7 \\ 2 \end{array}\right) \left(\frac{1}{13}\right)^2 \left(\frac{12}{13}\right)^5$.

d) In this problem, you should consider the spades and diamonds to be one "group" of 26 cards. You need to deal three from this group and three from 13 hearts and two from the 13 clubs. So this is a partition problem:

$$\frac{C(13,3)C(13,2)C(26,3)}{C(52,8)}$$

3.4 Fall 2012 Exam 2

1. Suppose X and Y are discrete, integer-valued r.v.s with joint density

$$f_{X,Y}(x,y) = \begin{cases} C\left(\frac{2}{3}\right)^y & \text{if } 0 \le x \le y \\ 0 & \text{else} \end{cases}$$

   a) (4.1) Find the value of C.
   b) (4.1) Find the probability that $X \le 13$.

2. Suppose X and Y are continuous, real-valued r.v.s with joint density

$$f_{X,Y}(x,y) = \begin{cases} x+y & \text{if } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

   a) (5.1) Determine (with proof) whether or not X and Y are independent.
   b) (5.1) Find the probability that both X and Y are less than or equal to $\frac{1}{2}$.
   c) (5.4) Let $W = XY$. Find a density function of W.

3. (8.1) Suppose X is a continuous, real-valued r.v. with density

$$f_X(x) = \frac{1}{3\sqrt{2\pi}} \exp\left[\frac{-(x+2)^2}{18}\right]$$

   Use the Chebyshev inequality to find an upper bound on $P(X \ge 9)$.

4. Let X be a continuous, real-valued r.v. with density

$$f_X(x) = \begin{cases} \frac{1}{x} & \text{if } 1 \le x \le e \\ 0 & \text{else} \end{cases}$$

   a) (6.1) Find the mean of X.
   b) (6.2) Find the variance of X.

5. a) (6.2) Let A, B and C be real-valued r.v.s, each with finite mean and finite variance. Prove:

$$Cov(A+B,C) = Cov(A,C) + Cov(B,C).$$

   b) (6.2) Suppose X is a Poisson r.v. with parameter 4, and let Y be a Poisson r.v. with parameter 3. Prove that there does not exist a joint distribution of X and Y for which $X+Y$ and $X-Y$ are independent.

Solutions

1. a) Since the distribution is discrete, we have

$$
\begin{aligned}
1 = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f_{X,Y}(x,y) &= \sum_{x=0}^{\infty} \sum_{y=x}^{\infty} C \left(\frac{2}{3}\right)^y \\
&= C \sum_{x=0}^{\infty} \left(\frac{2}{3}\right)^x \sum_{y=0}^{\infty} \left(\frac{2}{3}\right)^y \\
&= C \sum_{x=0}^{\infty} \left(\frac{2}{3}\right)^x [\frac{1}{1-\frac{2}{3}}] \\
&= 3C \sum_{x=0}^{\infty} \left(\frac{2}{3}\right)^x \\
&= 3C \left(\frac{1}{1-\frac{2}{3}}\right) = 9C \qquad \Rightarrow C = \frac{1}{9}.
\end{aligned}
$$

b) The set of points with $X \leq 13$ is a trapezoidal region bounded by the $y-$axis, the line $y = x$, and the line $x = 13$ with corner points $(0,0)$ and $(13,13)$. We add up the values of the density function in this region to find the probability:

$$
\begin{aligned}
P(X \leq 13) &= \sum_{x=0}^{13} \sum_{y=x}^{\infty} \frac{1}{9} \left(\frac{2}{3}\right)^y \\
&= \frac{1}{9} \sum_{x=0}^{13} \left(\frac{2}{3}\right)^x \left[\frac{1}{1-\frac{2}{3}}\right] \\
&= \frac{1}{9} \cdot 3 \sum_{x=0}^{13} \left(\frac{2}{3}\right)^x = \frac{1}{3} \left[\frac{1-\left(\frac{2}{3}\right)^{14}}{1-\frac{2}{3}}\right] = 1 - \left(\frac{2}{3}\right)^{14}.
\end{aligned}
$$

2. a) First, we find the marginal densities. For $x \in [0,1]$, we have

$$
f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^1 (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2}
$$

so

$$
f_X(x) = \begin{cases} x + \frac{1}{2} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}
$$

Also, for $y \in [0,1]$ we have

$$
f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^1 (x+y)\,dx = \left[\frac{x^2}{2} + xy\right]_0^1 = y + \frac{1}{2}
$$

so

$$f_Y(y) = \begin{cases} y + \frac{1}{2} & \text{if } 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

Finally for $x \in [0,1]$, $y \in [0,1]$, we have

$$f_X(y) f_Y(y) = (x + \frac{1}{2})(y + \frac{1}{2}) \neq x + y = f_{X,Y}(x,y)$$

so X and Y are not independent.

b) The region of points with $X \le \frac{1}{2}, Y \le \frac{1}{2}$ is a square with vertices $(0,0)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0)$. Integrate over this region to find the probability:

$$\begin{aligned} P(X + Y \le \frac{1}{2}) &= \int_0^{1/2} \int_0^{1/2} (x+y)\, dy\, dx \\ &= \int_0^{1/2} \left[xy + \frac{y^2}{2} \right]_0^{1/2} dx \\ &= \int_0^{1/2} \left[\frac{1}{2}x + \frac{1}{8} \right] dx \\ &= \left[\frac{1}{4}x^2 + \frac{1}{8}x \right]_0^{1/2} = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}. \end{aligned}$$

c) Let $W = XY$. Find a density function of W. First, we find F_W, the distribution function of W. To do this, note that the range of W is $[0,1]$ so when $w < 0$, $F_W(w) = 0$ and when $w \ge 1$, $F_W(w) = 1$. For $w \in [0,1]$,

$$F_W(w) = P(W \le w) = P(XY \le w) = P(Y \le \frac{w}{X}) = 1 - P(Y > \frac{w}{X}).$$

Computing this (the setup of these integrals comes from a picture which isn't shown on these solutions), we obtain

$$\begin{aligned} 1 - P(Y > \frac{w}{X}) &= \int_w^1 \int_{w/x}^1 (x+y)\, dy\, dx \\ &= 1 - \int_w^1 \left[xy + \frac{y^2}{2} \right]_{w/x}^1 dx \\ &= 1 - \int_w^1 \left[x + \frac{1}{2} - w - \frac{w^2}{2x^2} \right] dx \\ &= 1 - \left[\frac{1}{2}x^2 - \frac{1}{2}x - wx + \frac{w^2}{2x} \right]_w^1 \\ &= 1 - \left[\left(\frac{1}{2} - \frac{1}{2} - w + \frac{1}{2}w^2 \right) - \left(\frac{1}{2}w^2 - \frac{1}{2}w - w^2 + \frac{1}{2}w \right) \right] \\ &= 1 - w^2 + w. \end{aligned}$$

So

$$F_W(w) = \begin{cases} 0 & \text{if } w < 0 \\ 1 - w^2 + w & \text{if } 0 \leq w < 1 \\ 1 & \text{if } w \geq 1 \end{cases}.$$

Differentiating to obtain the density function of W, we get

$$f_W(w) = \frac{d}{dx} F_W(w) = \begin{cases} 1 - 2w & \text{if } 0 \leq w \leq 1 \\ 0 & \text{else} \end{cases}.$$

3. Since this is a normal density, we can read off the expected value and variance from the density function: in particular, $\mu = EX = -2$ and $\sigma^2 = Var(X) = 9$ since the density must have form

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]$$

Now by the Chebyshev inequality,

$$P(X \geq 9) = P(X + 2 \geq 11) \leq P(|X + 2| \geq 11) \leq \frac{9}{11^2} = \frac{9}{121}.$$

4. a) By definition,

$$EX = \int_1^e x \frac{1}{x}\, dx = \int_1^e 1\, dx = x|_1^e = e - 1.$$

b) First, by the change of variable formula for expected value,

$$EX^2 = \int_1^e x^2 \frac{1}{x}\, dx = \int_1^e x\, dx = \left.\frac{x^2}{2}\right|_1^e = \frac{e^2}{2} - \frac{1}{2}.$$

So the variance of X is

$$Var(X) = EX^2 - (EX)^2 = \left(\frac{e^2}{2} - \frac{1}{2}\right) - (e-1)^2.$$

5. a) This follows by direct calculation, using the linearity of covariance and expected value, and using the formula $Cov(X, Y) = E[XY] - EX \cdot EY$:

$$\begin{aligned} Cov(A + B, C) &= E[(A + B)C] - E[A + B]EC \\ &= E[AC + BC] - (EA + EB)\, EC \\ &= E[AC] + E[BC] - EA \cdot EC - EB \cdot EC \\ &= E[AC] - EA \cdot EC + E[BC] - EB \cdot EC \\ &= Cov(A, C) + Cov(B, C). \end{aligned}$$

b) We compute the covariance between $X+Y$ and $X-Y$. No matter what the joint distribution of X and Y is,

$$\begin{aligned}
Cov(X+Y, X-Y) &= Cov(X,X) + Cov(Y,X) - Cov(X,Y) - Cov(Y,Y) \\
&= Var(X) + Cov(X,Y) - Cov(X,Y) - Var(Y) \\
&= Var(X) - Var(Y) \\
&= 4 - 3 = 1.
\end{aligned}$$

(In the last line, we use the fact that the variance of a Poisson r.v. with parameter λ is λ.) Since this covariance is not zero, $X+Y$ and $X-Y$ cannot be uncorrelated, hence are not independent.

3.5 Fall 2013 Exam 2

1. Suppose X and Y are independent, that X is geometric with parameter $\frac{1}{2}$ and that Y is geometric with parameter $\frac{3}{4}$.

   a) (4.1) Find the probability that $X = 6$ and $Y = 12$.

   b) (4.1) Find the probability that $Y - X = 15$.

2. (5.3) Suppose X is exponential with parameter 4, and that given $X = x$, Y is uniform on $[x, x+2]$. Find the conditional density of X given $Y = 6$.

3. (5.1) Grandpa drives two cars: a sedan and a pickup truck. Suppose that Grandpa causes X thousand dollars worth of damage annually with his sedan and that he causes Y thousand dollars worth of damage annually with his pickup truck. If the joint density of the continuous r.v.s X and Y is

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4000}(50 - 2x) & \text{if } 0 \le x \le 10, 0 \le y \le 10 \\ 0 & \text{else} \end{cases},$$

   write an expression (possibly involving sums and/or integrals) that computes the probability that the total amount of damage caused by Grandpa in the next year is more than 4 thousand dollars.

4. Suppose X and Y are continuous and have joint density

$$f_{X,Y}(x,y) = \begin{cases} cy^3e^{-y} & \text{if } 0 \le x \le y \\ 0 & \text{else} \end{cases}.$$

   a) (5.1) Find the value of c.

   b) (5.3) Compute the conditional density of X given Y. Identify this conditional density $X|Y$ as a common one, giving parameters if necessary.

   c) (5.4) Let $U = X + Y$ and $V = Y/X$. Compute the joint density of U and V.

Solutions

1. a) This is a direct calculation:

$$\begin{aligned}
P(X=6, Y=12) &= f_{X,Y}(6,12) \\
&= f_X(6)f_Y(12) \quad \text{(since } X \perp Y\text{)} \\
&= \frac{1}{2}\left(1-\frac{1}{2}\right)^6 \cdot \frac{3}{4}\left(1-\frac{3}{4}\right)^{12} \quad \text{(since } X, Y \text{ geometric)} \\
&= \left(\frac{1}{2}\right)^7 \frac{3}{4}\left(\frac{1}{4}\right)^{12}. \\
&= \frac{3}{2^{33}}.
\end{aligned}$$

   b) The set of points satisfying $Y - X = 15$ lie on the line $Y = X + 15$, a line with slope 1 and $y-$intercept 15. Thus

$$\begin{aligned}
P(Y - X = 15) &= \sum_{x=0}^{\infty} f_{X,Y}(x, x+15) \\
&= \sum_{x=0}^{\infty} f_X(x) f_Y(x+15) \\
&= \sum_{x=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^x \frac{3}{4}\left(\frac{1}{4}\right)^{x+15} \\
&= \frac{3}{8}\left(\frac{1}{4}\right)^{15} \sum_{x=0}^{\infty}\left(\frac{1}{8}\right)^x \\
&= \frac{3}{8}\left(\frac{1}{4}\right)^{15} \cdot \frac{1}{1-1/8} \\
&= \frac{3}{7}\left(\frac{1}{4}\right)^{15}.
\end{aligned}$$

2. First, compute the joint density:

$$f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|x) = 4e^{-4x} \cdot \frac{1}{2} = 2e^{-4x}.$$

   Now, compute the marginal density Y when $Y = 6$. When $Y = 6$, X ranges from 4 to 6 so

$$f_Y(6) = \int_4^6 f_{X,Y}(x,y)\,dx = \int_4^6 2e^{-4x}\,dx = \left[\frac{-1}{2}e^{-4x}\right]_4^6 = \frac{1}{2}(e^{-16} - e^{-24}).$$

   Last, compute the conditional density of X given $Y = 6$:

$$f_{X|Y}(x|6) = \frac{f_{X,Y}(x,6)}{f_Y(6)} = \frac{2e^{-4x}}{\frac{1}{2}(e^{-16} - e^{-24})} = \frac{4}{e^{-16} - e^{-24}} e^{-4x}.$$

3. Given this joint density, we are asked to compute $P(X + Y \geq 4)$. The complement of the region of possible points in the (X, Y) plane satisfying this inequality is a triangle with vertices $(0, 0)$, $(0, 4)$ and $(4, 0)$; call this triangle E. E is bounded below by the line $Y = 0$ and above by the line $Y = 4 - X$. Therefore

$$\begin{aligned} P(X + Y \geq 4) &= P(E^C) = 1 - P(E) \\ &= 1 - \int\int_E f_{X,Y}(x, y)\, dA \\ &= 1 - \int_0^4 \int_0^{4-x} \frac{1}{4000}(50 - 2x)\, dy\, dx. \end{aligned}$$

4. Suppose X and Y are continuous and have joint density

$$f_{X,Y}(x, y) = \begin{cases} cy^3 e^{-y} & \text{if } 0 \leq x \leq y \\ 0 & \text{else} \end{cases}.$$

a) The joint density must integrate to 1. Note that if you try to integrate with respect to y on the inside and with respect to x on the outside, you will get stuck. However, if you integrate with respect to y on the outside and with respect to x on the inside, you get

$$\begin{aligned} 1 &= \int_0^\infty \int_0^y f_{X,Y}(x, y)\, dx\, dy \\ 1 &= \int_0^\infty \int_0^y cy^3 e^{-y}\, dx\, dy \\ 1 &= c\int_0^\infty \left[y^3 e^{-y} x\right]_0^y\, dy \\ 1 &= c\int_0^\infty y^4 e^{-y}\, dy \\ 1 &= c\frac{\Gamma(5)}{1^5} \quad \text{(Gamma Integral Formula with } r = 5, \lambda = 1) \\ 1 &= c\frac{4!}{1} = 24c \\ \frac{1}{24} &= c. \end{aligned}$$

b) First, find the marginal density Y:

$$f_Y(y) = \int_0^x f_{X,Y}(x, y)\, dx = \int_0^x \frac{1}{24} y^3 e^{-y}\, dx = \frac{1}{24} y^4 e^{-y}.$$

Now for the conditional density:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{\frac{1}{24} y^3 e^{-y}}{\frac{1}{24} y^4 e^{-y}} = \frac{1}{y}$$

This holds when $0 \leq x \leq y$, i.e $X|Y = y$ is uniform on $[0, y]$.

c) Let $\varphi(x,y) = (x+y, y/x) = (u,v)$. Compute the Jacobian of φ:

$$J(\varphi) = \det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \det\begin{pmatrix} 1 & 1 \\ \frac{-y}{x^2} & \frac{1}{x} \end{pmatrix}$$
$$= \frac{1}{x} + \frac{y}{x^2} = \frac{x+y}{x^2}.$$

Now, by the transformation theorem

$$f_{U,V}(u,v) = \frac{1}{|J(\varphi)|} f_{X,Y}(x,y) = \frac{x^2}{x+y}\frac{1}{24}y^3 e^{-y}.$$

Next, we need to substitute in for x and y. Since $u = x+y$ and $v = y/x$, we see $y = vx$ so $u = x + vx$ and therefore $x = \frac{u}{v+1}$ and $y = vx = \frac{uv}{v+1}$. Now by substituting in the transformation theorem, we obtain

$$f_{U,V}(u,v) = \frac{\left(\frac{u}{v+1}\right)^2}{u}\frac{1}{24}\left(\frac{uv}{v+1}\right)^3 e^{-\left(\frac{uv}{v+1}\right)} = \frac{u^4v^3}{24(v+1)^5}\exp\left(\frac{-uv}{v+1}\right).$$

(This holds when $u \geq 0$ and $v \geq 1$; $f_{U,V}(u,v) = 0$ otherwise.)

3.6 Fall 2014 Exam 2

1. Suppose X is a real-valued r.v. whose cumulative distribution function is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{16}x + \frac{1}{16} & \text{if } 0 \le x < 2 \\ 1 - \frac{1}{x} & \text{if } x \ge 2 \end{cases}$$

   a) (3.2) Find $P(X = 0)$, find $P(X = 1)$ and find $P(X = 2)$.
   b) (3.2) Find $P(X > 3)$.
   c) (3.2) Find $P(X \in [0, 2))$.
   d) (3.2) Find $P(X < 1 \,|\, X \le 2)$.

2. Suppose X is a continuous, real-valued r.v. with density function

$$f_X(x) = \begin{cases} \frac{c}{\sqrt{x}} & \text{if } 0 \le x \le 16 \\ 0 & \text{else} \end{cases}$$

   where c is a constant.

   a) (3.1) Find c.
   b) (3.1) Find $P(X > 4)$.
   c) (3.3) Let $W = X^2$. Find a density function of W.

3. a) (3.6) Suppose that the weights of adult male gorillas are modeled by a normal random variable with $\mu = 350$ lbs and $\sigma^2 = 400$. Find the probability that a randomly chosen adult male gorilla has weight between 320 and 360 lbs.
   b) (3.3) Suppose that the size of a property damage claim X and the size of a medical claim Y are modeled by choosing a point (X, Y) uniformly from the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, 4)$. Find the density function of the difference between the size of the medical claim and the size of the property damage claim.

4. Suppose that the times lightning strikes the ground in Mecosta County are given by a Poisson process with daily rate $\lambda = \frac{1}{5}$.

   a) (3.4) Find the probability that there are exactly 40 lightning strikes in a 30 day period.
   b) (3.4) Find the probability that at least three lightning strikes occur in a day.

c) (3.4) Find the probability that the time to the next lightning strike is between 1 and 3 days.

d) (3.4) If there are 20 lightning strikes in the next 35 days, what is the probability that 5 of those 20 strikes occur within ten days? Simplify your answer.

Solutions

1. a) $P(X=0) = F_X(0) - \lim_{x\to 0^-} F_X(x) = \frac{1}{16} - 0 = \frac{1}{16}$.
   $P(X=1) = F_X(1) - \lim_{x\to 1^-} F_X(x) = \frac{1}{8} - \frac{1}{8} = 0$.
   $P(X=2) = F_X(2) - \lim_{x\to 2^-} F_X(x) = \frac{1}{2} - \frac{3}{16} = \frac{5}{16}$.

   b) $P(X>3) = 1 - P(X\le 3) = 1 - F_X(3) = 1 - (1 - \frac{1}{3}) = \frac{1}{3}$.

   c) $P(X\in[0,2)) = P(X\in(0,2]) - P(X=2) + P(X=0) = F_X(2) - F_X(0) - P(X=2) + P(X=0) = \frac{1}{2} - \frac{1}{16} - \frac{5}{16} + \frac{1}{16} = \frac{3}{16}$.

   d) $P(X<1\,|\,X\le 2) = \frac{P(X<1)}{P(X\le 2)} = \frac{F_X(1) - P(X=1)}{F_X(2)} = \frac{\frac{1}{8}-0}{\frac{1}{2}} = \frac{1}{4}$.

2. a) $1 = \int_0^{16} \frac{c}{\sqrt{x}} = [2c\sqrt{x}]_0^{16} = 8c - 0$ so $c = \frac{1}{8}$.

   b) $P(X>4) = \int_4^{16} \frac{1}{8\sqrt{x}}\,dx = \left[\frac{1}{4}\sqrt{x}\right]_4^{16} = \frac{1}{4}(4) - \frac{1}{4}(2) = \frac{1}{2}$.

   c) W is continuous with range $[0,256]$. Let $w\in[0,256]$; then

$$\begin{aligned} F_W(w) = P(W\le w) = P(X^2\le w) &= P(X\le\sqrt{w}) \\ &= \int_0^{\sqrt{w}} \frac{1}{8\sqrt{x}}\,dx \\ &= \left[\frac{1}{4}\sqrt{x}\right]_0^{\sqrt{w}} = \frac{1}{4}w^{1/4}. \end{aligned}$$

   Then,

$$f_W(w) = \frac{d}{dw}F_W(w) = \begin{cases} \frac{1}{16}w^{-3/4} & \text{if } w\in[0,256] \\ 0 & \text{else} \end{cases}$$

3. a) Let X be the gorilla's weight. X is $n(350,400)$ so $X = 350 + \sqrt{400}Z = 350 + 20Z$ where Z is standard normal. Now

$$\begin{aligned} P(320\le X\le 360) &= P(320\le 350+20Z\le 360) \\ &= P(-30\le 20Z\le 10) \\ &= P(-1.5\le Z\le .5) = \Phi(.5) - \Phi(-1.5). \end{aligned}$$

   (This could also be written as $\Phi(.5) + \Phi(1.5) - 1$.)

   b) Let $W = Y - X$; the problem is to find f_W. First, W is continuous with range $[-4,4]$ so when $w<-4$, $F_W(w)=0$ and when $w\ge 4$, $F_W(w)=1$. Now let $w\in[-4,4]$ and define E to be the set of points in the triangle satisfying $W = Y - X \le w$.

   - When $w\in[-4,0]$, E is a triangle with vertices $(w,0)$, $(4,0)$ and $(\frac{4-w}{2}, \frac{w+4}{2})$. So it has area $\frac{1}{2}bh = \frac{1}{2}(4+w)(\frac{w+4}{2}) = \frac{1}{4}(w+4)^2$.
   - When $w\in[0,4]$, E^C is a triangle with vertices $(0,w)$, $(0,4)$ and $(\frac{4-w}{2}, \frac{w+4}{2})$. So E^C has area $\frac{1}{2}bh = \frac{1}{2}(\frac{4-w}{2})(4-w) = \frac{1}{4}(4-w)^2$.

Therefore

- for $w \in [-4,0]$, $F_W(w) = P(E) = \frac{area(E)}{area(\Omega)} = \frac{1}{32}(w+4)^2$.
- for $w \in [0,4]$, $F_W(w) = 1 - P(E^C) = 1 - \frac{area(E^C)}{area(\Omega)} = 1 - \frac{1}{32}(4-w)^2$.

Finally,

$$f_W(w) = \frac{d}{dw}F_W(w) = \begin{cases} \frac{1}{16}(w+4) & \text{if } w \in [-4,0] \\ \frac{1}{16}(4-w) & \text{if } w \in [0,4] \\ 0 & \text{else} \end{cases}$$

4. a) $P(X_{30} = 40) = P(Pois(30 \cdot \frac{1}{5}) = 40) = P(Pois(6) = 40) = \frac{e^{-6}6^{40}}{40!}$.

   b) Using the complement rule,

$$\begin{aligned} P(X_1 \geq 3) &= P(Pois(\frac{1}{5}) \geq 3) \\ &= 1 - P(Pois(\frac{1}{5}) = 0) - P(Pois(\frac{1}{5}) = 1 - P(Pois(\frac{1}{5}) = 2) \\ &= 1 - e^{-1/5} - \frac{1}{5}e^{-1/5} - \frac{1}{50}e^{-1/5} \\ &= 1 - \frac{61}{50}e^{-1/5}. \end{aligned}$$

   c) The waiting time W is exponential with parameter $\frac{1}{5}$ so its distribution function is $F_W(w) = 1 - e^{-w/5}$; we therefore have

   $P(1 \leq W \leq 3) = F_W(3) - F_W(1) = [1 - e^{-3/5}] - [1 - e^{-1/5}] = e^{-1/5} - e^{-3/5}$.

   d) This is a conditional probability problem; if 5 strikes occur within the first ten days, the other 15 must occur in the remaining 25 days. Therefore we have

$$\begin{aligned} P(X_{10} = 5 \,|\, X_{35} = 20) &= \frac{P(X_{10} = 5, X_{25} = 15)}{P(X_{35} = 20} \\ &= \frac{P(X_{10} = 5)P(X_{25} = 15)}{P(X_{35} = 20} \\ &= \frac{P(Pois(10 \cdot \frac{1}{5}) = 5) \cdot P(Pois(25 \cdot \frac{1}{5}) = 15)}{P(Pois(35 \cdot \frac{1}{5}) = 20)} \\ &= \frac{P(Pois(2) = 5) \cdot P(Pois(5) = 15)}{P(Pois(7) = 20)} \\ &= \frac{\frac{e^{-2}2^5}{5!} \cdot \frac{e^{-5}5^{15}}{15!}}{\frac{e^{-7}7^{20}}{20!}} \\ &= \frac{e^{-2}2^5}{5!} \cdot \frac{e^{-5}5^{15}}{15!} \cdot \frac{20!}{e^{-7}7^{20}} = \begin{pmatrix} 20 \\ 5 \end{pmatrix} \frac{2^5 5^{15}}{7^{20}}. \end{aligned}$$

   Notice that this answer is $b(20, \frac{2}{7}, 5)$.

3.7 Fall 2013 Exam 3

1. Parts (a) and (b) of this question are unrelated.

   a) (6.1) Suppose X is a continuous random variable whose density is

   $$f_X(x) = \begin{cases} 3x^2 & \text{if } x \in [0,1] \\ 0 & \text{else} \end{cases}$$

   Let $Y = X - X^3$; find EY.

   b) (6.2) Suppose U and V represent the amounts an insurance company will have to pay two people involved in a traffic accident. If the variance of $U+V$ is 15, and the variance of $U-V$ is 7, find the covariance between U and V.

2. Suppose X and Y are continuous random variables with joint density

   $$f_{X,Y}(x,y) = \begin{cases} 4x & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le x^2 \\ 0 & \text{else} \end{cases}.$$

   a) (6.2) Find $Cov(X,Y)$.

   b) (6.3) Find the conditional expectation of Y given $X = \frac{1}{2}$.

3. Suppose X is a real-valued r.v. whose moment generating function is

   $$M_X(t) = \left(1 - \frac{t}{2}\right)^{-3}.$$

   a) (7.2) Find the variance of X.

   b) (7.2) Find the moment generating function of $Z = 3X + 1$.

   c) (7.2) Suppose Y is a $\Gamma(2, \frac{1}{2})$ r.v. which is independent of X. Find the moment generating function of $X + Y$.

4. Suppose X and Y have a joint normal distribution. Suppose also that

   $$EX = 0 \quad EY = -2 \quad Var(X) = 2 \quad Var(Y) = 8 \quad \rho(X,Y) = \frac{1}{2}$$

   a) (7.3) Find the joint density of X and Y.

   b) (7.3) Find the density function of $W = 5X - 2Y$.

   c) (7.3) Find $E[X \mid Y]$.

Solutions

1. a) By the formula for expected value of a change of variable,

$$EY = \int_0^1 (x - x^3) f_X(x)\, dx = \int_0^1 (x - x^3) 3x^2\, dx = \int_0^1 (3x^3 - 3x^5)\, dx$$
$$= \left[\frac{3}{4}x^4 - \frac{1}{2}x^6\right]_0^1 = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}.$$

b) We are given the following two equations:

$$15 = Var(U + V) = Var(U) + Var(V) + 2Cov(U, V)$$

$$7 = Var(U - V) = Var(U) + Var(-V) + 2Cov(U, -V)$$
$$= Var(U) + Var(V) - 2Cov(U, V).$$

Subtracting the second equation from the first, we get $8 = 4Cov(U, V)$, i.e. $Cov(U, V) = 2$.

2. a) We have to compute three expected values:

$$EX = \int_0^1 \int_0^{x^2} x \cdot 4x\, dy\, dx = \int_0^1 4x^4\, dx = \frac{4}{5}.$$
$$EY = \int_0^1 \int_0^{x^2} y \cdot 4x\, dy\, dx = \int_0^1 \left[2xy^2\right]_0^{x^2} = \int_0^1 2x^5 = \frac{1}{3}.$$
$$E[XY] = \int_0^1 \int_0^{x^2} xy \cdot 4x\, dy\, dx = \int_0^1 \left[2x^2y^2\right]_0^{x^2} = \int_0^1 2x^6 = \frac{2}{7}.$$

Now

$$Cov(X, Y) = E[XY] - EX \cdot EY = \frac{2}{7} - \frac{4}{5} \cdot \frac{1}{3} = \frac{2}{7} - \frac{4}{15} = \frac{2 \cdot 15 - 4 \cdot 7}{15 \cdot 7} = \frac{2}{105}.$$

b) First, compute the density of the marginal X:

$$f_X(x) = \int_0^{x^2} f_{X,Y}(x, y)\, dy = \int_0^{x^2} 4x\, dy = 4x^3.$$

Therefore the conditional density of Y given X is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{4x}{4x^3} = \frac{1}{x^2}.$$

So the conditional expectation is

$$E[Y|X] = \int_0^{x^2} y f_{Y|X}(y|x)\, dy = \int_0^{x^2} \frac{y}{x^2}\, dy = \left[\frac{y^2}{2x^2}\right]_0^{x^2} = \frac{1}{2}x^2.$$

Therefore $E[Y \mid X = \frac{1}{2}] = \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{8}$.

3. a) First, compute EX and EX^2 by differentiation:

$$EX = M_X'(0) = -3\left(1-\frac{t}{2}\right)^{-4}\cdot\frac{-1}{2}\bigg|_{t=0} = \frac{3}{2}\left(1-\frac{t}{2}\right)^{-4}\bigg|_{t=0} = \frac{3}{2}$$

$$EX^2 = M_X''(0) = \frac{3}{2}\cdot(-4)\left(1-\frac{t}{2}\right)^{-5}\cdot\frac{-1}{2}\bigg|_{t=0} = 3\left(1-\frac{t}{2}\right)^{-5}\bigg|_{t=0} = 3$$

So $Var(X) = EX^2 - (EX)^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$.

b) By a theorem from class, $M_{3X+1}(t) = e^{1t}M_X(3t) = e^t\left(1-\frac{3t}{2}\right)^{-3}$.

c) Since $X \perp Y$,

$$M_{X+Y}(t) = M_X(t)M_Y(t) = \left(1-\frac{t}{2}\right)^{-3}\left(\frac{1/2}{1/2-t}\right)^2.$$

4. a) First, the covariance $Cov(X,Y)$ satisfies

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\cdot Var(Y)}} = \frac{Cov(X,Y)}{\sqrt{8\cdot 2}} = \frac{Cov(X,Y)}{4}$$

so $Cov(X,Y) = \sigma_{XY} = 4\rho(X,Y) = 4\cdot(1/2) = 2$. Now the covariance matrix is

$$\Sigma = \begin{pmatrix} Cov(X,X) & Cov(X,Y) \\ Cov(Y,X) & Cov(Y,Y)\end{pmatrix} = \begin{pmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y)\end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 8\end{pmatrix}.$$

Now

$$\Sigma^{-1} = \frac{1}{\det\Sigma}\begin{pmatrix} 8 & -2 \\ -2 & 2\end{pmatrix} = \frac{1}{12}\begin{pmatrix} 8 & -2 \\ -2 & 2\end{pmatrix} = \begin{pmatrix} 2/3 & -1/6 \\ -1/6 & 1/6\end{pmatrix}.$$

Thus the joint density of X and Y is

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{1}{(2\pi)^{2/2}\sqrt{\det\Sigma}}\exp\left[\frac{-1}{2}\left(\overrightarrow{x}-\overrightarrow{\mu}\right)^T\Sigma^{-1}\left(\overrightarrow{x}-\overrightarrow{\mu}\right)\right] \\ &= \frac{1}{2\pi\sqrt{12}}\exp\left[\frac{-1}{2}\begin{pmatrix} x & y+2\end{pmatrix}\begin{pmatrix} 2/3 & -1/6 \\ -1/6 & 1/6\end{pmatrix}\begin{pmatrix} x \\ y+2\end{pmatrix}\right] \\ &= \frac{1}{2\pi\sqrt{12}}\exp\left[\frac{-1}{2}\left(\frac{2}{3}x^2-\frac{1}{3}xy+\frac{1}{6}y^2-\frac{2}{3}x+\frac{2}{3}y+\frac{2}{3}\right)\right] \\ &= \frac{1}{2\pi\sqrt{12}}\exp\left[\frac{-1}{3}x^2+\frac{1}{6}xy-\frac{1}{12}y^2+\frac{1}{3}x-\frac{1}{3}y-\frac{1}{3}\right]. \end{aligned}$$

b) Let $\overrightarrow{b} = (5, -2)$. Then $W = \overrightarrow{b} \cdot (X, Y)$ is normal with parameters

$$EW = \overrightarrow{b} \cdot \overrightarrow{\mu} = (5, -2) \cdot (0, -2) = 4$$

and

$$Var(W) = \overrightarrow{b}^T \Sigma \overrightarrow{b} = \begin{pmatrix} 5 & -2 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 & -2 \end{pmatrix} \begin{pmatrix} 6 \\ -6 \end{pmatrix} = 42.$$

Thus W is normal $n(4, 42)$ so

$$f_W(w) = \frac{1}{\sqrt{42}\sqrt{2\pi}} \exp\left(\frac{-(w-4)^2}{42} \right).$$

c) By the formula derived in class,

$$E[X \mid Y](y) = \mu_X + \frac{\sigma_{XY}}{\sigma_Y^2}(y - \mu_Y) = 0 + \frac{2}{8}(y+2) = \frac{1}{4}(y+2).$$

3.8 Fall 2014 Exam 3

1. Suppose that a insurance policyholder makes two types of claims: major and minor. If X is the number of major claims and Y is the number of minor claims, then X and Y are modeled by discrete random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{20}\left(\frac{2}{5}\right)^x\left(\frac{3}{4}\right)^y & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{else} \end{cases}$$

   a) (4.1) Find the probability that 4 major claims and 4 minor claims are filed.
   b) (4.1) Find the probability that a total of 16 claims are filed.
   c) (4.1) Find the probability that at least 8 major claims are filed.

2. Suppose that X is uniform on the interval $[0, 2]$, and that given $X = x$, Y is uniform on the interval $[0, x^2]$.

   a) (5.3) Find the density function of Y.
   b) (5.3) Find the probability that $X > \frac{1}{2}$, given that $Y = \frac{1}{9}$.
   c) (5.4) Let $Z = X^2 - Y$. Find the density function of Z.

3. Suppose X and Y are continuous, real-valued r.v.s with joint density

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x-y} & \text{if } 0 \leq x \leq y \\ 0 & \text{else} \end{cases}$$

   a) (5.1) Find the probability that $Y \leq 1$.
   b) (5.4) Let $U = \frac{Y}{X}$ and $V = X + Y$. Compute the joint density of U and V.

Solutions

1. a) $P(X=4, Y=4) = f_{X,Y}(4,4) = \frac{3}{20}\left(\frac{2}{5}\right)^4\left(\frac{3}{4}\right)^4$.

   b) Compute the probability by adding values of the joint density function:

$$\begin{aligned} P(X+Y=16) = \sum_{x=0}^{16} f_{X,Y}(x, 16-x) &= \sum_{x=0}^{16} \frac{3}{20}\left(\frac{2}{5}\right)^x\left(\frac{3}{4}\right)^{16-x} \\ &= \frac{3}{20}\left(\frac{3}{4}\right)^{16}\sum_{x=0}^{16}\left(\frac{8}{15}\right)^x \\ &= \frac{3}{20}\left(\frac{3}{4}\right)^{16}\left[\frac{1-(8/15)^{17}}{1-8/15}\right] \\ &= \frac{3^{18}}{7\cdot 4^{17}}\left[1-(8/15)^{17}\right] \end{aligned}$$

   c) Compute the probability by adding values of the joint density function:

$$\begin{aligned} P(X \geq 8) = \sum_{x=8}^{\infty}\sum_{y=0}^{\infty} f_{X,Y}(x,y) &= \sum_{x=8}^{\infty}\sum_{y=0}^{\infty}\frac{3}{20}\left(\frac{2}{5}\right)^x\left(\frac{3}{4}\right)^y \\ &= \frac{3}{20}\sum_{x=8}^{\infty}\left(\frac{2}{5}\right)^x\sum_{y=0}^{\infty}\left(\frac{3}{4}\right)^y \\ &= \frac{3}{20}\left[\left(\frac{2}{5}\right)^8\left(\frac{1}{1-2/5}\right)\right]\left(\frac{1}{1-3/4}\right) \\ &= \frac{3}{20}\left(\frac{2}{5}\right)^8\left(\frac{5}{3}\right)4 \\ &= \left(\frac{2}{5}\right)^8. \end{aligned}$$

2. First, we are given $f_X(x) = \frac{1}{1-0} = 1$ for $0 < x < 1$ and $f_{Y|X}(y|x) = \frac{1}{x^2-0} = \frac{1}{x^2}$ for $0 < y < x^2$. Thus the joint density of X and Y is

$$f_{X,Y}(x,y) = f_X(x)f_{Y|X}(y|x) = 1\cdot\frac{1}{x^2} = \frac{1}{x^2} \text{ when } 0 < x < 1, 0 < y < x^2.$$

   Notice also that (X,Y) is being chosen from a curvilinear triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$; the top of the triangle is the curve $y = x^2$, a.k.a. $x = \sqrt{y}$.

   a) Integrate the joint density with respect to x:

$$f_Y(y) = \int_{\sqrt{y}}^1 \frac{1}{x^2}\,dx = \left.\frac{-1}{x}\right|_{\sqrt{y}}^1 = -1 + \frac{1}{\sqrt{y}}.$$

   This holds when $0 < y < 1$; the density is zero otherwise.

b) First, we need to compute the conditional density of X given $Y = \frac{1}{9}$. When $Y = \frac{1}{9}$, the joint density is $f_{X,Y}(x, \frac{1}{9}) = \frac{1}{x^2}$ and the marginal density of Y is $f_Y(\frac{1}{4}) = -1 + \frac{1}{\sqrt{1/9}} = 2$. Thus

$$f_{X|Y}\left(x|\frac{1}{9}\right) = \frac{f_{X,Y}(x, \frac{1}{9})}{f_Y(\frac{1}{9})} = \frac{\frac{1}{x^2}}{2} = \frac{1}{2x^2}.$$

Finally,

$$\begin{aligned} P\left(X > \frac{1}{2} \,|\, Y = \frac{1}{9}\right) &= \int_{1/2}^{1} f_{X|Y}(x|\frac{1}{9})\, dx \\ &= \int_{1/2}^{1} \frac{1}{2x^2}\, dx = \left.\frac{-1}{2x}\right|_{1/2}^{1} = \frac{-1}{1} + 1 = \frac{1}{2}. \end{aligned}$$

c) Z is largest when $X = 1$, $Y = 0$ (i.e. $Z = 1$) and Z is smallest when $X = Y^2$ (i.e $Z = 0$). Thus Z is continuous with range $[0, 1]$. Let $z \in [0, 1]$; then

$$\begin{aligned} F_Z(z) = P(X^2 - Y \le z) = P(Y - X^2 \ge -z) &= P(Y \ge X^2 - z) \\ &= 1 - P(Y \le X^2 - z) \end{aligned}$$

and this is

$$\begin{aligned} 1 - \int_{\sqrt{z}}^{1} \int_{0}^{x^2 - z} \frac{1}{x^2}\, dy\, dx &= 1 - \int_{\sqrt{z}}^{1} \frac{x^2 - z}{x^2}\, dx \\ &= 1 - \int_{\sqrt{z}}^{1} \left(1 - \frac{z}{x^2}\right) dx \\ &= 1 - \left[x + \frac{z}{x}\right]_{\sqrt{z}}^{1} \\ &= 1 - \left[1 + z - 2\sqrt{z}\right] \\ &= 2\sqrt{z} - z. \end{aligned}$$

Thus

$$F_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ 2\sqrt{z} - z & \text{if } 0 \le z \le 1 \\ 1 & \text{if } z > 1 \end{cases}$$

Differentiating this, we obtain

$$f_Z(z) = \begin{cases} \frac{1}{\sqrt{z}} - 1 & \text{if } z \in [0, 1] \\ 0 & \text{else} \end{cases}$$

3. a) Compute this probability by performing a double integral:

$$\begin{aligned}
P(Y \leq 1) &= \int_0^1 \int_x^1 2e^{-x}e^{-y}\,dy\,dx \\
&= \int_0^1 \left[-2e^{-x}e^{-y}\right]_x^1 \,dx \\
&= \int_0^1 \left(-2e^{-x-1} + 2e^{-2x}\right)\,dx \\
&= \left[2e^{-x-1} - e^{-2x}\right]_0^1 \\
&= \left[2e^{-2} - e^{-2}\right] - \left[2e^{-1} - 1\right] = 1 - 2e^{-1} + e^{-2}.
\end{aligned}$$

b) Define $\varphi(x, y) = (\frac{y}{x}, x + y)$ so that $\varphi(X, Y) = (U, V)$. First, compute the Jacobian of φ:

$$J(\varphi) = \det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \det\begin{pmatrix} \frac{-y}{x^2} & \frac{1}{x} \\ 1 & 1 \end{pmatrix} = \frac{-y}{x^2} - \frac{1}{x} = \frac{-(x+y)}{x^2}.$$

Next, we need to solve for x and y in terms of u and v. Note that $y = ux$ so $v = x + ux$ so $x = \frac{v}{1+u}$ and $y = \frac{uv}{1+u}$. Now, by the main transformation theorem, we have

$$\begin{aligned}
f_{U,V}(u, v) &= \frac{1}{|J(\varphi)|} f_{X,Y}(x, y) \\
&= \frac{x^2}{x+y} 2e^{-(x+y)} \qquad \text{if } 0 \leq x \leq y \\
&= \frac{\left(\frac{v}{1+u}\right)^2}{v} 2e^{-v} \qquad \text{if } 0 \leq \frac{v}{1+u} \leq \frac{uv}{1+u} \\
&= \frac{2v}{(1+u)^2} e^{-v} \qquad \text{if } u \geq 1 \text{ and } v \geq 0.
\end{aligned}$$

The joint density is zero otherwise.

3.9 Fall 2014 Exam 4

1. a) (3.4) Suppose that the number of typos on one page of a manuscript is a Poisson r.v. with variance $\frac{1}{10}$. Suppose further that the number of typos on any one page is independent of the number of typos on any other page. If the manuscript is 360 pages long, what is the probability that the manuscript contains exactly 22 typos?

   b) (6.1) Suppose X is a continuous random variable with density function

   $$f_X(x) = \begin{cases} \frac{9}{4}x^{-3} & \text{if } 1 \leq x \leq 3 \\ 0 & \text{else} \end{cases}$$

   Find the expected value of X.

   c) (7.2) Suppose X is a gamma random variable with parameters $r = 4$ and $\lambda = 2$, and Y is an exponential random variable with mean $\frac{1}{3}$. If $X \perp Y$, find the moment generating function of $Z = 3X + Y$.

   d) (6.2) Suppose U and W are random variables such that $Var(U) = 2$, $Var(W) = 8$ and $Var(U+W) = 8$. Find $\rho(U, W)$, the correlation between U and W.

2. Suppose X and Y are continuous random variables with joint density

   $$f_{X,Y}(x, y) = \begin{cases} x + y & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$$

   In this problem, you may assume without computation that $EX = EY = \frac{7}{12}$.

   a) (6.2) Compute the covariance between X and Y.

   b) (6.2) Based on your answer to part (a), would you expect Y to increase or decrease as X increases? Explain.

   c) (5.3) Find the conditional expectation of Y given X.

3. Suppose that an insurance company models the size of each of the medical claims it has to pay by a random variable with mean $\$1000$ and standard deviation $\$200$. Suppose also that the size of any claim is independent of the size of any other claim.

   a) (8.1) Use Chebyshev's inequality to find an upper bound on the probability that a given claim is at least $\$1600$.

   b) (8.3) Suppose that the company will have to pay 20 medical claims. Use the Central Limit Theorem to estimate the probability that the company has to pay a total of between $\$21500$ and $\$22000$ (leave your answers in terms of Φ, the cumulative distribution function of the standard normal).

4. Choose problem (a) or (b).

   a) (7.2) Suppose Z is a standard normal random variable. Let $X = Z^2 - Z$; find the expected value and variance of X.

   b) (7.2) Suppose Y is a normal random variable with mean μ and variance σ^2. Let $X = e^Y$; find the expected value and variance of X.

Solutions

1. a) Let E_j be the number of typos on page j; since E_j is $Pois(\frac{1}{10})$, the total number of errors is $E = E_1 + E_2 + ... + E_{360}$ which is $Pois(\frac{1}{10} + ... + \frac{1}{10}) = Pois(360(\frac{1}{10})) = Pois(36)$ since the sum of i.i.d. Poisson r.v.s. is Poisson. So the probability is given by the Poisson density:

$$P(E = 22) = f_E(22) = \frac{e^{-36}36^{22}}{22!}.$$

b) By the usual formula:

$$EX = \int x f_X(x)\, dx = \int_1^3 x \frac{9}{4} x^{-3}\, dx = \frac{9}{4}\int_1^3 x^{-2}\, dx = \frac{9}{4}\cdot\frac{2}{3} = \frac{3}{2}.$$

c) First, the parameter of Y is the reciprocal of its mean, i.e. 3. Now, calculate $M_Z(t)$ using properties of moment generating functions:

$$\begin{aligned}
M_Z(t) = M_{3X+Y}(t) &= M_{3X}(t)M_Y(t) \quad \text{(since } X \perp Y\text{)} \\
&= M_X(3t)M_Y(t) \\
&= \left(\frac{2}{2-3t}\right)^4 \left(\frac{3}{3-t}\right) \\
&\qquad \text{(since } X \text{ is } \Gamma(4,2) \text{ and } Y \text{ is } Exp(3)\text{)}
\end{aligned}$$

d) First, we need to find the covariance:

$$\begin{aligned}
Var(U+W) &= Var(U) + Var(W) + 2Cov(U,W) \\
8 &= 2 + 8 + 2Cov(U,W) \\
-2 &= 2Cov(U,W) \\
-1 &= Cov(U,W).
\end{aligned}$$

Now, by the formula for correlation:

$$\rho(U,W) = \frac{Cov(U,W)}{\sqrt{Var(U)\cdot Var(W)}} = \frac{-1}{\sqrt{2\cdot 8}} = \frac{-1}{4}.$$

2. a) We use the covariance formula $Cov(X,Y) = E[XY] - EX\,EY$:

$$\begin{aligned}
E[XY] = \int_0^1\int_0^1 xy(x+y)\, dy\, dx &= \int_0^1\int_0^1 (x^2y + xy^2)\, dy\, dx \\
&= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_0^1 dy \\
&= \int_0^1 \left(\frac{1}{2}x^2 + \frac{1}{3}x\right) dx \\
&= \left[\frac{x^3}{6} + \frac{x^2}{6}\right]_0^1 = \frac{1}{3}.
\end{aligned}$$

Therefore, applying the given information,

$$Cov(X,Y) = E[XY] - EX \cdot EY = \frac{1}{3} - \left(\frac{7}{12}\right)^2 = \frac{1}{3} - \frac{49}{144} = \frac{-1}{144}.$$

b) Since $Cov(X,Y) < 0$, we expect Y to decrease as X increases.

c) First, we compute the density of the marginal X:

$$f_X(x) = \int_0^1 (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2}.$$

Now the conditional density is $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{x+y}{x+\frac{1}{2}}$. Now for the conditional expectation:

$$\begin{aligned} E[Y|X] = \int_0^1 y\frac{x+y}{x+\frac{1}{2}}\,dy &= \frac{1}{x+\frac{1}{2}}\int_0^1 (xy+y^2)\,dy \\ &= \frac{1}{x+\frac{1}{2}}\left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_0^1 \\ &= \frac{1}{x+\frac{1}{2}}\left(\frac{x}{2} + \frac{1}{3}\right) \\ &= \frac{2}{2x+1} \cdot \frac{3x+2}{6} = \frac{3x+2}{6x+3}. \end{aligned}$$

3. a) Let X be the size of the claim. We have $\mu = 1000$ and $Var(X) = \sigma^2 = 40000$. Now by Chebyshev's inequality,

$$\begin{aligned} P(X > 1600) = P(X - \mu > 600) &\le P(|X-\mu| > 600) \\ &\le \frac{Var(X)}{600^2} = \frac{40000}{360000} = \frac{1}{9}. \end{aligned}$$

b) Let S_{20} be the sum of 20 independent copies of X. We are asked to find $P(21500 \le S_{20} \le 23000)$. By the Central Limit Theorem, since $S_n \approx n(\mu n, \sigma^2 n)$, we have

$$\begin{aligned} P(21500 \le S_{20} \le 23000) &\approx P(21500 \le n(1000(20),(200)^2(20)) \le 22000) \\ &= P(21500 \le n(20000, 800000) \le 22000) \\ &= P(21500 \le 20000 + \sqrt{800000}Z \le 22000) \\ &\quad \text{(where } Z \text{ is } n(0,1)\text{)} \end{aligned}$$

This is in turn equal to

$$\begin{aligned} P(1500 \leq 400\sqrt{5}Z \leq 2000) &= P\left(\frac{1500}{400\sqrt{5}} \leq Z \leq \frac{2000}{400\sqrt{5}}\right) \\ &= P\left(\frac{3}{4}\sqrt{5} \leq Z \leq \sqrt{5}\right) \\ &= \Phi\left(\sqrt{5}\right) - \Phi\left(\frac{3}{4}\sqrt{5}\right). \end{aligned}$$

4. a) The moment-generating function of Z is $M_Z(t) = e^{t^2/2}$. First, compute some moments of Z:

$$\begin{aligned} M_Z'(t) = te^{t^2/2} &\Rightarrow EZ = M_Z'(0) = 0 \\ M_Z''(t) = e^{t^2/2} + t^2e^{t^2/2} &\Rightarrow EZ^2 = M_Z''(0) = 1 \\ M_Z'''(t) = 3te^{t^2/2} + t^3e^{t^2} &\Rightarrow EZ^3 = M_Z'''(0) = 0 \\ M_Z''''(t) = 3e^{t^2/2} + 6t^2e^{t^2/2} + t^4e^{t^2/2} &\Rightarrow EZ^4 = M_Z''''(0) = 3 \end{aligned}$$

Therefore $EX = EZ^2 - EZ = 1 - 0 = 1$. Last, by the variance formula:

$$\begin{aligned} Var(X) = EX^2 - (EX)^2 &= E[Z^2 - Z]^2 - \left(E[Z^2 - Z]\right)^2 \\ &= E[Z^4 - 2Z^3 + Z^2] - \left(EZ^2 - EZ\right)^2 \\ &= [3 - 2(0) + 1] - (1 - 0)^2 \\ &= 4 - 1 \\ &= 3. \end{aligned}$$

b) Use the moment-generating function of Y to compute the first and second moments of X:

$$EX = E[e^Y] = E[e^{1Y}] = M_Y(1) = \exp\left(\mu + \frac{\sigma^2}{2}\right).$$

$$EX^2 = E[e^{2Y}] = M_Y(2) = \exp\left(2\mu + 2\sigma^2\right).$$

EX is $e^{\mu+\sigma^2/2}$ from above; last, apply the variance formula:

$$\begin{aligned} Var(X) = EX^2 - (EX)^2 &= \exp\left(2\mu + 2\sigma^2\right) - \left[\exp\left(\mu + \frac{\sigma^2}{2}\right)\right]^2 \\ &= e^{2\mu}e^{2\sigma^2} - e^{2\mu}e^{\sigma^2}. \end{aligned}$$

3.10 Fall 2012 Final Exam

1. Suppose a sack contains 20 red beads, 30 yellow beads, 40 green beads and 60 white beads (this makes a total of 150 beads in the sack).

   a) (2.3) Suppose 18 beads are drawn from the sack without replacement. What is the probability that of the 18 beads, 10 are white, 3 are red, 3 are yellow and 2 are green?

   b) (4.2) Suppose 18 beads are drawn from the sack with replacement. What is the probability that of the 18 beads, 10 are white, 3 are red, 3 are yellow and 2 are green?

   c) (6.1) Suppose 200 beads are drawn from the sack with replacement. How many white beads would you expect to draw?

   d) (2.3) Suppose 80 beads are drawn from the sack without replacement. What is the probability that 10 of the beads drawn are yellow?

   e) (2.4) Suppose beads are drawn from the sack, one at a time, with replacement. What is the probability that the sixth time you draw a red bead is on the 55^{th} draw?

   f) (2.3) Suppose you divide the beads into two groups of 75. What is the probability that the first group has more red beads in it than the second group?

2. Suppose X is a continuous, real-valued r.v. with density function

$$f_X(x) = \begin{cases} Cx & \text{if } 0 \leq x \leq 4 \\ 1 & \text{else} \end{cases}$$

   where C is a constant.

   a) (3.1) Find C.

   b) (3.1) Find the probability that $X > 2$.

   c) (3.1) Find $P(X > 3 \,|\, X > 2)$.

   d) (3.3) Let $Z = \sqrt{X}$. Find a density function of Z.

3. Suppose X and Y are continuous, real-valued r.v.s with joint density

$$f_{X,Y}(x, y) = \begin{cases} e^{-x} & \text{if } 0 \leq y \leq x \\ 0 & \text{else} \end{cases}$$

   a) (5.1) Find the probability that $X < 1$.

   b) (5.1) Find the density of the marginal Y.

c) (5.4) Let $Z = Y/X$. Find a density function of Z; describe Z as a common random variable (include all relevant parameters).
Hint: $\int_0^\infty xe^{-x}\,dx = 1$.

4. Let X and Y be discrete, integer-valued r.v.s with joint density

$$f_{X,Y}(x,y) = \begin{cases} \frac{12}{25} \cdot \frac{2^x}{5^{x+y}} & \text{if } 0 \le x \text{ and } 0 \le y \\ 0 & \text{else} \end{cases}$$

a) (5.1) Show that $X \perp Y$.

b) (5.1) Find $P(X \geq 1000)$.

c) (5.1) Find $P(X + Y = 12)$.

5. Suppose that X is a real-valued r.v. with moment generating function

$$M_X(t) = \left(\frac{3}{4}e^t + C\right)^{16}.$$

a) (7.2) Find C.

b) (7.2) Find the expected value of X.

c) (7.2) Find the variance of X.

6. Suppose X and Y have a bivariate normal distribution with $EX = EY = 2$ and covariance matrix

$$\Sigma = \begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}.$$

a) (7.3) Find the exact probability that $X \leq 4$. Leave your answer in terms of Φ, the cumulative distribution function of the standard normal random variable.

b) (7.3) Let $V = 4X + 3Y$. Find the mean and variance of V.

7. Suppose three women drive a car: a grandmother, a mother, and a daughter. Suppose further that:

- the number of accidents the grandmother will cause in the next year is a Poisson r.v. with parameter $\lambda = 1$;
- the number of accidents the mother will cause in the next year is a Poisson r.v. with parameter $\lambda = \frac{1}{2}$;
- the number of accidents the daughter will cause in the next year is a Poisson r.v. with parameter $\lambda = \frac{5}{2}$.

Finally, suppose that the number of accidents each woman causes is independent of the number of accidents the other women cause.

a) (7.2) Find the probability that the mother and daughter cause exactly six accidents in the next year.

b) (7.2) Find the mean and variance of the total number of accidents caused by the three women in the next year.

Solutions

1. a) This is hypergeometric: $\frac{C(60,10)C(20,3)C(30,3)C(40,2)}{C(150,18)}$

   b) This is multinomial: $\frac{18!}{10!3!3!2!}\left(\frac{2}{5}\right)^{10}\left(\frac{2}{15}\right)^{3}\left(\frac{1}{5}\right)^{3}\left(\frac{4}{15}\right)^{2}$

   c) The number of white beads is binomial with $n = 200$, $p = \frac{60}{150} = \frac{2}{5}$. Thus the expected number of white beads is $np = 200 \cdot 25 = 80$.

   d) This is hypergeometric: $\frac{C(30,10)C(120,70)}{C(150,80)}$.

   e) This is the probability that a negative binomial r.v. with $r = 6$ and $p = \frac{20}{150} = \frac{2}{15}$ takes the value $x = 49$ (this is 49 because if the sixth red comes on the 55^{th} draw, we need $55 - 6 = 49$ failures before the sixth red). This probability is

$$C(x+r-1,x)p^r(1-p)^x = C(54,49)\left(\frac{2}{15}\right)^6\left(\frac{13}{15}\right)^{49}.$$

   f) There are three disjoint possibilities:

$$A = \text{first group has more red beads}$$

$$B = \text{groups both have 10 red beads in them}$$

$$C = \text{second group has more red beads}$$

   Since there is really no difference between the "first" and "second" groups, $P(A) = P(C)$. Since these possibilities are disjoint and comprise all possible scenarios, $P(A) + P(B) + P(C) = 1$ so $2P(A) + P(B) = 1$ and $P(A) = \frac{1}{2}[1 - P(B)]$. The quantity asked for in this problem is $P(A)$, but we'll find $P(B)$ since it is easier (given by a hypergeometric random variable):

$$P(B) = \frac{C(20,10)C(130,65)}{C(150,75)}.$$

   Therefore $P(A) = \frac{1}{2}[1 - P(B)] = \frac{1}{2}[1 - \frac{C(20,10)C(130,65)}{C(150,75)}]$.

2. a) We have $1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^4 Cx\,dx = \frac{Cx^2}{2}\Big|_0^4 = 8C$ so $C = \frac{1}{8}$.

   b) $P(X > 2) = \int_2^{\infty} f_X(x)\,dx = \int_2^4 \frac{1}{8}x\,dx = \frac{x^2}{16}\Big|_2^4 = 1 - \frac{1}{4} = \frac{3}{4}$.

   c) $P(X > 3 \cap X > 2) = P(X > 3) = \int_3^4 \frac{1}{8}x\,dx = \frac{x^2}{16}\Big|_3^4 = 1 - \frac{9}{16} = \frac{7}{16}$. Therefore $P(X > 3 \,|\, X > 2) = \frac{P(X>3 \cap X>2)}{P(X>2)} = \frac{7/16}{3/4} = \frac{7}{12}$.

   d) Z is continuous with range $[0,2]$. Thus when $Z \le 0$, $F_Z(z) = 0$ and when $Z \ge 2$, $F_Z(z) = 1$. When $z \in [0,2]$,

$$F_Z(z) = P(Z \le z) = P(\sqrt{X} \le z) = P(X \le z^2) = \int_0^{z^2} \frac{1}{8}x\,dx = \frac{x^2}{16}\Big|_0^{z^2} = \frac{z^4}{16}.$$

Therefore, by differentiating to get a density function, we see

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \begin{cases} \frac{z^3}{4} & \text{if } z \in [0,2] \\ 0 & \text{else} \end{cases}.$$

3. a) Integrate the joint density function:

$$\begin{aligned} P(X<1) &= \int_0^1 \int_0^x e^{-x}\,dy\,dx \\ &= \int_0^1 \left[ye^{-x}\right]_0^x\,dx \\ &= \int_0^1 xe^{-x}\,dx \\ &= \left[-xe^{-x} - e^{-x}\right]_0^1 \\ &= (-e^{-1} - e^{-1}) - (-1) = 1 - \frac{2}{e}. \end{aligned}$$

b) Integrate the joint density with respect to the opposite variable:

$$\begin{aligned} f_Y(y) &= \int_y^\infty e^{-x}\,dx \\ &= \left[-e^{-x}\right]_y^\infty \\ &= 0 - (-e^{-y}) = e^{-y}. \end{aligned}$$

This is valid when $y \geq 0$; when $y < 0$, $f_Y(y) = 0$. (In particular, Y is exponential with parameter 1, not that this was asked.)

c) Z is continuous. Since $0 \leq Y \leq X$, the range of Z is $[0,1]$, so when $z < 0$, $F_Z(z) = 0$ and when $z \geq 1$, $F_Z(z) = 1$. When $z \in [0,1]$,

$$\begin{aligned} F_Z(z) = P(Z \leq z) &= \int_0^\infty \int_0^{zx} e^{-x}\,dy\,dx \\ &= \int_0^\infty \left[ye^{-x}\right]_0^{zx}\,dx \\ &= \int_0^\infty zxe^{-x}\,dx \\ &= z\int_0^\infty xe^{-x}\,dx \\ &= z \cdot 1 = z. \end{aligned}$$

Then

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \begin{cases} 1 & \text{if } z \in [0,1] \\ 0 & \text{else} \end{cases}$$

so since the density function of Z is constant, Z must be uniform (on $[0,1]$).

4. a) Compute the marginals (when either x or y are negative, the respective marginal is zero; when they are nonnegative they are as below):

$$f_X(x) = \sum_{y=0}^{\infty} \frac{12}{25} \cdot \frac{2^x}{5^{x+y}} = \frac{12}{25}\left(\frac{2}{5}\right)^x \sum_{y=0}^{\infty}\left(\frac{1}{5}\right)^y = \frac{12}{25}\left(\frac{2}{5}\right)^x \left[\frac{1}{1-1/5}\right] = \frac{3}{5}\left(\frac{2}{5}\right)^x.$$

$$f_Y(y) = \sum_{x=0}^{\infty} \frac{12}{25} \cdot \frac{2^x}{5^{x+y}} = \frac{12}{25}\left(\frac{1}{5}\right)^y \sum_{x=0}^{\infty}\left(\frac{2}{5}\right)^x = \frac{12}{25}\left(\frac{1}{5}\right)^y \left[\frac{1}{1-2/5}\right] = \frac{4}{5}\left(\frac{1}{5}\right)^y.$$

Finally,

$$f_X(x)f_Y(y) = \frac{3}{5}\left(\frac{2}{5}\right)^x \frac{4}{5}\left(\frac{1}{5}\right)^y = \frac{12}{25}\frac{2^x}{5^x 5^y} = f_{X,Y}(x,y)$$

so $X \perp Y$ as desired.

b) Use the marginal found in part (a):

$$P(X \geq 1000) = \sum_{x=1000}^{\infty} f_X(x) = \sum_{x=1000}^{\infty} \frac{3}{5}\left(\frac{2}{5}\right)^x = \frac{3}{5}\left(\frac{2}{5}\right)^{1000} \sum_{x=0}^{\infty}\left(\frac{2}{5}\right)^x$$
$$= \frac{3}{5}\left(\frac{2}{5}\right)^{1000}\left[\frac{1}{1-\frac{2}{5}}\right] = \left(\frac{2}{5}\right)^{1000}.$$

c) Sum appropriate values of the joint density function:

$$\begin{aligned} P(X+Y=12) &= \sum_{x=0}^{12} f_{X,Y}(x, 12-x) \\ &= \sum_{x=0}^{12} \frac{12}{25}\frac{2^x}{5^{x+12-x}} \\ &= \frac{12}{25}\sum_{x=0}^{12}\frac{2^x}{5^{12}} \\ &= \frac{12}{25\cdot 5^{12}}\sum_{x=0}^{12} 2^x \\ &= \frac{12}{5^{14}}\left[\frac{1-2^{13}}{1-2}\right] \\ &= \frac{12}{5^{14}}(2^{13}-1). \end{aligned}$$

5. a) Since $M_X(0) = 1$, we have $1 = \left(\frac{3}{4}e^0 + C\right)^{16}$ so $\frac{3}{4} + C = 1$ so $C = \frac{1}{4}$.

b) $EX = M_X'(0) = 16\left(\frac{3}{4}e^t + \frac{1}{4}\right)^{15}\left(\frac{3}{4}e^t\right)\Big|_{t=0} = 16 \cdot 34 = 12.$

c) First,

$$\begin{aligned}
EX^2 = M''_X(0) &= 16 \cdot 15 \left(\frac{3}{4}e^t + \frac{1}{4}\right)^{14} \left(\frac{3}{4}e^t\right)^2 + 16 \left(\frac{3}{4}e^t + \frac{1}{4}\right)^{15} \left(\frac{3}{4}e^t\right)\Bigg|_{t=0} \\
&= 16 \cdot 15 \cdot \left(\frac{3}{4}\right)^2 + 16 \cdot \frac{3}{4} \\
&= 15 \cdot 9 + 12 = 147.
\end{aligned}$$

Then, $Var(X) = EX^2 - (EX)^2 = 147 - 12^2 = 3$.

6. a) From the given information, X is normal with $\mu = EX = 2$ and $\sigma^2 = Var(X) = 3$. Therefore $X = \mu + \sigma Z = 2 + \sqrt{3}Z$ where Z is the standard normal. Thus $P(X \leq 4) = P(2 + \sqrt{3}Z \leq 4) = P(Z \leq \frac{2}{\sqrt{3}}) = \Phi\left(\frac{2}{\sqrt{3}}\right)$.

   b) $EV = 4EX + 3EY = 4 \cdot 2 + 3 \cdot 2 = 14$;
   $Var(V) = Var(4X + 3Y) = Var(4X) + Var(3Y) + 2Cov(4X, 3Y) = 16Var(X) + 9Var(Y) + 24Cov(X, Y) = 16 \cdot 3 + 9 \cdot 5 + 24 \cdot -1 = 48 + 45 - 24 = 69$.

7. Let G be the number of accidents the grandmother causes; G is Poisson with $\lambda = 1$; let M be the number of accidents the mother causes; M is Poisson with $\lambda = \frac{1}{2}$; let D be the number of accidents the daughter causes; D is Poisson with $\lambda = \frac{5}{2}$.

   a) This is asking $P(M + D = 6)$. By result from class, since $M \perp D$, $M + D$ is Poisson with parameter $\frac{1}{2} + \frac{5}{2} = 3$. So this is asking the probability a Poisson r.v. with parameter 3 is equal to 6; this probability is $\frac{e^{-3}3^6}{6!}$.

   b) By result from class, the total number of accidents caused is $G + M + D$, a Poisson r.v. with parameter $\frac{1}{2} + \frac{5}{2} + 1 = 4$. The mean and variance of a Poisson r.v. are its parameter, so the mean and variance of $G + M + D$ are both 4.

3.11 Fall 2013 Final Exam

1. a) (1.4) Suppose A and B are events such that $P(A) = P(B) = \frac{1}{2}$ and $P(A \cup B) = \frac{7}{8}$. Find $P(A \,|\, B)$.

   b) (1.3) A survey of 200 people regarding their TV habits reveals:
   - 135 people watch *The Big Bang Theory*;
   - 110 people watch *SportsCenter*;
   - 65 people watch *Game of Thrones*;
   - 65 people watch *The Big Bang Theory* and *SportsCenter*;
   - 45 people watch *The Big Bang Theory* and *Game of Thrones*;
   - 25 people watch *SportsCenter* and *Game of Thrones*; and
   - 20 people watch none of these three programs.

   How many people watch all three of these programs?

   c) (1.5) Suppose that 65% of all Windows computers crash within one year of purchase, and 25% of all Mac computers crash within one year of purchase. If 70% of all computers are Windows machines (assume the others are Macs), what is the probability that a computer is a Mac, given that it crashes within one year of purchase?

2. A fair die is rolled repeatedly (each roll is independent of past and future rolls).

   a) (2.4) What is the probability that of the first 9 rolls, exactly 2 of those rolls are 3s?

   b) (2.4) What is the probability that the sixth time a 5 is rolled is on the twentieth roll?

   c) (2.4) What is the probability that of the first ten rolls, the smallest number rolled is 2?

3. Suppose that the amount of damage caused in an accident is a random variable whose density function is

$$f(x) = \begin{cases} Cx^3 & \text{if } 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

   a) (3.1) Find the value of C.

   b) (3.1) Find the probability that the damage caused is at least $\frac{2}{3}$, given that it is at least $\frac{1}{2}$.

   c) (6.2) Find the variance of the amount of damage casued.

   d) (3.1) The *median* of a continuous random variable X is a number m such that $P(X \leq m) = \frac{1}{2}$. Find the median amount of damage caused in the accident.

e) (3.3) Suppose that the amount A that an insurance company has to pay out is exactly $1/2$ of the damage caused in the accident. Find the density function of A.

4. Suppose that the number of lightning strikes in an area is a Poisson process with daily rate 3.

   a) (3.4) What is the probability that there are at most 3 lightning strikes in a given day?
   b) (3.4) What is the probability that there are exactly 80 lightning strikes in a given 30 day time period?
   c) (3.4) What is the probability that the area will experience its next lightning strike between 36 and 48 hours from now?
   d) (3.4) Let X be the time (in days) until the 9th lightning strike in the area. Find the density function of X.

5. a) (4.4) Suppose W is a discrete random variable whose density function is given in the following table:

w	2	3	4
$f_W(w)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

   Let V be the sum of two independent copies of W. What is the probability that V is even?

   b) (3.4) Suppose the lifespan of a pet fish is exponentially distributed with mean 2 months. Find the probability that the fish lives at most 1 month.
   c) (7.2) Suppose X is a random variable with moment generating function $M_X(t) = \frac{e^{2t}}{1-t^2}$. Find the expected value of X.

6. Suppose X and Y are continuous random variables whose joint density is

$$f_{X,Y}(x,y) = \begin{cases} 3e^{-2x-y} & \text{if } 0 < x < y \\ 0 & \text{else} \end{cases}$$

   a) (5.1) Find the density of the marginal X. Identify X as a common random variable, giving parameters if necessary.
   b) (5.4) Find the density of $M = \frac{Y}{X}$.

7. Suppose X and Y have a bivariate normal density such that $EX = 6$ and $EY = 4$, X and Y have variance 4, and $Cov(X,Y) = -1$.

   a) (7.3) Find the mean and variance of $4X + Y$.
   b) (7.3) Find $P(X \geq Y)$. Leave your answer in terms of Φ, the cumulative distribution function of the standard normal r.v.

Solutions

1. a) By Inclusion-Exclusion,

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{1}{2} + \frac{1}{2} - \frac{7}{8} = \frac{1}{8}.$$

Therefore

$$P(A \,|\, B) = \frac{P(A \cap B)}{P(B)} = \frac{1/8}{1/2} = \frac{1}{4}.$$

b) Since 20 people watch none of the three programs, $180 - 20$ people watch at least one of the programs. Let A, B and C be the sets of people who watch *The Big Bang Theory*, *SportsCenter* and *Game of Thrones*, respectively. By Inclusion-Exclusion,

$$\begin{aligned}\#(A \cup B \cup C) &= \#(A) + \#(B) + \#(C) - \#(A \cap B) - \#(A \cap C) - \#(B \cap C) \\ &\quad + \#(A \cap B \cap C) \\ 180 &= 135 + 110 + 65 - 65 - 45 - 25 + \#(A \cap B \cap C) \\ 180 &= 175 + \#(A \cap B \cap C) \\ 5 &= \#(A \cap B \cap C).\end{aligned}$$

c) Let M be the event that a computer is a Mac, and let E be the event that a computer crashes. By Bayes' Law,

$$P(M \,|\, E) = \frac{P(E \,|\, M)P(M)}{P(E \,|\, M)P(M) + P(E \,|\, M^C)P(M^C)} = \frac{(.25)(.3)}{(.25)(.3) + (.65)(.7)}.$$

(This simplifies to $\frac{15}{106}$.)

2. a) This is binomial: $b(9, \frac{1}{6}, 2) = \begin{pmatrix} 9 \\ 2 \end{pmatrix} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^7$.

b) This is negative binomial; if the sixth success is on the twentieth roll, we want 14 successes before the sixth success. The answer is therefore

$$P(NB(6, \frac{1}{6}) = 14) = \begin{pmatrix} 19 \\ 14 \end{pmatrix} \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^{14}.$$

c) Let M be the smallest number rolled in the first ten rolls. Notice

$$P(M \geq 2) = P(0 \text{ ones rolled}) = b(10, \frac{1}{6}, 0) = \left(\frac{5}{6}\right)^{10}$$

and

$$P(M \geq 3) = P(0 \text{ ones or twos rolled}) = b(10, \frac{1}{3}, 0) = \left(\frac{2}{3}\right)^{10}$$

Therefore

$$P(M=2)=P(M\geq 2)-P(M\geq 3)=\left(\frac{5}{6}\right)^{10}-\left(\frac{2}{3}\right)^{10}.$$

3. a) $1=\int_{-\infty}^{\infty} f_X(x)\,dx=\int_0^1 Cx^3\,dx=\frac{C}{4}x^4|_0^1=\frac{C}{4}$ so $C=4$.
   b) By the definition of conditional probability:

$$\begin{aligned}P(X\geq \frac{2}{3}\,|\,X\geq \frac{1}{2}) &= \frac{P(X\geq \frac{2}{3}\cap X\geq \frac{1}{2})}{P(X\geq \frac{1}{2})}\\ &= \frac{P(X\geq \frac{2}{3})}{P(X\geq \frac{1}{2})}\\ &= \frac{\int_{2/3}^1 4x^3\,dx}{\int_{1/2}^1 4x^3\,dx}\\ &= \frac{1-(2/3)^4}{1-(1/2)^4}.\end{aligned}$$

   c) First, find the expected value:

$$EX=\int_{-\infty}^{\infty} xf_X(x)\,dx=\int_0^1 4x^4\,dx=\frac{4}{5}x^5|_0^1=\frac{4}{5}.$$

   Next, find the second moment:

$$EX^2=\int_{-\infty}^{\infty} x^2f_X(x)\,dx=\int_0^1 4x^6\,dx=\frac{2}{3}x^6|_0^1=\frac{2}{3}.$$

   Finally, $Var(X)=EX^2-(EX)^2=\frac{2}{3}-\left(\frac{4}{5}\right)^2=\frac{2}{3}-\frac{16}{25}=\frac{2}{75}$.
   d) We have

$$\frac{1}{2}=P(X\leq m)=\int_0^m 4x^3\,dx=x^4|_0^m=m^4$$

   so $m=\sqrt[4]{\frac{1}{2}}$.
   e) $A=\frac{1}{2}X$ is continuous with range $(0,\frac{1}{2})$. Thus $F_A(a)=0$ for $a\leq 0$ and $F_A(a)=1$ for $a\geq \frac{1}{2}$. Now for $a\in(0,\frac{1}{2})$,

$$F_A(a)=P(A\leq a)=P(\frac{1}{2}X\leq a)=P(X\leq 2a)=\int_0^{2a} 4x^3\,dx=x^4|_0^{2a}=16a^4.$$

   Therefore

$$f_A(a)=\frac{d}{da}F_A(a)=\begin{cases} 64a^3 & \text{if } 0<a<\frac{1}{2}\\ 0 & \text{else}\end{cases}$$

4. The setup of the problem indicates that if X_t is the number of lightning strikes that occur in t days, $\{X_t\}$ is a Poisson process with rate 3.

   a) X_1 is Poisson with parameter 3, so

   $$P(X_1 \leq 3) = \sum_{x=0}^{3} f_{X_1}(x) = \frac{e^{-3}3^0}{0!} + \frac{e^{-3}3^1}{1!} + \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!} = 13e^{-3}.$$

   b) X_{30} is Poisson with parameter $30 \cdot 3 = 90$, so

   $$P(X_{30} = 80) = \frac{e^{-90}(90)^{80}}{80!}.$$

   c) In a Poisson process, the waiting time to the next event is exponential with parameter equal to the rate, so W, the waiting time, is exponential with parameter 3. Thus (converting the hours to days),

   $$P(1.5 \leq W \leq 2) = \int_{1.5}^{2} f_W(w)\,dw = \int_{1.5}^{2} 3e^{-3w}\,dw = [-e^{-3w}]_{1.5}^{2} = e^{-4.5} - e^{-6}.$$

   d) In a Poisson process, the waiting time to the 9th event is $\Gamma(9, \lambda)$ which in this case is $\Gamma(9, 3)$. Thus the density of X is

   $$f_X(x) = \frac{3^9}{\Gamma(9)} x^8 e^{-3x} \quad \text{for } x \geq 0.$$

5. a) Let (W_1, W_2) be the independent copies of W. Then

   $$\begin{aligned} P(V \text{ is even}) &= P(2,2) + P(2,4) + P(3,3) + P(4,2) + P(4,4) \\ &= \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{5}{8}. \end{aligned}$$

   b) Let the lifespan of the fish be L. If L is exponential with $EL = 2$, then L is exponential with parameter $\frac{1}{2}$. Thus

   $$P(L \leq 1) = F_L(1) = 1 - e^{-\frac{1}{2}(1)}.$$

   c) The derivative of M_X, by the quotient rule, is $\frac{2e^{2t}(1-t^2) - 2te^{2t}}{(1-t^2)^2}$. Thus $EX = M_X'(0) = \frac{2-0}{(1-0)^2} = 2$.

6. a) Integrate the joint density:

$$
\begin{aligned}
f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy &= \int_x^\infty 3e^{-2x-y}\,dy \\
&= 3e^{-2x}\int_x^\infty e^{-y}\,dy \\
&= 3e^{-2x}(-e^{-y})|_x^\infty \\
&= 3e^{-2x}(0 - -e^{-x}) = 3e^{-3x}.
\end{aligned}
$$

Thus X is exponential with parameter 3.

b) M is continuous. Since $Y > X$, $Y/X > 1$ so M has range $(1,\infty)$, so $F_M(m) = 0$ for $m \le 1$. If $m > 1$,

$$
\begin{aligned}
F_M(m) = P(M \le m) = P(\frac{Y}{X} \le m) &= P(Y \le mX) \\
&= \int_0^\infty \int_x^{mx} 3e^{-2x-y}\,dy\,dx \\
&= \int_0^\infty 3e^{-2x}(-e^{-y})|_x^{mx}\,dx \\
&= \int_0^\infty 3e^{-2x}(e^{-x} - e^{-mx})\,dx \\
&= \int_0^\infty (3e^{-3x} - 3e^{-(2+m)x})\,dx \\
&= \int_0^\infty 3e^{-3x}\,dx - 3\int_0^\infty e^{-(2+m)x}\,dx \\
&= 3\cdot\frac{\Gamma(1)}{3^1} - 3\cdot\frac{\Gamma(1)}{(m+2)^1} \\
&= 1 - \frac{3}{m+2}.
\end{aligned}
$$

Therefore

$$
f_M(m) = \frac{d}{dm}F_M(m) = \begin{cases} \frac{3}{(m+2)^2} & \text{if } m > 1 \\ 0 & \text{else} \end{cases}
$$

7. Note that the mean vector of this joint normal distribution is $\begin{pmatrix} 6 \\ 4 \end{pmatrix}$ and the covariance matrix is $\Sigma = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$.

a) $W = 4X + Y = (4,1)\cdot(X,Y)$ is normal; by formulas developed in class we have

$$EW = (4,1)\cdot(EX,EY) = 4\cdot 6 + 1\cdot 4 = 28$$

and

$$Var(W) = (4,1)^T\Sigma(4,1) = \begin{pmatrix} 4 & 1 \end{pmatrix}\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \end{pmatrix}\begin{pmatrix} 15 \\ 0 \end{pmatrix} = 60.$$

b) Let $U = X - Y$. $P(X \geq Y) = P(X - Y \geq 0) = P(U \geq 0)$. Now since (X, Y) is joint normal, U is normal and

$$EU = (1, -1) \cdot (EX, EY) = 1 \cdot 6 - 1 \cdot 4 = 6 - 4 = 2;$$

$$Var(U) = (1, -1)^T \Sigma (1, -1) = 10.$$

Therefore U is normal $n(2, 10)$ so $U = 2 + \sqrt{10}Z$ where Z is the standard normal. So

$$\begin{aligned} P(U \geq 0) = P(2 + \sqrt{10}Z \geq 0) &= P(\sqrt{10}Z \geq -2) \\ &= P(Z \geq \frac{-2}{\sqrt{10}}) \\ &= 1 - \Phi\left(\frac{-2}{\sqrt{10}}\right). \end{aligned}$$

3.12 Fall 2014 Final Exam

1. The parts of this problem are unrelated.

   a) (1.4) Let E and F be events in a probability space. If $P(E \cap F) = \frac{2}{3}$ and $P(E \,|\, F) = \frac{3}{4}$, find $P(F^C)$.

   b) (1.4) Suppose 90% of all households have a dishwasher, and 80% of all households with dishwashers also have trash compactors. If having a trash compactor is independent of having a dishwasher, what percent of households have neither a dishwasher nor a trash compactor?

   c) (1.5) Suppose that an insurance company classifies its policyholders as "high risk", "medium risk" or "low risk" (these categories are mutually exclusive). Suppose further that 10% of its policyholders are high risk, 50% are medium risk and 40% are low risk. A high risk policyholder has a 40% chance of filing a claim, a medium risk policyholder has a 10% chance of filing a claim, and a low risk policyholder has a 2% chance of filing a claim. What is the probability that a policyholder who files a claim is a medium risk policyholder?

2. A bag contains 60 marbles, of which 20 are red, 10 are blue and 30 are yellow.

   a) (2.3) If 12 marbles are drawn from the bag without replacement, what is the probability that 5 are red, 3 are blue and 4 are yellow?

   b) (4.2) If 12 marbles are drawn from the bag with replacement, what is the probability that 5 are red, 3 are blue and 4 are yellow?

   c) (2.4) If marbles are drawn from the bag with replacement, what is the probability that the third time a blue marble is drawn is on the tenth draw?

   d) (2.4) Suppose someone has psychic powers, so that when they draw from the bag they are twice as likely to draw any particular yellow marble as they are to draw any particular non-yellow marble. If this person draws 14 marbles from the bag with replacement, what is the probability they draw 3 red marbles?

3. Suppose W is a continuous random variable with density function

$$f_W(w) = \begin{cases} 5w^{-6} & \text{if } w \geq 1 \\ 0 & \text{else} \end{cases}$$

   a) (3.1) Find $P(W < 3 \,|\, W < 6)$.

   b) (3.3) Let $X = \sqrt{W}$. Find a density function of X.

   c) (6.2) Let $Y = W^2$. Find the variance of Y.

4. Suppose X and Y are discrete, integer-valued random variables with joint density
$$f_{X,Y}(x,y) = \begin{cases} c\left(\frac{1}{2}\right)^x\left(\frac{3}{4}\right)^y & \text{if } 0 \le y \le x \\ 0 & \text{else} \end{cases}$$
where c is a constant.

   a) (4.1) Show that $c = \frac{5}{16}$.
   b) (4.1) Find $P(Y \ge 4)$.
   c) (4.1) Find $P(X = 10)$.
   d) (4.4) Find a density function of $Z = X - Y$.

5. The parts of this problem are unrelated.

   a) (7.3) Suppose X and Y have a bivariate normal distribution such that $E(Y \mid X) = \frac{11}{4} + \frac{1}{8}x$, $E(X \mid Y) = \frac{4}{3} + \frac{2}{9}y$ and $Var(X \mid Y) = \frac{35}{4}$. Find the variance of X.
   b) (7.1) Let V be a random variable whose probability generating function is $G_V(t) = e^{3t-3}$. Find $P(V = 7)$.
   c) (3.4) Suppose that the time until the channel is changed on a certain TV is an exponential random variable with variance 9. Find the probability that the channel will be changed within the next 2 units of time.

6. The parts of this problem are unrelated.

   a) (8.3) Suppose that the amount of time a battery lasts is a normal random variable with mean 3 years and standard deviation 2 years. Find the probability that 30 independently chosen batteries will last a total of at least 87 and at most 95 years (leave your answer in terms of Φ, the cumulative distribution function of the standard normal r.v.).
   b) (1.2) Suppose that a point (X,Y) is chosen uniformly from the interior of the triangle whose vertices are $(0,0)$, $(6,0)$ and $(0,6)$. Find the probability that $Y < X^2$.

7. Suppose that X is an exponential random variable with parameter 1 and that given $X = x$, Y is exponential with parameter $2x$.

   a) (5.4) Let $U = X + Y$ and $V = X - Y$. Find the joint density of U and V.
   b) (6.3) Find the conditional expectation of X given Y.

Solutions

1. a) First, by definition of conditional probability,

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} \quad \text{i.e.} \quad \frac{3}{4} = \frac{\frac{2}{3}}{P(F)}$$

so $P(F) = \frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$. Thus $P(F^C) = 1 - P(F) = \frac{1}{9}$.

b) Let D and T be the households with dishwashers and trash compactors, respectively. Then $P(D) = .9$, $P(T) = .8$ and since $D \perp T$, $P(D \cap T) = (.9)(.8) = .72$. Now by Inclusion-Exclusion, $P(D \cup T) = P(D) + P(T) - P(D \cap T) = .9 + .8 - .72 = .98$. Finally, we are asked $P(D^C \cap D^C) = P((D \cup T)^C)$ which is $1 - .98 = .02$.

c) Let H, M and L be the high risk, medium risk and low risk policyholders, and let K be the policyholders who file claims. We are given:

$$P(H) = .1 \; P(M) = .5 \; P(L) = .4 \; P(K \mid H) = .4 \; P(K \mid M) = .1 \; P(K \mid L) = .02$$

so by Bayes' Law, since $\{H, M, L\}$ forms a partition of Ω,

$$\begin{aligned} P(M \mid K) &= \frac{P(K|M)P(M)}{P(K|H)P(H) + P(K|M)P(M) + P(K|L)P(L)} \\ &= \frac{(.1)(.5)}{(.4)(.1) + (.1)(.5) + (.02)(.4)} \\ &= \frac{25}{129}. \end{aligned}$$

2. a) This is hypergeometric: the probability is

$$\frac{\binom{20}{5}\binom{10}{3}\binom{30}{4}}{\binom{60}{12}}.$$

b) This is multinomial: the probability is

$$\frac{12!}{5!\,4!\,3!}\left(\frac{20}{60}\right)^5\left(\frac{10}{60}\right)^3\left(\frac{30}{60}\right)^4 = \frac{12!}{5!\,4!\,3!}\left(\frac{1}{3}\right)^5\left(\frac{1}{6}\right)^3\left(\frac{1}{2}\right)^4.$$

c) This is negative binomial: to get the third success (success here is drawing a blue marble which has probability $\frac{10}{60} = \frac{1}{6}$) on the tenth draw, we need seven failures before the third success. Thus the probability is

$$P\left(NB(3, \frac{1}{6}) = 7\right) = \binom{9}{7}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^7.$$

d) Let p be the probability the psychic draws any one non-yellow marble; then $2p$ is the probability she draws any one yellow marble. Since there are 30 yellow and 30 non-yellow marbles, we have $30(p) + 30(2p) = 1$, i.e. $p = \frac{1}{90}$. That means the probability of drawing a red marble on any one draw is $20p = \frac{20}{90} = \frac{2}{9}$. Now this probability is binomial:

$$P\left(b(14, \frac{2}{9}) = 3\right) = \left(\begin{array}{c} 14 \\ 3 \end{array} \right) \left(\frac{2}{9}\right)^3 \left(\frac{7}{9}\right)^{11}.$$

3. a) Compute using the definition of conditional probability:

$$\begin{aligned} P(W < 3 \,|\, W < 6) &= \frac{P(W < 3 \cap W < 6)}{P(W < 6)} \\ &= \frac{P(W < 3)}{P(W < 6)} \\ &= \frac{\int_1^3 5w^{-6}\,dw}{\int_1^6 5w^{-6}\,dw} = \frac{[-w^{-5}]_1^3}{[-w^{-5}]_1^6} = \frac{-3^{-5}+1}{-6^{-5}+1} = \frac{1-3^{-5}}{1-6^{-5}}. \end{aligned}$$

b) X is continuous with range $[1, \infty)$. Now let $x \geq 1$:

$$\begin{aligned} F_X(x) = P(X \leq x) = P(\sqrt{W} \leq x) = P(W \leq x^2) &= \int_1^{x^2} 5w^{-6}\,dw \\ &= [-w^{-5}]_1^{x^2} \\ &= -x^{-10} + 1. \end{aligned}$$

Differentiating to find the density function of X, we see

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} 10x^{-11} & \text{if } x \geq 1 \\ 0 & \text{else} \end{cases}$$

c) By the formula for expected value of a transformation, we have

$$\begin{aligned} EY = E[W^2] &= \int_1^\infty w^2 f_W(w)\,dw = \int_1^\infty w^2 5w^{-6}\,dw \\ &= \int_1^\infty 5w^{-4}\,dw \\ &= \left[\frac{-5}{3}w^{-3}\right]_1^\infty = 0 - \frac{-5}{3} = \frac{5}{3}. \end{aligned}$$

Also, by the same formula

$$\begin{aligned} EY^2 = E[W^2]^2 = EW^4 &= \int_1^\infty w^4 f_W(w)\,dw = \int_1^\infty w^4 5w^{-6}\,dw \\ &= \int_1^\infty 5w^{-2}\,dw \\ &= \left[\frac{-5}{w}\right]_1^\infty = 0 - (-5) = 5. \end{aligned}$$

Finally, by the variance formula $Var(Y) = EY^2 - (EY)^2 = 5 - \left(\frac{5}{3}\right)^2 = \frac{20}{9}$.

4. a) The density function must sum to 1:

$$\begin{aligned}
1 = \sum_{y=0}^{\infty}\sum_{x=y}^{\infty} f_{X,Y}(x,y) &= \sum_{y=0}^{\infty}\sum_{x=y}^{\infty} c\left(\frac{1}{2}\right)^x\left(\frac{3}{4}\right)^y \\
&= c\sum_{y=0}^{\infty}\left(\frac{3}{4}\right)^y\sum_{x=y}^{\infty}\left(\frac{1}{2}\right)^x \\
&= c\sum_{y=0}^{\infty}\left(\frac{3}{4}\right)^y\left(\frac{1}{2}\right)^y\left(\frac{1}{1-\frac{1}{2}}\right) \\
&= c\left(\frac{1}{1-\frac{1}{2}}\right)\sum_{y=0}^{\infty}\left(\frac{3}{8}\right)^y \\
&= c\left(\frac{1}{1-\frac{1}{2}}\right)\left(\frac{1}{1-\frac{3}{8}}\right) \\
&= c\,(2)\left(\frac{8}{5}\right) = \frac{16}{5}c
\end{aligned}$$

Thus $c = \frac{5}{16}$.

b) This probability is

$$\begin{aligned}
P(Y \geq 4) = \sum_{y=4}^{\infty}\sum_{x=y}^{\infty} f_{X,Y}(x,y) &= \sum_{y=4}^{\infty}\sum_{x=y}^{\infty} \frac{5}{16}\left(\frac{1}{2}\right)^x\left(\frac{3}{4}\right)^y \\
&= \frac{5}{16}\sum_{y=4}^{\infty}\left(\frac{3}{4}\right)^y\sum_{x=y}^{\infty}\left(\frac{1}{2}\right)^x \\
&= \frac{5}{16}\sum_{y=4}^{\infty}\left(\frac{3}{4}\right)^y\left(\frac{1}{2}\right)^y\left(\frac{1}{1-\frac{1}{2}}\right) \\
&= \frac{5}{16}\left(\frac{1}{1-\frac{1}{2}}\right)\sum_{y=4}^{\infty}\left(\frac{3}{8}\right)^y \\
&= \frac{5}{16}\,(2)\left(\frac{3}{8}\right)^4\left(\frac{1}{1-\frac{3}{8}}\right) \\
&= c\,(2)\left(\frac{3}{8}\right)^4\left(\frac{8}{5}\right) = \left(\frac{3}{8}\right)^4.
\end{aligned}$$

c) By direct computation,

$$
\begin{aligned}
P(X=10) = \sum_{y=0}^{10} f_{X,Y}(10,y) &= \sum_{y=0}^{10} \frac{5}{16}\left(\frac{1}{2}\right)^{10}\left(\frac{3}{4}\right)^{y} \\
&= \frac{5}{16}\left(\frac{1}{2}\right)^{10}\sum_{y=0}^{10}\left(\frac{3}{4}\right)^{y} \\
&= \frac{5}{16}\left(\frac{1}{2}\right)^{10}\left(\frac{1-(3/4)^{11}}{1-3/4}\right) \\
&= \frac{5}{2^{12}}\left(1-\left(\frac{3}{4}\right)^{11}\right).
\end{aligned}
$$

d) Z is discrete, and since $X \geq Y$, Z takes values in $\{0,1,2,3,..\}$. Let $z \geq 0$ be an integer; then

$$
\begin{aligned}
f_Z(z) = P(Z=z) = P(X-Y=z) &= P(X=Y+z) \\
&= \sum_{y=0}^{\infty} f_{X,Y}(y+z,y) \\
&= \sum_{y=0}^{\infty} \frac{5}{16}\left(\frac{1}{2}\right)^{y+z}\left(\frac{3}{4}\right)^{y} \\
&= \frac{5}{16}\left(\frac{1}{2}\right)^{z}\sum_{y=0}^{\infty}\left(\frac{3}{8}\right)^{y} \\
&= \frac{5}{16}\left(\frac{1}{2}\right)^{z}\left(\frac{1}{1-\frac{3}{8}}\right) \\
&= \frac{5}{16}\left(\frac{8}{5}\right)\left(\frac{1}{2}\right)^{z} = \frac{1}{2}\left(\frac{1}{2}\right)^{z}.
\end{aligned}
$$

(In other words, Z is geometric with parameter $\frac{1}{2}$.)

5. a) Since (X,Y) is bivariate normal, we know X is normal so it is sufficient to find the mean and variance of X. First, we know

$$\frac{4}{3}+\frac{2}{9}y = E(X|Y) = \mu_X + \frac{\sigma_{XY}}{\sigma_Y^2}(y-\mu_Y) \Rightarrow \frac{\sigma_{XY}}{\sigma_Y^2} = \frac{2}{9} \Rightarrow \sigma_Y^2 = \frac{9}{2}\sigma_{XY}$$

$$\frac{11}{4}+\frac{1}{8}x = E(Y|X) = \mu_Y + \frac{\sigma_{XY}}{\sigma_X^2}(x-\mu_X) \Rightarrow \frac{\sigma_{XY}}{\sigma_X^2} = \frac{1}{8} \Rightarrow \sigma_X^2 = 8\sigma_{XY}$$

Therefore

$$\frac{35}{4} = Var(X|Y) = \sigma_X^2(1-\rho^2) = \sigma_X^2\left(1-\frac{\sigma_{XY}}{\sigma_X^2\sigma_Y^2}\right) = \sigma_X^2\left(1-\frac{\sigma_{XY}^2}{\frac{9}{2}\sigma_{XY}8\sigma_{XY}}\right) = \frac{35}{36}\sigma_X^2$$

Therefore $Var(X) = \sigma_X^2 = 9$.

b) By uniqueness of PGFs, V must be Poisson with parameter 3. Thus

$$P(V=7) = P(Pois(3)=7) = \frac{e^{-3}3^7}{7!}.$$

c) Let X be the time until the channel is changed. We have $Var(X) = 9 = \frac{1}{\lambda^2}$ so the parameter of X is $\lambda = \frac{1}{3}$. Now

$$P(X \le 2) = F_X(2) = 1 - e^{-(1/3)2} = 1 - e^{-2/3}.$$

6. a) Let X_j be the life of the j^{th} battery; let S_{30} be the sum of 30 i.i.d. copies of X_1; this is $n(30 \cdot 3, 30 \cdot (2)^2) = n(90, 120)$. We want to know $P(87 \le S_{30} \le 95)$; this is

$$\begin{aligned} P(87 \le n(90,120) \le 95) &= P(87 \le 90 + \sqrt{120}Z \le 95) \\ &= P\left(\frac{-3}{\sqrt{120}} \le Z \le \frac{5}{\sqrt{120}}\right) \\ &= \Phi\left(\frac{5}{\sqrt{120}}\right) - \Phi\left(\frac{-3}{\sqrt{120}}\right). \end{aligned}$$

b) Let Ω be the triangle from where we are picking points; notice that the top of this triangle is the line $y = 6 - x$. Let E be the set of points in the triangle satisfying $y < x^2$; the curve $y = x^2$ intersects $y = 6 - x$ at the point $(2,4)$ so the complement of E is a curvilinear triangle with vertices $(0,0)$, $(2,4)$ and $(0,6)$. Then

$$\begin{aligned} P(E) = 1 - \frac{area(E^C)}{area(\Omega)} &= 1 - \frac{\int_0^2 (6 - x - x^2)\, dx}{\frac{1}{2}(6)(6)} \\ &= 1 - \frac{1}{18}\int_0^2 \left(6 - x - x^2\right) dx \\ &= 1 - \frac{1}{18}\left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_0^2 \\ &= 1 - \frac{1}{18}\left(12 - 2 - \frac{8}{3}\right) \\ &= 1 - \frac{1}{18} \cdot \frac{22}{3} = 1 - \frac{11}{27} = \frac{16}{27}. \end{aligned}$$

7. First, we need to find the joint density. We are given $f_X(x) = e^{-x}$ and $f_{Y|X}(y|x) = 2xe^{-2xy}$. Therefore

$$f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|x) = \begin{cases} 2xe^{-x}e^{-2xy} & \text{if } 0 \le x, 0 \le y \\ 0 & \text{else} \end{cases}$$

a) Let $\varphi(x,y) = (u,v) = (x+y, x-y)$. First,

$$\begin{cases} u = x+y \\ v = x-y \end{cases} \Rightarrow u+v = 2x \Rightarrow x = \frac{u+v}{2} \Rightarrow y = u - x = \frac{u-v}{2}.$$

We have $J(\varphi) = \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2$ so by the transformation theorem,

$$\begin{aligned} f_{U,V}(u,v) = \frac{1}{|J(\varphi)|} f_{X,Y}(x,y) &= \frac{1}{2} \cdot 2xe^{-x}e^{-2xy} \\ &= \left(\frac{u+v}{2}\right) e^{-(u+v)/2} e^{-2(u+v)/2\cdot(u-v)/2} \\ &= \frac{1}{2}(u+v)e^{-\frac{1}{2}(u+v)}e^{-\frac{1}{2}(u^2-v^2)}. \end{aligned}$$

This holds on the range of U and V, which is $u \geq |v|$ (the joint density of U and V is zero otherwise).

b) First, we need the density of the Y marginal (use the Gamma integral formula to evaluate the integral):

$$\begin{aligned} f_Y(y) = \int_0^\infty f_{X,Y}(x,y)\,dx = \int_0^\infty 2xe^{-x}e^{-2xy}\,dx &= 2\int_0^\infty x^{2-1}e^{(-(1+2y)x}\,dx \\ &= 2\frac{\Gamma(2)}{(1+2y)^2} = \frac{2}{(1+2y)^2} \end{aligned}$$

Now the conditional expectation of X given Y is

$$\begin{aligned} E(X|Y) = \int_0^\infty x f_{X|Y}(x|y)\,dx &= \int_0^\infty x\frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx \\ &= \int_0^\infty x\frac{2xe^{-1x}e^{-2xy}}{\frac{2}{(1+2y)^2}}\,dx \\ &= (1+2y)^2\int_0^\infty x^{3-1}e^{-(1+2y)x}\,dx \\ &= (1+2y)^2\frac{\Gamma(3)}{(1+2y)^3} \\ &= \frac{2}{1+2y}. \end{aligned}$$

(This holds when $y \geq 0$.)

Chapter 4

Exams from 2015 to 2019

4.1 Fall 2015 Exam 1

1. A point (X, Y) is chosen uniformly from the triangle whose vertices are $(0, 0)$, $(2, 2)$, and $(6, 0)$.
   a) (1.2) Find $P(X \geq 4)$.
   b) (1.2) Find $P(X \leq 1 \,|\, Y \leq 1)$.
   c) (1.2) Find $P(4Y > X)$.
2. Suppose X is a random variable taking values in $\{1, 2, 3\}$ such that $f_X(x) = \frac{c}{x}$ for some constant c.
   a) (2.2) Find c.
   b) (2.2) Find $P(X \neq 2)$.
3. a) (2.3) A survey of customers at the Mongolian Grill at the Rock finds the following:
      - 20 people like pork;
      - 30 people like chicken;
      - 25 people like shrimp;
      - 18 people like pork and chicken;
      - 17 people like pork and shrimp;
      - 20 people like chicken and shrimp;
      - 15 people like pork, chicken and shrimp.

      Assuming that all those surveyed like at least one of pork, chicken and/or shrimp, how many people were surveyed?

b) (1.4) Let F and G be events in a probability space with $P(F) = .75$ and $P(G) = .65$. If F^C and G^C are disjoint, find $P(G \mid F)$.

c) (1.4) Let E_1, E_2, E_3, E_4, E_5 and E_6 be six mutually independent events in a probability space. If $P(E_j) = \frac{1}{2}$ for all j, find

$$P\left(\bigcap_{j=1}^{6} E_j\right) \quad \text{and} \quad P\left(\bigcup_{j=1}^{6} E_j\right).$$

4. A bag contains 5 dice. Three of the dice are normal, fair dice, but two of them are "loaded": the loaded dice are rigged so that they throw the number six twice as often as any other individual number.

a) (1.5) If you pick a die uniformly from the bag and roll it, what is the probability you will roll a 6?

b) (1.5) If you rolled a die, chosen uniformly from this bag, and got a 6, what is the probability the die is loaded?

5. A cookie jar contains the following 24 cookies:

- 6 sugar cookies;
- 3 chocolate chip cookies;
- 4 oatmeal raisin cookies;
- 8 pecan cookies;
- 3 peanut butter cookies.

a) (2.3) If you draw 5 cookies from the jar without replacement, what is the probability that you get 3 pecan cookies and 2 sugar cookies?

b) (2.3) If you draw 5 cookies from the jar without replacement, what is the probability that all 5 cookies you draw are the same type?

c) (2.4) If you draw 5 cookies from the jar with replacement, what is the probability that you get 3 pecan cookies?

d) (2.4) Suppose you draw cookies from the jar with replacement. What is the probability that the first time you get an oatmeal raisin cookie is on the eighth draw?

e) (2.4) Suppose you draw cookies from the jar with replacement. What is the probability that the fourth time you get a pecan cookie is on the tenth draw?

Solutions

1. Let Ω be the indicated triangle; the area of Ω is $\frac{1}{2}(base)(height) = \frac{1}{2}(6)(2) = 6$ so the probability of any region E is $P(E) = \frac{area(E)}{area(\Omega)} = \frac{1}{6}area(E)$.

   A point (X, Y) is chosen uniformly from the triangle whose vertices are $(0, 0)$, $(2, 2)$, and $(6, 0)$.

   a) The region where $X \geq 4$ is a triangle with vertices $(4, 0)$, $(4, 1)$ and $(6, 0)$. This triangle has area $\frac{1}{2}(2)(1) = 1$ so its probability is $\frac{1}{6} \cdot 1 = \frac{1}{6}$.

   b) By definition of conditional probability,

   $$P(X \leq 1 \,|\, Y \leq 1) = \frac{P(X \leq 1 \cap Y \leq 1)}{P(Y \leq 1)}.$$

   For the numerator, the region of points where $X \leq 1$ and $Y \leq 1$ is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$. This triangle has area $\frac{1}{2}(1)(1) = \frac{1}{2}$ so its probability is $\frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$.
   For the denominator, the region of points where $Y \leq 1$ is a trapezoid with vertices $(0, 0)$, $(6, 0)$, $(4, 1)$ and $(1, 1)$. This trapezoid has area $\frac{1}{2}(6 + 3)(1) = \frac{9}{2}$ so its probability is $\frac{1}{6} \cdot \frac{9}{2} = \frac{3}{4}$.
   Finally, $P(X \leq 1 \,|\, Y \leq 1) = \frac{P(X \leq 1 \cap Y \leq 1)}{P(Y \leq 1)} = \frac{1/12}{3/4} = \frac{1}{9}$.

   c) It is easier to find the probability of the complement: the region of points where $4Y \leq X$ is a triangle with vertices $(0, 0)$, $(6, 0)$ and $(4, 1)$; this triangle has area $\frac{1}{2}(6)(1) = 3$ so its probability is $\frac{1}{6}(3) = \frac{1}{2}$. Therefore by the complement rule, $P(4Y > X) = 1 - P(4Y \leq X) = 1 - \frac{1}{2} = \frac{1}{2}$.

2. a) We know $1 = P(1) + P(2) + P(3) = \frac{c}{1} + \frac{c}{2} + \frac{c}{3} = \frac{11}{6}c$, so $c = \frac{6}{11}$.

   b) $P(X \neq 2) = 1 - P(X = 2) = 1 - f_X(2) = 1 - \frac{6/11}{2} = 1 - \frac{3}{11} = \frac{8}{11}$.

3. a) Let P, C and S be the customers who like pork, chicken and shrimp, respectively. By the three-way Inclusion-Exclusion Law,

   $$\begin{aligned} \#(P \cup C \cup S) &= \#(P) + \#(C) + \#(S) - \#(P \cap C) - \#(P \cap S) - \#(C \cap S) \\ &\qquad + \#(P \cap C \cap S) \\ &= 20 + 30 + 25 - 18 - 17 - 20 + 15 \\ &= 35. \end{aligned}$$

   b) We know $F^C \cap G^C = \emptyset$. This means nothing belongs to neither F nor G, i.e. everything belongs to F or G, i.e. $F \cup G = \Omega$. Therefore $P(F \cup G) =$

$P(\Omega) = 1$. Next, by Inclusion-Exclusion,

$$\begin{aligned} P(F \cup G) &= P(F) + P(G) - P(F \cap G) \\ 1 &= .75 + .65 - P(F \cap G) \\ 1 &= 1.4 - P(F \cap G) \\ .4 &= P(F \cap G) \end{aligned}$$

Last, $P(G \mid F) = \frac{P(G \cap F)}{P(F)} = \frac{.4}{.75} = \frac{40}{75} = \frac{8}{15}$.

c) i. By independence,

$$P\left(\bigcap_{j=1}^{6} E_j\right) = \prod_{j=1}^{6} P(E_j) = \prod_{j=1}^{6} \frac{1}{2} = \left(\frac{1}{2}\right)^6 = \frac{1}{64}.$$

ii. Since the E_j are mutually independent, the E_j^C are also mutually independent. Therefore

$$P\left(\bigcap_{j=1}^{6} E_j^C\right) = \prod_{j=1}^{6} P(E_j^C) = \prod_{j=1}^{6} \left(1 - \frac{1}{2}\right) = \left(\frac{1}{2}\right)^6 = \frac{1}{64}.$$

Now, by using DeMorgan's Law,

$$P\left(\bigcup_{j=1}^{6} E_j\right) = 1 - P\left(\bigcap_{j=1}^{6} E_j^C\right) = 1 - \frac{1}{64} = \frac{63}{64}.$$

4. Let L be choosing a loaded die; we have $P(L) = \frac{2}{5}$ and $P(L^C) = \frac{3}{5}$. Now let S be rolling a six; we have $P(S \mid L^C) = \frac{1}{6}$. We need to figure out $P(S \mid L)$: we know that for a loaded die $1 = P(1) + ... + P(6) = x + x + x + x + x + 2x = 7x$ so $x = \frac{1}{7}$ and the probability of rolling a six with a loaded die is therefore $P(S \mid L) = \frac{2}{7}$.

a) By the Law of Total Probability,

$$\begin{aligned} P(S) &= P(S \mid L)P(L) + P(S \mid L^C)P(L^C) \\ &= \frac{2}{7} \cdot \frac{2}{5} + \frac{1}{6} \cdot \frac{3}{5} \quad \text{(this is fine as an answer)} \\ &= \frac{4}{35} + \frac{1}{10} \\ &= \frac{3}{14}. \end{aligned}$$

b) By Bayes' Law,

$$\begin{aligned} P(L \mid S) &= \frac{P(S \mid L)P(L)}{P(S \mid L)P(L) + P(S \mid L^C)P(L^C)} \\ &= \frac{\frac{2}{7} \cdot \frac{2}{5}}{\frac{2}{7} \cdot \frac{2}{5} + \frac{1}{6} \cdot \frac{3}{5}} \quad \text{(this is fine as an answer)} \\ &= \frac{56}{105}. \end{aligned}$$

If you rolled a die, chosen uniformly from this bag, and got a 6, what is the probability the die is loaded?

5. a) Sampling without replacement is hypergeometric:

$$P(3\text{ pecan}, 2\text{ sugar}) = \frac{\binom{8}{3}\binom{6}{2}}{\binom{24}{5}}.$$

b) There are two ways to do this: by drawing 5 sugar cookies or by drawing 5 pecan cookies. Therefore the probability is

$$P(5\text{ sugar}) + P(5\text{ pecan}) = \frac{\binom{6}{5}}{\binom{24}{5}} + \frac{\binom{8}{5}}{\binom{24}{5}}.$$

c) Since we are sampling with replacement, each draw constitutes a trial in a Bernoulli experiment (where success means drawing a pecan cookie) with $p = \frac{8}{24} = \frac{1}{3}$. Therefore the probability of three successes is

$$b(5, \frac{1}{3}, 3) = \binom{5}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^2.$$

d) Defining success as drawing an oatmeal raisin cookie, we have $p = \frac{4}{24} = \frac{1}{6}$. The probability of the first success being on the eighth draw is the probability that a $Geom(p)$ r.v. takes the value 7 (since you need seven failures before the first success), i.e. is

$$p(1-p)^x = \frac{1}{6}\left(\frac{5}{6}\right)^7.$$

e) Now define success to be drawing a pecan cookie; we have $p = \frac{8}{24} = \frac{1}{3}$. For the fourth success to be on the tenth draw, we need a negative binomial r.v. with parameters $r = 4$ and $p = \frac{1}{3}$ to have value 6 (since there would be six failures before the fourth success). This probability is

$$\begin{pmatrix} x+r-1 \\ r-1 \end{pmatrix} p^r(1-p)^x = \begin{pmatrix} 9 \\ 3 \end{pmatrix} \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^6 .$$

4.2 Fall 2015 Exam 2

1. Suppose X is a real-valued r.v. whose cumulative distribution function is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{6}x + \frac{1}{12} & \text{if } 0 \leq x < 4 \\ \frac{3}{4} & \text{if } 4 \leq x < 6 \\ 1 & \text{if } x \geq 6 \end{cases}$$

   a) (3.2) Find $P(X = 0)$, find $P(X = 4)$ and find $P(X = 6)$.
   b) (3.2) Find $P(X \geq 1)$.
   c) (3.2) Find $P(0 < X < 5)$.
   d) (3.2) Find $P(0 \leq X \leq 5)$.

2. Suppose X is a continuous, real-valued r.v. with density function

$$f_X(x) = \begin{cases} \frac{x}{8} & \text{if } 0 \leq x \leq b \\ 0 & \text{else} \end{cases}$$

   where b is a positive constant.

   a) (3.1) Find b.
   b) (3.1) Find $P(X = 1)$.
   c) (3.1) Find $P(X > 2)$.
   d) (3.1) Find $P(X > 1 \,|\, X \leq 2)$.

3. a) (3.6) Suppose that the grades of students on a standardized test are modeled by a normal r.v. with mean $\mu = 500$ and variance $\sigma^2 = 10000$. Find the probability that a randomly chosen student will have a score between 520 and 670. Leave your answer in terms of Φ, the cumulative distribution function of the standard normal r.v.
   b) (3.4) Suppose Y is an exponential r.v. with the property that $P(Y \leq 3) = \frac{1}{7}$. Find $P(Y > 5)$.
   c) Suppose that the number of calls to a 911 operator in a fixed time period is given by a Poisson random variable with parameter 12.
      i. (3.4) Find the probability that there are 7 calls to the 911 operator in this fixed time period.
      ii. (3.4) Find the probability that there are at least 2 calls to the 911 operator in this fixed time period.

4. a) (3.3) Suppose X is a random variable whose cumulative distribution function is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{x+1} & \text{if } x \geq 0 \end{cases}$$

   Let $Y = \sqrt[3]{X}$. Find the density function of Y.

b) (3.3) Suppose a point (X, Y) is chosen uniformly from a rectangle with vertices $(0, 0)$, $(2, 0)$, $(2, 1)$ and $(0, 1)$. Let $Z = \frac{Y}{X}$; find the density function of Z.

Solutions

1. a) $P(X = 0) = F_X(0) - \lim_{x \to 0^-} F_X(x) = \frac{1}{12} - 0 = \frac{1}{12}$.
   $P(X = 4) = F_X(4) - \lim_{x \to 4^-} F_X(x) = \frac{3}{4} - \left(\frac{1}{6} \cdot 4 + \frac{1}{12}\right) = \frac{3}{4} - \frac{3}{4} = 0$.
   $P(X = 6) = F_X(6) - \lim_{x \to 6^-} F_X(x) = 1 - \frac{3}{4} = \frac{1}{4}$.

   b) Since F_X is cts at 1, we have $P(X \geq 1) = P(X > 1) = 1 - P(X \leq 1) = 1 - F_X(1) = 1 - \left(\frac{1}{6} \cdot 1 + \frac{1}{12}\right) = \frac{3}{4}$.

   c) $P(0 < X < 5) = P(X < 5) - P(X \leq 0) = F_X(5) - F_X(0)$ since F_X is cts at 5. This works out to be $\frac{3}{4} - \frac{1}{12} = \frac{2}{3}$.

   d) $P(0 \leq X \leq 5) = P(X \leq 5) - P(X < 0) = F_X(5) - \lim_{x \to 0^-} F_X(x) = \frac{3}{4} - 0 = \frac{3}{4}$.

2. a) $1 = \int_0^b \frac{x}{8} = \left.\frac{x^2}{16}\right|_0^b = \frac{b^2}{16}$ so $b^2 = 16$, i.e. $b = 4$.

   b) $P(X = 1) = 0$ since X is cts.

   c) $P(X > 2) = \int_2^4 f_X(x)\,dx = \int_2^4 \frac{x}{8}\,dx = \left.\frac{x^2}{16}\right|_2^4 = 1 - \frac{1}{4} = \frac{3}{4}$.

   d) Apply the definition of conditional probability:

$$
\begin{aligned}
P(X > 1 \mid X \leq 2) = \frac{P(1 < X \leq 2)}{P(X \leq 2)} &= \frac{\int_1^2 f_X(x)\,dx}{\int_0^2 f_X(x)\,dx} \\
&= \frac{\int_1^2 \frac{x}{8}\,dx}{\int_0^2 \frac{x}{8}\,dx} = \frac{\left.\frac{x^2}{16}\right|_1^2}{\left.\frac{x^2}{16}\right|_0^2} = \frac{\frac{1}{4} - \frac{1}{16}}{\frac{1}{4} - 0} = \frac{3}{4}.
\end{aligned}
$$

3. a) Let X be the student's score. Since X is $n(500, 10000)$, we have $X = 500 + \sqrt{10000}Z = 500 + 100Z$ where Z is standard normal. Therefore

$$
\begin{aligned}
P(520 < X < 670) &= P(520 < 500 + 100Z < 670) \\
&= P(20 < 100Z < 170) \\
&= P(.2 \leq Z \leq 1.7) = \Phi(1.7) - \Phi(.2).
\end{aligned}
$$

   b) By the complement rule and since Y is cts, $P(Y \geq 3) = 1 - \frac{1}{7} = \frac{6}{7}$. By the Hazard Law for exponential r.v.s, this must be equal to $e^{-\lambda(3)}$ so we have $e^{-3\lambda} = \frac{6}{7}$, i.e. $\lambda = \frac{-1}{3} \ln \frac{6}{7}$. Applying the Hazard Law again, we have $P(Y > 5) = P(Y \geq 5) = e^{-5\lambda} = e^{-5\left(\frac{-1}{3} \ln \frac{6}{7}\right)} = e^{\frac{5}{3} \ln \frac{6}{7}} = \left(\frac{6}{7}\right)^{5/3}$.

c) Let the number of calls be N. $N \sim Pois(12)$ so $f_N(n) = P(N = n) = \frac{e^{-12}12^n}{n!}$.

i. This is asking for $P(N = 7) = f_N(7) = \frac{e^{-12}12^7}{7!}$.

ii. By the complement rule, $P(N \geq 2) = 1 - P(N = 0) - P(N = 1) = 1 - f_N(0) - f_N(1) = 1 - \frac{e^{-12}12^0}{0!} - \frac{e^{-12}12^1}{1!} = 1 - 13e^{-12}$.

4. a) Since the range of X is $[0, \infty)$, that is also the range of $Y = \sqrt[3]{x}$. That means $F_Y(y) = 0$ for $y < 0$. Now take $y \geq 0$.

$$F_Y(y) = P(Y \leq y) = P(\sqrt[3]{X} \leq y) = P(X \leq y^3) = F_X(y^3) = \frac{y^3}{y^3 + 1}.$$

Differentiating this to get f_Y, we see

$$f_Y(y) = \begin{cases} \frac{3y^2(y^3+1) - 3y^2(y^3)}{(y^3+1)^2} = \frac{3y^2}{(y^3+1)^2} & \text{if } y > 0 \\ 0 & \text{else} \end{cases}$$

b) Z is continuous with range $[0, \infty)$, so $F_Z(z) = 0$ when $z < 0$. Now let $z \geq 0$; for such a z,

$$F_Z(z) = P(Z \leq z) = P(\frac{Y}{X} \leq z) = P(Y \leq zX).$$

- When $0 < z < \frac{1}{2}$, the region of points E where $Y \leq zX$ is a triangle with vertices $(0, 0)$, $(2, 0)$ and $(2, 2z)$, so it has area $\frac{1}{2}(2)(2z) = 2z$. Its probability is therefore $\frac{area(E)}{area(\Omega)} = \frac{2z}{2} = z$.
- When $z \geq \frac{1}{2}$, the complement E^C of the region of points E where $Y \leq zX$ is a triangle with vertices $(0, 0)$, $(0, 1)$ and $(\frac{1}{z}, 1)$. Therefore its area is $\frac{1}{2}(1)(\frac{1}{z}) = \frac{1}{2z}$, and its probability is therefore $\frac{area(E^C)}{area(\Omega)} = \frac{1/2z}{2} = \frac{1}{4z}$. The probability of E is therefore $1 - \frac{1}{4z}$.

To summarize,

$$F_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ z & \text{if } 0 \leq z < \frac{1}{2} \\ 1 - \frac{1}{4z} & \text{if } z \geq \frac{1}{2} \end{cases}$$

Differentiate the distribution function to get the density function of Z:

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ 1 & \text{if } 0 \leq z < \frac{1}{2} \\ \frac{1}{4}z^{-2} & \text{if } z \geq \frac{1}{2} \end{cases}$$

4.3 Fall 2015 Exam 3

1. Suppose X and Y are discrete, integer-valued random variables whose joint density function is

$$f_{X,Y}(x,y) = \begin{cases} c\left(\frac{2}{3}\right)^x \left(\frac{1}{2}\right)^y & \text{if } 0 \leq x \leq y \\ 0 & \text{else} \end{cases}$$

   a) (4.1) Find c.

   b) (4.1) Find $P(Y = 2X)$.

   c) (4.1) Find $P(X \leq 50)$.

2. Suppose X and Y are independent exponential random variables, where X has parameter $\frac{1}{2}$ and Y has parameter 2.

   a) (5.2) Write an expression involving one or more integrals and/or double integrals that would give the probability that $X - Y \leq 6$.

   b) (5.4) Let $W = X + Y$. Find the density function of W.

   c) (5.4) Find the joint density function of $U = X + Y$ and $V = \frac{Y}{X}$.

3. Suppose the amount of time (in hours) X it takes for a programmer to make a coding error is modeled by a continuous random variable whose density function is

$$f_X(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x \leq 2 \\ 0 & \text{else} \end{cases}$$

   Suppose further that if the coding error occurs at time x, then the amount of time (in hours) Y it takes to fix the error is uniform on the interval $[0, x^3]$.

   a) (5.3) Find the density of Y.

   b) (5.3) Find the conditional probability that the error was made in the first hour, given that it takes $\frac{1}{8}$ hour to fix the error.

Solutions

1. a) The joint density must sum to 1:

$$\begin{aligned}1 &= \sum_{x=0}^{\infty}\sum_{y=x}^{\infty} c\left(\frac{2}{3}\right)^x\left(\frac{1}{2}\right)^y \\ &= c\sum_{x=0}^{\infty}\left(\frac{2}{3}\right)^x \sum_{y=x}^{\infty}\left(\frac{1}{2}\right)^y \\ &= c\sum_{x=0}^{\infty}\left(\frac{2}{3}\right)^x\left(\frac{1}{2}\right)^x\left(\frac{1}{1-\frac{1}{2}}\right) \\ &= 2c\sum_{x=0}^{\infty}\left(\frac{1}{3}\right)^x \\ &= 2c\cdot\frac{1}{1-\frac{1}{3}} = 2c\cdot\frac{3}{2} = 3c. \text{ Therefore } c = \frac{1}{3}.\end{aligned}$$

b) Add up values of the density function:

$$\begin{aligned}P(Y=2X) = \sum_{x=0}^{\infty} f_{X,Y}(x,2x) &= \sum_{x=0}^{\infty}\frac{1}{3}\left(\frac{2}{3}\right)^x\left(\frac{1}{2}\right)^{2x} \\ &= \sum_{x=0}^{\infty}\frac{1}{3}\left(\frac{1}{6}\right)^x = \frac{1}{3}\cdot\frac{1}{1-\frac{1}{6}} = \frac{2}{5}.\end{aligned}$$

c) Again, add up values of the density function:

$$\begin{aligned}P(X\le 50) = \sum_{x=0}^{50}\sum_{y=x}^{\infty} f_{X,Y}(x,y) &= \sum_{x=0}^{50}\sum_{y=x}^{\infty}\frac{1}{3}\left(\frac{2}{3}\right)^x\left(\frac{1}{2}\right)^y \\ &= \frac{1}{3}\sum_{x=0}^{50}\left(\frac{2}{3}\right)^x\sum_{y=x}^{\infty}\left(\frac{1}{2}\right)^y \\ &= \frac{1}{3}\sum_{x=0}^{50}\left(\frac{2}{3}\right)^x\cdot\left(\frac{1}{2}\right)^x\left(\frac{1}{1-\frac{1}{2}}\right) \\ &= \frac{1}{3}\cdot 2\sum_{x=0}^{50}\left(\frac{1}{3}\right)^x \\ &= \frac{2}{3}\cdot\frac{1-(1/3)^{51}}{1-1/3} \\ &= 1-(1/3)^{51}.\end{aligned}$$

2. First, using the given information, the joint density is

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{2}e^{-(1/2)x}\cdot 2e^{-2y} = e^{-x/2}e^{-2y}$$

when $x \ge 0$ and $y \ge 0$.

a) The region of points (X, Y) satisfying $X - Y \leq 6$ is the set of points above the line $Y = X - 6$; to find the probability of this region we would need a double integral:

$$\int_0^\infty \int_0^{y+6} e^{-x/2} e^{-2y}\, dx\, dy$$

b) First, the range of W is $[0, \infty)$, so when $w < 0$, $F_W(w) = 0$. Now let $w \geq 0$:

$$\begin{aligned}
F_W(w) = P(W \leq w) &= P(X + Y \leq w) \\
&= P(Y \leq -X + w) \\
&= \int_0^w \int_0^{w-x} f_{X,Y}(x, y)\, dy\, dx \\
&= \int_0^w \int_0^{w-x} e^{-x/2} e^{-2y}\, dy\, dx \\
&= \int_0^w e^{-x/2} \left[\frac{-1}{2} e^{-2y} \right]_0^{w-x} dx \\
&= \int_0^w e^{-x/2} \left[\frac{1}{2} - \frac{1}{2} e^{-2(w-x)} \right] dx \\
&= \int_0^w \left(\frac{1}{2} e^{-x/2} - \frac{1}{2} e^{-2w} e^{3x/2} \right) dx \\
&= \left[-e^{-x/2} - \frac{1}{3} e^{-2w} e^{3x/2} \right]_0^w \\
&= \left[-e^{-w/2} - \frac{1}{3} e^{-2w} e^{3w/2} \right] - \left[-1 - \frac{1}{3} e^{-2w} \right] \\
&= 1 + \frac{1}{3} e^{-2w} - \frac{4}{3} e^{-w/2}.
\end{aligned}$$

Last, differentiate F_W to get f_W:

$$f_W(w) = \frac{d}{dw} F_W(w) = \begin{cases} \frac{2}{3} e^{-w/2} - \frac{2}{3} e^{-2w} & \text{if } w \geq 0 \\ 0 & \text{else.} \end{cases}$$

c) Define $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\varphi(x, y) = (u, v) = (x + y, \frac{y}{x})$. First, compute the Jacobian of φ:

$$J(\varphi) = \det \begin{pmatrix} 1 & 1 \\ \frac{-y}{x^2} & \frac{1}{x} \end{pmatrix} = \frac{1}{x} + \frac{y}{x^2} = \frac{x + y}{x^2} = \frac{u}{x^2}.$$

Next, back-solve for x and y in terms of u and v: if $v = \frac{y}{x}$, then $y = vx$. Substituting into the equation for u, we get $u = x + y = x + vx = (v + 1)x$ so $x = \frac{u}{v+1}$. Then $y = vx = \frac{uv}{v+1}$. Now by the transformation theorem, the joint density of U and V is

$$
\begin{aligned}
f_{U,V}(u,v) &= \frac{1}{|J(\varphi)|} f_{X,Y}(x,y) \\
&= \frac{x^2}{u} e^{-x/2} e^{-2y} \\
&= \frac{\left(\frac{u}{v+1}\right)^2}{u} \exp\left[-\frac{u}{2(v+1)} - \frac{2uv}{v+1}\right] \\
&= \frac{u}{(v+1)^2} \exp\left[-\frac{u+4uv}{2(v+1)}\right].
\end{aligned}
$$

(This holds when $u \geq 0$, $v \geq 0$; the joint density is zero otherwise.)

3. First, by the multiplication principle the joint density function is

$$f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|x) = \frac{x}{2} \cdot \frac{1}{x^3} = \frac{1}{2x^2}.$$

This holds when $0 \leq x \leq 2$ and $0 \leq y \leq x^3$; the joint density is zero otherwise.

a) Integrate the joint density with respect to x:

$$f_Y(y) = \int_{\sqrt[3]{y}}^{2} \frac{1}{2x^2}\, dx = \left.\frac{-1}{2x}\right|_{\sqrt[3]{y}}^{2} = \frac{1}{2\sqrt[3]{y}} - \frac{1}{4}.$$

(This holds when $0 \leq y \leq 8$; $f_Y(y) = 0$ otherwise.)

b) We are asked to find $P(X \leq 1 \,|\, Y = \frac{1}{8})$. This is computed using conditional densities. First,

$$f_{X|Y}\left(x|\frac{1}{8}\right) = \frac{f_{X,Y}(x,\frac{1}{8})}{f_Y(\frac{1}{8})} = \frac{\frac{1}{2x^2}}{\frac{1}{2\sqrt[3]{1/8}} - \frac{1}{4}} = \frac{\frac{1}{2x^2}}{\frac{3}{4}} = \frac{2}{3x^2}.$$

Now, integrate this conditional density to get the answer:

$$
\begin{aligned}
P(X \leq 1 \,|\, Y = \frac{1}{8}) &= \int_{1/2}^{1} f_{X|Y}(x|\frac{1}{8})\, dx \\
&= \int_{1/2}^{1} \frac{2}{3x^2}\, dx \\
&= -\frac{2}{3} x^{-1}\Big|_{1/2}^{1} = -\frac{2}{3} + \frac{4}{3} = \frac{2}{3}.
\end{aligned}
$$

4.4 Fall 2015 Exam 4

1. For each random variable X given below, find the expected value of X:

   a) (6.1) X has density function $f_X(x) = \frac{2}{9}x$ for $0 < x < 3$ ($f_X(x) = 0$ otherwise).

   b) (7.2) X has moment generating function $M_X(t) = \frac{1}{\sqrt{1-7t}}$.

   c) (6.1) $X = 8U - 5V$ where $EU = 2$, $EV = 3$ and $Cov(U, V) = -2$.

   d) (6.2) X is a Poisson random variable whose standard deviation is 4.

   e) **(Bonus)** (6.1) X has distribution function

   $$F_X(x) = \begin{cases} \frac{x^2}{x^2+1} & x \geq 0 \\ 0 & \text{else} \end{cases}.$$

2. Suppose X and Y are continuous random variables whose joint density function is

   $$f_{X,Y}(x, y) = \begin{cases} 6x^2y & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$$

   a) (6.2) Find the variance of XY.

   b) (6.3) Find the conditional expectation of X given Y.

3. a) (7.2) Suppose that the number of service interruptions a business encounters in a day is a geometric random variable with mean 3. Suppose also that the number of service interruptions in one day is independent of the number of service interruptions in any other day. Find the probability that during a 20 day period, the business encounters exactly 7 service interruptions.

   b) (6.3) Suppose V and W are random variables such that $\rho(V, W) = \frac{-2}{3}$, $Var(V) = 8$ and $Var(W) = 2$. Find $Var(3V + 5W)$.

   c) (7.1) Suppose X is a random variable whose probability generating function is

   $$G_X(t) = .2 + .3t + .2t^2 + .1t^3 + c\,t^4$$

   where c is a constant. Find c and find $P(X = 1)$.

4. Suppose that the temperature of two objects X and Y are given, respectively, by random variables X and Y which have this bivariate normal density:

   $$f_{X,Y}(x, y) = \frac{1}{\pi\sqrt{2}} \exp\left[\frac{-1}{2} \begin{pmatrix} x-2 & y-3 \end{pmatrix} \begin{pmatrix} 2 & -4 \\ -4 & 9 \end{pmatrix} \begin{pmatrix} x-2 \\ y-3 \end{pmatrix}\right]$$

   a) (7.3) Find EX and $Var(X)$.

b) (7.3) Find $E[Y|X]$.

c) (7.3) Let A be the average temperature of objects X and Y. Find the density function of A.

Solutions

1. a) $EX = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_0^3 \frac{2}{9}x^2\, dx = \left[\frac{2}{27}x^3\right]_0^3 = 2.$

   b) $EX = M_X'(0)$. By the Chain Rule, $M_X'(t) = \frac{-1}{2}(1-7t)^{-3/2} \cdot -7 = \frac{7}{2}(1-7t)^{-3/2}$ so $EX = \frac{7}{2}(1-7\cdot 0)^{-3/2} = \frac{7}{2}$.

   c) $EX = E[8U - 5V] = 8EU - 5EV = 8(2) - 5(3) = 1.$
   Note: the fact that $Cov(U,V) = -2$ is irrelevant in this problem.

   d) The variance of X is $4^2 = 16$. Since X is Poisson, this means its parameter $\lambda = 16$ and therefore $EX = \lambda = 16$.

   e) The survival function is

   $$H_X(x) = 1 - F_X(x) = 1 - \frac{x^2}{x^2+1} = \frac{x^2+1}{x^2+1} - \frac{x^2}{x^2+1} = \frac{1}{x^2+1}.$$

   Therefore

   $$EX = \int_0^{\infty} H_X(x)\, dx = \int_0^{\infty} \frac{1}{x^2+1}\, dx = \arctan x|_0^{\infty} = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

2. a) By the variance formula, $Var(XY) = E[(XY)^2] - (E[XY])^2 = E[X^2Y^2] - (E[XY])^2$. Next, compute each of these terms by the formula for the expected value of a transformation:

   $$\begin{aligned} E[X^2Y^2] = \int\int_{\Omega} x^2y^2 f_{X,Y}(x,y)\, dA &= \int_0^1 \int_0^1 x^2y^2 6x^2y\, dy\, dx \\ &= \int_0^1 \int_0^1 6x^4y^3\, dy\, dx \\ &= \int_0^1 \left[\frac{3}{2}x^4y^4\right]_0^1 dx \\ &= \int_0^1 \frac{3}{2}x^4\, dx = \frac{3}{10}x^5\Big|_0^1 = \frac{3}{10}. \end{aligned}$$

   $$\begin{aligned} E[XY] = \int\int_{\Omega} xy f_{X,Y}(x,y)\, dA &= \int_0^1 \int_0^1 xy6x^2y\, dy\, dx \\ &= \int_0^1 \int_0^1 6x^3y^2\, dy\, dx \\ &= \int_0^1 \left[2x^3y^3\right]_0^1 dx \\ &= \int_0^1 2x^3\, dx = \frac{1}{2}x^4\Big|_0^1 = \frac{1}{2}. \end{aligned}$$

Finally, $Var(XY) = \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$.

b) First, the density of the marginal Y:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^1 6x^2y\,dx = 2x^3y|_0^1 = 2y$$

Now, the conditional density of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{6x^2y}{2y} = 3x^2$$

and the conditional expectation of X given Y is

$$E[X|Y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx = \int_0^1 x\,3x^2\,dx = \int_0^1 3x^3\,dx = \frac{3}{4}x^4\Big|_0^1 = \frac{3}{4}.$$

Remark: The fact that $E[X|Y]$ is a constant, rather than a function of y as it would be in general, shows that $X \perp Y$.

3. a) Let X be the number of daily business interruptions. Letting $X \sim Geom(p)$, we have

$$EX = \frac{1-p}{p} = 3 \;\Rightarrow\; 3p = 1-p \;\Rightarrow\; p = \frac{1}{4}.$$

Next, using the fact that the sum of 20 independent $Geom(p)$ r.v.s is $NB(20,p)$, we know that the number of business interruptions in a 20 day period is $NB(20, \frac{1}{4})$. Finally, the answer is

$$\begin{aligned} P(NB\left(20, \frac{1}{4}\right) = 7) &= \binom{20+7-1}{7}\left(\frac{1}{4}\right)^{20}\left(1-\frac{1}{4}\right)^7 \\ &= \binom{26}{7}\left(\frac{1}{4}\right)^{20}\left(\frac{3}{4}\right)^7. \end{aligned}$$

b) By the definition of correlation, we can solve for the covariance between V and W:

$$\begin{aligned} \rho(V,W) &= \frac{Cov(V,W)}{\sqrt{Var(V)\cdot Var(W)}} \\ \frac{-2}{3} &= \frac{Cov(V,W)}{\sqrt{8\cdot 2}} \\ \frac{-2}{3} &= \frac{Cov(V,W)}{4} \\ \frac{-8}{3} &= Cov(V,W). \end{aligned}$$

Now, by properties of variance we have

$$\begin{aligned} Var(3V+5W) &= Var(3V) + Var(5W) + 2Cov(3V,5W) \\ &= 3^2 Var(V) + 5^2 Var(W) + 2\cdot 3\cdot 5 Cov(V,W) \\ &= 9\cdot 8 + 25\cdot 2 + 30\cdot \frac{-8}{3} \\ &= 42. \end{aligned}$$

c) i. For any probability generating function, $G_X(1) = 1$. In this case, that means $.2 + .3 + .2 + .1 + c = 1$, i.e. $c = .2$.

ii. $P(X = x) = f_X(x)$ is always the coefficient on the t^x term in the probability generating function. Here $P(X = 1) = .3$, the coefficient on the t^1 term in $G_X(t)$.

4. a) Given the form of the density function, the mean vector is clearly $\mu = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ so $EX = 2$ (and $EY = 3$; we'll need this in parts (b) and (c)). Next, from the given form of the density function, the covariance matrix Σ has inverse

$$\Sigma^{-1} = \begin{pmatrix} 2 & -4 \\ -4 & 9 \end{pmatrix}$$

so

$$\Sigma = \begin{pmatrix} 2 & -4 \\ -4 & 9 \end{pmatrix}^{-1} = \frac{1}{2\cdot 9 - (-4)^2} \begin{pmatrix} 9 & 4 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{9}{2} & 2 \\ 2 & 1 \end{pmatrix}.$$

Therefore $Var(X) = \frac{9}{2}$ (also, $Cov(X,Y) = 2$ and $Var(Y) = 1$; we'll need these in parts (b) and (c)).

b) From the theorem derived in class,

$$E[Y|X] = \mu_Y + \frac{\sigma_{XY}}{\sigma_X^2}(x - \mu_X) = 3 + \frac{2}{9/2}(x-2) = 3 + \frac{4}{9}(x-2).$$

c) $A = \frac{1}{2}(X+Y) = \frac{1}{2}X + \frac{1}{2}Y$. Since (X,Y) is bivariate normal, any linear combination of X and Y (such as A) is normal; so to write the density of A we need to compute $\mu_A = EA$ and $\sigma_A^2 = Var(A)$:

$$\mu_A = EA = \frac{1}{2}EX + \frac{1}{2}EY = \frac{1}{2}(2) + \frac{1}{2}(3) = \frac{5}{2};$$

$$\begin{aligned} \sigma_A^2 = Var(A) = Var\left[\frac{1}{2}(X+Y)\right] &= \frac{1}{4}Var(X+Y) \\ &= \frac{1}{4}\left[Var(X) + Var(Y) + 2Cov(X,Y)\right] \\ &= \frac{1}{4}\left[\frac{9}{2} + 1 + 2(2)\right] = \frac{19}{8}. \end{aligned}$$

Finally, using the formula for density of normal random variables, the density of A is

$$\begin{aligned} f_A(a) = \frac{1}{\sigma_A\sqrt{2\pi}} \exp\left[\frac{-(a-\mu_A)^2}{2\sigma_A^2}\right] &= \frac{1}{\sqrt{\frac{19}{8}2\pi}} \exp\left[\frac{-(a-\frac{5}{2})^2}{\frac{19}{4}}\right] \\ &= \frac{2}{\sqrt{19\pi}} \exp\left[\frac{-4}{19}(a-\frac{5}{2})^2\right]. \end{aligned}$$

4.5 Fall 2015 Final Exam

1. a) (1.4) Let F and G be events in a probability space. Suppose $P(F) = .65$ and $P(G) = .2$. If $P(F \,|\, G) = P(F^C \,|\, G)$, what is $P(F \cup G)$?

   b) (1.3) In a group of 140 college students, 45 are enrolled in a math class, 70 are enrolled in a statistics class, and 68 are enrolled in a biology class. 28 students are taking both math and statistics, 38 are taking statistics and biology, and 20 are taking math and biology. If 8 students are taking all three courses, how many students in the group are taking neither math nor statistics nor biology?

   c) (1.5) A board game player wins 50% of the time she plays Monopoly, wins 80% of the time when she plays Clue and wins 60% of the time when she plays Risk. Suppose she is a member of a board game club which plays Monopoly 10% of the time, Clue 40% of the time and Risk 50% of the time. If the player wins a game at this club, what is the probability that the game played was Risk?

2. Bag A contains 30 coins, of which 17 are real and 13 are counterfeit. Bag B contains 20 coins, of which 15 are real and 5 are counterfeit.

   a) (2.3) If 8 coins are drawn from Bag A without replacement, what is the probability that 5 of the 8 coins drawn are real?

   b) (2.4) If 8 coins are drawn from Bag A with replacement, what is the probability that 5 of the 8 coins drawn are real?

   c) (2.4) If coins are drawn from Bag B with replacement, what is the probability that the third time a real coin is drawn is on the seventh draw?

   d) (2.3) Suppose you draw two coins at once from Bag B and, without looking at them, drop them into Bag A. Then you mix up the coins in Bag A and draw 5 coins simultaneously from Bag A. What is the probability that, of the 5 coins you draw from Bag A, you draw 3 real coins?

3. Suppose Z is an exponential random variable whose expected value is $\frac{1}{4}$.

   a) (8.1) Use Chebyshev's Inequality to find an upper bound on $P(Z > \frac{3}{4})$.

   b) (3.4) Find $P(Z > 9 \,|\, Z > 4)$.

   c) (3.4) Let $X = Z^4$. Find a density function of X.

   d) (3.4 or 7.2) Let Y be the sum of eight independent copies of Z. Find a density function of Y.

4. a) (6.2) Suppose X is a continuous random variable whose density function is

$$f_X(x) = \begin{cases} a + bx & \text{if } 0 \le x \le 2 \\ 0 & \text{else} \end{cases}$$

where a and b are constants. If $EX = \frac{13}{12}$, find the variance of X. (Your answer should be a number.)

b) (7.2) Suppose Y is a continuous random variable whose density function is

$$f_Y(y) = \begin{cases} \frac{1}{e-1}e^y & \text{if } 0 \le y \le 1 \\ 0 & \text{else} \end{cases} .$$

Compute the moment generating function of Y.

5. Suppose X and Y are independent geometric random variables, where X has parameter $\frac{1}{5}$ and Y has parameter $\frac{2}{5}$.

   a) (4.3) Find $P(X = 8, Y = 3)$.
   b) (4.3) Find $P(9 \le X \le 15)$.
   c) (4.3) Find $P(X - Y = 20)$.

6. a) (8.3) A student guesses the answer to every question on a multiple-choice test. If each question has four choices to choose from, and the test consists of 60 questions, use normal approximation (i.e. the Central Limit Theorem) to estimate the probability that the student gets at least 12 questions correct.

   b) (6.3) Suppose that X is a gamma r.v. with parameters $r = 4$ and $\lambda = 3$ (i.e. $X \sim \Gamma(4, 3)$). Suppose also that given $X = x$, $Y \sim \Gamma(5, x)$. Find the conditional expectation of X given Y.

7. Suppose that a point (X, Y) is chosen uniformly from a triangle with vertices $(0, 0)$, $(0, 2)$ and $(2, 0)$.

   a) (5.1) Are X and Y independent? Explain (either with a proof or heuristic argument).
   b) (5.1) Find the probability that $X \le 1$.
   c) (6.1) Find $E[X^2Y]$.
   d) (5.4) Let $W = X + Y$; find a density function of W.
   e) (5.4) Find the joint density of W and Y, where W is as in part (c) of this problem.

Bonus. A lake contains four types of fish. Suppose that every time a fisherman casts his line, he catches one of the four types of fish (each type has probability $\frac{1}{4}$ of being caught on any cast). Let X be the number of casts necessary to catch at least one of each type of fish (the order the types are caught in does not matter). Compute the expected value and variance of X.

Hint: If you have no idea where to start, first try doing the problem in a simpler situation where there are only two types of fish instead of four. Then try it with three types of fish (if you can do three types, you can do four).

Solutions

1. a) First, by multiplying through the equation $P(F \mid G) = P(F^C \mid G)$ by $P(G)$ on both sides, we get $P(F \cap G) = P(F^C \cap G)$. But by additivity, we have $.2 = P(G) = P(F \cap G) + P(F^C \cap G) = 2P(F \cap G)$ so $P(F \cap G) = .1$. Therefore, by Inclusion-Exclusion, $P(F \cup G) = P(F) + P(G) - P(F \cap G) = .65 + .2 - .1 = .75$.

   b) Let M, S and B be the sets of students taking math, statistics and biology, respectively. By three-way Inclusion-Exclusion, we have

$$\begin{aligned}
\#(M \cup S \cup B) &= \#(M) + \#(S) + \#(B) - \#(M \cap S) - \#(M \cap B) - \#(B \cap S) \\
&\quad + \#(M \cap S \cap B) \\
&= 45 + 70 + 68 - 28 - 38 - 20 + 8 \\
&= 183 - 86 + 8 \\
&= 105.
\end{aligned}$$

   By the complement rule, the number of students taking none of the courses is $140 - 105 = 35$.

   c) Let M, C and R be the events that the club plays Monopoly, Clue and Risk. Let W be the event that the player wins the game. By Bayes' Law,

$$\begin{aligned}
P(R \mid W) &= \frac{P(W \mid R)P(R)}{P(W \mid R)P(R) + P(W \mid M)P(M) + P(W \mid C)P(C)} \\
&= \frac{(.6)(.5)}{(.6)(.5) + (.5)(.1) + (.8)(.4)} \\
&= \frac{30}{67}.
\end{aligned}$$

2. Bag A contains 30 coins, of which 17 are real and 13 are counterfeit. Bag B contains 20 coins, of which 15 are real and 5 are counterfeit.

   a) The number of real coins is $Hyp(30, 17, 8)$; the probability that this number is 5 is $\frac{C(17,5)C(13,3)}{C(30,8)}$.

   b) Defining a success to be drawing a real coin, we have a Bernoulli experiment with $p = \frac{17}{30}$. In this setting, the number of real coins drawn is binomial$(8, p)$ so the answer is $b(8, \frac{17}{30}, 5) = \begin{pmatrix} 8 \\ 5 \end{pmatrix} \left(\frac{17}{30}\right)^5 \left(\frac{13}{30}\right)^3$.

   c) Defining a success to be drawing a real coin, we have a Bernoulli experiment with $p = \frac{15}{20} = \frac{3}{4}$. We want the probability of four failures before the third success, which is $P(NB(3, \frac{3}{4}) = 4) = \begin{pmatrix} 6 \\ 4 \end{pmatrix} \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right)^4$.

d) Let E_j be the event that of the two coins drawn from Bag B, j of them are real. We have, using hypergeometric formulas (or other reasoning),

$$P(E_j) = \frac{C(15,j)C(5,2-j)}{C(20,2)}.$$

Now, if event E_j happens, then we are drawing 5 coins from a bag with $17+j$ real coins and $13+(2-j)$ counterfeit coins. Letting F be the event that 3 of the 5 coins are drawn, we have

$$P(F \mid E_j) = \frac{C(17+j,3)C(13+2-j,2)}{C(32,5)}.$$

Finally, by the Law of Total Probability, we have

$$\begin{aligned} P(F) &= \sum_{j=0}^{2} P(F \mid E_j)P(E_j) \\ &= \frac{C(17,3)C(15,2)}{C(32,5)} \cdot \frac{C(15,0)C(5,2)}{C(20,2)} \\ &\quad + \frac{C(18,3)C(14,2)}{C(32,5)} \cdot \frac{C(15,1)C(5,1)}{C(20,2)} \\ &\quad\quad + \frac{C(19,3)C(13,2)}{C(32,5)} \cdot \frac{C(15,2)C(5,0)}{C(20,2)}. \end{aligned}$$

This simplifies to

$$\frac{C(17,3)C(15,2)C(5,2) + 75\,C(18,3)C(14,2) + C(19,3)C(13,2)C(15,2)}{C(32,5)C(20,2)}.$$

3. First, since $EZ = \frac{1}{4} = \frac{1}{\lambda}$, we have $\lambda = 4$.

a) $Var(Z) = \frac{1}{\lambda^2} = \frac{1}{16}$. Now $P(Z > \frac{3}{4}) = P(Z - \frac{1}{4} > \frac{1}{2}) \le P(|Z - EZ| > \frac{1}{2}) \le \frac{Var(Z)}{(1/2)^2} = \frac{1/16}{1/4} = \frac{1}{4}$.

b) Since Z is exponential, it is memoryless so $P(Z > 9 \mid Z > 4) = P(Z > 5) = H_Z(5) = e^{-4(5)} = e^{-20}$.

c) $X = Z^4$ is continuous with range $[0, \infty)$. Let $x \ge 0$; then $F_X(x) = P(X \le x) = P(Z^4 \le x) = P(Z \le \sqrt[4]{x}) = F_Z(\sqrt[4]{x}) = 1 - e^{-4\sqrt[4]{x}}$. Last,

$$f_X(x) = \frac{d}{dx}F_X(x) = \begin{cases} e^{-4\sqrt[4]{x}}x^{-3/4} & \text{if } x \ge 0 \\ 0 & \text{else} \end{cases}$$

d) By facts about sums of independent random variables, $Y \sim \Gamma(8,4)$. Therefore $f_Y(y) = \frac{4^8}{\Gamma(8)}y^{8-1}e^{-4y} = \frac{4^8}{7!}y^7e^{-4y}$ (when $y \ge 0$).

4. a) Since the density function must integrate to 1, we have

$$1 = \int_0^2 (a + bx)\, dx = \left[ax + \frac{1}{2}bx^2\right]_0^2 = 2a + 2b.$$

Since the mean is $\frac{4}{3}$, we have

$$\frac{13}{12} = \int_0^2 x(a + bx)\, dx = \left[\frac{1}{2}ax^2 + \frac{1}{3}bx^3\right]_0^2 = 2a + \frac{8}{3}b.$$

This gives us two equations in two variables:

$$\begin{cases} 1 = 2a + 2b \\ \frac{13}{12} = 2a + \frac{8}{3}b \end{cases}$$

Subtract the second equation from the first to get $\frac{-1}{12} = \frac{-2}{3}b$, i.e. $b = \frac{1}{8}$. Then $a = \frac{3}{8}$ so the density function is $f_X(x) = \frac{3}{8} + \frac{1}{8}x$. Now, find the second moment:

$$EX^2 = \int_0^2 x^2 f_X(x)\, dx = \int_0^2 \left(\frac{3}{8}x^2 + \frac{1}{8}x^3\right)\, dx = \left[\frac{1}{8}x^3 + \frac{1}{32}x^4\right]_0^2 = \frac{3}{2}.$$

Last, the variance is $Var(X) = EX^2 - (EX)^2 = \frac{3}{2} - \left(\frac{13}{12}\right)^2 = \frac{47}{144}$.

b) By direct calculation:

$$M_Y(t) = E[e^{tY}] = \int_0^1 e^{ty} f_Y(y)\, dy = \frac{1}{e-1}\int_0^1 e^{ty}e^y\, dy = \left[\frac{e^{(t+1)y}}{(e-1)(t+1)}\right]_0^1$$
$$= \frac{e^{t+1} - 1}{(e-1)(t+1)}.$$

5. From the given information, $f_X(x) = \frac{1}{5}\left(\frac{4}{5}\right)^x$, $f_Y(y) = \frac{2}{5}\left(\frac{3}{5}\right)^y$ and by independence, $f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{2}{25}\left(\frac{4}{5}\right)^x\left(\frac{3}{5}\right)^y$.

a) $P(X = 8, Y = 3) = f_{X,Y}(8,3) = \frac{2}{25}\left(\frac{4}{5}\right)^8\left(\frac{3}{5}\right)^3$.

b) Using the hazard law for geometric random variables, we have $P(9 \le X \le 15) = P(X \ge 9) - P(X \ge 16) = \left(\frac{4}{5}\right)^9 - \left(\frac{4}{5}\right)^{16}$.

c) The given condition is a single equation, so requires a single summation:

$$\begin{aligned} P(X-Y=20) &= \sum_{y=0}^{\infty} f_{X,Y}(y+20,y) \\ &= \sum_{y=0}^{\infty} \frac{2}{25}\left(\frac{4}{5}\right)^{y+20}\left(\frac{3}{5}\right)^{y} \\ &= \frac{2}{25}\left(\frac{4}{5}\right)^{20} \sum_{y=0}^{\infty}\left(\frac{12}{25}\right)^{y} \\ &= \frac{2}{25}\left(\frac{4}{5}\right)^{20} \frac{1}{1-12/25} \\ &= \frac{2}{13}\left(\frac{4}{5}\right)^{20}. \end{aligned}$$

6. a) Let $X_j = 1$ if the student guesses correctly and let $X_j = 0$ otherwise. We have $\mu = EX_j = \frac{1}{4}$ and $\sigma^2 = Var(X_j) = \frac{1}{4}(1-\frac{1}{4}) = \frac{3}{16}$. We want to know $P(S_{60} \geq 12)$:

$$\begin{aligned} P(S_{60} \geq 12) &\approx P(n(60\cdot\frac{1}{4}, 60\cdot\frac{3}{16}) \geq 12) \\ &= P(n(15, \frac{45}{4}) \geq 12) \\ &= P(15 + \frac{\sqrt{45}}{2}Z \geq 12) \\ &= P(Z \geq \frac{-6}{\sqrt{45}}) \\ &= 1 - \Phi\left(\frac{-2}{\sqrt{5}}\right) = \Phi\left(\frac{2}{\sqrt{5}}\right). \end{aligned}$$

b) We are given

$$f_X(x) = \frac{3^4}{\Gamma(4)}x^3e^{-3x} \quad f_{Y|X}(y|x) = \frac{x^5}{\Gamma(5)}y^4e^{-xy}.$$

Therefore the joint density is

$$f_{X,Y}(x,y) = f_X(x)f_{Y|X}(y|x) = \frac{3^4}{\Gamma(4)}x^3e^{-3x}\cdot\frac{x^5}{\Gamma(5)}y^4e^{-xy} = \frac{3^4}{\Gamma(4)\Gamma(5)}x^8y^4e^{-x(y+3)}.$$

Now the Y marginal (using the Gamma Integral Formula) is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^{\infty} \frac{3^4y^4}{\Gamma(4)\Gamma(5)}x^8e^{-x(y+3)}\,dx = \frac{3^4y^4}{\Gamma(4)\Gamma(5)}\cdot\frac{\Gamma(9)}{(y+3)^9}.$$

Next, compute the conditional density of X given Y:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{3^4}{\Gamma(4)\Gamma(5)}x^8y^4e^{-x(y+3)}}{\frac{3^4y^4}{\Gamma(4)\Gamma(5)} \cdot \frac{\Gamma(9)}{(y+3)^9}} = \frac{1}{\Gamma(9)}(y+3)^9x^8e^{-x(y+3)}.$$

Finally, the conditional expectation:

$$\begin{aligned} E(X|Y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx &= \int_0^\infty \frac{(y+3)^9}{\Gamma(9)} x \cdot x^8 e^{-x(y+3)}\,dx \\ &= \frac{(y+3)^9}{\Gamma(9)} \cdot \frac{\Gamma(10)}{(y+3)^{10}} = \frac{9}{y+3}. \end{aligned}$$

7. a) Not knowing anything about Y, X could range from 0 to 2. But if Y is close to 2, then X cannot also be close to 2 so X and Y are clearly not independent.

b) The region where $X \le 1$ is a trapezoid with corner points $(0,0)$, $(1,0)$, $(1,1)$ and $(0,2)$. This region has area $\frac{3}{2}$ so its probability is $\frac{3/2}{2} = \frac{3}{4}$.

c) First, the joint density is $f_{X,Y}(x,y) = \frac{1}{area(\Omega)} = \frac{1}{2}$ when x is in the triangle. Now,

$$\begin{aligned} E[X^2Y] = \int\int_\Omega x^2 y f_{X,Y}(x,y)\,dA &= \int_0^2 \int_0^{2-x} \frac{1}{2}x^2y\,dy\,dx \\ &= \int_0^2 \left[\frac{1}{4}x^2y^2\right]_0^{2-x} dx \\ &= \int_0^2 \frac{x^2}{4}(2-x)^2\,dx \\ &= \frac{1}{4}\int_0^2 (4x^2 - 4x^3 + x^4)\,dx \\ &= \frac{1}{4}\left[\frac{4}{3}x^3 - x^4 + \frac{1}{5}x^5\right]_0^2 \\ &= \frac{1}{4}\left(\frac{32}{3} - 16 + \frac{32}{5}\right) = \frac{4}{15}. \end{aligned}$$

d) W is continuous and ranges from 0 to 2. Now when $0 \le w \le 2$, we have

$$F_W(w) = P(W \le w) = P(X+Y \le w) = P(E)$$

where E is a triangle with vertices $(0,0)$, $(0,w)$ and $(w,0)$. This triangle has area $\frac{1}{2}w^2$ so its probability is $\frac{area(E)}{area(\Omega)} = \frac{1}{4}w^2$. Thus

$$F_W(w) = \begin{cases} 0 & \text{if } w \le 0 \\ \frac{1}{4}w^2 & \text{if } 0 \le w \le 2 \\ 1 & \text{if } w \ge 2 \end{cases}.$$

Differentiating, we get

$$f_W(w) = \frac{d}{dw}F_W(w) = \begin{cases} \frac{1}{2}w & \text{if } 0 \le w \le 2 \\ 0 & \text{else} \end{cases}.$$

e) Let $(W, Y) = \varphi(X, Y) = (X + Y, Y)$. Note that $x = w - y$. We have

$$J(\varphi) = \det \begin{pmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{pmatrix} = \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1$$

so by the transformation theorem,

$$f_{W,Y}(w, y) = \frac{1}{|J(\varphi)|} f_{X,Y}(x, y) = f_{X,Y}(w - y, y) = \frac{1}{2}.$$

The rest of this problem is to figure out the range of W and Y. We know that since $W = X+Y$, $0 \le W \le 2$ and since $X \ge 0$, $0 \le Y \le X+Y = W$. Thus we have

$$f_{W,Y}(w, y) = \begin{cases} \frac{1}{2} & \text{if } 0 \le w \le 2, 0 \le y \le w \\ 0 & \text{else} \end{cases}.$$

Bonus. It will take you 1 cast to catch the first type of fish.

Now define "success" to be catching a type of fish other than the type you caught on the first cast. After the first catch, your probability of success is $\frac{3}{4}$ so if you let X_1 be the number of failures before the first success, $X_1 \sim Geom(\frac{3}{4})$ so $EX_1 = \frac{1-3/4}{3/4} = \frac{1}{3}$ and $Var(X_1) = \frac{1-3/4}{(3/4)^2} = \frac{4}{9}$.

After the X_1 failures, you will use 1 cast to catch the second type of fish.

Now redefine "success" to be catching a type of fish you haven't yet caught; since you have now caught two types of fish, your probability of success is $\frac{1}{2}$ so if you let X_2 be the number of failures before the next success, $X_2 \sim Geom(\frac{1}{2})$ so $EX_2 = \frac{1-1/2}{1/2} = 1$ and $Var(X_2) = \frac{1-3/4}{(1/2)^2} = 2$.

After the X_2 failures, you will use 1 cast to catch the third type of fish.

Now, again redefine "success" to be catching a type of fish you haven't yet caught; since you have now caught three types of fish, your probability of success is $\frac{1}{4}$ so if you let X_3 be the number of failures before you catch the last type of fish, $X_3 \sim Geom(\frac{1}{4})$ so $EX_3 = \frac{1-1/4}{1/4} = 3$ and $Var(X_3) = \frac{1-1/4}{(1/4)^2} = 12$.

Last, it will take you 1 final cast to catch the last type of fish.

Putting this all together, we have $X = 1 + X_1 + 1 + X_2 + 1 + X_3 + 1 = X_1 + X_2 + X_3 + 4$. This means

$$EX = EX_1 + EX_2 + EX_3 + 4 = \frac{1}{3} + 1 + 3 + 4 = \frac{25}{3}.$$

Since X_1, X_2, X_3 all have to do with separate casts, they are independent, so their variances add (and the constant an be dropped since it only shifts the random variable), so

$$Var(X) = Var(X_1) + Var(X_2) + Var(X_3) = \frac{4}{9} + 2 + 12 = \frac{130}{9}.$$