1. Consider a Markov chain with state space $\mathcal{S} = \{0,1\}$, where $p = P(0,1)$ and $q = P(1,0)$; compute the following in terms of p and q:
   (a) $P(X_3 = 0 \,|\, X_2 = 0)$
   (b) $P(X_0 = 0 \,|\, X_1 = 0)$
   (c) $P(X_1 = 0 \,|\, X_0 = X_2 = 0)$

2. The weather in a city is always one of two types: rainy or dry. If it rains on a given day, then it is 25% likely to rain again on the next day. If it is dry on a given day, then it is 10% likely to rain the next day. If it rains today, what is the probability it rains the day after tomorrow?

3. Suppose we have two boxes and $2d$ marbles, of which d are black and d are red. Initially, d of the balls are placed in Box 1, and the remainder are placed in Box 2. At each trial, a ball is chosen uniformly from each of the boxes; these two balls are put back in the opposite boxes. Let X_0 denote the number of black balls initially in Box 1, and let X_t denote the number of black balls in Box 1 after the t^{th} trial. Find the transition function of the Markov chain $\{X_t\}$.

4. A Markov chain on the state space $\mathcal{S} = \{1,2,3,4,5\}$ has transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

   and initial uniform distribution on $\mathcal{S}$.
   (a) Find the distribution of X_2.
   (b) Find $P(X_3 = 5 \,|\, X_2 = 4)$.
   (c) Find $P(X_4 = 2 \,|\, X_2 = 3)$.
   (d) Find $P(X_4 = 5, X_3 = 2, X_1 = 1)$.
   (e) Find $P(X_8 = 3 \,|\, X_7 = 1 \text{ and } X_9 = 5)$

5. Consider the Ehrenfest chain with $d = 4$.
   (a) Find P, P^2 and P^3.
   (b) If the initial distribution is uniform, find the distributions at times $1, 2$ and 3.
   (c) Find $P_x(T_0 = n)$ for all $x \in \mathcal{S}$ and when $n = 1, 2$ and 3.

6. A dysfunctional family has six members (named Al, Bal, Cal, Dal, Eal, and Fal) who have trouble passing the salt at the dinner table. The family sits around a circular table in clockwise alphabetical order. This family has the following quirks:

   - Al is twice as likely to pass the salt to his left than his right.
   - Cal and Dal alway pass the salt to their left.
   - All other family members pass the salt to their left half the time and to their right half the time.

   (a) If Al has the salt now, what is the probability Bal has the salt 3 passes from now?

   (b) If Al has the salt now, what is the probability that the first time he gets it back is on the 4th pass?

   (c) If Bal has the salt now, what is the probability that Eal can get it in at most 4 passes?

7. Consider the Markov chain with $\mathcal{S} = \{1, 2, 3\}$ whose transition matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1-p & 0 & p \\ 0 & 1 & 0 \end{pmatrix}.$$

   (a) Find P^2.

   (b) Show $P^4 = P^2$.

   (c) Find P^n for all $n \geq 1$.

   (d) If the initial distribution is $(\frac{2}{5}, \frac{2}{5}, \frac{1}{5})$, find the time 200 distribution.

   (e) If the initial distribution is $(\frac{2}{5}, \frac{2}{5}, \frac{1}{5})$, find the time 111111 distribution.

8. Consider a Markov chain with state space $\mathcal{S} = \{0, 1\}$, where $p = P(0, 1)$ and $q = P(1, 0)$. (Assume that neither p nor q are either 0 or 1.) Compute, for each n, the following in terms of p and q:

   (a) $P_0(T_0 = n)$

   (b) $P_1(T_0 = n)$

   (c) $P_0(T_1 = n)$

   (d) $P_1(T_1 = n)$

   (e) $P_0(T_0 < \infty)$
   Hint: Add up the values of $P_x(T_0 = n)$ from $n = 1$ to ∞.

(f) $P_1(T_0 < \infty)$

(g) $P_0(T_1 < \infty)$

(h) $P_1(T_1 < \infty)$

9. Consider a Markov chain with state space $\mathcal{S} = \{0, 1, 2, 3, ...\}$ such that $P(x, x+1) = p$ for all $x \in \mathcal{S}$ and $P(x, 0) = 1 - p$ for all $x \in \mathcal{S}$.

   (a) Explain why this chain is irreducible by showing, for arbitrary states x and y, a sequence of steps which could be followed to get from x to y.

   (b) Determine whether this chain is recurrent or transient.

10. Consider a Markov chain whose state space is $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7\}$ and whose transition matrix is

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

   (a) Find all closed subsets of $\mathcal{S}$.

   (b) Find all communicating classes.

   (c) Determine which states are recurrent and which states are transient.

   (d) Find $f_{x,y}$ for all $x, y \in \mathcal{S}$.

11. Consider a Markov chain with state space $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ whose transition matrix is

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

   (a) Determine which states are recurrent and which states are transient.

   (b) Find $f_{x,1}$ for all $x \in \mathcal{S}$.

12. Consider a Markov chain with state space $\mathcal{S} = \{0, 1, 2, 3, ...\}$ and transition function defined by

$$P(x, y) = \begin{cases} \frac{1}{2} & \text{if } x = y \\ \frac{1}{2} & \text{if } x > 0 \text{ and } y = x - 1 \\ \left(\frac{1}{2}\right)^{y+1} & \text{if } x = 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases}.$$

    Show this Markov chain is irreducible. Is this chain recurrent or transient?

13. Consider a Markov chain with state space $\mathcal{S} = \{0, 1, 2, 3, ...\}$ and transition function defined by

$$P(x, y) = \begin{cases} \frac{1}{7} & \text{if } y = 0 \\ \frac{2}{7} & \text{if } y \in \{x + 2, x + 4, x + 6\} \\ 0 & \text{otherwise} \end{cases}.$$

    Classify the states of this Markov chain as recurrent or transient, and find all communicating classes (if any).

14. Prove Lemma 2.24 in the lecture notes.

15. Prove that $\{Z_t\}$ is a martingale, where $\{Z_t\}$ is as in Lemma 2.27 of the lecture notes.

16. A gambler makes a series of independent $\$1$ bets. He decides to quit betting as soon as his net winnings reach $\$25$ or his net losses reach $\$50$. Suppose the probabilities of his winning and losing each bet are each equal to $\frac{1}{2}$.

    (a) Find the probability that when he quits, he will have lost $\$50$.

    (b) Find the expected amount he wins or loses.

    (c) Find the expected number of bets he will make before quitting.

17. A typical roulette wheel has 38 numbered spaces, of which 18 are black, 18 are red, and 2 are green. A gambler makes a series of independent $\$1$ bets, betting on red each time (such a bet pays him $\$1$ if the ball in the roulette wheel ends up on a red number). He decides to quit betting as soon as his net winnings reach $\$25$ or his net losses reach $\$50$.

    (a) Find the probability that when he quits, he will have lost $\$50$.

    (b) Find the expected amount he wins or loses.

    (c) Find the expected number of bets he will make before quitting.

18. Suppose two friends, George the Genius and Ichabod the Idiot, play a game that has some elements of skill and luck in it. Because George is better at the game than Ichabod, George wins 55% of the games they play and Ichabod wins the other 45% (the result of each game is independent of each other game). Suppose George and Ichabod both bring $\$100$ to bet with, and they agree to play until one of them is broke.

    (a) Suppose George and Ichabod wager $\$1$ on each game. What is the probability that George ends up with all the money?

    (b) Suppose George and Ichabod wager $\$5$ on each game. What is the probability that George ends up with all the money?

    (c) Suppose George and Ichabod wager $\$25$ on each game. What is the probability that George ends up with all the money?

    (d) Suppose George and Ichabod wager $\$100$ on each game. What is the probability that George ends up with all the money?

    (e) Based on the answers to parts (a),(b) and (c), determine which of the following statements is true:

    Statement I: The more skilled player benefits when the amount wagered on each game increases.

    Statement II: The more skilled player is harmed when the amount wagered on each game increases.

    (f) Suppose you had $\$1000$ and needed $\$2000$ right away, and you therefore decided to go to a casino and turn your $\$1000$ into $\$2000$ by gambling on roulette. In light of your answer to the previous question, which of these strategies gives you the highest probability of ending up with $\$2000$: betting $\$1000$ on red on one spin of the wheel, or betting $\$1$ on red repeatedly, trying to work your way up to $\$2000$ without going broke first?

19. Prove Wald's Third Identity (Theorem 2.31 in the lecture notes).

20. Finish the proof of Gambler's Ruin by writing out the case where $a < x$.

21. Consider an irreducible, simple random walk X_t starting at zero, where $r = 0$.

    (a) Find the probability that $X_t = -2$ for some $t > 0$.

    (b) Find p such that $P(X_t = 4 \text{ for some } t > 0) = \frac{1}{2}$.

22. Consider a Markov chain with state space $\{1, 2, 3\}$ whose transition matrix is

$$\begin{pmatrix} .4 & .4 & .2 \\ .3 & .4 & .3 \\ .2 & .4 & .4 \end{pmatrix}.$$

Find all stationary distributions of this Markov chain.

23. Let $\{X_t\}$ be a Markov chain that has a stationary distribution π. Prove that if $\pi(x) > 0$ and $x \to y$, then $\pi(y) > 0$.

24. Find all stationary distributions of the Ehrenfest chain where $d = 5$.

25. Find the stationary distribution of a Markov chain with transition function

$$P = \begin{pmatrix} 1/9 & 0 & 4/9 & 4/9 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 2/9 & 2/9 & 2/9 & 1/3 & 0 \\ 0 & 0 & 1/9 & 8/9 & 0 \\ 0 & 0 & 7/9 & 2/9 & 0 \end{pmatrix}.$$

Hint: The answer is $\pi = \left(\frac{9}{221}, \frac{8}{221}, \frac{36}{221}, \frac{160}{221}, \frac{8}{221}\right)$.

26. Find the stationary distribution of the Markov chain described in problem 3.

Hint: You will need the following identity, which can be assumed without proof:

$$\sum_{j=0}^{d} \binom{d}{j}^2 = \binom{2d}{d}.$$

27. Show that the Markov chain introduced in problem 7 has a unique stationary distribution (and find this stationary distribution).

28. A transition matrix of a Markov chain is called **doubly stochastic** if its columns add to 1 (recall that for any transition matrix, the rows must add to 1). Find a stationary distribution of a finite state-space Markov chain with a doubly stochastic transition matrix (the way you do this is by "guessing" the answer, and then showing your guess is stationary).

Note: It is useful to remember the fact you prove in this question.

29. Let $\pi_1, \pi_2, ...,$ be a finite or countable list of stationary distributions for a Markov chain $\{X_t\}$. Let $\alpha_1, \alpha_2, ...$ be nonnegative numbers whose sum is 1, and let $\pi = \sum_j \alpha_j \pi_j$. Prove π is a stationary distribution for $\{X_t\}$.

30. Let $\{X_t\}$ be an irreducible, aperiodic, positive recurrent Markov chain with state space $\mathcal{S}$ and unique stationary distribution π (the initial distribution of $\{X_t\}$ need not be the stationary one, however). Let $\{Y_t\}$ be a Markov chain, independent of $\{X_t\}$ with the same state space and transition function as $\{X_t\}$, where the initial distribution of $\{Y_t\}$ is the stationary distribution π.

Let $b \in \mathcal{S}$ and let $T = \min\{t \geq 1 : X_t = Y_t = b\}$ (if there is no such t, set $T = \infty$. Prove that $P(T < \infty) = 1$.

Hints: The first step would be to prove a lemma which says that if $\{X_t\}$ is aperiodic, then for every pair of states x and y, there is a number N such that $P^n(x,y) > 0$ for all n larger than N. This lemma comes from a basic number theory argument and can be assumed without proof here.

Next, construct a new Markov chain $\{Z_t\}$ with state space $\mathcal{S} \times \mathcal{S}$ by setting $Z_t = (X_t, Y_t)$, where the coordinates are independent. This makes $\{Z_t\}$ a Markov chain with transition probabilities defined by $P_Z((x,y),(x',y')) = P_X(x,x')P_Y(y,y')$. Explain why $\{Z_t\}$ is positive recurrent by describing a stationary distribution for $\{Z_t\}$. Then explain why $\{Z_t\}$ is irreducible (use the result of the lemma above which you don't have to prove). Last, explain why all this implies that $P(T < \infty) = 1$.

31. Consider the Markov chain introduced in Problem 10.
    (a) Find the stationary distribution concentrated on each of the communicating classes.
    (b) Characterize all stationary distributions of the Markov chain.
    (c) For each $x, y \in \mathcal{S}$, find $\lim\limits_{n\to\infty} \frac{E_x(V_{n,y})}{n}$.

32. Consider the Markov chain with state space $\{1,2,3\}$ whose transition matrix is

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

    (a) Find the period of this Markov chain.
    (b) Find the stationary distribution.
    (c) Is this distribution steady-state?
    (d) Suppose the initial distribution is uniform on $\mathcal{S}$. Estimate the time 10^6 distribution.
    (e) Find $\lim\limits_{n\to\infty} P^n$.

33. Find the Cesàro limit of the sequence of numbers $\{0,1,0,1,0,1,...\}$ (justify your answer).

34. Consider the Markov chain with state space $\{1,2,3,4,5\}$ whose transition matrix is

$$\begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find the period of this Markov chain.

(b) Find the stationary distribution.

(c) Is this distribution steady-state?

(d) Describe P^n for n large (there is more than one answer depending on the relationship n and the period d).

(e) Suppose the initial distribution is uniform on $\mathcal{S}$. Estimate the time n distribution for large n (there are cases depending on the value of n).

(f) Find $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} P^k$.

(g) Find m_1 and m_2.

35. Fix nonnegative constants $p_0, p_1, ...$ such that $\sum\limits_{y=0}^{\infty} p_y = 1$ and let X_t be a Markov chain on $\mathcal{S} = \{0, 1, 2, \cdots\}$ with transition function P defined by

$$P(x,y) = \begin{cases} p_y & \text{if } x = 0 \\ 1 & \text{if } x > 0, y = x - 1 \\ 0 & \text{else} \end{cases}$$

(a) Show this chain is recurrent.

(b) Calculate, in terms of the p_y, the mean return time to 0.

(c) Under what conditions on the p_y is the chain positive recurrent?

(d) Suppose this chain is positive recurrent. Find $\pi(0)$, the value that stationary distribution assigns to state 0.

(e) Suppose this chain is positive recurrent. Find the value the stationary distribution π assigns to an arbitrary state x.

36. Let $\{X_t\}$ be the Ehrenfest chain with $d = 4$ and $X_0 = 0$ (i.e. there are no particles in the left-hand chamber).

(a) Find the approximate distribution of X_t when t is large and even.

(b) Find the approximate distribution of X_t when t is large and odd.

(c) Find the expected amount of time until there are again no particles in the left-hand chamber.

37. Consider a Markov chain on $\mathcal{S} = \{0, 1, 2, 3\}$ with transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{5} & 0 & \frac{4}{5} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{5} & 0 & \frac{4}{5} & 0 \end{pmatrix}.$$

(a) Find the Cesàro limit of P^n.

(b) Find m_0 and m_2.

38. Consider a Markov chain $\{X_t\}$ on $\mathcal{S} = \{0, 1, 2, ...\}$ with transition function

$$P(x, y) = \begin{cases} 2^{-y-1} & \text{if } x \leq 3 \\ 1/4 & \text{if } x > 3 \text{ and } y \leq 3 \\ 0 & \text{if } x > 3 \text{ and } y > 3 \end{cases}$$

(a) Show the chain is positive recurrent. *Hint:* Consider a Markov chain $\{Y_t\}$ defined by $Y_t = X_t$ if $X_t \leq 3$ and $Y_t = 4$ if $X_t \geq 4$. Show $\{Y_t\}$ is positive recurrent; why does this imply $\{X_t\}$ is positive recurrent?

(b) Find all stationary distributions of $\{X_t\}$. *Hint:* The stationary distribution of Y_t (from part (a)) tells you something about the stationary distribution of X_t.

(c) Suppose you start in state 2. How long would you expect it to take for you to return to state 2 for the fifth time?

39. Suppose a fair die is thrown repeatedly. Let S_n represent the sum of the first n throws. Compute

$$\lim_{n \to \infty} P(S_n \text{ is a multiple of } 13),$$

justifying your reasoning.

40. Your professor owns 3 umbrellas, which at any time may be in his office or at his home. If it is raining when he travels between his home and office, he carries an umbrella (if possible) to keep him from getting wet.

(a) If on every one of his trips, the probability that it is raining is p, what is the long-term proportion of journeys on which he gets wet?

(b) What p as in part (a) causes the professor to get wet most often?

41. Prove that

$$\lim_{n \to \infty} \left(1 + \frac{t}{n}\right)^n = e^t.$$

Hint: Rewrite the expression inside the limit using natural exponentials and logarithms, then use L'Hôpital's Rule.

42. Consider a continuous-time Markov chain $\{X_t\}$ with with state space $\{0, 1, 2\}$ and $Q-$matrix (a.k.a. infinitesimal matrix)

$$Q = \begin{pmatrix} -4 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{pmatrix}.$$

(a) Find the time t transition matrix $P(t)$.

(b) Find the jump matrix Π.

(c) Find the probability that $X_5 = 2$ given that $X_2 = 2$.

(d) Suppose the initial distribution is $\pi_0 = (1/8, 1/2, 3/8)$. What is the distribution at time 2?

(e) Classify the recurrent and transient states. Are the recurrent states positive recurrent or null recurrent?

(f) Find all stationary distributions. Are any of them steady-state?

43. Suppose $\{X_t\}$ is a continuous-time Markov chain with state space $\{0, 1\}$ and time t transition matrix

$$P(t) = \frac{1}{9}\begin{pmatrix} 1 + 6te^{-3t} + 8e^{-3t} & 6 - 6e^{-3t} & 2 - 6te^{-3t} - 2e^{-3t} \\ 1 - 3te^{-3t} - e^{-3t} & 6 + 3e^{-3t} & 2 + 3te^{-3t} - 2e^{-3t} \\ 1 + 6te^{-3t} - e^{-3t} & 6 - 6e^{-3t} & 2 - 6te^{-3t} + 7e^{-3t} \end{pmatrix}.$$

(a) Find the infinitesimal matrix of this process.

(b) What is the probability that $X_2 = 0$, given that $X_0 = 0$?

(c) What is the probability that $X_t = 0$ for all $t < 2$, given that $X(0) = 0$?

(d) Find the steady-state distribution π.

(e) Find the mean return time to each state.

(f) Suppose you let time pass from $t = 0$ to $t = 1,200,000$. What is the expected amount of time in this interval for which $X(t) = 1$?

44. Suppose $\{X_t\}$ is a continuous-time Markov chain with finite state space $\mathcal{S}$ and infinitesimal matrix Q.

(a) Prove that if π is stationary (i.e. $\pi P(t) = \pi$ for all $t \geq 0$), then $\pi Q = \mathbf{0}$.

(b) Prove that if $\pi Q = \mathbf{0}$, then π is stationary.

45. Consider a continuous-time Markov chain $\{X_t\}$ with state space $\mathcal{S} = \{1, 2, 3\}$ with holding rates $q_1 = q_2 = 1, q_3 = 3$ and jump probabilities $\pi_{13} = \frac{1}{3}$, $\pi_{23} = \frac{1}{4}$ and $\pi_{31} = \frac{2}{5}$.

(a) Use linear approximation to estimate $P_{31}(.001)$ and $P_{22}(.06)$.

(b) What is the probability that $X_t = 2$ for all $t < 4$, given that $X_0 = 2$?

(c) What is the probability that your first two jumps are to states 3 and 2, given that you start in state 1?

(d) Find all stationary distributions. Are any of them steady-state?

46. Let $\{X_t\}$ be an irreducible birth-death chain with $\mathcal{S} = \{0, 1, 2, 3, ...\}$. Show that if for all $x \geq 1$, $p_x \leq q_x$, then the chain is recurrent.

47. Let $\{X_t\}$ be an irreducible birth-death chain with $\mathcal{S} = \{0, 1, 2, 3, ...\}$ such that

$$\frac{q_x}{p_x} = \left(\frac{x}{x+1}\right)^2 \qquad \text{for all } x \geq 1.$$

(a) Is this chain recurrent or transient?

(b) Compute $f_{x,0}$ for all $x \geq 1$. *Hint:* $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

48. Consider a birth and death chain on $\mathcal{S} = \{0, 1, 2, ...\}$ with

$$p_x = \frac{1}{2^{x+1}} \, \forall x; \; q_x = \frac{1}{2^{x-1}} \, \forall x > 1; \; q_1 = \frac{1}{2}.$$

Show this chain is positive recurrent, find the stationary distribution, and find the mean return time to state 2.

49. Prove Lemma 5.3 in the lecture notes.

50. Consider a pure death process on $\{0, 1, 2, ...\}$.

(a) Write the forward equation of this process.

(b) Find $P_{x,x}(t)$.

(c) Solve the differential equation from part (a) for $P_{x,y}(t)$ in terms of $P_{x,y+1}(t)$.

(d) Find $P_{x,x-1}(t)$.

51. Consider a birth-death process $\{X_t\}$ with $\mathcal{S} = \{0, 1, 2, 3, ...\}$ where $\lambda_x = \lambda x$ and $\mu_x = \mu x$ for constants $\lambda, \mu \geq 0$.

(a) Write the forward equation of this process.

(b) Let $g_x(t) = E_x(X_t)$. Use the forward equation to show $g_x'(t) = (\lambda - \mu)g_x(t)$.

(c) Find a formula for $g_x(t)$.

52. Consider a birth-death process $\{X_t\}$ on $\{0, 1, 2, 3, ...\}$ whose death rates are given by $\mu_x = x$ for all $x \in \mathcal{S}$.

(a) Determine whether the process is transient, null recurrent or positive recurrent if the birth rates are $\lambda_x = x + 1$ for all $x \in \mathcal{S}$;

(b) Determine whether the process is transient, null recurrent or positive recurrent if the birth rates are $\lambda_x = x + 2$ for all $x \in \mathcal{S}$.

53. Consider a discrete-time branching chain where the number of offspring is geometric with parameter p. Compute the extinction probability (there will be different cases depending on the value of p).

54. Consider a population of creatures with the following rule of (asexual) reproduction: an individual that is born has probability q of surviving long enough to produce offspring. If the individual does produce offspring, it produces one or two offspring (with equal probability), then dies.

    (a) For which values of q is it guaranteed that the population will eventually die out?

    (b) If $q = .99$, what is the probability that the population lives forever, if there are four creatures alive initially?

55. Suppose $A_1, ..., A_d$ are independent exponential r.v.s with respective parameters $\lambda_1, ..., \lambda_d$. Prove that $M = \min(A_1, ..., A_d)$ is exponential with parameter $\lambda_1 + ... + \lambda_d$.

    Hint: This is a transformation problem; the first step is to compute $F_M(m) = P(M \leq m)$. It is easiest to compute this probability by computing the probability of its complement.

56. Suppose d particles are distributed into two boxes, A and B. Each particle in box A remains in that box for a random length of time that is exponentially distributed with parameter μ before moving to box B. Each particle in box B remains in that box for a random length of time that is exponentially distributed with parameter λ before moving to box A. All particles act independently of one another. For each $t \geq 0$, let X_t be the number of particles in box A at time t. Then $\{X_t\}$ is a birth-death process on $\mathcal{S} = \{0, 1, 2, ..., d\}$.

    (a) This setup be thought of as a continuous version of what discrete-time Markov chain?

    (b) Find the birth and death rates.

    (c) Find $P_{x,d}(t)$ for all $x \in \mathcal{S}$. *Hint:* Think of each particle as generating its own CTMC, where state zero corresponds to being in box B and state 1 corresponds to being in box A. This is a two-state CTMC, so its transition probabilities were derived in class. From these transition probabilities, you can get the probability that any one fixed particle is in box A at time t. Multiply these together to get $P_{x,d}(t)$.

(d) Find $E_x(X_t)$. *Hint:* Write $X_t = A_t + B_t$ where A_t is the number of particles in box A that started in box A and B_t is the number of particles in box A at time t that started in box B. If $X_0 = x$, then A_t and B_t are both binomial, defined in terms of x and the transition function of the two-state birth-death process described in the hint for part (c).

(e) Compute the steady-state distribution for this process; identify this distribution as a common r.v. (stating the parameters).

(f) Verify that as $t \to \infty$, $E_x(X_t)$ converges to the expected value of the steady-state distribution.

57. Suppose customers call a technical support line according to a Poisson process with parameter $\lambda > 0$. They are provided with technical support by N agents where N is a positive integer (N is a constant, not a r.v.). Suppose that the amount of time it takes an agent to solve a customer's problem is exponentially distributed with parameter μ (and that these times are independent of the Poisson process and all independent of one another). Last, assume that whenever there are more than N customers calling the technical support line, the excess customers get placed on hold until one of the N agents is available. Let X_t represent the number of people on the phone with technical support (including those on hold) at time t. $\{X_t\}$ is called the $N-$*server queue* or the $(M/M/N)-$*queue*.

(a) Explain why $\{X_t\}$ is a birth and death process.

(b) Find the birth and death rates of $\{X_t\}$.

(c) Show that $\lambda < N\mu$ if and only if $\{X_t\}$ is positive recurrent.

(d) Show that $\lambda > N\mu$ if and only if $\{X_t\}$ is transient.

58. Suppose $\{W_t\}$ is a Brownian motion with parameter $\sigma^2 = 3$.

(a) Find $P(W_4 \geq 1)$.

(b) Find $P(W_9 - W_2 \leq -2)$.

(c) Find $P(W_8 < 2 \,|\, W_2 = W_1 = 3)$.

(d) Find $P(W_7 > W_5)$.

(e) Find the variance of W_8.

(f) Find $Cov(W_3, W_7)$.

(g) Find $Var(W_8 + W_9)$.

59. You own one share of stock whose price is approximated by a Brownian motion with parameter $\sigma^2 = 10$ (time $t = 1$ here corresponds to the passage of one day). You bought the stock when its price was $\$15$, but now it is worth

$20. You decide to sell the stock the next time its price hits $15 or after ten days, whichever happens first. What is the probability that when you sell your stock, you will have to sell it for $15?

60. Let $\{W_t\}$ be a Brownian motion with parameter σ^2 and let $M(t) = \max\{W_s : 0 \leq s \leq t\}$. Show $M(t)$ is a continuous r.v. (this implies $M(t) > 0$ with probability one) and find the density function of $M(t)$.

61. Prove Theorem 6.15 in the lecture notes.

62. Prove Corollary 6.16 in the lecture notes.

63. Suppose that the value of a commodity is measured by a Brownian motion with drift with $\sigma^2 = 2$ and $\mu = \frac{1}{5}$. If the commodity is presently valued at $30, what is the probability that its value reaches $45 before its value drops to $20?

64. Prove that if $\{W_t\}$ and $\{\widehat{W_t}\}$ are independent Brownian motions with respective parameters σ^2 and $\widehat{\sigma}^2$, then for any constants b_1 and b_2, the process $\{b_1 W_t + b_2 \widehat{W_t}\}$ is also a Brownian motion. Find its parameter in terms of b_1, b_2, σ and $\widehat{\sigma}$.

    Note: The result of this problem generalizes: any linear combination of a finite number of independent BMs is also a BM (although you don't have to prove this).

65. Let $\{W_t\}$ be a Brownian motion with parameter σ^2 and let $a \leq s$ and $a \leq t$. Prove that
$$E[(W_s - W_a)(W_t - W_a)] = \sigma^2 \min(s - a, t - a).$$

66. Let $\{W_t\}$ be standard Brownian motion; find the mean and covariance functions of $\{X_t\}$ in each of these cases:

    (a) $X_t = -W_t$
    (b) $X_t = tW_{1/t}$
    (c) $X_t = W_t - tW_1$
    (d) $X_t = e^{-\alpha t} W_{\exp(2\alpha t)}$ (where α is a constant)
    (e) $X_t = aW_{t/a^2}$ (where a is a positive constant)

67. Which of the processes $\{X_t\}$ in Problem 66 are also standard Brownian motions? (You may assume without proof that all the processes in # 66 are Gaussian.)

68. Let $\{W_t\}$ be standard Brownian motion and let $X_t = (W_t)^2$ for all t.

(a) Is $\{X_t\}$ a Gaussian process? Explain your answer.

(b) Find the mean function of $\{X_t\}$.

(c) Find the joint moment generating function of W_s and W_t.

(d) Use your answer to part (b) to find $E[W_s^2 W_t^2]$.

(e) Find the covariance function of $\{X_t\}$.

69. Prove Theorem 6.21 in the lecture notes.

70. Give an example of a stochastic process $\{X_t\}$ whose mean function is $\mu_X(t) = 0$ and whose covariance function is $r_X(s,t) = \min(s,t)$, but which is not a Brownian motion.

71. Let $\{W_t\}$ and $\{\widehat{W_t}\}$ be independent, standard BMs and let a be a positive constant.

(a) Prove that $P(W_t = a\widehat{W_t} \text{ for infinitely many } t) = 1$.

(b) What is the probability that $W_t = \widehat{W_t} + a$ for infinitely many t? Prove your answer.